A System for Modeling Strategy Change, Demonstrated with the Ultimatum Game

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Abstract: A system has been created that provides a new framework for analyzing Ultimatum Games based on modeling strategy change throughout a game. It models strategy change of individuals, groups, or populations and characterizes dynamics by quantifying newly-defined rates, accelerations, and stability indices of strategy change. To better model real situations, the system has been generalized to ultimatum bargaining games with multi-dimensional strategies. It also applies to experimental environments.

1. INTRODUCTION

1.1. BACKGROUND

A system has been created whose purpose is to provide a new framework for analyzing Ultimatum Games based on modeling strategy *change throughout* a game. This approach fundamentally differs from the more common focus of game theory on strategies and outcomes. It models strategy change of individuals, groups, or populations and characterizes dynamics by quantifying newly-defined rates, accelerations, and stability of strategy change.

To better model real situations, the system is generalized to ultimatum bargaining games with multi-dimensional strategies. The framework also applies to experimental environments. The Ultimatum Game, selected for its simplicity, models bargaining between two entities, e.g., people, businesses, or nations (Guth, Schmittberger, & Schwarze, 1982).

This system describes strategy change without using dynamical systems or differential equations. It is a simpler approach that is based on an essential aspect of strategy change: an entity cannot have an instantaneous rate of strategy change because it must employ a strategy at all times.

Smith discusses rates (1976), but only in terms of rates of evolution, which describe a population's fitness relative to its environment. The system described in this paper focuses on strategy change irrespective of whether fitness, in evolutionary terms, improves or not. This work also differs from replicator dynamics (Hofbauer, & Sigmund, 1998), a mathematical tool of evolutionary game theorists, because the system assumes a finite, not infinite, population and focuses on the entirety of a game, not just its outcomes or interim strategies. Maximum stability, defined later in this paper, is equivalent to the classic stability state.

This work is organized into seven sections plus references. In the first section, the Ultimatum Game introduces the definitions for a strategy dimension and basic game parameters. The fundamental parameters and theorems underlying the analytical system are then developed; these establish the new concepts of strategy points and the strategy graph. Next, the construction of a strategy graph is outlined. Section four introduces the new properties of strategy distance, rate, and acceleration, and new definitions for stability. The fifth section describes Ultimatum Game experiments that were conducted to demonstrate these properties. The Discussion section highlights significant ramifications and applications of the system. The paper closes with a summary of conclusions and ideas for future research based on the unique perspectives offered by the system.

1.2. The Ultimatum Game

The Ultimatum Game.is played by a "Proposer" and a "Responder," who aim to split a total "pie" (Π) or a pot of money. The game begins with the Proposer temporarily getting the entire pie and the Responder getting nothing. The Proposer then makes an offer of part of the pie to the Responder. The Responder can accept or reject this offer. If the Responder accepts the offer, the pie is split accordingly. However, if the Responder rejects the offer, neither player gets anything in that round, i.e., both players receive a payoff of zero from that pie. (Guth, Schmittberger, & Schwarze, 1982).

Strategies to a given game have one or more *dimensions*, where each dimension describes a category of action chosen by a player during the game. An Ultimatum Game strategy is two-dimensional, having an offer value, called a p value, and an acceptance limit, called a q value (Shayan, 2003). Thus, the notation for an Ultimatum Game strategy is (p,q).

In an Ultimatum Game, the experimenter sets certain parameters for players. ε is the minimum amount that a player can offer or accept (Guth, Schmittberger, & Schwarze, 1982), which is set at $\varepsilon > 0$ for this work. Another parameter, increment of play, i in element-units, describes the increment of the unit allowed for an offer or acceptance, such as amounts rounded to one hundred dollar increments.

2. THE ANALYTICAL SYSTEM – FUNDAMENTALS

The analytical system deals with a finite number of pure strategies for a given game. *Strategy points* are graphable abstractions that contain strategies, based on the following definitions:
1. Dimensions and sets: A strategy point is composed of sets, with one set per dimension. Each set consists of a category of actions – up to all the actions – for its dimension. A particular strategy at a point is specified by one element from each dimensional set.
2. Size of sets: Each dimensional set has the same size, m elements per set. $\therefore$ Strategy points for a given game all contain a uniform number of elements. $m \geq 1$, $m \in$ integers – or else the strategy points would cease to exist. *m-space* is the set size and width of a point in element-units, based on the difference between the last and first element of a dimension or = i for $m = 1$. *m-space* $\geq i$. *m-space* shows how much strategy space is contained in a point.
3. Element order: All elements that can be in a dimension are sequenced in an ascending arithmetic progression within each dimensional set for $m > 1$, with difference = i element-units. If the difference were smaller than i, then some strategies in that dimension would be unavailable to players. If it were larger than i, then usable strategies would not be accounted for.
4. Element correspondence: Each element in a dimensional set has a corresponding element by position in every other dimensional set for every point (from definitions 2 and 4).

The aggregation of strategy points for a game comprises the strategy map, also known as the *strategy graph*, for that game. The dimensional sets that define each strategy point are the coordinates of that point on the strategy graph. The coordinates for strategy point R_x in the Ultimatum Game are (p_x , q_x). A valid strategy graph of a game is complete, i.e., all possible strategies of the game must be accounted for and so must exist somewhere within the strategy points. The graph depicts all strategies, meaning that it is player-independent – players are mapped to strategy points on the strategy graph.

To determine the number of points in a strategy graph, let:

- Ω be the set of all strategies; Ω_x be the number of strategy points necessary to represent all elements of set Ω on a complete and valid strategy graph for a particular game.
- z be the number of distinct sets that can become any dimensional set $\forall$ strategy points in a game
- y be the number of dimensions to a strategy point, such that $\Omega_x , z , y \in$ Integers and $\Omega_x , z , y > 0$

***Theorem** 1:* $\Omega_x = z^y$

Proof: This is a permutation with replacement. $z \bullet z \bullet z ... y$ times = z^y each dimension offers z possibilities. Distinct strategy points are generated ∍ ∃ z sets for each dimension and each dimension is independent. ∴ $\Omega_x = z^y$ □

$\Omega_x = z^2$ in the standard Ultimatum Game because the Ultimatum Game always has two-dimensional strategies, (p , q). If m grows to where all elements are in one set, then $z = 1$ ∴ $\Omega_x = 1$ strategy point.

z can either be specified or derived. Let:

- Π be the maximum value of an element in a game, in element-units in the Ultimatum Game.
- $\frac{\pi}{i}$ = the total number of elements for the game; $\frac{\pi}{i} \geq m$; ε > 0; $m, z, y, \Omega_x \in$ integers

Theorem 2: $(\Omega_x)^{1/y} = \frac{\pi}{im}$

Proof: Since $\frac{\pi}{i}$ = number of elements in a game; divide number elements by number elements per set:

$$\frac{\pi}{im} = z \text{ , so } z^y = \frac{\pi^y}{(im)^y} \text{ and } \Omega_x = z^y \therefore (\Omega_x)^{1/y} = \frac{\pi}{im}$$ □

The incremental change between strategy points is measured by a distance unit. Let:

- 1 *strategy meter =* the magnitude *of m-space,* measured between corresponding elements
- y_d = any dimension of strategy point R; y_{dm} = incremental change of m units to y_d
- $y_d \pm y_{dm}$ are both 1 strategy meter away from y_d
- *A, B* be different points on a strategy graph, which are one strategy meter apart
- λ = {a_1,…,a_m} and ρ = {b_1,…,b_m} are sets in the same dimension for points *A* and *B*, respectively
- Ψ represents all other dimensional sets, which are held constant; *A* = (λ , Ψ) and *B* = (ρ , Ψ)

Postulate 1: For any dimension in a *y*-dimensional strategy game, addition or subtraction on one dimensional set of a point has no effect on the other dimensional sets of that point. Dimensions are independent of one another from a tool perspective, as opposed to a player perspective.

Theorem 3: Points that are one strategy meter apart on a strategy graph are adjacent, with a gap of *i* between them.

Proof: Since there are only *m* elements per set and *A* and *B* are one strategy meter apart,
$a_1 + m = b_1$ and $a_m + m = b_m$, where $b_1 \notin \lambda$, $b_1 \in \rho$ and $b_m \notin \lambda$, $b_m \in \rho$
Elements in the open interval (a_1, a_m) will also yield corresponding elements within ρ when *m* is added: {a_1+m,…, a_m+m} = {b_1,…, b_m} . No element of a set can be outside of a point, meaning that *A* and *B* are adjacent with a gap of *i* between them, the difference between the consecutive elements a_m and b_1 along the dimension changed. Subtraction of *m* from a dimensional set is equivalent to going from any element of ρ to the corresponding element of λ. ∴ Points on a strategy map that are one strategy meter apart are adjacent, with a gap of *i* between them. □

Corollary 1: There is only one network for the strategy graph of a given game and a given *m*. Adjacency between two adjacent strategy points is based on the dimension that has changed, not the dimensions held constant. Thus, adjacent points have a specific orientation relative to one another.

Corollary 2: There exist two incremental operations of *m*, plus and minus. If μ = the number of pathways from a specific strategy point to adjacent points on a strategy graph, then $y \leq \mu \leq 2y$. The minimum number of links is y, because for all y dimensions, the minimum adjacency is either addition or subtraction per dimension and the maximum is addition and subtraction per dimension.

Corollary 3: If $y \geq 2$, then all the pathways along the strategy graph will be closed. There will be no isolated points, since all points will have at least two pathways (theorem 1, corollary 2).

Corollary 4: Each strategy point for a given game is unique, since at least one corresponding dimension between any two points is represented by different sets.

3. CONSTRUCTION OF THE STRATEGY GRAPH

One constructs the strategy graph by computing the adjacencies for every point. By definition, adjacent points are most similar to each other. Based on the properties of theorem 3 and its third corollary, the Incremental function I(R) generates one adjacent point for every dimensional set of a point. It does this through the addition or subtraction of *m,* while holding the other dimensional sets constant. The resulting table establishes every adjacency in the graph.

$I(R) = \{R \pm y_{1m}, \ldots, R \pm y_{ym}\}$, where y_1 and y_y are the 1st and y^{th} dimensions to a y-dimensional strategy point, respectively, and $R \pm y_{ym}$ is the increment of m to the dimensional set for y of a strategy point R. The restriction on I(R) is that the incremental operations must result in existing strategy points, not points above or below the specified parameters of the game ("domain of the game"). Table 1 shows I(R) for a standard, two-dimensional Ultimatum Game.

R	R + p_m	R – p_m	R + q_m	R – q_m
R_1 = (1,1)	(2,1) = R_3	O.D. – e.n.p.	(1,2) = R_2	O.D. – e.n.p.
R_2 = (1,2)	(2,2) = R_4	O.D. – e.n.p.	O.D. – e.n.p.	(1,1) = R_1
R_3 = (2,1)	O.D. – e.n.p.	(1,1) = R_1	(2,2) = R_4	O.D. – e.n.p.
R_4 = (2,2)	O.D. – e.n.p.	(1,2) = R_2	O.D. – e.n.p.	(2,1) = R_3

TABLE 1. I(R) output for Ultimatum Game: $i = 1$, $m=1$, and $z=2$; 1 and 2 are elements of different sets. O.D. – e.n.p. means this operation is outside the domain and so there exists no point on this strategy graph for that operation.

$\Omega_x = 4$ (R_1 through R_4). $y = \{p,q\}$, a cardinality of 2. $\therefore\ \exists$ four operations of I(R) for every R. Each row depicts the strategy points one strategy meter from the corresponding R in that row. If players in a game have different, non-overlapping strategy sets (different set Ω), then they are mapped on different strategy graphs.

Strategy points of a strategy graph can be depicted as congruent polygons with $2y$ vertices (corollary 2 to theorem 3) – or as line segments for $y = 1$. For each point, the ordered elements of every dimensional set can be represented as an axis, with the lowest and highest elements at opposite vertices. Figure 1 is the strategy graph for the Ultimatum Game constructed from the output of I(R) in Table 1.

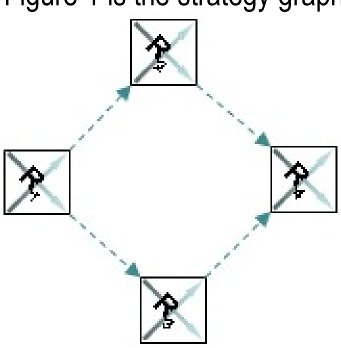

Figure 1. Strategy graph for Ultimatum Game: $y = 2$; arrows *do not* indicate an oriented graph, they indicate ascending order along a dimension; gaps = i

4. SYSTEM PROPERTIES

4.1 Distance

A distance between any two points is defined as the shortest path between those points. The pathways and distances between strategies can be determined visually by inspecting a strategy graph, counting hops from one point to the next point along the shortest path in strategy meters, or algorithmically by associating each strategy with its strategy point.

For a given game, specify: y dimensions, z number of dimensional sets, m elements per set, minimum value ε of the initial element of a dimension, and i increments of play; a is the value for some element of a set. A pathway from one strategy point to any other point in the graph could trace movement in one to up to y dimensions.

Based on theorem 1 and postulate 1, the distance is the sum of the net displacement in each dimension. If one starts with strategies, the point-to-point distance is calculated in strategy meters using the *Strategy Integer operator, SI*, a piece-wise function using the quotient and remainder for $\frac{a}{m}$.

Step 1: Identify the dimensional set to which element a belongs using SI:

$$SI = \begin{cases} \left\lfloor \left(\frac{a}{m} \right) \right\rfloor & \forall \text{ elements of a dimensional set, except} \\ \left\lfloor \left(\frac{a}{m} \right) \right\rfloor - 1 & \text{for condition 1} \end{cases}$$

- 1 for condition 1: There are ε (mod m) exceptions per dimensional set, starting with the element where $a \equiv 0$ (mod m) and continuing to the m^{th} element, when ε (mod m) > 0.

Step 2: The shortest distance between two points in the dimension generating the movement = Δ SI

Thus, net displacement distance in strategy meters across all the dimensions = $\sum_{j=1}^{y} \Delta \, SI_j$.

A relation between the L_1Norm and the strategy meter validates the strategy metric. Let:

- *U, V* be two strategy points on a strategy graph, each composed of y dimensional sets
- c_j, d_j be corresponding elements of dimensional sets in U and V for dimension j; elements c_j and d_j must correspond in the L_1 Norm N there is only one element per dimension per point; for consistency, they correspond for the strategy meter
- $g_j = |c_j - d_j|$, the distance traveled from point U to point V in dimension j, in number of element-units
- m-space in element-units; y = the number of dimensions for a strategy point

Theorem 4: *strategy metric* in strategy meters = $\frac{L_1\text{Norm}}{m - space}$

Proof: For both systems, the distance from point U to point V in dimension j can be evaluated individually for each j and then summed with the distance computed for the all other dimensions.

L_1 Norm distance between points U and $V = \sum_{j=1}^{y} g_j$ *each g_j* is evaluated individually.

The strategy distance between points U and $V = \frac{1}{m} \sum_{j=1}^{y} g_j$; $\frac{1}{m - space}$ is used to convert the metric from element-units to strategy meters. To convert the L_1 Norm distance to strategy meters, divide by m-

space. ∴ *strategy metric* in strategy meters = $\dfrac{L_1\text{Norm}}{m-space}$ □

The remaining system properties relate to mapping entities onto the strategy graph for a given game. An entity can be a population, group, or individual.

4.2 Rate

Strategy points never move; players "travel" from point to point along the graph as their strategies change from round to round. There are two rates of strategy change for an entity. Both rates are based on distances between an entity's strategy points identified from an earlier and later round. The choice of rounds depends on the experimenter's objectives. The rates are:
Displacement rate: the net distance between the two points divided by number of rounds elapsed.
Distance rate: the cumulative distance traveled divided by number of rounds elapsed.

If a player moved away from an initial point and then moved back towards it, the *displacement rate* over these rounds would be less than the *distance rate*.

The rate of change for each rate is a strategy acceleration, defined as *displacement-acceleration and distance-acceleration* for the displacement rate and distance rate, respectively.

4.3 Time

An entity's affinity for a point is a time-related attribute called *t-space*. It is a probability, computed as the sum of an entity's visits to a point divided by the total number of visits during some number of rounds. Since there is only one visit per round, the denominator equals the number of rounds elapsed. A visit must include the entire entity. Each point has multiple t-spaces, based on the entities.

t-space is an indicator of similarity between entities, when entities have the same *t-spaces* for multiple points. If a point's *t-space* approaches some non-low percentage as time continues, then that *t-space* could indicate part of the stability distribution for that entity.

4.4 Stability

An evolutionary stable strategy is a strategy that will eventually exist within the population with a frequency of one and remain that way even if a mutant deviant enters the population; an evolutionary stable state is a stable distribution of strategies throughout a population (Hofbauer, & Sigmund, 1998). The typical stability definition is equivalent to the maximum stability of the system, defined below.
Postulate 2: Stability in this new system is: If and only if an entity persists on one strategy point or it persists in a constant distribution among strategy points or it oscillates between a constant distribution of strategy points, then that entity is stable. Stability states can be broken; therefore stability can exist for finite periods of time.

Postulate 2's conditions are satisfied when displacement rate of an entity consistently equals zero for some number of successive rounds. Thus, movement within a point is stable. An entity's proximity to a stable state is tracked by graphing time versus displacement. Its stability is measured by a stability index, β:

β = $\dfrac{1}{m}$, where $\dfrac{1}{\pi} \le \beta \le 1\ m\ m_{\max} = \pi$ from theorem 2 and $m_{\min} = 1$, by definition.

Lower β-values result in a less stable dynamic, since the entity's stability is relative to strategy points that incorporate more strategies. For example, the four-point strategy graph in Figure 1 for m = 1 becomes a one-point graph containing all strategies for m = 2. To find the stability value, a new graph does not have to be drawn. The experimenter simply determines the strategy space that must be occupied by the points in order for the stability definition to be satisfied and then plugs into the stability equation.

5. DEMONSTRATION WITH THE ULTIMATUM GAME

5.1 Experimental Method

To demonstrate the validity and utility of the system, two Ultimatum Game experiments were performed. These experiments maintained the features of anonymity, subjects' inexperience with the Ultimatum Game, and non-influential external environments. These settings were implemented to test whether people play a game differently based on whom they play and whether they share some common characteristic with their opponent.

A pen and paper method was created to perform these experiments. Payoffs were mentally computed in real-time during an experiment. The first experiment used four players and the second experiment used six. At the outset of an experiment, players filled out a social orientation scale (Van Lange, Otten, De Bruin, & Joireman, 1997). Since the number of players for each experiment was predetermined, random pairings and roles for every round of the experiments could be made ahead of time. Strategy sheets were created for each round, in which players would indicate their player number, the round number, the amount each would offer if selected to be the Proposer, and the amount each would minimally accept if selected to be the Responder.

Both experiments consisted of seven rounds. Only one strategy dimension for each player was used per round, which was the player's pre-selected role. After collecting the strategy sheets, payoff sheets for each round were tabulated per pairing and then distributed to the corresponding players. The payoff sheets were then collected and the players advanced to the next round. To ensure that subjects did not write down random numbers for their strategies, the top two subjects with the most total payoff at the end of

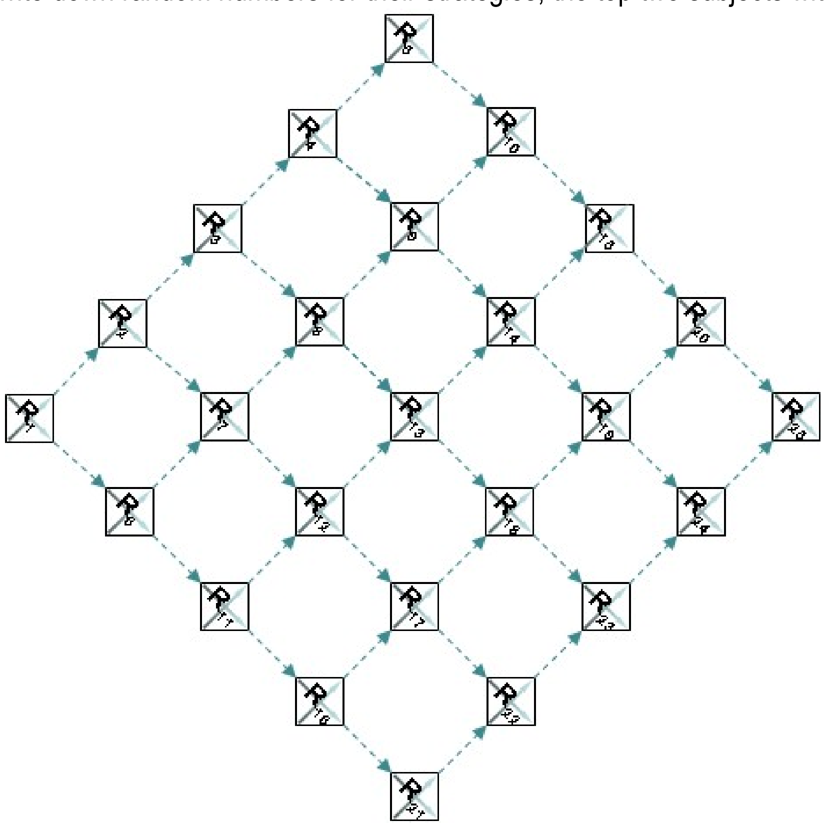

Figure 2. 25-point strategy graph: created from the I(R) function.

seven rounds received a prize, donated by local merchants. The strategy graph is shown in Figure 2.

The independent variables for these experiments: π total per round (aka Π "pie") = 10; y number of dimensions = 2 for offer and acceptance; i increment of play = 1; ε minimum offer or acceptance = 1; m number of elements per set = 2. ∴ z = 5 sets and z^y=25 points in the strategy graph. The measured dependent variables: record of players' offer and acceptance strategies per round.

5.2 Experimental Results

The dependent experimental variables are based on the properties of the strategy system: strategy distance, displacement and distance rates, acceleration, t-space, β, and strategy point frequency per round. The graphs below are derived from data of the six-player experiment. They demonstrate how the system provides a framework for analyzing an Ultimatum Game's dynamics, using the various property variables.

Simply based on network connectivity, one might expect strategy point frequencies to be generally less for points with fewer links. Based on inspection of the 25-point strategy graph in figure 2, the least probable frequencies would be associated with the corner points R_1, R_2, R_3, and R_4 – each with the minimum of two links. The most likely frequencies would be associated with all points within the perimeter of the graph -- each with the maximum of four links. This is shown by the frequency graph in figure 3, in which the highest frequencies do occur on points with four links and the next highest frequencies occur at R_6 and R_{20}, points with three links each.

Figure 4 shows displacement and distance rates versus round for each player. Although the displacement rates show an interesting trend towards convergence, the stability for the population works out to a consistently low β-value of 0.1. This occurs because some rates decrease, while others increase.

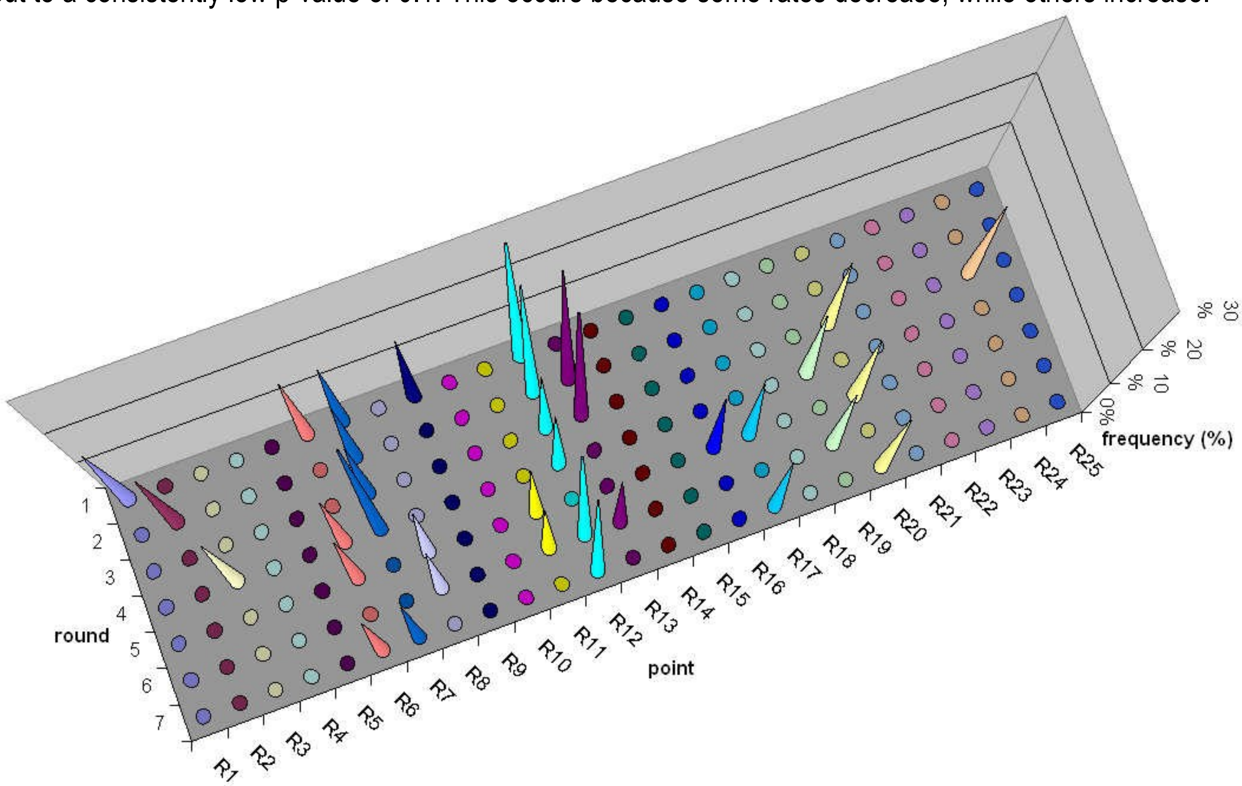

Figure 3. Frequency graph: for each of the seven rounds of the six-player experiment. Overall, R_{12} = (5-6, 3-4) had the highest frequencies; R_7 and R_{13} were also high.

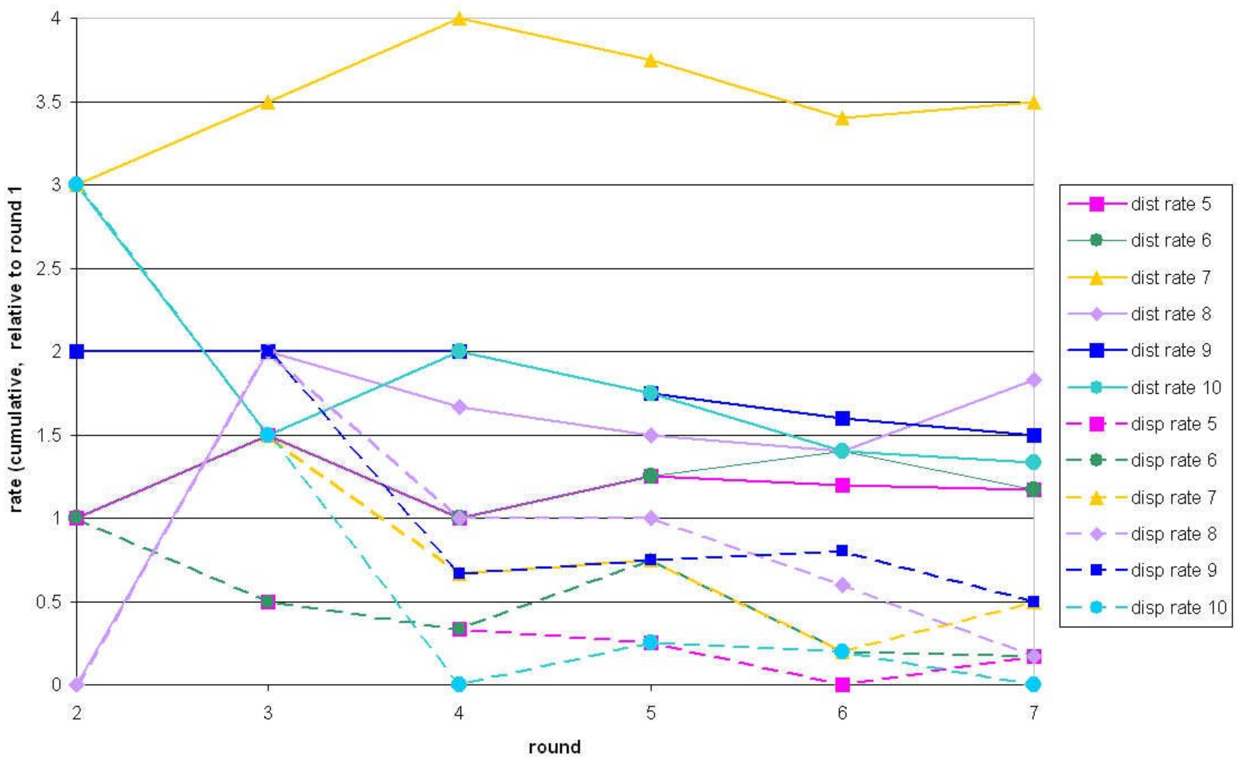

Figure 4. Displacement and distance strategy rates: per player for each round, calculated relative to round 1. This graph suggests a trend of rate convergence among players for both rate types. In the legend, numbers five through ten identify the six players.

Other system-related plots include overlays for each player of documented and projected strategy pathways, *t-space* vs. round, displacement rate and distance rate vs. payoff, and β-stability plots for different values of *m*. For some experiments, though not for the ones described above, these plots might correlate with more standard plots of payoff vs. round, average per round for offer, acceptance, payoff, and distance, or graphs involving social groupings or experience.

6. DISCUSSION

6.1 Applications and Modifications to the Reverse Ultimatum Game

Bargaining is typically not a one-shot situation. Parties usually go back and forth numerous times until an agreement is reached. This makes the Ultimatum Game, a one-shot game, a simplified bargaining model. Gneezy et. al.'s Reverse Ultimatum Game depicts the process more realistically (Geezy, Haruvy, & Roth, 2003). In their Reverse Ultimatum Game, there is one Proposer, one Responder, and a pie to be divided between them. This game can be, and usually is, multistage rather than one-shot. If the Proposer's offer is rejected, the Proposer can make another offer as long as it is strictly higher than the previous one. The game ends when either the offer is accepted or the Proposer decides to walk away. Therefore, the number of stages to the game is not known ahead of time. In Gneezy et. al.'s paper, the Reverse Ultimatum Game is investigated both with and without a time deadline. This paper modifies the reverse ultimatum game to make it even more general and then applies the analytical system.

Bargaining situations that end in agreements don't always follow the stipulation of increasing offers. In our modified version of the Reverse Ultimatum Game, the requirement of strictly increasing offers is removed to allow for more generality. For instance, as a deadline nears, a party might make a final "take it or leave it" offer which is less than a previous offer. Some might argue that such an offer is irrational if a previous offer has been rejected. However, the proposing party makes such an offer indicating that the opportunity of a better one has been lost. The modified Reverse Ultimatum Game has a deadline setting not of time but of stages. Before the game begins, a stage limit is specified. Both players know the stage limit. If there is no agreement by the final stage, both players receive a payoff of zero. Because of the stage deadline setting, the "walk away" element of the Proposer's strategy space need not be included. A Proposer can propose any offer value at any stage and therefore if she is not willing to offer more than p, she can continue to offer p or less at later stages. This implies that she is willing to walk away and have both players receive a payoff of zero if p is not accepted in the final stage.

Therefore, Proposers have all elements of all z sets in their strategy sets. Responders also have all elements of all z sets in their strategy sets because they can change their acceptance threshold at various stages to either demand more or accommodate lesser proposals in trying to reach an agreement. Proposers don't know whether their offer will be accepted early on or not. Responders don't know whether they will get an acceptable offer early on or not. Thus, players must have a strategy encompassing all potential stages of the game. If roles are not fixed, then strategy points have double the dimensions of the maximum number of stages in the game – just as in the classic one-shot Ultimatum Game.

It is assumed that players are "sufficiently rational." This is defined to mean that in the case that roles are not fixed, players do not have the same utility goal for both bargaining roles in a given round. This allows the total strategy space to be defined. For example, the Ultimatum Game strategy (p,q) = (1,2) for π = 4 exists. If this player is a Proposer and his offer is accepted, he will be left with 3. If this player is a responder, he is willing to accept 2. Sufficiently rational proposers will offer p:

$\varepsilon \leq p \leq \pi - i$ and sufficiently rational responders have an acceptance threshold q: $q \geq \varepsilon$.

6.2 Factors that affect strategy change in this evaluative model:

1. Payoff: Players react to payoff in some way. Even if they are satisfied with their own realized payoffs and/or payoffs of others in a given round, players may experiment to see how playing another strategy will change their payoff. Therefore, payoff affects the dynamics of the game indirectly. Such experimenting is a version of trial and error learning. (Young, 2008).
2. Player matchings and population sizes: Players will probably not interact with all other strategies or all other players. The reactions and experiments that a player makes will be based on the players and strategies that that player has come into contact with. The size of the population will affect player matchings because a larger population means it is less likely for a player to come into contact with every other player. Multiple populations coming into contact with one another also affect player matchings by nature of a larger, new population being created. However multiple populations do not complicate the analysis of the analytical system because the system is player-independent.
3. Players implement fictitious play and information updating.
4. Imitation: "Strategies are chosen in proportion to how frequently they were played in the population in the previous period." Direct quote from Levine & Pesendorfer (2005).
5. Innovation: "Strategies are picked regardless of how widely used they are, or how successful they are." Direct quote from Levine & Pesendorfer (2005).

6.3 Connectivity of the Strategy Graph and Nash Equilibria

For bargaining games, the equilibrium strategy point(s) have the lowest possible connectivity and exist on the outside of the strategy graph. This is true because its coordinates are the extremes: low offer value and low acceptance threshold. This does not mean that all points of lowest connectivity in bargaining games are Nash Equilibria.

An experimenter using the analytical system described in this research would set m at some intermediate level. There are two competing motivations for adjusting m up or down, one to increase the level of detail observed and one to decrease the detail observed in order to evaluate stability.

SI demonstrates that movement within a point for a given m gets rounded down to zero – there is no change in points, even though there is a strategy change. To evaluate movement between strategies contained within a point and to increase the detail of the analysis, reduce m to generate a related strategy graph having more points and containing fewer elements per dimension. This can influence the experimental design, since ε or i should be set sufficiently small. Movement in the recast strategy graph could be a full strategy meter or more. $\lim_{m \to 1}$ (number of strategies per strategy point) = 1 describes the greatest detail, since a strategy is defined as one element per dimensional set and $m_{\min} = 1$.

Alternatively, when stability has not been achieved, the experimenter might increase m to the minimum value necessary for attaining stability, based on a strategy graph recast for the higher m. The analytical cost of increasing m is a lower stability, as measured by β. If stability has been achieved, then one can decrease m to the lowest value where stability still exists, based on graphs of an entity's displacement rate versus time. Change in the levels of stability can be tracked with a plot of time versus β by re-computing β during an experiment at different times, while using a consistent time increment over which to define stability. Such a plot could be used to better predict stability and a rate of stability.

For constant displacement or distance acceleration of an entity – even if it is non-zero – a locus of points and branching pathways for succeeding rounds can be forecast by extrapolating an entity's positions on a strategy graph. The probability of two or more players residing on the same strategy point in subsequent rounds can be more accurately predicted using this approach. A change in the trend of an entity's rate or acceleration might indicate the effects of some factor. The magnitude of the change would suggest the sensitivity of strategy rate or acceleration to the change in that factor.

The strategy change system might be applied to population dynamics, since a behavior or trait can be called a strategy. Individual changes and thus individual dynamics could be explored with this system to describe how an evolutionary dynamic of a population occurs. For example, punctuated equilibrium can be depicted in a stability graph of time versus β. Where a plot for population would show a sharp change in β-value, stability plots at the group or individual level might identify the individual(s) causing the change. An overlay of these individuals' plots with the population plot would show the time lag between the causes at the individual level and the effect at the population level (Smith, 1976).

A player employing a mixed strategy has probabilities for using each pure strategy (Siegfried, 2006). Mixed strategies can be represented in a strategy graph by the t-spaces for an entity. If $t\text{-}space < 1$ and stability exists, then a player has a mixed strategy and oscillates amongst strategy points.

Most games have one- or two-dimensional strategies. As more dimensions are involved, games become more complex. The more dimensions a game has, the more that model approaches reality. The analytical system developed by this research has been generalized to a finite, y-dimensions and so supports analysis of multi-dimensional games.

7. CONCLUSIONS AND FUTURE RESEARCH

A new system has been established that models strategy change throughout a bargaining game for individuals, groups, and populations. It provides game experimentalists and theorists with a unique framework for analyzing relationships and trends in data from new or documented experiments, relative to the newly-quantified system properties of distance, rates, acceleration, t-space, and the β stability index. For example, the framework can explain how some strategies have a higher probability of implementation than others.

One can forecast probable strategy pathways for any timeframe based on a constant strategy acceleration of an entity and its corresponding locus of points. A stability index has been defined that expresses how close an entity is to maximum stability. Since stability is quantified, time to maximum stability – or any β value – can be forecast based on a trend throughout rounds of a repeated game.

To better model real situations, the system has been generalized to bargaining games of multi-dimensional strategies. A logical next step for future research will focus on whether the system can be further generalized to other games, such as games with unquantifiable actions. Another research direction will address whether additional system properties and relationships derive from graph theory. For example, does the number of links to a strategy point correlate with strategy frequency or t-space?

Further investigations could also pursue questions raised by the demonstration experiments: Do entities' rates converge at any time (see figure 4)? Is there a geometry to strategy change, meaning do players of various types travel along the strategy graph in specific patterns? Under imitation behaviors, do different time delays exist for different possible geometries of strategy change? Can accelerations become constant and, if so, do they decrease uniformly on the way to stability states?

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