ON THE EXISTENCE OF NASH EQUILIBRIA IN DISCONTINUOUS AND QUALITATIVE GAMES

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Abstract. We show that compact games have pure strategy Nash equilibria if conditions C and Q are satisfied. Condition C states that, whenever a profile of strategies x is not an equilibrium, there exists an open neighborhood V of x and well behaved maps φ_i, one for each player i, mapping V to each player's strategies, and satisfying the following property: for any profile of strategies y in the neighborhood V, there exists one player i such that $\varphi_i(y)$ belongs to player i's strict upper contour set, while $\varphi_j(y)$ is unrestricted for players j other than i. For other profiles $\hat{y}$ in V, the chosen player need not be player i. Condition Q is a weakening of own-strategy quasiconcavity. This result unifies and generalizes results establishing existence of pure strategy Nash equilibria in the literatures on discontinuous quasiconcave games and on qualitative convex games.

Keywords: Nash equilibrium, discontinuous game, qualitative game, compact game, diagonal transfer continuity, better reply security

1. Introduction

The known sufficient conditions for existence of Nash equilibria in games fall into two literatures. On the one hand, for games with possibly discontinuous payoff functions, Baye, Tian and Zhou [3] and Reny [28] provide conditions on the allowed discontinuities of the payoff functions and on the convexity of the preferences that guarantee the existence of a Nash equilibrium. On the other hand, for games with possibly incomplete and non transitive preferences (i.e. qualitative games), there is a large literature (captured by the monographs of Tarafdar and Chowdhuri [33] and Yuan [39]) providing conditions on the allowed discontinuities of the strict upper contour sets and on the convexity of preferences guaranteeing existence of a Nash equilibrium.

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In this paper we identify a condition that unifies and generalizes the allowed discontinuities in both literatures, and allows us to prove existence of Nash equilibria under a weak notion of convexity of preferences. Our notion of convexity is weaker than the standard quasiconcavity of payoff functions and the convexity/irreversibility condition imposed on strict upper contour sets in qualitative games. Moreover, since discontinuities are the sole responsible for lack of existence of Nash equilibria (c.f. Dasgupta and Maskin [8]), our result provides the weakest set of assumptions guaranteeing existence of Nash equilibria.

By analyzing a game with payoff functions from the perspective of the strict upper contour sets, and by applying the "better reply security" logic in Reny [28] to qualitative games, we are able to identify the underlying conditions on the allowed discontinuities of payoff functions and strict upper contour sets, and then to propose our generalization.

The better reply security logic, which can be traced back to the logic behind "diagonal transfer continuity" in Baye, Tian and Zhou [3], goes as follows. Whenever a profile of strategies is not an equilibrium, there must exist a player whose preferences are sufficiently well behaved locally. This means that there exists a strategy that "secures" strictly more than a given payoff limit in an open neighborhood of the opponents' strategies. In a loose sense, this securing strategy is a constant selection of the strict upper contour set of the given player.

Our approach starts off by allowing more general selections of the strict upper contour sets. One can think of continuous selections of a particular player's upper contour set. Or even semicontinuous selections. The problem with those selections, for a given player, is that the sort of discontinuities found in economic games will not necessarily be taken into account. As emphasized by Dasgupta and Maskin [8] the discontinuities in economic games are likely to be complementary. In fact, in any zero-sum discontinuous game (like war of attrition), whenever the payoff of one player jumps down it must be that the payoff of the other jumps up (by definition of a zero-sum game). In more elaborated models about competing players, similar discontinuities arise naturally.

We keep the complementarity idea and allow different players to secure payoffs locally. Whenever complementary discontinuities emerge, we switch the player that is required to secure a payoff. This is captured by our **condition** C: whenever a profile of strategies x is not an equilibrium, there must exist an open neighborhood V_x of the profile x and, for each player i, a well behaved correspondence $\varphi_{i,x}$ mapping strategies in this neighborhood to strategies of player i with the property that for any profile of strategies $y \in V_x$ there exists a player j that is "activated", that is, whose image $\varphi_{j,x}(y)$ lies on the strict upper contour set of his preferences. For other profiles $\hat{y} \in V_x$, the activated player may differ. This switching among activated players reflects the complementarity in the discontinuities in economic games.

The kinds of discontinuities allowed by the idea above are quite permissive. For instance, in Example 2.3, we have a game where the strict upper contour sets of the players are erratic, jumping from one end to the other of the strategy space (the interval $[0, 1]$) as profiles of strategies are composed entirely of rational numbers or not. There are obviously no continuous or semicontinuous selections, and yet condition C is satisfied.

In addition to addressing complementary discontinuities, the more general selections allowed by condition C also address games where discontinuities can be offset by coalitional deviations, in the sense that a subset of the group of players has a well behaved "path" of strategies that guarantees a higher payoff to at least one of the players in the group, for each profile of strategies in the neighborhood of a given non equilibrium strategy profile.

Condition C and convexity of preferences guarantee the existence of a pure strategy Nash equilibrium. This result generalizes every result on existence of pure strategy Nash equilibria in quasiconcave games known to us: better reply secure and diagonal transfer continuous games, and qualitative games with lower hemicontinuous strict upper contour sets all satisfy condition C.

Actually, the results proven here are more general than the description above. Borrowing from the qualitative games literature, we allow the "securing" strategies to lie outside of the strict upper contour set of the activated player, say player i, provided that they lie on a correspondence B_i that majorizes i's strict upper contour set while still satisfying the "convexity/irreversibility" condition $x_i \notin coB_i(x)$ (where x_i is the ith coordinate of the strategy profile x). Because this latter condition is only required to hold for the activated player, our convexity requirement is weaker than the usual own-strategy quasiconcavity condition.

We also provide sufficient conditions for games with non-compact strategy sets. Formally, the results described above require the strategy spaces to be compact and convex. Using a fixed point result due to Yuan [39], we generalize the results for paracompact and convex strategy spaces. Apart from one minor technical condition that is not required in the compact case, the conditions are immediate extensions of the conditions described above. For this reason, and because the results in the literature focus on compact and convex strategy spaces, we present the non compact case separately. The fact that any metrizable space is paracompact, on the other hand, adds considerable appeal to our results: if strategy spaces are metrizable, no assumption other than convexity is required on the strategies, a result that is bound to be useful in applications, where boundedness of strategy spaces is sometimes a restrictive assumption.

For applications, we also provide a generalization of the sufficient conditions for better reply security identified in Reny [28] and Bagh and Jofre [2], as follows: a "generalized payoff secure" game is a game where each player can secure a payoff at most $\varepsilon > 0$ below any given payoff, for every $\varepsilon > 0$, where securing a payoff means that there exists a well behaved correspondence mapping strategies in an open neighborhood of the others' strategies to strategies of the given player that yield the required payoff. We show that a generalized payoff secure and "weakly reciprocal upper semicontinuous" game is generalized better reply secure, and *a fortiori* satisfies condition C. Recently, Carmona [6] identified an alternative pair of sufficient conditions for existence of pure strategy Nash equilibria in quasiconcave games: "weak payoff security" and "weak upper semicontinuity". On the one hand weak payoff security is weaker than generalized payoff security because it does not require well behaved "securing" correspondences, and on the other hand weak upper semicontinuity is stronger than weak reciprocal upper semicontinuity. We show that condition C is implied by these two conditions.

The paper proceeds as follows. Our main results, Theorems 2.4 and 2.6, are presented in Section 2, together with an explanation of the way the proofs work. In Sections 3 and 4, we prove that condition C generalizes every known condition on the allowed discontinuities of either strict upper contour sets or payoff functions. In particular we provide generalized versions of better reply security and diagonal transfer continuity (and of the recent condition "weak transfer continuity", introduced by Nessah and Tian [24]), and show that these generalized versions are still special cases of condition C. In Section 5, we take a brief pass on the mixed extension of a game, providing a generalized version of uniform payoff security (c.f. Carbonell-Nicolau and Ok [5] and Monteiro and Page [20]) that is sufficient for the mixed extension to be generalized payoff secure. In Section 6, we present two economic applications of our results: the first is an application of our ideas to existence of equilibria in abstract economies, and the second is a general model of multi-principal multi-agent games, where discontinuities that arise naturally are allowed by condition C and not by the other conditions in the literature. Finally, the Appendix provides the proofs of the results.

2. Existence of Nash Equilibria

Let N be the set of players. Each player $i \in N$ has a pure strategy set X_i, which is a **nonempty and convex** subset of a Hausdorff locally convex topological vector space, and a preference relation $\succ_i$ defined on $X \times X$, where $X = \times_{i \in N} X_i$. Product sets are endowed with the product topology and we use X_{-i} to denote $\times_{j \neq i} X_j$, with typical element x_{-i}. Let $P_i(x) = \{y_i : (y_i, x_{-i}) \succ_i (x_i, x_{-i})\}$ denote player i's strict upper contour set, also called player i's preference correspondence. A **qualitative game** is given by $G = (X_i, P_i)_{i \in N}$. A (possibly) **discontinuous game** is the special case when each $\succ_i$ can be represented by a payoff function $u_i : X \to \mathbb{R}$, in which case $P_i(x) = \{y_i : u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}$. For ease of notation, in this section $G = (X_i, P_i)_{i \in N}$ will be used to denote both a qualitative and a discontinuous game, and we will simply refer to G as a "game".

A pure strategy Nash equilibrium of $G = (X_i, P_i)_{i \in N}$ is a profile $x^* \in X$ such that $P_i(x^*) = \varnothing$ for all $i \in N$. Let $E \subset X$ denote the set of pure strategy Nash equilibria of G.

Let $\Phi : X \rightrightarrows Y$ be a correspondence between two topological vector spaces. We say that Φ is **upper hemicontinuous** if the set $\{x \in X : \Phi(x) \subset V\}$ is open for every open set $V \subset Y$. We say that Φ has non empty, convex, compact values if $\Phi(x)$ is non empty, convex, compact for each $x \in X$. When Φ is compact valued, upper hemicontinuity is equivalent to the closedness of the graph of Φ, given by $\{(x, y) \in X \times Y : y \in \Phi(x)\}$. Since we will repeatedly use such correspondences, we say that a correspondence Φ is **well behaved** if it is non empty, convex and compact valued, and upper hemicontinuous. For any set $K \subset X$, let coK denote the convex hull of K.

Definition 2.1. *A game $G = (X_i, P_i)_{i \in N}$ satisfies **Condition** C if whenever $x \notin E$, there exist an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$, for $i \in N$, such that for each $y \in V_x$ there exists $i \in N$ with $\varphi_{i,x}(y) \subset B_i(y)$, where $B_i : X \rightrightarrows X_i$ is a convex valued correspondence such that $coP_i(x) \subset B_i(x)$ for every $x \in X$.*

If condition C is satisfied, we say that player i is **activated** at $y \in V_x$ if $\varphi_{i,x}(y) \subset B_i(y)$.

Definition 2.2. *A game $G = (X_i, P_i)_{i \in N}$ satisfies **Condition** Q if $x_i \notin B_i(x)$ for every $i \in N$ that is activated at x.*

The use of the "majorizing" convex-valued correspondence $B_i : X \rightrightarrows X_i$ instead of the convex hull coP_i comes from the use of the various majorizing classes in the qualitative games literature.[1] It provides an added degree of generality, in that it does not require that the mappings $\varphi_{i,x}$ be contained in coP_i.

Condition Q is a weakening of the standard convexity/irreversibility condition $x_i \notin coP_i(x)$, for every $x \in X$ and $i \in N$ in the qualitative games literature, which in turn is the counterpart of the standard own strategy quasiconcavity in games with payoff functions, that is, the assumption that $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasiconcave for each $x_{-i} \in X_{-i}$.

In order to appreciate the scope of condition C, consider the following example.

Example 2.3. *There are three players with strategy sets $X_i = [0,1]$, $i = 1,2,3$. The payoffs of player $i = 1,2,3$ are given by*

$$u_i(x_i, x_j, x_k) = \begin{cases} 0 & \text{if } x_i \in (0,1) \\ 1 & \text{if } x_i = 0 \text{ and } (x_j, x_k) \in \mathbb{Q}^2 \cap [0,1]^2 \\ 1 & \text{if } x_i = 1 \text{ and } (x_j, x_k) \notin \mathbb{Q}^2 \cap [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

where 0 is considered a rational number.

The strict upper contour sets of the players are quite erratic. For instance, for $x_i \in (0,1)$, $P_i(x_i, x_j, x_k) = \{0\}$ if both x_j and x_k are rational numbers and $P_i(x_i, x_j, x_k) = \{1\}$ if at least one of them is an irrational number. Hence, there do not exist continuous selections of P_i. Nevertheless, for a fixed pair (x_j, x_k), $P_i(x_i, x_j, x_k)$ is a convex set (if non-empty), so condition Q is satisfied. More importantly, condition C is also satisfied: the unique equilibrium is given by $x_i = x_j = x_k = 0$, so any profile $x = (x_i, x_j, x_k)$ with at least one non-zero coordinate is not an equilibrium. For any such profile x, let i be a player with $x_i > 0$ and V_x be an open neighborhood not containing the point $(0,0,0)$, and put $\varphi_{i,x}(y) = \{0\}$, $\varphi_{j,x}(y) = \varphi_{k,x}(y) = \{1\}$ for all $y \in V_x$. Pick any $y \in V_x$: if the jth and kth coordinates are rational, then

[1] For instance, let $\Phi : X \rightrightarrows Y$ be a correspondence between a topological space $X = \times_{i \in I} X_i$ and a subset Y of a vector space, and for a given $x \in X$ let $\Phi_x : X \rightrightarrows Y$ and V_x be an open neighborhood of x such that (a) for each $z \in V_x$, $\Phi(z) \subset \Phi_x(z)$, (b) $z_i \notin co\Phi_x(z)$, and (c) for each $y \in Y$ $\Phi_x^{-1}(y) = \{z \in X : y \in \Phi_x(z)\}$ is open in X. The pair (Φ_x, V_x) is called an L_C-majorant of Φ at x. We say that Φ is L_C-majorized if for each $x \in X$ with $\Phi(x) \neq \varnothing$ there exists an L_C-majorant (Φ_x, V_x). From Theorem 4.5.20 in Yuan [39], we know that whenever X is a regular topological space, Φ is L_C-majorized and the domain of Φ is open and paracompact, then there exists a correspondence (the majorant) $\Psi : X \rightrightarrows Y$ such that (a) $\Phi(x) \subset \Psi(x)$ for every $x \in X$, (b) $x_i \notin co\Psi(x)$, and (c) Ψ^{-1} is open in X.

player i is activated; if the jth coordinate is irrational, then player j is activated if at least one of the other two coordinates is irrational, otherwise it is player k that is activated; and finally if the jth coordinate is rational and the kth is irrational, then player j is activated.

In Sections 3 and 4, we show that Example 2.3 violates all of the conditions proposed in the literature. And, in Section 6, we provide a class of multi-principal multi-agent games that also satisfy condition C and violate all of the conditions proposed in the literature.

Our main result is the following:

Theorem 2.4. *Let $G = (X_i, P_i)_{i \in N}$ be a game such that X_i is compact for each $i \in N$ and conditions C and Q are satisfied. Then there exists a pure strategy Nash equilibrium.*

The idea of the proof is the following. If there is no equilibrium, then the open neighborhoods figuring in condition C form an open covering of the compact space X. There then exists a partition of unity subordinated to a locally finite open refinement of the open covering. This means that we can glue the mappings $\varphi_{i,x}$ together and construct, for each player $i \in N$, a well behaved mapping $\phi_i : X \rightrightarrows X_i$. The mapping $\phi : X \rightrightarrows X$, given by $\phi = \times_{i \in N} \phi_i$ is well behaved and must have a fixed point $x^* \in X$. A fortiori, $x_i^* \in \phi_i(x^*)$ for every i. Now condition C states that one player must be activated at x^*, and this will contradict condition Q if it is also the case that $\phi_i(x^*) \in B_i(x^*)$. The main step of the proof is in showing this fact. That is, in guaranteeing that the open neighborhoods can be refined in such a way that there is never a case that a player must be activated at x^* using $\varphi_{i,x}$ and cannot be activated at x^* with some other $\varphi_{i,z}$. The refinement used is what is known as a Δ-refinement of an open covering: a new open covering such that the family of open sets formed by the "stars" of each point refines the given open covering, where a star of a point is the union of all of the open sets in the covering that contain the point. Notice that we can use the open neighborhoods in the new open covering as the ones figuring in condition C. Hence, it can never be the case that a player must be activated at x^* according to some $\varphi_{i,x}$ and cannot be activated at x^* according to some $\varphi_{i,z}$, because the corresponding open sets are in the star of x^*, which in turn is entirely contained in one of the original open neighborhoods V_x.

2.1. **Non Compact Strategy Sets.** In applications, it is often the case that strategy spaces are not compact. We now provide an extension of Condition C that allows for non compact strategy spaces. We say that a correspondence $A : X \rightrightarrows X$ satisfies the **coercive condition** if there exist a non empty compact and convex subset X_0 of X and a non empty compact subset K of X such that for each $y \in X \backslash K$ there exists $x \in co(X_0 \cup \{y\})$ with $y \notin \overline{X \backslash A^{-1}(x)}$. As noted by Yuan [39], any correspondence $A : X \rightrightarrows X$ satisfies the coercive condition whenever X is compact and convex.

Notice that the main step of the proof of Theorem 2.4 described above allows us to define an "activating" mapping $I : X \rightrightarrows N$ indicating the players that are activated at a given profile $x \in X$, independently of which neighborhood we consider. With such mapping, we can define an **aggregator**, given by a correspondence $A : X \rightrightarrows X$ with $A(x) = \times_{i \in N} A_i(x)$, where for each $i \in N$

$$A_i(x) = \begin{cases} B_i(x) & \text{if } i \in I(x) \\ X_i & \text{otherwise} \end{cases} \tag{2.1}$$

where $B_i : X \rightrightarrows X_i$ is the majorizing mapping figuring in condition C. The following is a strengthening of condition C that allows us to deal with non compact strategy spaces:

Definition 2.5. *A game $G = (X_i, P_i)_{i \in N}$ satisfies **Condition** C^{nc} if (i) whenever $x \notin E$, there exist an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$, for $i \in N$, such that for each $y \in V_x$ there exists i with $\varphi_{i,x}(y) \subset B_i(y)$; (ii) $B_i : X \rightrightarrows X_i$ is a convex valued correspondence with $coP_i(x) \subset B_i(x)$ for every $x \in X$; (iii) $X \backslash B_i(y)$ is compact whenever $\varphi_{i,x}(y) \subset B_i(y)$; the aggregator $A = \times_{i \in N} A_i$, where A_i is given by (2.1) using the implied activating mapping I, satisfies the coercive condition.*

Theorem 2.6. *Let $G = (X_i, P_i)_{i \in N}$ be a game such that X_i is paracompact for each $i \in N$ and condition C^{nc} and Q are satisfied. Then there exists a pure strategy Nash equilibrium.*

Theorem 2.6 does not fully encompass Theorem 2.4 in the compact case because of the assumption that $X \backslash B_i(y)$ is compact whenever player i is activated at $y \in X$. On the other hand, the local convexity assumption of the underlying topological vector spaces is not needed in Theorem 2.6.

3. Qualitative Games

Theorems 2.4 and 2.6 strictly generalize a host of results providing sufficient conditions for existence of equilibria in the qualitative games literature, and their corresponding results for generalized games and abstract economies. For instance, in the case that each X_i is compact (paracompact) and convex, and to the best of our knowledge, Theorem 2.4 (Theorem 2.6) improves upon every result in the literature.[2] What is at stake here is that condition C does not require that the strict upper contour sets (or the majorants B_i) have continuous selections (c.f. Example 2.3 above). In contrast, the typical result in the literature assumes that each P_i satisfies some continuity condition. For instance, each P_i has to have open inverses or to satisfy the local intersection property, or to belong to some majorized class, forcing the majorants B_i to satisfy some continuity condition (in the case of L_C-majorants, for instance, each B_i would be required to have open inverses, as explained in footnote 1.)

To illustrate further, consider the following example taken from Bagh and Jofre [2]:

Example 3.1. *There are two players with strategy sets $X_i = [0, 1]$, $i = 1, 2$. The game is zero-sum and the payoffs of player 1 are given by*

$$u_i(x_i, x_j) = \begin{cases} 10 & \text{if } x_1 < x_2 \text{ or } x_1 = x_2 > \frac{1}{2} \\ 1 & \text{if } x_1 = x_2 \leq \frac{1}{2} \\ -10 & \text{otherwise} \end{cases}$$

The profile of strategies $(x_1, x_2) = (0, 0)$ is a pure strategy Nash equilibrium, and the payoff functions are quasiconcave in the owner's strategy. The strict upper contour set of player 1 does not have open lower sections: for instance, the lower section of the strategy $x_1 = \frac{3}{4}$ is given by $\{(x_1, x_2) : x_1 > \frac{3}{4}, x_2 \geq \frac{3}{4}\}$, hence not open in $[0, 1]^2$. Hence the sufficient conditions set forth in the literature on qualitative games do not apply. And condition C is easily verified: it suffices to activate player 1 below the diagonal and activate player 2 above (and including) the diagonal.

4. Discontinuous Games

In this section, we will restrict the analysis to games with payoff functions, so for consistency we will write $G = (X_i, u_i)_{i \in N}$. Let $\Gamma = \{(x, u) \in X \times \mathbb{R}^N : u(x) = u\}$ be the graph of the game's vector payoff function, and let $\overline{\Gamma}$ be its closure. Reny [28] defined the following condition:

Definition 4.1. *A game $G = (X_i, u_i)_{i \in N}$ is called **better reply secure** if whenever $(x^*, u^*) \in \overline{\Gamma}$ and x^* is not an equilibrium, there exists a player i and a strategy $\bar{x}_i$ such that $u_i(\bar{x}_i, y_{-i}) \geq \alpha > u_i^*$, for some $\alpha \in \mathbb{R}$ and all y_{-i} in some open neighborhood V of x^*_{-i}.*

[2] For a summary of the results, see the monographs of Tarafdar and Chowdhuri [33] and Yuan [39].

Notice that it is without loss of generality to use a neighborhood V of x^* in the definition above, since the ith coordinate of $y \in V$ does not matter for the inequality $u_i(\bar{x}_i, y_{-i}) \geq \alpha > u_i^*$.

Baye, Tian and Zhou [3] defined the following condition:

Definition 4.2. *A game $G = (X_i, u_i)_{i\in N}$ is called **diagonal transfer continuous** if for each non equilibrium strategy profile x, there exists a profile $\bar{x}$ such that $\sum_{i=1}^N u_i(\bar{x}_i, y_{-i}) > \sum_{i=1}^N u_i(y)$, for all y in some open neighborhood V_x of x.*

The following example (taken from Carmona [6]) illustrates a class of games where condition C improves on diagonal transfer continuity and better reply security.

Example 4.3. *A **diagonal game** is a two-player game with strategy sets $X_i = [0, 1]$, $i = 1, 2$ and payoff functions given by*

$$u_i(x_i, x_j) = \begin{cases} \phi_i(x) & \text{if } x_i = x_j \\ f_i(x) & \text{if } x_i \neq x_j \end{cases}$$

where $\phi_i, f_i : [0, 1]^2 \to \mathbb{R}$ are continuous functions. In addition, assume that the functions ϕ_i, f_i are such that u_i is quasiconcave in X_i and either $\phi_i(x_j, x_j) = sup_{x_i\in X_i} u_i(x_i, x_j)$ or there exists $\bar{x}_i \in X_i$ such that $f_i(\bar{x}_i, x_i) > f_i(x_j, x_j)$.

The continuity of the functions ϕ_i, f_i ensure that the game satisfies condition C, trivially. In fact, the strict upper contour sets of the players are continuous, so condition C and the sufficient conditions in the qualitative games literature are satisfied. As Carmona [6] shows, the functions ϕ_i, f_i can be chosen so as to violate diagonal transfer continuity and/or better reply security. Consider a weakening of the conditions: for player i, the functions ϕ_i, f_i remain continuous, $\phi_i(x_j, x_j) = \sup_{x_i\in X_i} u_i(x_i, x_j)$, and for player j, $\phi_j(x_i, x_i) < \sup_{x_j\in X_j} u_j(x_j, x_i)$, and the function ϕ_j is discontinuous in such a way that the strict upper contour set is not lower hemicontinuous. Then condition C is still satisfied because the diagonal $x_i = x_j$ is the graph of player i's best response correspondence, and for any pair (x_i, x_j) with $x_i \neq x_j$, continuity of f_i ensures that player i can secure payoffs with a continuous function mapping a small enough open neighborhood to the diagonal.

Finally, recently Nessah and Tian [24] introduced the following condition:

Definition 4.4. *A game $G = (X_i, u_i)_{i\in N}$ is called **weakly transfer continuous** if for each non equilibrium strategy profile x, there exist an open neighborhood V_x of x, a player i and a strategy $\bar{x}_i$ such that $u_i(\bar{x}_i, y_{-i}) > u_i(y)$ for every $y \in V_x$.*

Let us now propose generalized versions of the three conditions presented above. In Proposition 4.8 below we show that these generalized versions are intermediate steps between each of the conditions and condition C.

Definition 4.5. *A game $G = (X_i, u_i)_{i\in N}$ is called* ***generalized better reply secure*** *if whenever $(x^*, u^*) \in \overline{\Gamma}$ and x^* is not an equilibrium, there exists a player i and a triple $(\varphi_i, V_x \ni x^*, \alpha_i > u_i^*)$ where $\varphi_i : V \rightrightarrows X_i$ is a well behaved correspondence, and V_x is open, such that $u_i(z_i, x_{-i}) \geq \alpha_i$ for every $z_i \in \varphi_i(x)$ and all $x \in V_x$.*

Definition 4.6. *A game $G = (X_i, u_i)_{i\in N}$ is called* ***generalized diagonal transfer continuous*** *if for each non equilibrium strategy profile x, there exists an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : X \rightrightarrows X_i$ such that $\sum_{i=1}^N u_i(z_i, y_{-i}) > \sum_{i=1}^N u_i(y)$, for all $y \in V_x$ and all $z_i \in \varphi_{i,x}(y)$.*

Definition 4.7. *A game $G = (X_i, u_i)_{i\in N}$ is called* ***generalized weakly transfer continuous*** *if for each non equilibrium strategy profile x, there exist an open neighborhood V_x of x, a player i and a well behaved correspondence $\varphi_{i,x} : X \rightrightarrows X_i$ such that $u_i(z_i, y_{-i}) > u_i(y)$ for all $y \in V_x$ and every $z_i \in \varphi_{i,x}(y)$.*[3]

The idea behind the generalized versions is that the "securing" strategies are allowed to vary as we vary the profiles y in the open neighborhood. The following result shows that condition C is more general than these generalized versions.

Proposition 4.8. *If a game $G = (X_i, u_i)_{i\in N}$ is either generalized better reply secure, generalized diagonal transfer continuous or generalized weakly transfer continuous, then condition C is satisfied.*

The latter two implications follow directly from the definitions. In order to show the former, i.e., that generalized better reply security implies condition C, we proceed as follows. For any given non equilibrium profile of strategies $x \in X$, we collect the set of payoff limits that each player can secure according to generalized better reply security, and compute the supremum of this set.[4] If condition C is violated, then we can find a sequence of profiles y converging to x such that each player's

[3] In an early version of this project, we introduced a generalization of weak transfer continuity, allowing the securing strategy $\bar{x}_i$ to vary continuously as y varied in V_x. The version below allows for semicontinuous variations. As explained in the Introduction, such condition does not capture complementary discontinuities, like those in Example 3.1, which is the fact that led us to propose condition C.

[4] As argued by Reny [28], it is without loss to have bounded payoff functions, so that this supremum is finite.

corresponding "securing" strategy does not secure more than the payoffs $u_i(y)$ associated with y. But since y converges to x, for any payoff limit $u = (u_i)_{i \in N}$ with $u_i = \lim_{y \to x} u_i(y)$ for each $i \in N$, resulting from this sequence, there must exist a player i than can secure a payoff strictly above u_i. But this is impossible because we started off with the supremum of the set of payoffs that a given player can secure. The details are in the Appendix.

Collecting the results above, we have:

Corollary 4.9. *Let $G = (X_i, u_i)_{i \in N}$ be a compact game. Then there exists a pure strategy Nash equilibrium if the game is either generalized better reply secure, generalized diagonal transfer continuous or generalized weakly transfer continuous, and the associated condition Q is satisfied.*

Therefore, whenever X is a *Hausdorff locally convex* topological vector space, we have a strict generalization of Theorem 3.1 in Reny [28] and an extension of Theorem 1 in Baye, Tian and Zhou [3] and Theorem 3.1 in Nessah and Tian [24].[5] It is only an extension of Theorems 1 in Baye, Tian and Zhou [3] and 3.1 in Nessah and Tian [24] because the convexity condition used in these theorems, namely diagonal transfer quasiconcavity, is quite weak. In fact, it is a necessary condition for existence of pure strategy Nash equilibria. On the other hand, in addition to allowing securing strategies to vary, Corollary 4.9 shows that Theorem 1 in Baye, Tian and Zhou [3] and Theorem 3.1 in Nessah and Tian [24] are valid under condition Q, which neither implies nor is implied by diagonal transfer quasiconcavity.[6]

Recall that Example 4.3 satisfies condition C and violates both better reply security and diagonal transfer continuity. It is simple to verify that it satisfies generalized better reply security, generalized diagonal transfer continuity and generalized weak transfer continuity.[7] But Example 2.3 violates generalized better reply security and generalized weak transfer continuity because the strict upper contour sets are erratic,

[5]In Reny [28], X is not required to be either Hausdorff or locally convex.

[6]It is important to stress that our generalization is in terms of the allowed discontinuities, not in terms of the convexity requirements. A number of papers in the qualitative games literature, and Baye, Tian and Zhou [3] and Nessah and Tian [24] explore the route of relaxing the convexity requirements. In fact, Nessah and Tian [24] covers a lot of ground in that direction, with the following condition: for every finite set $F \subset X$, there exists a profile $x \in X$ such that $u_i(y_i, x_{-i}) \leq u_i(x)$, for every player i and every $y \in F$. Nessah and Tian [24] shows that this condition can be used in the place of quasiconcavity in Theorem 3.1 in Reny [28], and in the place of diagonal transfer quasiconcavity in Theorem 1 in Baye, Tian and Zhou [3], also dropping the assumption that X_i be convex.

[7]Example 3.1 does not satisfy generalized weak transfer continuity.

and it also violates generalized diagonal transfer continuity, because of the following argument. At the non equilibrium profile $x = (1, 0, 1)$, the only two candidate profiles to "secure" a sum of payoffs strictly above 1 are $(0, 1, 0)$ and $(0, 0, 0)$. Any open neighborhood of x will include profiles of the form $y = (1, r, 1-r)$, where r is a small irrational number. For such profiles, we have $\sum_{i=1}^{3} u_i(z_i, y_{-i}) \leq 1 = \sum_{i=1}^{3} u_i(y)$, where z is either $(0, 1, 0)$ or $(0, 0, 0)$, so the game is not generalized diagonal transfer continuous. It follows that condition C is strictly weaker than the generalized versions of better reply security, diagonal transfer continuity and weak transfer continuity.

4.1. **Payoff Security.** One of the distinctive features of better reply security is that Reny [28] identified two simple and easily verifiable sufficient conditions for better reply security. A game is called **payoff secure** if for each $x \in X$ and each $\varepsilon > 0$, there exists an open neighborhood V of x and a strategy $\bar{x}_i$ such that $u_i(\bar{x}_i, y_{-i}) \geq u_i(x) - \varepsilon$ for all $y \in V$. A game is **weakly reciprocal upper semicontinuous** (*wrusc*) if whenever $(x^*, u^*) \in \overline{\Gamma} \backslash \Gamma$, there exists a player i and a strategy $\bar{x}_i$ such that $u_i(\bar{x}_i, x^*_{-i}) > u^*_i$. As shown in Bagh and Jofre [2] (extending the argument in Reny [28]), a payoff secure and *wrusc* game is better reply secure. Let us propose the following extension, allowing the securing strategies to vary:

Definition 4.10. *A game is called* ***generalized payoff secure*** *if for each $x \in X$, each $i \in N$ and each $\varepsilon > 0$, there exists an open neighborhood V of x and a well behaved correspondence $\varphi_i : V \rightrightarrows X_i$ such that $u_i(z_i, y_{-i}) \geq u_i(x) - \varepsilon$ for every $z_i \in \varphi_i(y)$ and all $y \in V$.*

Adapting the argument in Bagh and Jofre [2] we have:

Proposition 4.11. *A generalized payoff secure and wrusc game $G = (X_i, u_i)_{i \in N}$ is generalized better reply secure, and a fortiori satisfies condition C.*

Hence, generalized payoff security and *wrusc* are sufficient conditions for existence of pure strategy Nash equilibria in compact and quasiconcave games. Recently, Carmona [6] introduced the following two conditions. A game is **weakly payoff secure** if for every $i \in N$, $\varepsilon > 0$ and $x \in X$ there exists an open neighborhood V of x such that for each $z \in V$ there exists a y_i such that $u_i(y_i, z_{-i}) \geq u_i(x) - \varepsilon$; and a game $G = (X_i, u_i)_{i \in N}$ is **weakly upper semicontinuous** if for every (x, y, u) in the frontier of the graph of $\overline{u} : X \times X \to \mathbb{R}$, where $\overline{u}(x, y) = \{u_i(x_i, y_{-i})\}_{i \in N}$, there exists an $\bar{x}_i$ such that $u_i(\bar{x}_i, y_{-i}) > u_i$. In a compact, metric and quasiconcave game, Carmona [6] showed that weak payoff security and weak upper semicontinuity are sufficient conditions for existence of a pure strategy Nash equilibrium.

It is clear that weak payoff security is weaker than generalized payoff security: generalized payoff security allows the "securing" strategy to vary, so it is more general than payoff security, but it only allows some forms of semicontinuous variations, whereas weak payoff security does not restrict the kinds of variations allowed. Also, weak upper semicontinuity is stronger than *wrusc* (*wrusc* has bite only when $y = x$ in the definition of weak upper semicontinuity). The last example in Carmona [6] shows that we do not have room to improve on Proposition 4.11: it is an example of a compact, quasiconcave, weakly payoff secure and *wrusc*[8] game with no pure strategy Nash equilibrium.

Nevertheless, we have the following:

Proposition 4.12. *A compact game $G = (X_i, u_i)_{i \in N}$, where each X_i is metric, satisfying weak payoff security and weak upper semicontinuity satisfies condition C.*

Proposition 2.4 then shows that a compact, metric, weak payoff secure and weak upper semicontinuous game satisfying condition Q has a pure strategy Nash equilibrium.

As a final remark, let us mention that Nessah and Tian [24] provided some sufficient conditions for weak transfer continuity, introducing the following two notions. A game is **weakly transfer upper continuous** if whenever x is not an equilibrium, there exists a player i, a strategy $\bar{x}_i$ and an open neighborhood V_x of x such that $u_i(\bar{x}_i, x_{-i}) > u_i(y)$ for every $y \in V_x$; and a game is **weakly transfer lower continuous** if whenever x is not an equilibrium, there exists a player i, a strategy $\bar{x}_i$ and an open neighborhood V_x of x such that $u_i(\bar{x}_i, y_{-i}) > u_i(x)$ for every $y \in V_x$. It is straightforward to extend such conditions to allow the securing strategies to vary according to a well behaved correspondence, in the lines of generalized payoff security and provide sufficient conditions for generalized weak transfer payoff security, and *a fortiori* to condition C. The details are left for the reader.

5. Mixed Strategies

As advanced in the Introduction, our results strictly generalize the known sufficient conditions for existence of Nash equilibria under standard convexity assumptions. The traditional justification for convexity assumptions is the presumption that players may randomize their choices. Formally, for any given compact and

[8] In fact, the sum of payoffs in the example is smooth.

Hausdorff game $G = (X_i, u_i)_{i \in N}$, the **mixed extension** is obtained by using the spaces of Borel probability measures $\Delta(X_i)$ as the choice spaces for each player, and assuming that each u_i is a Borel measurable function so that expected utilities can be computed. By construction, the mixed extension is a compact and convex game with linear payoff functions. Therefore, we have existence of mixed strategy Nash equilibria whenever the mixed extension satisfies condition C.

Carbonell-Nicolau and Ok [5] and Monteiro and Page [20] introduced a sufficient condition for a game to have a payoff secure mixed extension. Namely, a game is called **uniform payoff secure** if if for each player i, each $x_i \in X_i$ and each $\varepsilon > 0$, there exists $\bar{x}_i$ such that for every $x_{-i} \in X_{-i}$ there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} with $u_i(\bar{x}_i, y_{-i}) \geq u_i(x_i, x_{-i}) - \varepsilon$ for all $y_{-i} \in V_{x_{-i}}$.

Let us propose an extension that allows the securing strategies to vary:

Definition 5.1. *A game is called* ***generalized uniform payoff secure*** *if for each player i, each $x_i \in X_i$ and each $\varepsilon > 0$, there exists a well behaved correspondence $\varphi_i : X_{-i} \rightrightarrows X_i$ such that for every $x_{-i} \in X_{-i}$ there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i(z_i, y_{-i}) \geq u_i(x_i, x_{-i}) - \varepsilon$ for every $z_i \in \varphi_i(y_{-i})$ and all $y_{-i} \in V_{x_{-i}}$.*

A straightforward extension of the arguments in Carbonell-Nicolau and Ok [5] and Monteiro and Page [20] shows that a generalized uniform payoff secure game has a generalized payoff secure mixed extension. Hence, existence of mixed strategy equilibria is guaranteed if in addition the mixed extension satisfies *wrusc*. A sufficient condition for which is that the sum of the payoffs be upper semicontinuous, as shown in Reny [28]. As an example, a zero sum game that is generalized uniform payoff secure has a Nash equilibrium in mixed strategies.

A different approach to existence of mixed strategy Nash equilibria is taken by Simon and Zame [32]. They allow payoff functions to be highly discontinuous and determined only in equilibrium. More precisely, Simon and Zame [32] define a **game with endogenous sharing rules** as $(X_i, U)_{i \in N}$, where $U : X \rightrightarrows \mathbb{R}^N$ represents the universe of possible payoff functions. A solution is a measurable selection $u : X \to \mathbb{R}$ from U and a profile of mixed strategies that form an equilibrium of the game $(X_i, u_i)_{i \in N}$. The main result in Simon and Zame [32] is as follows: if X_i is a compact metric space, and U is a non empty, bounded, convex, and upper hemicontinuous correspondence, then the game with endogenous sharing rules has a solution. Even though condition C in principle allows payoff functions that are highly discontinuous, and possibly not even measurable, the conditions in Simon and Zame [32] neither imply nor are implied by by condition C on the mixed extension of a game. On the other hand, a quite successful application of Simon and Zame [32] is in ensuring existence of a mixed strategy (subgame perfect) Nash equilibrium in two period games where a group of players moves in the first period, and the rest of the players moves in the second period, in particular games of competing mechanisms: for the first period movers, the realized payoffs can be viewed as a selection of an universe of possible payoffs that are a function of what happens in the second period. In the next section, we present an economic application of such games where conditions C and Q are satisfied and the conditions in Simon and Zame [32] are not.

6. Economic Applications

Summarizing, conditions C and Q are sufficient for existence of pure strategy Nash equilibria in compact, convex games, and C^{nc} in the place of C works for paracompact (in particular metric) games. In this section we show that conditions C and Q allow us to ascertain existence of Nash equilibria in important classes of economic games.

6.1. **Abstract Economies.** To begin with, the standard application of qualitative games is in showing existence of a general competitive equilibrium. The approach was pioneered by Mas-Colell [16], and followed by many others.[9] The basic idea is that a competitive economy gives rise to what is called an "abstract economy", which is easily seem as a "generalized game", that is, a qualitative game with feasibility constraints. Formally, a **generalized game** is given by $(X_i, P_i, F_i)_{i\in N}$, where for each agent $i \in N$, X_i is the consumption set, $P_i : X \rightrightarrows X_i$ is the preference correspondence, and $F_i : X \rightrightarrows X_i$ is the feasibility correspondence. An equilibrium is a profile $x \in X$ such that $x_i \in F_i(x)$ and $P_i(x) \cap F_i(x) = \varnothing$ for each $i \in N$.

An immediate extension of condition C is to consider that, whenever x is not an equilibrium, there exist an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$, one for each $i \in N$, such that for each $y \in V_x$ there exists $i \in N$ with $\varphi_{i,x}(y) \subset B_i(y)$, where $B_i : X \rightrightarrows X_i$ is a convex-valued majorant of $coP_i \cap F_i$.

And an immediate extension of condition Q is that $y_i \notin B_i(y)$ whenever i is activated at y, where activation means that $\varphi_{i,x}(y) \subset B_i(y)$. Following the steps leading to the proof of Theorem 2.4 it is straightforward to verify that an equilibrium exists whenever these conditions are satisfied and X_i is a compact and convex subset

[9]The list is too numerous to cite; the early contributions include Gale and Mas-Colell [11], Shafer and Sonnenschein [30], McKenzie [18], Yannelis and Prabhakar [38]. Again, see the monographs Tarafdar and Chowdhuri [33] and Yuan [39] and the references therein for a more complete account of the recent developments.

of a Hausdorff LCTVS. The extension to non compact spaces is also a straightforward adaptation of the steps leading to the proof of Theorem 2.6.

The extension above by-passes the issue of feasibility by implicitly resolving it in the proposed extension of condition C. An alternative approach goes as follows. Let $F = \{x \in X : x_i \in F_i(x) \text{ for every } i \in N\}$ and put $W_i = \{x \in F : P_i(x) \cap F_i(x) \neq \varnothing\}$. Assume the following extension of condition C:

Definition 6.1. *A generalized game $(X_i, P_i, F_i)_{i \in N}$ satisfies **condition** C_g if, whenever $x \in W_i$ for some i, then there exists an open neighborhood V_x of x, with $V_x \subset F$, and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$ with $\varphi_{i,x}(y) \subset F_i(y)$ for all $y \in V_x$, such that for each $y \in V_x$ there exists $i \in N$ with $\varphi_{i,x}(y) \subset B_i(y)$, where $B_i : X \rightrightarrows X_i$ is a convex-valued majorant of $coP_i \cap F_i$.*

And define the **condition** Q_g as above: $y_i \notin B_i(y)$ whenever i is activated at y, where activation means that $\varphi_{i,x}(y) \subset B_i(y)$. We can now show that a generalized game has an equilibrium whenever each F_i is a well behaved correspondence, each X_i is a compact and convex subset of a Hausdorff LCTVS, and conditions C_g and Q_g are satisfied. In fact, if there was no equilibrium, then we would have $W_i = F$ for each i, and since F is a closed subset of X, it is paracompact, and the steps leading to the proof of Theorem 2.4 would generate a contradiction, guaranteeing the existence of an equilibrium. Again, an analogous analysis can be applied to show existence of an equilibrium in the non compact case.

As with qualitative games, the improvement relative to the known results in the literature is that the extensions of condition C used above do not require that the strict upper contour sets (or their majorants) have continuous selections.

6.2. **Multi-Principal Multi-Agent Games.** Let us turn now to games with payoff functions. Since condition C strictly generalizes the conditions in Dasgupta and Maskin [9], Simon [31], Baye, Tian and Zhou [3] and Reny [28], among others, the economic applications considered in these works are examples of applications of our results. And one can in principle relax some of the continuity requirements in those applications. Instead of following this route, we present a class of multi-principal multi-agent games where the discontinuities that arise naturally are allowed by condition C and not allowed by the previous conditions used in the literature. The class encompasses and generalizes the following widely studied models:

(i) The original principal-agent model
(ii) Models with many principals and one agent, known as *common agency* games (see Bernheim and Whinston [4] and Martimort [14])

(iii) Models with one principal and many agents such as in Segal [31].

There is a set N of principals and a set M of agents. Let i denote the typical element of N and m the typical element of M. The game takes place in two stages. First the principals move simultaneously, each principal choosing a collection of contracts, one to each agent. The principals' offers are publicly announced to the agents, who then simultaneously take actions that are feasible given the principals' choices.

The set of available contracts of principal $i \in N$ is denoted by X_i, with typical element x_i. Let $X = \times_{i \in N} X_i$ denote the set of contract profiles available. Each agent has a set of available actions A_m, which may or may not depend on the principals' choices in the previous period. Let a_m denote a typical element of A_m, and $A = \times_{m \in M} A_m$ be the set of outcomes, with typical element a. Assume that each X_i and A_m is a compact and convex subset of a Hausdorff LCTVS.

This framework allows principals to use more complex mechanisms. Very few papers investigate existence of equilibria with such generality. McAfee [17] considers only direct mechanisms, and the principals in Prat and Rustichini [27] are allowed to choose only transfer schedules such that the money an agent receives is contingent on the action chosen by that particular agent. For a discussion on the issues that arise when one expands the set of allowed mechanisms, see Epstein and Peters [10] and Martimort and Stole [15].

As for the preferences, we allow for externalities among agents, principals, and cross-externalities. Thus, agent m's payoff is $v_m : X \times A \to \mathbb{R}$. The payoff to principal i is $u_i : X \times A \to \mathbb{R}$. Finally, we will focus on subgame perfect equilibria of the two stage game $(X, A, (u_i)_{i \in N}, (v_m)_{m \in M})$.

Remark 6.2. *It is possible to extend this setting to allow for incomplete information. Carmona and Fajardo* [7] *generalize Simon and Zame* [32] *in that direction, by allowing the payoff correspondence representing the universe of possible payoffs of the principals to depend in a measurable way on the agents type. Along the same lines of Prat and Rustichini* [27]*, we avoid this route since keeping track of measurability of strategies would only add difficulties to the analysis without bringing any insightful conclusion. Likewise, it is possible to adapt the above model to allow contract offers to be private. The conclusions are virtually unchanged.*

6.2.1. *Equilibrium analysis.* For each given profile $x \in X$ of mechanisms chosen by the principals, we have a **continuation game** $G^x = (A_m, V_m^x)_{m \in M}$, where

$$V_m^x(a) = v_m(x, a).$$

Assume that, for each $x \in X$, the continuation game G^x satisfies conditions C and Q, so that it has an equilibrium. Note that we can allow the agents to use mixed strategies, so that condition Q is without loss of generality. In this case, A_m is the set of probability distributions over a given set of pure strategies, and v_m is agent m's expected payoff. The results are the same.

Given any choice $x \in X$ of the principals, let $\sigma(x)$ denote a possibly random selection from the set of equilibria of the agents' game, that is, $\sigma(x) \in \Delta(E(G^x))$, where $E(G^x)$ is the set of equilibria of the continuation game G^x. The mapping σ induces a game played by the principals in the first stage, in which each principal $i \in N$ has strategy space X_i and derives payoff

$$u_i^\sigma(x) = \int u_i(x, a)\sigma(x)(da).$$

If u^σ satisfies conditions C and Q, then Proposition 2.4 ensures that the induced game $G^\sigma = (X_i, u_i^\sigma)_{i \in N}$ has an equilibrium. Under this combination of assumptions, therefore, the original game has a subgame perfect equilibrium.

This framework generalizes the best available existence results on existence of equilibria in multi-principal multi-agent models. It is general enough to encompass externalities among agents and principals and to allow for a large set of available mechanisms - direct and indirect, exclusive and non-exclusive. The results in Prat and Rustichini [27] do not apply to this case because payoff functions are assumed separable, there are no externalities among principals and the mechanisms are restricted to monetary transfers between principals and agents.

The standard approach taken in the literature is to assume that the payoff functions of the principals are continuous and that the agents are allowed to use correlated strategies, so that the conditions in Simon and Zame [32] are satisfied and one can use their result to conclude that a mixed strategy equilibrium exists, with the agents' actions providing the endogenous sharing rule for the principals' game. There are two problems with this approach. First, it is not clear that the principals' payoffs should be continuous, especially if there are externalities among them. Second, it is still desirable to find equilibria in pure strategies. In fact, Prat and Rustichini [27] investigates the connection between pure-strategy equilibria and efficiency;[10] Pérez-Castrillo [26] investigates the connection between pure-strategy equilibria and the set of stable solutions and the core of a cooperative game.

[10]Although they show that with direct agent externality, such as in our framework, there is no connection between pure-strategy equilibria and efficiency.

Instead, we consider that the discontinuities arising from the response of the agents are of the form allowed by condition C. To illustrate, consider the following example of an coordination game among principals. There are two principals and M agents. Principal i's payoff is given by

$$u_i(x_i, x_j) = \begin{cases} f_i(a) & \text{if } x_1 = g(x_2) \\ 0 & \text{otherwise} \end{cases}$$

where $g : X_2 \to X_1$ is a continuous and injective function, and $f_i : A \to \mathbb{R}$ is a non negative function.

The idea is the if the principals manage to coordinate their choices (that is, if $x_1 = g(x_2)$), then each i can potentially obtain a positive payoff, provided that the agents respond with an equilibrium $a \in A$ such that $f_i(a) > 0$. The function f_i may capture how the agents coordinate their actions to visit the principals. For instance, if a given action $\overline{a}$ means that agents choose to contract only with principal 1, then $f_1(\overline{a}) > 0$ and $f_2(\overline{a}) = 0$. If other action $\tilde{a}$ implies that some agents visit principal 1 and other agents visit principal 2, then $f_1(\tilde{a}) > 0$ and $f_2(\tilde{a}) > 0$.

Note that the strict upper contour set of each principal is either a singleton or the empty set, so condition Q is satisfied. Now, for each fixed profile of continuation equilibria $\sigma : X \to A$, we see that condition C is satisfied for the game G^σ. It suffices to use $\varphi_{i,x} : X \to X_i$ with $\varphi_{1,x}(y) = g(y_2)$ and $\varphi_{2,x}(y) = g^{-1}(y_1)$, for every $x \in X$ that is not an equilibrium of the game G^σ, and for an open neighborhood V_x belonging to the complement of the graph of g. Therefore, there exist Nash equilibria, which are by construction subgame perfect.

The game is not better reply secure, diagonal transfer continuous or weakly transfer continuous. This is obvious because the function g need not allow for constant securing strategies. Therefore, the approach taken in Monteiro and Page [20] and Monteiro and Page [21] does not work for this example. Moreover, the game does not satisfy the conditions in Simon and Zame [32] either, because the correspondence $U : X \rightrightarrows \mathbb{R}^2$, given by the universe of possible payoff vectors for the principals, given the response of the agents, is not upper hemicontinuous. In fact, for profiles x^n approaching a profile x with $x_1 = g(x_2)$, and with $g(x_1^n) \neq g(x_2^n)$, the payoffs of both principals are equal to zero, and in the limit at least one may jump up to a strictly positive number, depending on the equilibrium response of the agents. Therefore, the approach taken in Monteiro and Page [22] and Carmona and Fajardo [7] does not work either for this example.

Finally, notice that the response of the agents does provide the sharing rule: for instance, it may well be that, for each x with $x_1 = g(x_2)$, there are two equilibria of the continuation game, where in the equilibrium $\sigma^1(x)$ every agent chooses the mechanism of principal 1, so $f_1(\sigma^1(x)) > 0$ and $f_2(\sigma^2(x)) = 0$, and in the equilibrium $\sigma^2(x)$ every agent chooses the mechanism of principal 2, so $f_1(\sigma^2(x)) = 0$ and $f_2(\sigma^2(x)) > 0$. So, in effect, the two equilibria of the continuation game act as a coin flip for the principals. Notice also that for a fixed profile of continuation equilibria $\sigma : X \to A$, the payoff function of a given principal may be highly discontinuous in X, with agents responding with different choices of principals for nearby profiles $x \in X$. This feature makes the strict upper contour sets erratic, and yet is a natural economic condition (consumers may all coordinate in one store when two competitors choose the same price for their products).

To summarize, given our assumptions guaranteeing existence of a continuation equilibrium $\sigma(x)$ for each $x \in X$, assume further that the game G^σ satisfies conditions C and Q as in the example above. Then we guarantee existence of a pure strategy subgame perfect equilbrium in a class of games where some of the discontinuities that arise naturally are allowed only by condition C.

7. Appendix

Recall the Fan-Glicksberg fixed point theorem:

Lemma 7.1. *Let X be a compact and convex subset of a Hausdorff locally convex topological vector space, and $\Phi : X \rightrightarrows X$ a well behaved correspondence. Then there exists $x \in \Phi(x)$.*

A topological space X is **paracompact** if every open covering $\mathcal{V}$ has a locally finite open refinement $\mathcal{U}$ that is still a covering of X. That is, for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ with $U \subset V$, the set $\{U \in \mathcal{U} : x \in U\}$ is finite, and $X = \bigcup\{U : U \in \mathcal{U}\}$. For a given open covering $\mathcal{U}$ of X and $x \in X$, let $Star(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$. If the open covering $\{Star(x,\mathcal{U}) : x \in X\}$ refines an open covering $\mathcal{V}$ of X, then we say that $\mathcal{U}$ is a **Δ-refinement** of $\mathcal{V}$. A topological space is called **fully normal** if every open covering has an open Δ-refinement. Stone's theorem states that a T_1 space is paracompact if and only if it is fully normal.[11] In particular, X is paracompact whenever it is either a compact Hausdorff space or a metric space.

A paracompact space admits a **partition of unity** subordinated to the locally finite open refinement $\mathcal{V} = \{V_\alpha\}_{\alpha \in D}$, which consists of a family of functions $\{g_\alpha\}_{\alpha \in D}$ such that:

(i) $g_\alpha : X \to [0, 1]$ is continuous for each $\alpha \in D$,

[11] A topological space X is called T_1 for every distinct points x, y of X there exist two open sets U and V, with $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. A Hausdorff space is T_1. For more on the topological concepts used here, see Howes [12].

(ii) $\overline{\{x : g_\alpha(x) > 0\}} \subset V_\alpha$ for each $\alpha \in D$,[12]
(iii) $\sum_{\alpha\in D} g_\alpha(x) = 1$ for each $x \in X$.

Partitions of unity allow one to obtain global properties from local properties. Formally, we have the following "Gluing Lemma":

Lemma 7.2. *Let X be a paracompact and convex subset of a topological vector space. If for each $x \in X$ there corresponds an open neighborhood V_x of x and a well behaved correspondence $\varphi_x : V_x \rightrightarrows X$, then the φ_x's can be glued together into a well behaved correspondence $\Phi : X \rightrightarrows X$.*

Proof. Because X is paracompact, there exists a locally finite open refinement $\{V_\alpha\}_{\alpha\in D}$ of the open cover $\{V_x\}_{x\in X}$ and a partition of unity $\{g_\alpha\}_{\alpha\in D}$ subordinated. Define $\Phi : X \rightrightarrows X$ by

$$\Phi(x) = \sum_{\alpha\in D} g_\alpha(x)\varphi_{x_\alpha}(x). \tag{7.1}$$

Now, because $\{V_\alpha\}_{\alpha\in D}$ is locally finite, each x has a neighborhood V that intersects only finitely many of the V_α's. Hence $\Phi(x)$ is a convex combination of finitely many φ_x's, and hence the values of Φ are non empty, convex and compact. Finite convex combinations of compact valued upper hemicontinuous correspondences are upper hemicontinuous[13], so we need to show that the correspondence $g\varphi : X \rightrightarrows X$, defined by $g(x)\varphi(x)$ for each $x \in X$, where $g : X \to [0, 1]$ is a continuous function, is upper hemicontinuous with compact values. The values of $g\varphi$ are compact because each $g(x)\varphi(x)$ is the continuous image of a compact set. Pick a net x^α converging to x and a corresponding net $y^\alpha \in g(x^\alpha)\varphi(x^\alpha)$. Then $y^\alpha = c^\alpha z^\alpha$ with $c^\alpha = g(x^\alpha)$ and $z^\alpha \in \varphi(x^\alpha)$. If $y^\alpha \to y$, then because $[0, 1]$ is compact, $c^\alpha \to c \in [0, 1]$, and a fortiori $z^\alpha \to z$ with $y = cz$. By upper hemicontinuity of φ, $z \in \varphi(x)$, and by continuity of g, $c = g(x)$. Hence $y \in g(x)\varphi(x)$. □

Let N be an indexing set let X_i be a convex subset of a topological vector space, for each $i \in N$. Let $X = \times_{i\in N} X_i$ and $E \subset X$ be a given (possibly empty) subset of X. For any set $K \in X$, let coK denote its convex hull.

Definition 7.3. ***Condition*** *A is satisfied if: (i) there exist correspondences $A_i : X \rightrightarrows X_i$, $i \in N$, such that for each $x \in X$ there exists $i \in N$ with $x_i \notin coA_i(x)$; (ii) whenever $x \notin E$, there exist an open neighborhood V_x of x and well behaved*

[12]For any set $K \subset X$, $\overline{K}$ denotes its closure in X.
[13]Aliprantis and Border [1] Theorem 17.32.

correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$*, for each* $i \in N$*, such that* $\varphi_{i,x}(y) \subset A_i(y)$ *for every* $y \in V_x$ *and every* $i \in N$*.*

Theorem 7.4. *For each* $i \in N$*, let* X_i *be a compact and convex subset of a Hausdorff locally convex topological vector space. Let* $X = \times_{i \in N} X_i$ *and* $E \subset X$ *satisfy condition* A*. Then* $E \neq \varnothing$*.*

Proof. By way of contradiction, if $E = \varnothing$, then $X = \{x \in X : \exists (V_x, \{\varphi_{i,x}\}_{i \in N})$ as in condition $A\}$. Because a compact Hausdorff space is paracompact, Lemma 7.2 applied to each $i \in N$ yields a non empty, convex and compact valued upper hemicontinuous correspondence $\Phi_i : X \rightrightarrows X_i$. By condition A, $\Phi_i(x) \subset coA_i(x)$ for every $x \in X$ and $i \in N$. Putting $\Phi : X \rightrightarrows X$ as $\Phi(x) = \times_{i \in N} \Phi_i(x)$, we have that Φ is a non empty, convex and compact valued upper hemicontinuous correspondence, and we must have $x \notin \Phi(x)$ for every $x \in X$, for otherwise $x_i \in coA_i(x)$ for every $i \in N$, contradicting condition A. But from Lemma 7.1 there must exist a fixed point of Φ. This contradiction establishes that $E \neq \varnothing$. □

Using $E \subset X$ as the set of pure strategy Nash equilibria, from Theorem 7.4 we immediately have the following:

Corollary 7.5. *Let* $G = (X_i, P_i)_{i \in N}$ *be a game such that* X_i *is compact for each* $i \in N$ *and condition* A *is satisfied. Then there exists a pure strategy Nash equilibrium.*

Corollary 7.5 has no counterpart in the literature. It requires no assumption on the allowed discontinuities of the preference correspondences or the payoff functions, and neither does it require convexity/quasiconcavity assumptions. On the other hand, it does not provide a useful way of verifying if a particular game has an equilibrium, for condition A is not tied down to conditions on the payoff functions of the players.

Let $I : X \rightrightarrows N$ be a non empty correspondence, and let $Gr(I) = \{(x, i) \in X \times N : i \in I(x)\}$. An **aggregator** is a correspondence $A : X \rightrightarrows X$ given by $A(x) = \times_{i \in N} A_i(x)$, where for each $i \in N$

$$A_i(x) = \begin{cases} B_i(x) & \text{if } i \in I(x) \\ X_i & \text{otherwise} \end{cases} \tag{7.2}$$

where $B_i : X \rightrightarrows X_i$ is a convex valued correspondence with $coP_i(x) \subset B_i(x)$ for every $x \in X$. The idea is that $i \in I(x)$ means that i is "activated" at x.

Definition 7.6. *A game $G = (X_i, P_i)_{i\in N}$ satisfies **Condition** B if whenever $x \notin E$, there exist an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$, for $i \in N$, such that $\varphi_{i,x}(y) \subset A_i(y)$ for every $y \in V_x$, for some aggregator A.*

Definition 7.7. *A game $G = (X_i, P_i)_{i\in N}$ satisfies **Condition** Q^B if $x_i \notin B_i(x)$ for every $(x, i) \in Gr(I)$.*

Using the aggregator A in the definition of condition A, it is immediate that:

Lemma 7.8. *If a game $G = (X_i, P_i)_{i\in N}$ satisfies conditions B and Q^B, then it satisfies condition A.*

Using Lemma 7.8 we have the following corollary of Theorem 7.4:

Corollary 7.9. *Let $G = (X_i, P_i)_{i\in N}$ be a game such that X_i is compact for each $i \in N$ and conditions B and Q^B are satisfied. Then there exists a pure strategy Nash equilibrium.*

It is clear that condition B implies condition C. The converse is not necessarily true, because we may have a situation where one player must be activated at a profile y from the perspective of $y \in V_x$ (that is, using the well behaved correspondence $\varphi_{i,x} : V_x \rightrightarrows X_i$) and that the same player cannot be activated at the same profile y from the perspective of $y \in V_z$, with $x \neq z$. In the proof of Theorem 2.4 below we show that these situations can be avoided.

7.1. **Proof of Theorem 2.4.** By way of contradiction, assume that the game satisfies conditions C and Q and $E = \varnothing$. Since X is paracompact and $X = \{x \in X : \exists (V_x, \{\varphi_{i,x}\}_{i\in N})$ as in condition $C\}$, the open covering $\mathcal{V} = \{V_x\}_{x\in X}$ has an open Δ-refinement $\mathcal{U}$. Assign to each $x \in X$ an open neighborhood $U_x \in \mathcal{U}$, and the associated well behaved correspondences $\varphi_{i,y}$ restricted to U_x for some $V_y \in \mathcal{V}$ with $U_x \subset V_y$. Now, since $Star(x,\mathcal{U}) \subset V$ for some $V \in \mathcal{V}$, it is impossible to have a situation where one player must be activated at a profile y from the perspective of $y \in U_x$ and that the same player cannot be activated at the same profile y from the perspective of $y \in U_z$, with $x \neq z$: both U_x and U_z are contained in $Star(y,\mathcal{U})$, which is in turn contained in some $V \in \mathcal{V}$, and the associated well behaved correspondences can be applied throughout $Star(y,\mathcal{U})$. Hence we can define the activating mapping $I : X \rightrightarrows N$ by declaring that $i \in I(x)$ whenever player i is activated from the perspective of all neighborhoods forming $Star(x,\mathcal{U})$. Now defining the aggregator as in (7.2), we have that conditions C and Q imply that conditions B and Q^B are satisfied, contradicting Corollary 7.9.

7.2. **Extension to Non-Compact Spaces.** Let $\Phi : X \rightrightarrows Y$ be a correspondence between two topological vector spaces. We say that Φ satisfies the **local intersection property** if whenever $\Phi(x) \neq \varnothing$, there exists an open neighborhood V_x of x such that $\bigcap_{z \in V_x} \Phi(z) \neq \varnothing$.[14] Given any correspondence $\Phi : X \rightrightarrows Y$ with non empty values and an open neighborhood U of the origin of X, the correspondence $\Phi_U : X \rightrightarrows Y$ defined by $\Phi_U(x) = \bigcup_{x' \in x+U} \Phi(x')$ satisfies the local intersection property.[15]

Yuan [39] proved the following fixed point theorem:

Lemma 7.10. *Let X be a convex subset of a Hausdorff topological vector space, and let $A : X \rightrightarrows X$ be a correspondence satisfying the coercive condition. Let $\Phi : X \rightrightarrows X$ be a non empty valued correspondence satisfying the local intersection property and with $\Phi(x) \subset A(x)$ for every $x \in X$. Then there exists $x \in coA(x)$.*

Let X_i be a convex subset of a topological vector space for each $i \in N$, $X = \times_{i \in N} X_i$ and $E \subset X$ be a given (possibly empty) subset of X.

Definition 7.11. ***Condition*** *A^{nc} is satisfied if: (i) there exist convex correspondences $A_i : X \rightrightarrows X_i$, $i \in N$, such that for each $x \in X$ there exists $i \in N$ with $x_i \notin A_i(x)$; (ii) the set $X \backslash A(x)$ is compact for each $x \in X$, where $A(x) = \times_{i \in N} A_i(x)$, and $A : X \rightrightarrows X$ satisfies the coercive condition; (iii) whenever $x \notin E$, there exist an open neighborhood V_x of x and well behaved correspondences $\varphi_{i,x} : V_x \rightrightarrows X_i$, for each $i \in N$, such that $\varphi_{i,x}(y) \subset A_i(y)$ for every $y \in V_x$ and every $i \in N$.*

Theorem 7.12. *For each $i \in N$, let X_i be a paracompact and convex subset of a Hausdorff topological vector space. Let $X = \times_{i \in N} X_i$ and $E \subset X$ satisfy condition A^{nc}. Then $E \neq \varnothing$.*

Proof. If $E = \varnothing$, then $\{V_x\}_{x \in X}$, where V_x is the open neighborhood of x figuring in condition A, is an open covering of X, and from paracompactness and Lemma 7.2 we have a well behaved correspondence $\Phi : X \rightrightarrows X$ with $\Phi(x) \subset A(x)$ for every $x \in X$, where $A(x) = \times_{i \in N} A_i(x)$. We claim that there exists an open neighborhood U of the origin such that $\Phi_U(x) \subset A(x)$ for every $x \in X$. If such U does not exist,

[14] Local intersection property is also called "transfer open inverse valuedness" in the literature.

[15] In fact, there exists a symmetric open neighborhood V of the origin such that $V \subset U$. Let $y \in \Phi(x)$. For any $z \in x + V$ we have $x \in z + V \subset z + U$, so $y \in \Phi_U(z)$ for every $z \in x + V$.

then we can construct a net $x^U \to x$ with a corresponding net $y^U \in \Phi(x^U)$ with $y^U \notin A(x)$. Because $X \backslash A(x)$ is compact, there is a convergent subnet $y^U \to y \notin A(x)$, contradicting upper hemicontinuity of Φ. Because Φ_U satisfies the local intersection property, Lemma 7.10 implies that there exists $x \in A(x)$, contradicting condition A^{nc}. This contradiction establishes that $E \neq \varnothing$. □

It is important to note that condition A^{nc} does not correspond to condition A when X is compact. Although the assumption that the correspondence A satisfies the coercive condition is trivially satisfied when X is compact and convex, the assumption that $X \backslash A(x)$ is compact requires assuming, in addition to condition A, that the correspondence A be open-valued. It is also important to notice that X need not be locally convex in Theorem 7.12, as opposed to Theorem 7.4. The reason is that the fixed point theorem employed is based on local intersection property, and not on upper hemicontinuity. The cost, as just noted, is that extra assumption of open-valuedness of the correspondence A.

The results in the compact case can be extended to the non-compact case in a simple way. As above, since Lemma 7.10 does not require local convexity, the underlying spaces are assumed to be Hausdorff topological vector spaces. From Theorem 7.12 we have:

Corollary 7.13. *Let $G = (X_i, P_i)_{i \in N}$ be a game such that X_i is paracompact for each $i \in N$ and condition A^{nc} is satisfied. Then there exists a pure strategy Nash equilibrium.*

7.3. **Proof of Theorem 2.6.** The intermediate steps with extensions of conditions B and Q^B are analogous to the ones presented above, so we can go directly to the general case. That is, conditions C^{nc} and Q imply condition A^{nc}, so Theorem 2.6 follows directly from Corollary 7.13.

7.4. **Proof of Proposition 4.8.** For the case of generalized weak transfer continuity the result follows immediately from the definitions.

For the case of generalized diagonal transfer continuity the result is almost immediate, for if $\sum_{i=1}^{N} u_i(z_i, y_{-i}) > \sum_{i=1}^{N} u_i(y)$ for all $y \in V_x$ and all $z_i \in \varphi_{i,x}(y)$, then it must be the case that for each $y \in V_x$ there exists a player i with $u_i(z_i, y_{-i}) > u_i(y)$ for every $z_i \in \varphi_{i,x}(y)$ (if this is not true for some z_i, we can shrink $\varphi_{i,x}(y)$ accordingly – it is surely true for at least one element of this image).

For the case of generalized better reply security, let x^* not be an equilibrium, and let $L(x^*) = \{u \in \mathbb{R}^N : u = \lim_{x' \to x^*} u(x') \text{ for some } x' \to x^*\}$ be the associated compact set of vectors of payoff limits.[16] For each $i \in N$, let $L^i(x^*) = \{u \in L(x^*) : \exists(\varphi_i, V_{x^*} \ni x^*, \alpha_i > u_i)$ s.t. $u_i(z_i, x_{-i}) \geq \alpha_i$ for every $z_i \in \varphi_i(x)$ and all $x \in V_{x^*}\}$ be the set vectors of payoff limits associated with x^* such that player i can secure a better reply (by generalized better reply security, for each $u \in L(x^*)$ there must exist at least one player that can secure a better reply). Let $L_i^i(x^*) = proj_i L^i(x^*)$.

Let $u_i^* = \sup\{u_i : u_i \in L_i^i(x^*)\}$ whenever $L_i^i(x^*) \neq \varnothing$, and $N^* = \{i \in N : u_i^* \in L_i^i(x^*)$ be the set of players for which the supremum u_i^* is achieved. For each $i \in N$ with $L_i(x^*) \neq \varnothing$ let $\{u_i^n\}_{n>0}$ be a sequence in $L_i^i(x^*)$ with $u_i^n \to u_i^*$, where $u_i^n = u_i^*$ for all $n > 0$ whenever $i \in N^*$. Let $(\varphi_i^n, V_{i,x^*}^n \ni x^*, \alpha_i^n > u_i^n)$ be the associated securing triple $u_i(z_i, x_{-i}) \geq \alpha_i^n$ for every $z_i \in \varphi_i^n(x)$ and all $x \in V_{i,x^*}^n$. Let $V^n = \bigcap_{\{i:L_i(x^*)\neq\varnothing\}} V_{i,x^*}^n$, and for each i with $L_i(x^*) = \varnothing$, define $\varphi_i^n(x) = \bar{x}_i$, for some arbitrary $\bar{x}_i$ and all $x \in V^n$.

Fix $n > 0$. If there exists an open neighborhood $U \ni x^*$ such that for each $y \in U$ there exists $i \in N$ with $u_i(z_i, y_{-i}) > u_i(y)$ for every $z_i \in \varphi_i^n(y)$, then condition C is obtained (using U and φ_i^n for all $i \in N$). Otherwise, we can find a directed system of neighborhoods $\mathcal{U}$ of x^* and a net $\{y^{n,U}\}_{U\in\mathcal{U}}$ with $y^{n,U} \to_{\mathcal{U}} x^*$ with the property that $u_i(z_i^{n,U}, y_{-i}^{n,U}) \leq u_i(y^{n,U})$, for some $z_i^{n,U} \in \varphi_i^n(y^{n,U})$ and every $i \in N$, along the net.

Repeat the argument for each $n > 0$. If there is an $n > 0$ such that we are able to find the open neighborhood U as above, then condition C is satisfied. If not, then we can construct a diagonal sequence $\{y^{n,U(n)}\}_{n>0}$ such that $y^{n,U(n)} \to x^*$, and a fortiori $u(y^{n,U(n)}) \to u \in L(x^*)$. There must exist a player i such that $u \in L^i(x^*)$.

If $i \in N^*$, then

$$u_i^* < \alpha_i^* \leq u_i(z_i^{n,U(n)}, y_{-i}^{n,U(n)}) \leq u_i(y^{n,U(n)})$$

which is impossible because $u_i(y^{n,U(n)}) \to u_i \leq u_i^*$.

If $i \notin N^*$, then

$$u_i^n < \alpha_i^n \leq u_i(z_i^{n,U(n)}, y_{-i}^{n,U(n)}) \leq u_i(y^{n,U(n)})$$

which is again impossible because $u_i^n \to u_i^*$ and $u_i(y^{n,U(n)}) \to u_i < u_i^*$ (because $i \notin N^*$).

Summing up, it has to be that there exists $n > 0$, an open neighborhood $U \ni x^*$ and well behaved correspondences $\varphi_i^n : U \rightrightarrows X_i$ such that for each $y \in U$ there exists $i \in N$ with $\varphi_i^n(y) \in P_i(y)$, so condition C is obtained.

[16]Without loss of generality, the payoff functions are assumed to be bounded. See Reny [28].

7.5. **Proof of Proposition 4.12.** From Theorem 1 in Carmona [6] we know that whenever a compact and metric game is weakly payoff secure and weakly upper semicontinuous, the best reply correspondences are non empty and compact valued, and upper hemicontinuous. Hence, picking any non equilibrium profile $x \in X$, there exists a player $i \in N$ for which $x \notin Gr(BR_i)$, where $BR_i : X_{-i} \rightrightarrows X_i$ is given by $BR_i(x_{-i}) = \arg\max_{y_i \in X_i} u_i(y_i, x_{-i})$ and $Gr(BR_i)$ is its graph ($Gr(BR_i) = \{x \in X : x_i \in BR_i(x_{-i}\})$). Since $Gr(BR_i)$ is closed, there exists an open neighborhood V_x of x contained in $X \backslash Gr(BR_i)$. Put $\varphi_{i,x} : V_x \rightrightarrows X_i$ as $\varphi_{i,x}(y) = coBR_i(y_{-i})$, a well behaved correspondence. For the remaining players $j \neq i$, let $\varphi_{j,x}$ be an arbitrary well behaved correspondence. Then, by construction, $\varphi_{i,x}(y) \in coP_i(y)$ for every $y \in V_x$, so condition C is verified.

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