AGREEING TO DISAGREE: THE NON-PROBABILISTIC CASE

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Abstract. A non-probabilistic generalization of Aumann's (1976) agreement theorem is proved. Early attempts at such a theorem were based on a version of the sure-thing principle which assumes an *intrapersonal-interstate* comparison of knowledge. But such comparisons are impossible in partition structures. The theorem proved here is based on a new version of the sure-thing principle that makes an *interpersonal-intrastate* comparison of knowledge.

1. Introduction

1.1. **Agreement theorems.** In his seminal paper "Agreeing to disagree" Aumann (1976) proved a probabilistic agreement theorem: Agents with a common prior cannot have common knowledge of their posterior probabilities for some given event, unless these posteriors coincide. In non-probabilistic agreement theorems the posteriors of the agents are replaced by abstract "decisions". Such theorems specify conditions on agents' decisions under which the agents cannot have common knowledge of their decisions unless the decisions coincide. A non-probabilistic agreement theorem generalizes the probabilistic one, if agents' posteriors of a given event satisfy the conditions required from decisions. For a survey on agreement theorems see Bonanno and Nehring (1997).

1.2. **Knowledge and decisions.** An agreement theorem is formally stated for a *knowledge structure* (*structure*, for short) $(\Omega, K_1, \dots, K_n)$, for n agents $(n \geq 1)$, where Ω is a nonempty set, of *states*, and K_i is a function $K_i \colon 2^\Omega \to 2^\Omega$ called agent i's *knowledge operator*. Subsets of Ω are called *events*. The *common knowledge* operator C is defined by $C(E) = \cap_{m=1}^{\infty} K^m(E)$, where $K(E) = \cap_i K_i(E)$, and K^m are powers of the operator K.

The structure is a *partition structure* if for each i there exists a partition of Ω, π_i, such that for each event E, $K_i(E) = \{\omega \mid \pi_i(\omega) \subseteq E\}$, where $\pi_i(\omega)$ is the element of the partition π_i that contains ω. Partitionality can be expressed in terms of the knowledge operators. A structure is a partition structure iff for each i, E, and F, $K_i(E) \subseteq E$, $K_i(E \cap F) = K_i(E) \cap K_i(F)$, and $\neg K_i(E) = K_i(\neg K_i(E))$. (See, Aumann (1999), Fagin et al. (1995) and Samet (2006).)

Let D be a nonempty set of *decisions*. A *decision function* for agent i is a function $\mathbf{d}_i \colon \Omega \to D$. A vector $\mathbf{d} = (\mathbf{d}_1, \dots, \mathbf{d}_n)$ is called a *decision function profile*. We denote by $[\mathbf{d}_i = d]$ the event $\{\omega \mid \mathbf{d}_i(w) = d\}$.

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1.3. **Early attempts.** Bacharach (1985) and Cave (1983) independently proved an agreement theorem for partition structures, in terms of *virtual decision functions* $\delta_i \colon 2^\Omega \to D$. For an event E, $\delta_i(E)$ is the decision made by i when her knowledge is given by E. The function δ_i satisfies the **sure-thing principle** when for any family of disjoint events $\mathcal{E}$ and decision d, if $\delta(E) = d$ for each $E \in \mathcal{E}$, then $\delta(\bigcup_{E\in\mathcal{E}} E) = d$.[1]

The decision function $\mathbf{d}_i$ is *derived* from δ_i if for each ω, $\mathbf{d}_i(\omega) = \delta(\pi_i(\omega))$. The agents are **like-minded** if all the decision functions $\mathbf{d}_i$ are derived from the same function δ.

Bacharach's and Cave's theorem states that if agents are like-minded and the virtual decision function from which the decision functions are derived satisfies the sure-thing principle, then whenever the decisions are common knowledge they coincide.

1.4. **Flaws and partial remedies.** The setup in Bacharach's and Cave's works brings together two *epistemic* structures: a partition structure and a virtual decision function. In the first, knowledge is explicitly expressed in terms of knowledge operators. In the second, knowledge is implicit. In particular the sure-thing principle is *not* expressed in terms of the knowledge operators of the structure. It is not surprising, therefore, that this conceptual mixture results in the following inconsistency.

The sure-thing principle is based on the idea that when her knowledge is given by E, an agent is *at least as knowledgeable* as she is when her knowledge is given by $E \cup F$. Unfortunately, the relation of being "at least as knowledgeable as" cannot be properly formalized for partition structures. Given any two states, either an agent's knowledge is the same in both, or else she knows in either state something she does not know in the other. To see this, suppose the agent knows a fact f in state ω and does not know it in state ω'. Then, she *knows* in ω' that she does not know f, while in ω, she *does not know* that she does not know f.

Thus, proper *intrapersonal-interstate* comparison of knowledge (the knowledge of *one* agent in *two* states) is impossible in partition structures. Hence, the sure-thing principle cannot be expressed in such structures.

The main focus in Moses and Nachum (1990) is criticism of the sure-thing principle along the lines suggested here. Moreover, in their paper, as well as in Aumann and Hart (2006), the sure-thing principle is rescued by restricting the relation of being "at least as knowledgeable" to certain facts only. This alone is not enough, and other requirements are added to prove an agreement theorem. These requirements are ad hoc and hard to defend, the proposed models perpetuate the conceptual mixture of knowledge structures with virtual decision functions, and finally, their agreement theorems do not generalize the probabilistic agreement theorem.

2. The generalized agreement theorem

2.1. **Comparison of knowledge.** We adopt here a sure-thing principle which is based on *interpersonal-intratstate* comparison of knowledge. That is, we make the conceptually innocuous comparison of the knowledge of *two* agents in *one* and the same state.

The event that agent j is **at least as knowledgeable as** i is

[1] The use of the term sure-thing principle, borrowed from Savage (1954), was introduced into the present context by Bacharach (1985).

$$[j \succeq i] := \bigcap_{E \in 2^{\Omega}} \Big(\neg K_i(E) \cup K_j(E)\Big). \tag{1}$$

At each $\omega \in [j \succeq i]$, j knows at ω every event that i knows there.

The event that agents i and j are **equally knowledgeable** is

$$[j \sim i] := [j \succeq i] \cap [i \succeq j].$$

An agent i is an **epistemic dummy** if it is always the case that all the agents are at least as knowledgeable as i. That is, for each agent j, $[j \succeq i] = \Omega$.[2]

2.2. **Properties of decision function profiles.** Our main assumption on a decision function profile $\mathbf{d} = (\mathbf{d}_1, \ldots, \mathbf{d}_n)$ is as follows,

Interpersonal sure-thing principle (ISTP): *For any pair of agents i and j, and decision d, if i knows that j is at least as knowledgeable as she is, and also knows that j's decision is d, then her decision is also d. That is,*

$$K_i\Big([j \succeq i] \cap [\mathbf{d}_j = d]\Big) \subseteq [\mathbf{d}_i = d].$$

The ISTP alone is not enough to state an agreement theorem. We need a stronger property that says that the ISTP can be preserved even if a new agent, who is an epistemic dummy, joins the agents.

Expandability: *A decision function profile* $\mathbf{d}$ *on* $(\Omega, K_1, \ldots, K_n)$ *is* **expandable** *if for any expanded structure $(\Omega, K_1, \ldots, K_n, K_{n+1})$, where $n+1$ is an epistemic dummy, there exists a decision function $\mathbf{d}_{n+1}$, such that $(\mathbf{d}_1, \ldots, \mathbf{d}_n, \mathbf{d}_{n+1})$ satisfies the* ISTP

Obviously if $\mathbf{d}$ is expandable it satisfies the ISTP.

2.3. **A statement of the theorem.**

The Generalized Agreement Theorem. *If* $\mathbf{d}$ *is an expandable decision function profile on a partition structure, then for any decisions $d_1, \ldots, d_n$ which are not identical, $C(\cap_i [\mathbf{d}_i = d_i]) = \emptyset$.*

3. Discussion

3.1. **The formulation of the theorem.** The theorem is formulated in purely syntactical terms. That is, using only set theoretic operations and knowledge operators without mentioning states or partitions.[3] The translation into formal language is straightforward. In particular, no use is made of virtual decision functions which cannot be described in terms of a partition structure.

Note, that unlike the sure-thing principle, the formulation of the ISTP involves no union of events, let alone the bewildering requirement of disjointness of these events.[4] Disjointness in our setup plays an important role, but it is not peculiar to the ISTP. Rather, it is a property of the elements of a partition that defines knowledge, and it is derived from the axioms of partitionality.

[2]The notion of an epistemic dummy is closely related to the notion of uninformed outsider defined by Nehring (2003) for probabilistic models.

[3]The required partitionality of the structure can be expressed in syntactic terms.

[4]See Moses and Nachum (1990) and Aumann et al. (2005).

3.2. **The implications of the ISTP.** The ISTP fuses the ideas underlying both the sure-thing principle and the like-mindedness in Bacharach's and Cave's theorem. Like the sure-thing principle, the ISTP reflects the idea that if some decision is invariably made when there is a lot of knowledge, then the same decision should be made when there is less knowledge. Like the like-mindedness assumption, the ISTP compares agents' knowledge and decisions. It is this comparison that ties together the decisions of different agents and provides common ground for the decision functions.

Not only does the ISTP resemble like-mindedness in spirit. It implies like-mindedness in a very precise sense that does not require the use of the questionable virtual decision functions.

Proposition 1. *If the decision function profile* $\mathbf{d}$ *satisfies the* ISTP*, then equally knowledgeable agents make the same decisions. That is,*

$$[i \sim j] \subseteq \bigcup_{d \in D} \big([\mathbf{d}_i = d] \cap [\mathbf{d}_j = d]\big).$$

The implicit assumption that the decisions are made by the agents in the same manner, except for the differences in information, is also manifested in the following implication of the ISTP. It is possible that an agent k knows that both i and j are at least as knowledgeable as he is, and he may also know their decisions. By the ISTP his decision is the same as both j and k. Thus j and k must make the same decision.

3.3. **Expandability.** For all its strength the ISTP is not enough for the agreement theorem. It may be satisfied vacuously without revealing that agents' decisions are in tune. In Bacharach's and Cave's theorem this problem is solved by the injection of knowledge external to the structure, through the virtual knowledge function, which helps to reveal the consistency of agents' decisions. Faithful to our interpersonal approach, finding the source of the required external knowledge here is simple: we allow the introduction of a new agent into the structure. Thus, we require not only that the decisions of the agents in the structure satisfy the ISTP, but that even if we add another agent to the structure we can endow her with a decision function such that the ISTP is still preserved for the larger set of agents.

Obviously, the theorem would hold a fortiori if we allowed the introduction of *any* agent, not necessarily an epistemic dummy one. Weakening expandability, of course, strengthens the theorem. But a deeper reason for this weakening is the implicit assumption contained in expandability. The decision functions $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ depend on agents' knowledge as described in the structure $(\Omega, K_1, \ldots, K_n)$. In the expanded structure $(\Omega, K_1, \ldots, K_n, K_{n+1})$ knowledge of the agents increases and it involves agent $n+1$'s knowledge. We assume, though, that the agents make the same decision as before. This demanding assumption becomes plausible when the added agent is and epistemic dummy.

3.4. **Generalizing Aumann's agreement theorem.** In Aumann's probabilistic agreement theorem for partition structures, decisions are real nonnegative numbers, and $\mathbf{d}_i$ assigns to each state ω, i's posterior probability of a given event E at ω. It is straightforward to show that $\mathbf{d}$ satisfies expandability (even a stronger version of it that allows the introduction of any agent, not necessarily an epistemic dummy one). Therefore Aumann's theorem is a special case of the generalized theorem.

4. Proofs

Lemma 1. $\omega \in [j \succeq i]$ *iff* $\pi_j(\omega) \subseteq \pi_i(\omega)$.

Proof: Suppose $\omega \in [j \succeq i]$. For $E = \pi_i(\omega)$ it follows from (1) that $\omega \in \neg K_i(\pi_i(\omega)) \cup K_j(\pi_i(\omega))$. As $\omega \in \pi_i(\omega) = K_i(\pi_i(\omega))$ it follows that $\omega \in K_j(\pi_i(\omega))$, and hence $\pi_j(\omega) \subseteq \pi_i(\omega)$.

Conversely, suppose the latter inclusion holds, and assume that for some E, $\omega \in K_i(E)$. Then $\pi_i(\omega) \subseteq E$ and therefore $\pi_j(\omega) \subseteq E$, which means that $\omega \in K_j(\omega)$. Hence for each E, $\omega \in \neg K_i(E) \cup K_j(E)$ which means that $\omega \in [j \succeq i]$. ∎

Lemma 2. $\omega \in K_i([j \succeq i])$ *iff* $\pi_i(\omega) = \cup_{\omega' \in \pi_i(\omega)} \pi_j(\omega')$.

Proof: $\omega \in K_i([j \succeq i])$ iff $\pi_i(\omega) \subseteq [j \succeq i]$. By Lemma 1 this holds iff for each $\omega' \in \pi_i(\omega)$, $\pi_j(\omega') \subseteq \pi_i(\omega)$ which is equivalent to $\pi_i(\omega) = \cup_{\omega' \in \pi_i(\omega)} \pi_j(\omega')$. ∎

Proof of Proposition 1: If $\omega \in [j \sim i]$, then by Lemma 1, $\pi_i(\omega) = \pi_j(\omega)$. Therefore, by Lemma 2, $\omega \in K_i([j \succeq i])$. Suppose $\mathbf{d}_j(\omega) = d$. Then, $\pi_i(\omega) = \pi_j(\omega) \subseteq [\mathbf{d}_j = d]$. Hence, $\omega \in K_i([\mathbf{d}_j = d])$. By ISTP this implies that $\omega \in [\mathbf{d}_i = d]$. ∎

Proof of The Generalized Agreement Theorem: Define π_{n+1} to be the finest partition, coarser than any of the partitions π_i. It is well known that the knowledge operator K_{n+1} defined by π_{n+1} is the common knowledge operator C. (See, Aumann (1999) and Fagin et al. (1995).) Note also that by the definition of π_{n+1} and Lemma 2, for each j, $K_{n+1}([j \succeq n+1]) = \Omega$ and therefore $[j \succeq n+1] = \Omega$. Thus, agent $n+1$ is epistemic dummy. By expandability, there exists $\mathbf{d}_{n+1}$, such that $(\mathbf{d}_1, \ldots, \mathbf{d}_n, \mathbf{d}_{n+1})$ satisfies ISTP.

Suppose $\omega \in C(\cap_i [\mathbf{d}_i = d_i]) = K_{n+1}(\cap_i [\mathbf{d}_i = d_i])$. Since

$$K_{n+1}(\cap_i [\mathbf{d}_i = d_i]) = \cap_i K_{n+1}([\mathbf{d}_i = d_i]),$$

it follows that for each j,

$$\omega \in K_{n+1}([\mathbf{d}_j = d_j]). \tag{2}$$

For each j, π_{n+1} is coarser than π_j, and thus

$$\pi_{n+1}(\omega) = \cup_{\omega' \in \pi_{n+1}(\omega)} \pi_j(\omega').$$

Hence, by Lemma 2,

$$\omega \in K_{n+1}([j \succeq i]). \tag{3}$$

By ISTP, it follows from (2) and (3) that for each j, $\omega \in [\mathbf{d}_{n+1} = d_j]$. Thus, all the decisions d_j coincide with $\mathbf{d}_{n+1}(\omega)$. ∎

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