A Dual Pre-Kernel Representation based on the Fenchel-Moreau Conjugation of the Characteristic Function

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Abstract

Like common pool resources, a central task in real life managerial problems concerns the sustainability of the resource. Especially in areas where multilateral agreements are non-binding, compliance becomes a crucial issue in order to avoid a destruction of a natural resource. Compliance can be achieved if agents exercise self-constraint and refrain from using their powers to exploit one another. Then a solution can be obtained that is acceptable for all participants - say a fair compromise. Such a compromise will be considered a fair outcome that produces a common virtual world where compliance is reality and obstruction be held to account. Rather than considering fairness as some cloudy concept, in order to advance our understanding of compliance on non-binding agreements, we study a fairness solution that is based on the cooperative game theory's pre-kernel. Other solutions such as the Shapley value are usually considered a more attractive concept for solving economical problems or for experimental studies, which might however originate in its simplicity of computation. In this paper, we review and improve an approach invented by Meseguer-Artola (1997) to compute the pre-kernel of a cooperative game by the indirect function. The indirect function is known as the Fenchel-Moreau conjugation of the characteristic function introduced by Martinez-Legaz (1996). Following and extending the approach proposed by Meseguer-Artola (1997) with the indirect function, we are able to characterize the pre-kernel of the grand coalition simply by the solution sets of a family of quadratic objective functions. Now, each function can be easily solved by means well known from analysis and linear algebra.

Keywords: Transferable Utility Game, Pre-Kernel, Convex Analysis, Fenchel-Moreau Conjugation, Indirect Function

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1 Introduction

Consider the situation that n-agents want jointly manage a natural resource like a fishery. In order to realize the gains which are available through mutual cooperation, the agents must act beyond their individual interest to overcome the problem of acting against their common interest. This is usually classified as a social dilemma. Concerning the sustainability of the natural resource the agents can make binding as well as non-binding agreements when they have the opportunity to communicate with each other. Especially, in the latter case, that is, in the absence of any institution that guarantees fulfillment of the agreement, compliance becomes the crucial issue. People will obstruct their promise, if they have the feeling of being treated unfair. Establishing fairness rules, which are accepted by all partners, so that powerful agents behave respectful and do not use their power to bully weak partners, is used as a mean to achieve compliance. As a consequence, obstruction becomes more unlikely and the destruction of the resource can be avoided. But what kind of rule can be considered a fair compromise that is acceptable for all participants? Various fairness rules have been already introduced and discussed in the literature (for an overview see Moulin (2003)). Each fairness concept has its merits as well as its flaws. In this paper, we want to focus on a fairness rule from cooperative game theory that did not find so much attention in the literature as a possible outcome to solve a social dilemma situation. The solution concept we want to study is known as the pre-kernel of a transferable utility game. This solution was not considered an attractive outcome for real life problems simply due its difficulty in computation. Here, for applied oriented researchers the Shapley value offers much more charm than the pre-kernel. Surprisingly, probably for many, one of our findings is that the pre-kernel can be calculated almost as simple as the Shapley value; and can be therefore considered a fair division rule for real life managerial problems. But before we discuss the details, we want to review certain aspects of the pre-kernel in order to allow the reader a better assessment of our arguments. By doing so, we focus first our attention to the kernel, and later to our actual subject of interest, the pre-kernel.

The kernel solution, which is related to the pre-kernel, was introduced by Davis and Maschler (1965) to study the bargaining set of a transferable utility game. The precise definition of the kernel is given in terms of a system of inequalities. But this definition lacks crucial information about the structure of the kernel. Moreover, it is almost impracticable to compute the kernel simply in applying its definition, even for games with a small number of players. This is due to the large number of inequalities and pre-imputations one has to consider in performing such a computation. Thus, there was a strong need to find out an alternative description of the kernel that offers more insight into its structure, and therefore a more tractable way to compute an element. A step forward in this direction was made by Maschler and Peleg (1966), as they found out that the kernel can be characterized by a representation formula based on a separation rule of players which have been induced by sets of coalitions. Although the content of such a characterization seems not to be clear at the moment, it provides, nevertheless, a clear geometrical description of the kernel. Namely that this characterization describes the kernel as a finite union of closed convex polyhedra. Relying on such a representation, Maschler and Peleg were able to give a complete algebraic proof that the kernel is a non-empty subset of the bargaining set. Furthermore, Maschler and Peleg (1967) were able to derive a set of rules that determine elements of the kernel by studying the properties of the representation formula. In this respect the solution concept of the pre-kernel was developed in the work of Maschler et al. (1972) in order to assist the study of the kernel for a certain class of games. The definition of the pre-kernel is now stated in terms of a system of equalities rather than a system of inequalities.

Although Maschler and Peleg (1967) established certain rules to determine kernel elements, the computation process was still difficult and required non-systematic short cuts. Therefore, it was not surprising to see that some early convergence algorithms could be developed, for instance by Aumann et al. (1965, 1966); Kopelowitz (1967); Stearns (1968) to specify systematically the kernel of a transferable utility game. More recent convergence algorithms have been introduced by Meseguer-Artola (1997); Faigle et al. (1998). Generically, a convergence process has several drawbacks. First, the convergence process could be very slow. Second, one has to assure that in any event where the process terminates, an element of the kernel was really found. Finally, any relationship to the structure of the game is lost.

Following an LP approach that was evolved by Kohlberg (1972) to compute the nucleolus, Wolsey (1976) illustrated that such an approach can also be successfully applied to the kernel computation for simple games. One of the latest LP method for computing the kernel stemmed from Meinhardt (2006), which emerged from an idea to use its geometrical properties, as it has been exhibited by the work conducted in Maschler et al. (1979). This LP approach relies on the fact that the kernel occupies a central position in the strong ϵ-core. Drawbacks related to an LP approach, are for instance, that an enormous number of constraints must be treated somehow, whenever the number of players is large. To this end, it was shown in Meinhardt (2006) that the kernel might not be completely represented by the solution sets of LPs.

The kernel characterization discussed so-far made use of concepts and methods from discrete mathematics. Despite the fact that the representation formula of the kernel can be expressed as the union of closed convex polyhedra, the intuitive meaning of this formula is difficult to understand. In consequence, a direct computation by hand for large and asymmetric games is complicated and requires a lot of experience. Fortunately, convex analysis provides us with the methods to solve this problem at least for the pre-kernel for every transferable utility game. This statement can be transmitted to the kernel solution for the class of zero-monotonic games. Since for this large class of cooperative games, the kernel coincides with the pre-kernel. The mathematical objects we derived to obtain an alternative characterization of the pre-kernel offer a clear and intuitive meaning in terms of solution sets. The first step towards this new pre-kernel characterization relies on a dual representation of a cooperative game invented by Martinez-Legaz (1996). In this paper, it was established that every cooperative game has a representation based on the Fenchel-Moreau conjugation, what he called the indirect function. It was shown that the concepts of cooperative game theory can also be formulated in terms of the indirect function. Some concepts can be substantially simplified, like monotonicity, but many, like convexity, can not. Although we get for most of the concepts more complicated expressions than in terms of the characteristic function, it was the merit of Meseguer-Artola (1997) to show that the indirect function approach is very useful to obtain a simplified pre-kernel representation. He recognized that the pre-kernel can be described as an overdetermined system of non-linear equations. From this overdetermined system an equivalent minimization problem can be constructed whose set of global minimums coalesce with the pre-kernel set. However, the structural form of the objective function remained unclear. Thus he was forced to develop a convergence algorithm that was based on a modified *Steepest Descent Method* for determining zeros of continuous functions in order to solve the resultant minimization problem. However, Meseguer-Artola did not recognize that imposing some additional conditions is enough to induce a simplified form of the objective function, which makes it possible to describe a practical method in the computation of the pre-kernel.

In particular, if we fix an arbitrary element of the pre-imputation set, the induced objective function is convex and quadratic. In addition, we can prove that whenever we have solved the reduced problem, we have also solved the original problem with its more complex objective function. This implies that we are able to calculate an element of the pre-kernel simply in applying methods well known from analysis and linear algebra. The solution set of a quadratic function is convex and it constitutes only a subset of the pre-kernel, if an additional condition is imposed, otherwise, we compute a minimum but not a pre-kernel element. Remind from the discussion above that the pre-kernel can be described by the union of convex polyhedra, hence the pre-kernel might be not a convex set. Thus, a unique convex solution set might not be enough to describe the pre-kernel. This statement can only be confirmed, for instance, for three person and convex games, but not in general. Now our main result should not be anymore so surprising that a union of arbitrary solution sets derived from a collection of quadratic objective functions completely characterize the pre-kernel of a transferable utility game. However, we can not guarantee in general that the condition on the solution set is satisfied to compute a point in the pre-kernel directly. Hence, we need to construct a sequence of optimization problems that converges in the limit to an optimization problem from which we can determine a pre-kernel point. We are able to show that such a sequence exists. Thus, we provide a practical method to derive the pre-kernel.

The present paper is organized as follows. Section 2 introduces the basic notations and definitions that enable us to study the pre-kernel solution. In section 3, we devote first our attention to concepts and methods well known from convex analysis and in the next step towards some preliminary results obtained by Martinez-Legaz (1996) and Meseguer-Artola (1997). The results obtained by these authors are the building tools to derive a new characterization of the pre-kernel that is based on solution sets which are derived from a family of convex and quadratic functions. A practical method in the pre-kernel computation is discussed in section 5 by providing an algorithm from which we can compute in the end an element of the pre-kernel. An numerical example from the literature is revised, to demonstrate the strength of the new algorithm. Using this new approach, a pre-kernel point can now be computed without any computer help and in a systematical way. We conclude this paper with some final remarks.

2 Basic Notations and Definitions

An n-person cooperative game with side payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1,2,\ldots,n\}$ represents the player set and v is the characteristic function with $v : 2^N \rightarrow \mathbb{R}$ and the convention that $v(\emptyset) := 0$. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. Formally, we identify a cooperative game by the vector $v := (v(S))_{S \subseteq N} \in \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$. A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a vector $\vec{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\vec{x}$ with the corresponding measure on S, such that $x(S) = \sum_{k \in S} x_k$. The set of vectors $\vec{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathcal{I}'(v) := \{\vec{x} \in \mathbb{R}^n \mid x(N) = v(N)\}, \tag{2.1}$$

where an element $\vec{x} \in \mathcal{I}'(v)$ is called a pre-imputation.

Given a vector $\vec{x} \in \mathcal{I}'(v)$, we define the **excess** of coalition S with respect to the imputation $\vec{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S,\vec{x}) := v(S) - x(S). \tag{2.2}$$

An non-negative (non-positive) excess of S at $\vec{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S, if the members of S do not accept the payoff distribution $\vec{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game v. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to the pre-imputation $\vec{x} \in \mathcal{I}'(v)$, is given by the maximum excess at $\vec{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\vec{x}, v) := \max_{S \in \mathcal{G}_{ij}} e^v(S,\vec{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.3}$$

The expression $s_{ij}(\vec{x}, v)$ describes the maximal amount at the pre-imputation $\vec{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\vec{x} \in \mathcal{I}'(v)$ that balance the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{P}r\mathcal{K}(v) := \{\vec{x} \in \mathcal{I}'(v) \mid s_{ij}(\vec{x}, v) = s_{ji}(\vec{x}, v) \quad \text{for all } i, j \in N, i \neq j\}. \tag{2.4}$$

3 Some Preliminary Results

Before, we introduce some preliminary results from convex analysis and cooperative game theory, we discuss first some additional notations, definitions and some concepts from matrix theory. We express the Euclidean scalar product of two vectors $\mathbf{x}$ and $\mathbf{x}^*$ as $\langle \mathbf{x}, \mathbf{x}^* \rangle = \mathbf{x}_1 \cdot \mathbf{x}_1^* + \cdots + \mathbf{x}_n \cdot \mathbf{x}_n^*$. The symbols $\mathbf{x}$ and $\mathbf{x}^*$ are regarded as column vectors, whereas the symbol Q is used to indicate a $(m \times n)$-matrix. The identity matrix is identified by $\mathbf{I}$ and the transpose of a vector or a matrix is denoted by the symbols $\mathbf{x}^T$ and Q^T respectively. Coefficient vectors are usually identified by $\mathbf{a}$ and the null vector by $\mathbf{0}$.

A vector $\mathbf{x}$ that satisfies $Q\,\mathbf{x} = \mathbf{a}$ is called a solution of the linear system $Q\,\mathbf{x} = \mathbf{a}$. We call a linear system consistent if it has a unique or more solutions. If a linear system has no solution, its is called inconsistent. The matrix Q of a consistent linear system is either invertible or non invertible. In the latter case the matrix Q is rectangular or singular for a square matrix, with the consequence that the solution of the linear system is not anymore unique. In order to give a characterization of the solution for a consistent linear system where the matrix is not invertible, it is useful to introduce the concept of a generalized inverse of a matrix. In matrix theory a generalized inverse Q^G for a $(m \times n)$-matrix Q is an $(n \times m)$-matrix with the property $Q\,Q^G\,Q = Q$. When Q is not invertible there exist in general infinite many different generalized inverses Q^G, but at least one. Although it is for our purpose enough to consider an arbitrary generalized inverse, we impose some additional conditions in order to have a unique generalized inverse. This is done with regard to get a unique characterization of a pre-kernel solution expressed in terms of a matrix and coefficient vector. Whenever we impose to the general condition $Q\,Q^G\,Q = Q$, three further algebraic constraints like a reflexive condition: $Q^G\,Q\,Q^G = Q^G$, a normalized condition: $(Q\,Q^G)^T = Q^G\,Q$, and a reversed normalized condition: $(Q^G\,Q)^T = Q\,Q^G$, then the generalized inverse Q^G of Q is unique. This generalized inverse is known under the name **Moore-Penrose** matrix or **pseudo-inverse**, which we denote by Q^{MP}.

Observe, in addition, that $Q\,Q^G$ and $Q^G\,Q$ respectively, are always idempotent, since $Q\,Q^G\,Q\,Q^G = (Q\,Q^G)(Q\,Q^G) = (Q\,Q^G\,Q)\,Q^G = Q\,Q^G$ and $Q^G\,Q\,Q^G\,Q = (Q^G\,Q)\,(Q^G\,Q) = Q^G\,(Q\,Q^G\,Q) = Q^G\,Q$ respectively. Thus, $Q\,Q^G$ is a $(m \times m)$-projection matrix, whereas $Q^G\,Q$ is an $(n \times n)$-projection matrix. Furthermore, a linear system $Q\,\mathbf{x} = \mathbf{a}$ can now be classified as consistent if and only if $Q\,Q^G\,\mathbf{a} = \mathbf{a}$ or equivalently if and only if $(\mathbf{I} - Q\,Q^G)\,\mathbf{a} = \mathbf{0}$ (cf. Harville (1997)).

In order to introduce the concept of a convex function, recall first that given $\mathbf{x}$ and $\mathbf{z}$ in $\mathbb{R}^n$, the defined vector $\mathbf{y} := \theta \cdot \mathbf{z} + (1-\theta) \cdot \mathbf{x}$ with $0 \leq \theta \leq 1$, is called a convex combination of $\mathbf{x}$ and $\mathbf{z}$. Let f be a real-valued function defined on a convex subset C in $\mathbb{R}^n$. The function f is called to be **convex** if for all $\mathbf{x}, \mathbf{z} \in C$ and $0 \leq \theta \leq 1$,

$$\theta \cdot f(\mathbf{z}) + (1-\theta) \cdot f(\mathbf{x}) \geq f(\theta \cdot \mathbf{z} + (1-\theta) \cdot \mathbf{x}).$$

The **convex conjugate** or **Fenchel transform** $f^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$) of a convex function f (cf. Rockafellar (1970, Section 12)) is defined by

$$f^*(\mathbf{x}^*) = \sup_{\mathbf{x} \in \mathbb{R}^n} \{\langle \mathbf{x}^*, \mathbf{x} \rangle - f(\mathbf{x})\} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n \tag{3.1}$$

Observe that the Fenchel transform f^* is the pointwise supremum of affine functions $p(\mathbf{x}^*) = \langle\, \mathbf{x}, \mathbf{x}^* \,\rangle - \mu$ such that $(\mathbf{x}, \mu) \in C \subseteq (\mathbb{R}^n \times \mathbb{R})$. Thus, the Fenchel transform f^* is again a convex function.

We can generalize the definition of a Fenchel transform (cf. Martinez-Legaz (1996)) by introducing a fixed nonempty subset K of $\mathbb{R}^n$, then the conjugate of a function $f : K \rightarrow \overline{\mathbb{R}}$ is $f^c : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, given by

$$f^c(\mathbf{x}^*) = \sup_{\mathbf{x} \in K} \{\langle\, \mathbf{x}^*, \mathbf{x} \,\rangle - f(\mathbf{x})\} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n, \tag{3.2}$$

which also known as the **Fenchel-Moreau conjugation**. On the other hand, the restriction of the conjugate g^* to the subset K of the function $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is denoted by $g^\gamma : K \rightarrow \overline{\mathbb{R}}$, which is defined by

$$g^\gamma(\mathbf{x}^*) = \sup_{\mathbf{x}^* \in \mathbb{R}^n} \{\langle\, \mathbf{x}^*, \mathbf{x} \,\rangle - g(\mathbf{x}^*)\} \qquad \forall \mathbf{x} \in K.$$

A vector $\mathbf{x}^*$ is said to be a subgradient of a convex function f at a point $\mathbf{x}$, if

$$f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \qquad \forall \mathbf{z} \in \mathbb{R}^n.$$

This condition states that the graph of the affine function $p(\mathbf{z}) := f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle$ is a non-vertical supporting hyperplane to the convex set $epi\,f$ at the point $(\mathbf{x}, f(\mathbf{x}))$, whereas the convex set $epi\,f$ is defined by

$$epi\,f := \left\{(\mathbf{x}, \mu) \in \left(\mathbb{R}^n \times \mathbb{R}\right) \mid f(\mathbf{x}) \leq \mu\right\}$$

Note that a differentiable function f can be described in terms of gradient vectors, which correspond to tangent hyperplanes to the graph f. In addition, the set of all subgradients of f at $\mathbf{x}$ is called the subdifferentiable of f at $\mathbf{x}$ and it is defined by

$$\partial f(\mathbf{x}) := \Big\{\mathbf{x}^* \in \mathbb{R}^n \mid f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \qquad (\forall \mathbf{z} \in \mathbb{R}^n)\Big\}.$$

The set of all subgradients $\partial f(\mathbf{x})$ is a closed convex set, which could be empty or may consists of just one point. The multivalued mapping $\partial f : \mathbf{x} \mapsto \partial f(\mathbf{x})$ is called the subdifferential of f. Moreover, a vector $\mathbf{x}^*$ is said to be an ϵ-subgradient of a convex function f a point $\mathbf{x} \in \mathbb{R}^n$ (where $\epsilon > 0$) if

$$f(\mathbf{z}) \geq (f(\mathbf{x}) - \epsilon) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \qquad \forall \mathbf{z} \in \mathbb{R}^n.$$

Similar to the definition of the subdifferential of f we define the ϵ-subdifferential of the function f at a vector $\mathbf{x}$ to be the set

$$\partial f_\epsilon(\mathbf{x}) := \Big\{\mathbf{x}^* \in \mathbb{R}^n \mid f(\mathbf{z}) \geq (f(\mathbf{x}) - \epsilon) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \qquad (\forall \mathbf{z} \in \mathbb{R}^n)\Big\}.$$

For a thorough discussion of this topic, we refer the reader to Rockafellar (1970, Section 23)).

Theorem 3.1 (Martinez-Legaz (1996)). *Let K be a nonempty subset of $\mathbb{R}^n$ such that ext co $K = K$ and assume that K is bounded. Then the mapping $f \mapsto f^c$ is a bijection from the set of bounded from below continuous functions $f : K \rightarrow \mathbb{R}$ onto the set of convex functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying*

$$\partial_\epsilon g(\mathbf{x}_0^*) \cap K \neq \emptyset \qquad \forall \mathbf{x}_0^* \in \mathbb{R}^n \quad \epsilon > 0 \tag{3.3}$$

and

$$K \subset \bigcup_{\mathbf{x}^* \in \mathbb{R}^n} \partial_1 g(\mathbf{x}^*), \tag{3.4}$$

with inverse $g \longrightarrow g^{\gamma}$. If, moreover, K is finite then (3.4) can be replaced by the stronger condition

$$K \subset \bigcup_{\mathbf{x}^* \in \mathbb{R}^n} \partial g(\mathbf{x}^*). \tag{3.5}$$

Theorem 3.2 (Martinez-Legaz (1996)). *The indirect function $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ of any n-person TU-game is a non-increasing polyhedral convex function such that*

(i) $\partial \pi(\vec{x}) \cap \{0,-1\}^n \neq \emptyset \qquad \forall \vec{x} \in \mathbb{R}^n$,

(ii) $\{0,-1\}^n \subset \bigcup_{\vec{x} \in \mathbb{R}^n} \partial \pi(\vec{x})$, *and*

(iii) $\min_{\vec{x} \in \mathbb{R}^n} \; \pi(\vec{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (i)-(iii) then there exists a unique n-person TU-game $\langle N, v \rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\vec{x} \in \mathbb{R}^n} \left\{ \pi(\vec{x}) + \sum_{k \in S} x_k \right\} \qquad \forall\, S \subset N. \tag{3.6}$$

According to the result above, we can express the **indirect function** π by:

$$\pi(\vec{x}) = \max_{S \subseteq N} \left\{ v(S) - \sum_{k \in S} x_k \right\} \qquad \forall \vec{x} \in \mathbb{R}^n. \tag{3.7}$$

It was worked out by Martinez-Legaz (1996) that the indirect function π is a dual representation of the characteristic function v. Furthermore, it was emphasized by Martinez-Legaz that the indirect function π is the generalized conjugate or Fenchel transform of the characteristic function v. To see this compare the expression (3.8) with the definition of a generalized conjugation of f as it was given in (3.2), and observe that we assigned to each $S \in 2^N$ its characteristic vector $-\mathbf{1}_S$. Recall that the characteristic vector for $\vec{x} \in \mathbb{R}^n$ is given by $x_k = 1$ if $k \in S$ and $x_k = 0$ whenever $k \notin S$. Then the indirect function takes the form

$$\pi(\vec{x}) = \max_{S \subseteq N} \left\{ v(-\mathbf{1}_S) + \langle\, -\mathbf{1}_S, \vec{x}\, \rangle \right\} \qquad \forall \vec{x} \in \mathbb{R}^n. \tag{3.8}$$

An economic interpretation in terms of a production problem was given in Martinez-Legaz (1996, p. 293). Here, we want to give an alternative interpretation related to an investment fund. A fund manager has the opportunity to invest in n-assets. The expression x_k is considered the amount, which can be invested in assets k. If he decides not to invest in all assets available in the market, he buys assets to form the sub-portfolio S. This can be interpreted, for instance, as an investment in a market-index portfolio. The term $\sum_{k \in S} \; x_k$ can be seen as the expenditure to buy the portfolio S that gives a total yield of $v(S)$. The expression in the bracket is the net profit made by the fund manager to invest in portfolio S. The fund manager will select now the market portfolio that gives him the highest net profit at $\vec{x}$. If we allow negative values for x_k, we have a more natural interpretation in terms of negative investment (selling short) instead of negative salaries as in Martinez-Legaz (1996).

The pre-imputation $\vec{x}^{\,i,j,\delta} \in \mathcal{I}'(v)$, with $\delta \geq 0$, is given by

$$\vec{x}^{\,i,j,\delta}_{N \backslash \{i,j\}} = \vec{x}_{N \backslash \{\,i,j\}}, \; x^{i,j,\delta}_i = x_i - \delta \quad \text{and} \quad x^{j,i,\delta}_j = x_j + \delta$$

Lemma 3.1 (Meseguer-Artola (1997)). *Let $\langle N, v\rangle$ be an n-person cooperative game with side payments. Let π and s_{ij} be the associated indirect function and the maximum surplus of player i against player j, respectively. If $\vec{x} \in \mathcal{I}'(v)$ then the equality:*

$$s_{ij}(\vec{x}, v) = \pi(\vec{x}^{\,i,j,\delta}) - \delta$$

holds for every $i, j \in N$, $i \neq j$, and for every $\delta \geq \delta_1(\vec{x}, v)$, where:

$$\delta_1(\vec{x}, v) := \max_{k \in N, S \subset N \setminus \{k\}} |v(S \cup \{k\}) - v(S) - x_k|.$$

Besides the fact that we restrict our attention in the analysis of the pre-kernel to the trivial coalition structure $\mathcal{B} = \{N\}$, leaving aside any problems involving non-trivial coalition structures, we cite the next crucial result, which was worked out by Meseguer-Artola (1997), in its most general form. The importance of this result cannot be overemphasized, especially, to provide an alternative representation of the pre-kernel and also with regard to a possible generalization of our forthcoming results to more complex coalition structures.

Proposition 3.1 (Meseguer-Artola (1997)). *For a TU-Game $\langle N, v\rangle$ with indirect function π, a pre-imputation $\vec{x} \in \mathcal{I}'(v)$ is in the pre-kernel of $\langle N, v\rangle$ for the coalition structure $\mathcal{B} = \{B_1, B_2, \ldots, B_l\}$, $\vec{x} \in \mathcal{P}r\mathcal{K}(v)$, if and only if, for every $k \in \{1, 2, \ldots, l\}$, every $i, j \in B_k$, $i > j$, and some $\delta \geq \delta_1(v, \vec{x})$, one has*

$$\pi(\vec{x}^{\,i,j,\delta}) = \pi(\vec{x}^{\,j,i,\delta}).$$

In order to restate a first characterization of the pre-kernel in terms of a solution set of a **minimization problem**, as it was derived by Meseguer-Artola (1997, cf. p.13), we make use of the result given by Proposition 3.1. Then, we can derive a system of nonlinear equations from which we can construct a minimization problem in order to characterize the pre-kernel of a TU-game $\langle N, v\rangle$. As already mentioned, we are only interested in the trivial coalition structure $\mathcal{B} = \{N\}$, thus the system associated with the characterization of the pre-kernel is given by

$$\begin{cases} f_{ij}(\vec{x}) = 0 \quad \forall i, j \in N, i > j \\ f_0(\vec{x}) = 0 \end{cases} \tag{3.9}$$

where, for some $\delta \geq \delta_1(\vec{x}, v)$,

$$f_{ij}(\vec{x}) := \pi(\vec{x}^{\,i,j,\delta}) - \pi(\vec{x}^{\,j,i,\delta}), \tag{3.10}$$

and

$$f_0(\vec{x}) := \sum_{k \in N} x_k - v(N).$$

In Meseguer-Artola (1997, p. 13), it was recognized that the system of equations introduced above, can be expressed as a minimization problem. Since, this system has $n \cdot (n-1)/2$ nonlinear equations. Furthermore, the system is overdetermined, since we have $n \cdot (n-1)/2 + 1 \geq n$. To any overdetermined system there is associated an equivalent minimization problem such that the set of global minima coincides with the solution set of the system. The solution set of such a minimization problem is the set of values for $\vec{x}$ which minimizes the following function

$$h(\vec{x}) := \sum_{\substack{i,j \in N \\ i > j}} (f_{ij}(\vec{x}))^2 + (f_0(\vec{x}))^2 \geq 0 \qquad \vec{x} \in \mathcal{I}'(v). \tag{3.11}$$

REMARK 3.1.
From Theorem 3.2 we know that the indirect function π is a non-increasing polyhedral convex function on the convex polyhedral set $\mathcal{I}'(v)$. In addition, the indirect function π is non-negative convex, since by condition (iii) we have $\min_{\vec{x}\in\mathbb{R}^n} \; \pi(\vec{x}) = 0$. According to its definition the function f_{ij} is the difference of two non-negative convex functions $(\pi(\vec{x}^{\,i,j,\delta}) - \pi(\vec{x}^{\,j,i,\delta}))$ on $\mathcal{I}'(v)$. If we take the square on f_{ij}, then $(f_{ij}(\vec{x}))^2$ is still the difference of two convex functions, since we can write $(f_{ij}(\vec{x}))^2 = (\pi(\vec{x}^{\,i,j,\delta}) - \pi(\vec{x}^{\,j,i,\delta}))^2$ on $\mathcal{I}'(v)$ as the difference of two convex functions given by $(f_{ij}(\vec{x}))^2 = 2\cdot((\pi(\vec{x}^{\,i,j,\delta}))^2 + (\pi(\vec{x}^{\,j,i,\delta})))^2 - (\pi(\vec{x}^{\,i,j,\delta}) + \pi(\vec{x}^{\,j,i,\delta}))^2$ for all $\vec{x} \in \mathcal{I}'(v)$ (see also the discussion in Meseguer-Artola (1997, Footnote 13)). In fact, instead of taking the sum of two convex functions, we take the difference, which implies that the structural form of the objective function h is at the moment ambiguous and not clear for us. We need more information to deduce that the function h is convex or not. $\diamond$

4 CHARACTERIZATION OF THE PRE-KERNEL BY SOLUTION SETS OF MINIMIZATION PROBLEMS

To obtain an alternative representation of the pre-kernel, it is useful to introduce the concept of the parametrized nest of level sets, which is defined as:

$$\operatorname{lev}_\beta f := \left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; f(\mathbf{x}) \leq \beta\right\}, \qquad \beta \in \mathbb{R}. \tag{4.1}$$

Similar, for $\underline{\beta} := \inf_{\mathbf{x}} \; f$ the minimum or solution set of the function f is defined as:

$$M_f := \left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; f(\mathbf{x}) \leq \underline{\beta}\right\}. \tag{4.2}$$

Observe first that for the objective function (3.11) the level set $\operatorname{lev}_\beta h$ is empty, if $\beta < \underline{\beta} = \inf_{\vec{x}} \; h$. It follows from Proposition 3.1 that the function h attains a minimum at 0, thus it holds $0 = \min_{\vec{x}} h$ and the non-empty minimum set denoted by M_h can be specified by $\{\vec{x} \in \mathcal{I}'(v) \mid h(\vec{x}) = 0\}$. As mentioned in Remark 3.1 the objective function h may be non-convex, therefore the minimum set M_h needs not necessarily to be a convex set.

REMARK 4.1.
According to the fact that $0 = \min_{\vec{x}} h$, the pre-kernel can be fully characterized by M_h. To see this, take a pre-kernel element, i.e. $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$, then the efficiency property is satisfied with $f_0(\vec{x}\,') = 0$ and the maximal surpluses $s_{ij}(\vec{x}\,', v)$ must be balanced for each distinct pair of players i, j, hence, we conclude that $f_{ij}(\vec{x}\,') = 0$ for all $i, j \in N, i > j$ and therefore $h(\vec{x}\,') = 0$. Thus, we get $\vec{x}\,' \in M_h$. To prove the converse, suppose that $\vec{x}\,' \in M_h$, then $h(\vec{x}\,') = 0$. But this implies that $f_{ij}(\vec{x}\,') = 0$. That means that the difference $f_{ij}(\vec{x}\,') = (\pi(\vec{x}\,'^{\,i,j,\delta}) - \pi(\vec{x}\,'^{\,j,i,\delta}))$ is equalized for each distinct pair of indices $i, j \in N, i > j$. Thus, $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$. It turns out that the minimum set coincides with the pre-kernel, i.e., we have:

$$M_h = \{\vec{x} \in \mathcal{I}'(v) \mid h(\vec{x}) = 0\} = \mathcal{P}r\mathcal{K}(v), \tag{4.3}$$

which can be also stated as: $\arg\min \; h(\vec{x}) = \mathcal{P}r\mathcal{K}(v)$, as it was done in Meseguer-Artola (1997). $\diamond$

We summarize this discussion in the following corollary

Corollary 4.1. *For a TU-Game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\vec{x}\,') = \sum_{\substack{i,j\in N\\ i>j}} (f_{ij}(\vec{x}\,'))^2 + (f_0(\vec{x}\,'))^2 = \min_{\vec{x}\in\mathcal{I}(v)} \; h(\vec{x}) = 0, \tag{4.4}$$

if and only if $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$.

The characterization of the pre-kernel by the minimum set M_h is convincing in its elementariness. Although, this new characterization is very striking through its simplification, it offers us no keen insight on the pre-kernel. From the discussion of Remark (3.1) it is known to us that the shape of the objective function h (3.11) is still unclear, therefore, we get no new idea about the geometrical shape of the pre-kernel by looking on the minimum set M_h. According to the work by Maschler and Peleg (1966, 1967), it is well known that the pre-kernel may be composed by more than one convex polyhedron. Thus, we would gain more if we would be able to decompose the minimum set in geometrical objects which are quite similar to the existing characterization of the pre-kernel by convex polyhedra. To this end, we want to replace the minimization problem with its rather complicated objective function h of the type (3.11) by an optimization problem with an objective function that is much more easier to treat and that solves besides the new optimization problem also the original one.

Proposition 4.1. *If we fix an arbitrary vector $\vec{\gamma}$ in the the pre-imputation set $\mathcal{I}'(v)$, then for a TU-Game $\langle N, v\rangle$ with indirect function π the objective function h of type* (3.11) *induces a quadratic function:*

$$h_{\gamma}(\vec{x}) = (1/2)\cdot\langle\, \vec{x}, Q\,\vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}\,\rangle + \alpha \qquad \vec{x} \in \mathcal{I}'(v) \tag{4.5}$$

where $\vec{a}$ is a column vector of coefficients, α is a scalar and Q is a symmetric $(n \times n)$-matrix with integer coefficients taken from the interval $[-n\cdot(n-1), n\cdot(n-1)]$. Furthermore, if the minimum set of h_γ on $\mathcal{I}'(v)$ is $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$ and $\vec{x}\,' \in M_{h_\gamma}$, then $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$.

Proof. If $\vec{\gamma} \in \mathcal{I}'(v)$, then $(f_0(\vec{\gamma}))^2 = 0$ and the function $h(\vec{\gamma})$ given by (3.11) simplifies to

$$h(\vec{\gamma}) := \sum_{\substack{i,j\in N\\ i>j}} (f_{ij}(\vec{\gamma}))^2 = \sum_{\substack{i,j\in N\\ i>j}} (\pi(\vec{\gamma}^{\,i,j,\delta}) - \pi(\vec{\gamma}^{\,j,i,\delta}))^2,$$

which is by Lemma 3.1

$$h(\vec{\gamma}) = \sum_{\substack{i,j\in N\\ i>j}} (s_{ij}(\vec{\gamma}, v) + \delta - s_{ji}(\vec{\gamma}, v) - \delta)^2 = \sum_{\substack{i,j\in N\\ i>j}} (s_{ij}(\vec{\gamma}, v) - s_{ji}(\vec{\gamma}, v))^2.$$

Next define the set

$$\mathcal{C}_{ij}(\vec{\gamma}) := \left\{ S \in \mathcal{G}_{ij} \;\middle|\; s_{ij}(\vec{\gamma}, v) = e^v(S, \vec{\gamma}) \qquad \forall S \in 2^N \backslash \{\emptyset, N\} \right\}.$$

Take a set $S_{ij} \in \mathcal{C}_{ij}(\vec{\gamma})$ for all $i, j \in N,\ i \neq j$, then we obtain

$$h_{\gamma}(\vec{x}) = \sum_{\substack{i,j\in N\\ i>j}} \left(e^v(S_{ij}, \vec{x}) - e^v(S_{ji}, \vec{x}) \right)^2 = \sum_{\substack{i,j\in N\\ i>j}} \left(v(S_{ij}) - \sum_{k\in S_{ij}} x_k - v(S_{ji}) + \sum_{k\in S_{ji}} x_k \right)^2. \tag{4.6}$$

It should be apparent from equation (4.6) that a minimum is attained on $\mathcal{I}'(v)$, if $h_\gamma(\vec{x}\,') = 0$. This happens whenever the maximal surplus is balanced for each distinct pair of players i, j. This condition is satisfied for pre-kernel elements, and we conclude that if the minimum set of h_γ on $\mathcal{I}'(v)$ is given by $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$ and it holds that $\vec{x}\,' \in M_{h_\gamma}$, then it must hold in addition that $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$.

Now let us define the characteristic vector denoted $\mathbf{1}_{S_{ij}}$ for all S_{ij} with $i, j \in N,\ i \neq j$ by

$$\mathbf{1}_{S_{ij}} := \begin{cases} 1 & k \in S_{ij} \\ 0 & k \notin S_{ij} \end{cases}$$

Then formula (4.6) can now be written as

$$h_\gamma(\vec{x}) = \sum_{\substack{i,j \in N \\ i>j}} \Big(v(S_{ij}) - v(S_{ji}) - \langle\, \vec{x}, \mathbf{1}_{S_{ij}} \,\rangle + \langle\, \vec{x}, \mathbf{1}_{S_{ji}} \,\rangle \Big)^2.$$

Moreover, defining $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$, we can simplify the above expression even further. We get

$$h_\gamma(\vec{x}) = \sum_{\substack{i,j \in N \\ i>j}} \Big(\alpha_{ij} + \langle\, \vec{x}, (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle \Big)^2.$$

Expanding the above term

$$h_\gamma(\vec{x}) = \sum_{\substack{i,j \in N \\ i>j}} \Big\{ \alpha_{ij}^2 + 2 \cdot \alpha_{ij} \cdot \langle\, \vec{x}, (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle + \langle\, \vec{x}, (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle^2 \Big\}.$$

and then collecting terms, it gives

$$h_\gamma(\vec{x}) = \sum_{\substack{i,j \in N \\ i>j}} \alpha_{ij}^2 + \langle\, \vec{x}, \sum_{\substack{i,j \in N \\ i>j}} 2 \cdot \alpha_{ij} \cdot (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle + \sum_{\substack{i,j \in N \\ i>j}} \langle\, \vec{x}, (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle^2.$$

Let now $\alpha := \sum_{\substack{i,j \in N \\ i>j}} \alpha_{ij}^2$, $\vec{a} := 2 \cdot \sum_{\substack{i,j \in N \\ i>j}} \alpha_{ij} \cdot (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})$, and observe that $\langle\, \vec{x}, (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}})\,\rangle^2 = \sum_{k \in N} \sum_{l \in N} q_{ij}^{k,l}\, x_k\, x_l = \langle\, \vec{x}, Q_{ij}\, \vec{x} \,\rangle$ with $q_{ij}^{k,l} \in \{-1, 0, 1\}$ if $k \neq l$, and for $k = l$, $q_{ij}^{k,k} \in \{0, 1\}$, then

$$h_\gamma(\vec{x}) = \sum_{\substack{i,j \in N \\ i>j}} \langle\, \vec{x}, Q_{ij}\, \vec{x} \,\rangle + \langle\, \vec{x}, \vec{a} \,\rangle + \alpha = \langle\, \vec{x}, \Big(\sum_{\substack{i,j \in N \\ i>j}} Q_{ij} \Big)\, \vec{x} \,\rangle + \langle\, \vec{x}, \vec{a} \,\rangle + \alpha,$$

where Q_{ij} is a symmetric $(n \times n)$-matrix, with off-diagonal elements of $-1, 0$ or 1, and on-diagonal elements of 0 and 1.

Write $\overline{Q} := \sum_{\substack{i,j \in N \\ i>j}} Q_{ij}$. It should be apparent that the matrix $\overline{Q}$ is also a symmetric $(n \times n)$-matrix. Moreover, we have in total $(n \cdot (n-1))/2$ matrices of type Q_{ij}, which are made of by coefficients of values of $-1, 0$ and 1, thus each coefficient of the matrix $\overline{Q}$ will adopt integer values from the interval of $[-(n \cdot (n-1))/2, (n \cdot (n-1))/2]$. Therefore, on-diagonal elements can adopt integer values of at most $(n \cdot (n-1))/2$. Multiplying each coefficient of the matrix $\overline{Q}$ by 2, we obtain $\overline{Q} = (1/2) \cdot Q$, therefore we receive the desired result

$$h_\gamma(\vec{x}) = (1/2) \cdot \langle\, \vec{x}, Q\, \vec{x} \,\rangle + \langle\, \vec{x}, \vec{a} \,\rangle + \alpha, \tag{4.5}$$

with coefficients taken from the intverval $[-n \cdot (n-1), n \cdot (n-1)]$. This argument concludes the proof. □

REMARK 4.2.
Obviously, we cannot guarantee in general that a quadratic function h_γ of the form (4.5) on $\mathcal{I}'(v)$ has a minimum set of $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$ for every possible parameter constellation $(Q, \vec{a}, \alpha)$. Generically, the minimum set on $\mathcal{I}'(v)$ is given by $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = \underline{\beta}\}$, where $\underline{\beta} \geq 0$. The reader may be aware that in the sequel, we have the former definition of the minimum set in mind, when we speak from a set M_{h_γ}. We will state it explicitly, if we make use of its general definition. Notice, that the emphasis on minimum sets is not only crucial to obtain a dual representation of the pre-kernel by the indirect function approach, it is also of great importance to construct a tractable algorithm to compute a pre-kernel point, as it will become more clear in the next section. Following this approach, we can construct a sequence of pre-imputation points induced from a sequence of quadratic minimization problems which are related to different parameters $(Q, \vec{a}, \alpha)$ and therefore to different minimum sets, which, in the end, converges to an element of a set M_{h_γ}, that is to say to a point in the pre-kernel. $\diamond$

Proposition 4.2. *If the minimum set of a convex objective function h_γ of type (4.5) on $\mathcal{I}'(v)$ is $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$, then a vector $\vec{x}\,' \in M_{h_\gamma}$ solves the minimization problem $\min_{\vec{x} \in \mathcal{I}'(v)} h_\gamma(\vec{x})$, and the original minimization problem $\min_{\vec{x} \in \mathcal{I}'(v)} h(\vec{x})$ as well. That is, we get the following inclusion of minimum sets: $M_{h_\gamma} \subseteq M_h$. As a consequence, the solution of the minimization problem $\min_{\vec{x} \in \mathcal{I}'(v)} h_\gamma(\vec{x})$ determines the pre-kernel element by*

$$\vec{x}\,' = -Q^{-1}\,\vec{a},$$

if Q is non-singular and $h_\gamma(\vec{x}\,') = 0$.

However, if the symmetric $(n \times n)$-matrix Q is singular and $h_\gamma(\vec{x}\,') = 0$ is given, then a particular solution given by

$$\vec{x}\,' = -Q^{MP}\,\vec{a},$$

determines a pre-kernel element.

Proof. If the minimum set M_{h_γ} on $\mathcal{I}'(v)$ is equal to $\{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$ and $\vec{x}\,' \in M_{h_\gamma}$, then the maximal surplus is balanced for all for all $i, j \in N, i > j$ due to equation (4.6). We can now conclude from $h_\gamma(\vec{x}\,') = 0$ that $(f_{ij}(\vec{x}\,'))^2 = 0$ is fulfilled for all $i, j \in N, i > j$, which implies $h(\vec{x}\,') = 0$. In other words, we get the desired result $\vec{x}\,' \in M_h$.

In order to determine a pre-kernel solution, take the directional derivative of the function h_γ at $\vec{x}$ relative to $\vec{z}$, which is

$$h'_\gamma(\vec{x}, \vec{z}) := \lim_{\lambda \to 0} \frac{h_\gamma(\vec{x} + \lambda \cdot \vec{z}) - h_\gamma(\vec{x})}{\lambda},$$

and then, we obtain for the first order condition for minimizing h

$$h'_\gamma(\vec{x}, \vec{z}) = \vec{a} + Q\,\vec{x} = \mathbf{0} \Longleftrightarrow \vec{x} = -Q^{-1}\,\vec{a},$$

if Q is non-singular and $h_\gamma(\vec{x}) = h_\gamma(-Q^{-1}\,\vec{a}) = 0$, then we have found an element of the pre-kernel.

Finally, let us consider the case that the matrix Q is singular. To this end, let Q^{MP} be the corresponding Moore-Penrose matrix of the singular and symmetric matrix Q. The objective function has minimum set $M_{h_\gamma} = \{\vec{x} \mid h_\gamma(\vec{x}) = 0\}$, therefore the linear system $Q\,\vec{x} = -\vec{a}$ has a solution, say $\vec{x}\,'$. This tell us that the linear system is consistent which is equivalent to $Q\,Q^{MP}(-\vec{a}) = -\vec{a}$. Then

$$Q\,Q^{MP}(-\vec{a}) = Q\,(Q^{MP}(-\vec{a})) = Q\,Q^{MP}\,(Q\,\vec{x}\,') = (Q\,Q^{MP}\,Q)\,\vec{x}\,' = Q\,\vec{x}\,' = -\vec{a}.$$

Thus, it holds $Q\,\vec{x}\,' = Q\,(Q^{MP}(-\vec{a}))$, it gives that $\vec{x}\,' = -Q^{MP}\,\vec{a}$ is a particular solution of the linear system $Q\,\vec{x} = -\vec{a}$. $\square$

Lemma 4.1. *Let Q be a non-singular symmetric $(n \times n)$-matrix and let $\vec{x}\,' = -Q^{-1}\,\vec{a}$ be a pre-kernel point, then for a TU-Game $\langle N, v\rangle$ with indirect function π the minimum value function H_{γ} at $\vec{x}\,' \in \mathcal{P}r\mathcal{K}(v)$ has the following general form*

$$H_{\gamma}(Q,\vec{a},\alpha) := \min_{\vec{x} \in \mathcal{I}'(v)} h_{\gamma}(\vec{x}) = (-1/2) \cdot \langle\, \vec{a}, Q^{-1}\,\vec{a}\,\rangle + \alpha \equiv 0. \tag{4.7}$$

Similar, whenever Q is a singular and symmetric $(n \times n)$-matrix and a pre-kernel point is specified by $\vec{x}\,' = -Q^{MP}\,\vec{a}$, then the minimum value function H_{γ} at $\vec{x}\,'$ is specified by

$$H_{\gamma}(Q,\vec{a},\alpha) := \min_{\vec{x} \in \mathcal{I}'(v)} h_{\gamma}(\vec{x}) = (-1/2) \cdot \langle\, \vec{a}, Q^{MP}\,\vec{a}\,\rangle + \alpha \equiv 0. \tag{4.8}$$

Proof. Take the vector $\vec{x}\,' = -Q^{-1}\,\vec{a} \in \mathcal{P}r\mathcal{K}(v)$ that solves the problem $\min_{\vec{x} \in \mathcal{I}'(v)} h_{\gamma}(\vec{x}) = 0$ and substituting this result in h_{γ}, then

$$h_{\gamma}(\vec{x}\,') = h_{\gamma}(-Q^{-1}\,\vec{a}) = (-1/2) \cdot \langle\, Q^{-1}\,\vec{a}, -Q\,Q^{-1}\,\vec{a}\,\rangle - \langle\, Q^{-1}\,\vec{a}, \vec{a}\,\rangle + \alpha \equiv 0,$$

which is

$$(-1/2) \cdot \langle\, \vec{a}, Q^{-1}\,\vec{a}\,\rangle + \alpha \equiv 0.$$

For the subcase that the symmetric matrix Q is singular, its associated Moore-Penrose matrix Q^{MP} is symmetric as well. A pre-kernel element can be characterized by $\vec{x}\,' = -Q^{MP}\,\vec{a}$, which implies that we get

$$(-1/2) \cdot \langle\, \vec{a}, Q^{MP}\,\vec{a}\,\rangle + \alpha \equiv 0.$$

□

In order to conclude that the function h_{γ} has besides a local minimum at $\vec{x}\,'$ also a global minimum at that point, we must show in a next step that the quadratic function h_{γ} is a convex function.

Proposition 4.3. *For a TU-Game $\langle N, v\rangle$ with indirect function π, the objective function h_{γ} of the form (4.5) on $\mathcal{I}'(v)$ with minimum set $M_{h_{\gamma}} = \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma}(\vec{x}) = 0\}$ is convex, i.e. the symmetric $(n \times n)$-matrix Q is positive semi-definite, thus the matrix Q satisfies*

$$\langle\, \vec{z}, Q\,\vec{z}\,\;\rangle \geq 0 \qquad \forall \vec{z} \in \mathcal{I}'(v). \tag{4.9}$$

Proof. In order to prove that the function h_{γ} is convex, we have to establish that for all $\vec{x}, \vec{z} \in \mathcal{I}'(v)$ and $0 \leq \theta \leq 1$ it holds that

$$\theta \cdot h_{\gamma}(\vec{z}) + (1-\theta) \cdot h_{\gamma}(\vec{x}) \geq h_{\gamma}(\vec{y}) = h_{\gamma}(\theta \cdot \vec{z} + (1-\theta) \cdot \vec{x}),$$

where $y := \theta \cdot \vec{z} + (1-\theta) \cdot \vec{x}$.

Assume next without loss of generality that $\vec{x}\,' \in M_{h_{\gamma}}$, and that $\vec{x}\,' = -Q^{-1}\,\vec{a}$, whenever the matrix is Q is non-singular or that $\vec{x}\,' = -Q^{MP}\,\vec{a}$ for the case that the symmetric matrix Q is singular. Thus, we have $h_{\gamma}(\vec{x}\,') = 0$. Moreover, observe that $h_{\gamma}(\vec{z}) \geq 0$ for all $\vec{z} \in \mathcal{I}'(v)$. To show convexity, we have to prove that the following inequality is satisfied:

$$\theta \cdot h_{\gamma}(\vec{z}) + (1-\theta) \cdot h_{\gamma}(\vec{x}\,') = \theta \cdot h_{\gamma}(\vec{z}) \geq h_{\gamma}(\vec{y}) = (1/2) \cdot \langle\, \vec{y}, Q\,\vec{y}\,\rangle + \langle\, \vec{y}, \vec{a}\,\rangle + \alpha,$$

which is

$$\theta \cdot h_{\gamma}(\vec{z}) \geq$$
$$(1/2) \cdot \langle\, (\theta \cdot \vec{z} + (1-\theta) \cdot \vec{x}\,'), Q\,(\theta \cdot \vec{z} + (1-\theta) \cdot \vec{x}\,')\,\rangle + \langle\, (\theta \cdot \vec{z} + (1-\theta) \cdot \vec{x}\,'), \vec{a}\,\rangle + \alpha.$$

We obtain after some calculation steps the inequality:

$$\theta \cdot h_{\gamma}(\vec{z}) \geq$$
$$(1/2) \cdot \theta \cdot (\theta - 1)\langle\, \vec{z}, Q\,\vec{z}\,\rangle + \theta \cdot h_{\gamma}(\vec{z}) - (1/2) \cdot \theta \cdot (1-\theta)\;\langle\, \vec{x}\,', Q\,\vec{x}\,'\,\rangle + \theta \cdot (1-\theta)\;\langle\, \vec{z}, Q\,\vec{x}\,'\,\rangle.$$

This expression is equivalent to

$$0 \geq (-1/2) \cdot \langle\, \vec{z}, Q\,\vec{z}\,\rangle - (1/2) \cdot \langle\, \vec{x}\,', Q\,\vec{x}\,'\,\rangle + \langle\, \vec{z}, Q\;\vec{x}\,'\rangle.$$

Substituting in the above inequality $-Q^{-1}\;\vec{a}$ for $\vec{x}\,'$ in the case that Q is non-singular, and $-Q^{MP}\;\vec{a}$ if Q is singular. We can simplify the above expression to

$$0 \leq (1/2) \cdot \langle\, \vec{z}, Q\,\vec{z}\,\rangle + (1/2) \cdot \langle\, \vec{a}, Q^{-1}\,\vec{a}\,\rangle + \langle\, \vec{z}, \vec{a}\,\rangle.$$

Using the result from Lemma 4.1 $\langle\, \vec{a}, Q^{-1}\;\vec{a}\,\rangle = 2 \cdot \alpha$ or $\langle\, \vec{a}, Q^{MP}\;\vec{a}\,\rangle = 2 \cdot \alpha$ respectively, we get

$$0 \leq (1/2) \cdot \langle\, \vec{z}, Q\,\vec{z}\,\rangle + \langle\, \vec{z}, \vec{a}\,\rangle + \alpha = h_{\gamma}(\vec{z}).$$

This argument establish convexity of the objective function h, since it holds $h_{\gamma}(\vec{z}) \geq 0$ for all $\vec{z} \in \mathcal{I}'(v)$. □

Proposition 4.4. *Let Q be a non-singular symmetric $(n \times n)$-matrix. If h_{γ} is the quadratic function of the form (4.5) on $\mathcal{I}'(v)$ with minimum set $M_{h_{\gamma}} = \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma}(\vec{x}) = 0\}$, then the Fenchel transform or the conjugation of the function h_{γ} is given by*

$$h^{*}_{\gamma}(\vec{x}^{\,*}) = (1/2) \cdot \langle\, \vec{x}^{\,*}, Q^{-1}\,\vec{x}^{\,*}\,\rangle - \langle\, \vec{x}^{\,*}, Q^{-1}\,\vec{a}\,\rangle \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^{n}. \tag{4.10}$$

In case that the symmetric $(n \times n)$-matrix Q is singular, the Fenchel transform has the form

$$h^{*}_{\gamma}(\vec{x}^{\,*}) = (1/2) \cdot \langle\, \vec{x}^{\,*}, Q^{MP}\,\vec{x}^{\,*}\,\rangle - \langle\, \vec{x}^{\,*}, Q^{MP}\,\vec{a}\,\rangle \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^{n}. \tag{4.11}$$

Proof. The Fenchel transform of the function h_{γ} as specified in (4.5) has the general form

$$h^{*}_{\gamma}(\vec{x}^{\,*}) = \sup_{\vec{x} \in \mathcal{I}'(v)} \left\{ \langle\, \vec{x}, \vec{x}^{\,*}\,\rangle - h_{\gamma}(\vec{x}) \right\} \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^{n}.$$

Plug in the above formula the structural form of h_{γ} that was given by equation (4.5), this yields

$$h^{*}_{\gamma}(\vec{x}^{\,*}) = \sup_{\vec{x} \in \mathcal{I}'(v)} \left\{ \langle\, \vec{x}, (\vec{x}^{\,*} - \vec{a})\,\rangle - (1/2) \cdot \langle\, \vec{x}, Q\;\vec{x}\,\rangle - \alpha \right\} \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^{n}.$$

Define $\vec{y}^{\,*} := \vec{x}^{\,*} - \vec{a}$, then

$$h^{*}_{\gamma}(\vec{y}^{\,*}) = \sup_{\vec{x} \in \mathcal{I}'(v)} \left\{ \langle\, \vec{x}, \vec{y}^{\,*})\,\rangle - (1/2) \cdot \langle\, \vec{x}, Q\;\vec{x}\,\rangle - \alpha \right\} \qquad \forall \vec{y}^{\,*} \in \mathbb{R}^{n}. \tag{4.12}$$

Next, we define the function

$$k(\vec{x}) := \langle\, \vec{x}, \vec{y}^{\,*} \,\rangle - (1/2) \cdot \langle\, \vec{x}, Q\, \vec{x} \,\rangle - \alpha.$$

Take the directional derivative of the function k at $\vec{x}$ relative to $\vec{z}$, which is

$$k'(\vec{x}, \vec{z}) := \lim_{\lambda \to 0} \frac{k(\vec{x} + \lambda \cdot \vec{z}) - k(\vec{x})}{\lambda},$$

and then, we obtain by first order condition

$$k'(\vec{x}, \vec{z}) = \vec{y}^{\,*} - Q\, \vec{x} = \mathbf{0} \iff \vec{x} = Q^{-1}\, \vec{y}^{\,*},$$

if Q is non-singular. Substituting this result in the Fenchel transform (4.12) of function h_γ, and we get

$$h^*_\gamma(\vec{y}^{\,*}) = \langle\, Q^{-1}\, \vec{y}^{\,*}, \vec{y}^{\,*}) \,\rangle - (1/2) \cdot \langle\, Q^{-1}\, \vec{y}^{\,*}, Q\, Q^{-1} \vec{y}^{\,*} \,\rangle - \alpha \qquad \forall \vec{y}^{\,*} \in \mathbb{R}^n.$$

Since, Q is a symmetric matrix, we can simplify to

$$h^*_\gamma(\vec{y}^{\,*}) = (1/2) \cdot \langle\, \vec{y}^{\,*}, Q^{-1} \vec{y}^{\,*} \,\rangle - \alpha.$$

Using $\vec{x}^{\,*} - \vec{a}$ for $\vec{y}^{\,*}$ and the result from Lemma 4.1 $\langle\, \vec{a}, Q^{-1}\, \vec{a} \,\rangle = 2 \cdot \alpha$, we finally obtain the following expression after some collection and rearrangement of terms

$$h^*_\gamma(\vec{x}^{\,*}) = (1/2) \cdot \langle\, \vec{x}^{\,*}, Q^{-1}\, \vec{x}^{\,*} \,\rangle - \langle\, \vec{x}^{\,*}, Q^{-1}\, \vec{a} \,\rangle \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^n. \tag{4.10}$$

Remind that if the matrix Q is singular, symmetric and positive semi-definite, its associated Moore-Penrose matrix Q^{MP} is symmetric and positive semi-definite. Using the second part from Lemma 4.1 $\langle\, \vec{a}, Q^{MP}\, \vec{a} \,\rangle = 2 \cdot \alpha$, then we receive a similar expression related to (4.10), which is specified by

$$h^*_\gamma(\vec{x}^{\,*}) = (1/2) \cdot \langle\, \vec{x}^{\,*}, Q^{MP}\, \vec{x}^{\,*} \,\rangle - \langle\, \vec{x}^{\,*}, Q^{MP}\, \vec{a} \,\rangle \qquad \forall \vec{x}^{\,*} \in \mathbb{R}^n. \tag{4.11}$$

□

REMARK 4.3.
A standard result of convex analysis states that a given vector $\vec{x}$ belongs to the solution set or minimum set of a convex function f if and only if the null vector $\mathbf{0}$ is a subgradient of a convex function f, that is $\mathbf{0} \in \partial f(\mathbf{x})$ (Rockafellar, 1970, p. 264). As it becomes more clear through the proof of Theorem 4.1 below, the minimum set of a convex function f is equal to the subdifferential of the conjugation f^* at $\mathbf{0}$, that is $\partial f^*(\mathbf{0})$. Proposition 4.3 states that the function h_γ is convex, this implies that the conjugation h^*_γ is a convex function, too. We can conclude that the subdifferential $\partial h^*_\gamma(\mathbf{0})$ that coincides with the minimum/solution set is a closed convex set, which may consists of just one point. It could not be empty, since there always exists a pre-kernel point for a TU-game. In fact, the minimum set $\partial h^*_\gamma(\mathbf{0})$ of an objective function h_γ is a closed and bounded convex set that can only completely characterize the pre-kernel of a game $\langle v, N\rangle$, whenever the pre-kernel is itself convex or consists of just a single point. Remind our discussion after Corollary 4.1 that the pre-kernel may contain more than one convex polyhedron (cf. Maschler and Peleg (1966, 1967) and Maschler (1992)). This implies that the structure of the pre-kernel may be disconnected or may even be a non-convex set. How can we now characterize the pre-kernel? It turns out by Theorem 4.1 that the pre-kernel can be composed by an arbitrary union of subdifferential $\partial h^*_\gamma(\mathbf{0})$ (closed and bounded convex sets) which had been induced by a collection of quadratic functions h_{γ_k} of the form (4.5).

Theorem 4.1. *Let $\{h_{\gamma_k} \mid k \in \mathcal{J}\}$ be a collection of quadratic functions h_{γ_k} of type (4.5) on $\mathcal{I}'(v)$ with minimum set $M_{h_{\gamma_k}} = \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma_k}(\vec{x}) = 0\}$ for each $k \in \mathcal{J}$. Then for a TU-Game $\langle N, v\rangle$ with indirect function π the pre-kernel is composed by an arbitrary union of closed and bounded convex minimum sets, that is*

$$\mathcal{P}r\mathcal{K}(v) = \bigcup_{k\in\mathcal{J}} \partial h^{*}_{\gamma_k}(\mathbf{0}) = \bigcup_{k\in\mathcal{J}} \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma_k}(\vec{x}) = 0\}, \tag{4.13}$$

where $\mathcal{J}$ is an arbitrary index set.

Proof. In the first step, we establish that the minimum set M_{h_γ} of the convex function h_γ coincides with the subdifferential of the Fenchel transform h^*_γ at the null vector $\mathbf{0}$, that is we want to show that $\partial h^*_\gamma(\mathbf{0}) = M_{h_\gamma}$. For this purpose, applying the definition of a subdifferential to the conjugate h^*_γ, then we get by definition:

$$\partial h^*_\gamma(\vec{x}^*) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; h^*_\gamma(\vec{z}^*) \geq h^*_\gamma(\vec{x}^*) + \langle\, \vec{x}, \vec{z}^* - \vec{x}^*\,\rangle \quad (\forall \vec{z}^* \in \mathbb{R}^n)\big\}$$
$$\partial h^*_\gamma(\vec{x}^*) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; \langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) \geq \langle\, \vec{x}, \vec{z}^*\,\rangle - h^*_\gamma(\vec{z}^*) \quad (\forall \vec{z}^* \in \mathbb{R}^n)\big\}$$
$$\partial h^*_\gamma(\vec{x}^*) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; \langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) \geq \sup_{\vec{z}^*\in\mathbb{R}^n} \{\langle\, \vec{x}, \vec{z}^*\,\rangle - h^*_\gamma(\vec{z}^*)\}\big\}$$
$$\partial h^*_\gamma(\vec{x}^*) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; \langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) \geq h_\gamma(\vec{x})\big\}.$$

For the second step, remind from convex analysis that a convex function f is called to be closed whenever the convex set $epi f$ is closed. Observe now that a quadratic objective function h_γ of the form (4.5) is closed, since it is continuous, that is, the $epi\ h_\gamma$ is closed. This implies that the assumption of Theorem 23.5 (ii) in Rockafellar (1970, p. 218) is satisfied, then $\langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) \geq h_\gamma(\vec{x})$ is equivalent to $\langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) = h_\gamma(\vec{x})$. Thus, we can simplify the subdifferential $\partial h^*_\gamma(\vec{x}^*)$ to

$$\partial h^*_\gamma(\vec{x}^*) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; \langle\, \vec{x}, \vec{x}^*\,\rangle - h^*_\gamma(\vec{x}^*) = h_\gamma(\vec{x})\big\}.$$

In addition, let $\vec{x}^* = \mathbf{0}$ and note that due to the result expressed in Proposition 4.4 we have $h^*_\gamma(\mathbf{0}) = 0$, then we get

$$\partial h^*_\gamma(\mathbf{0}) = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; \langle\, \vec{x}, \mathbf{0}\,\rangle - h^*_\gamma(\mathbf{0}) = h_\gamma(\vec{x})\big\} = \big\{\vec{x} \in \mathcal{I}'(v) \;\big|\; h_\gamma(\vec{x}) = 0\big\} = M_{h_\gamma}.$$

Notice, that the conjugation h^*_γ is a finite convex function for any $\vec{x}^*$ due to formula (4.10) and (4.11), then at each point $\vec{x}^*$ the subdifferentiable ∂h^*_γ is a non-empty closed, bounded and convex set (cf. Rockafellar (1970, p. 218)), thus we conclude that the subdifferentiable $\partial h^*_\gamma(\mathbf{0})$ is a non-empty closed bounded convex set.

Take now a collection $\{h_{\gamma_k} \mid k \in \mathcal{J}\}$ with minimum set $M_{h_{\gamma_k}} = \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma_k}(\vec{x}) = 0\}$ for each $k \in \mathcal{J}$, then the union of the subdifferentials $\partial h^*_{\gamma_k}$ must be equal to the union of the minimum sets $M_{h_{\gamma_k}}$, i.e.,

$$\bigcup_{k\in\mathcal{J}} \partial h^*_{\gamma_k}(\mathbf{0}) = \bigcup_{k\in\mathcal{J}} \{\vec{x} \in \mathcal{I}'(v) \mid h_{\gamma_k}(\vec{x}) = 0\}.$$

It remains to prove that $\mathcal{P}r\mathcal{K}(v) = \cup_{k\in\mathcal{J}}\, \partial h^*_{\gamma_k}(\mathbf{0}) = \cup_{k\in\mathcal{J}}\, M_{h_{\gamma_k}}$.

First assume that $\vec{x}\,' \in \cup_{k\in\mathcal{J}}\ \partial h^*_{\gamma_k}(\mathbf{0})$, then there exists at least one k such that $h_{\gamma_k}(\vec{x}\,') = 0$. The vector $\vec{x}\,'$ solves the problem $\min_{\vec{x}\in\mathcal{I}'(v)} h_{\gamma_k}$ and it solves by Proposition 4.2 the original problem $\min_{\vec{x}\in\mathcal{I}'(v)} h$, too. Hence, the differences in the indirect function values are balanced $\pi(\vec{x}\,'^{\,i,j,\delta}) =$

$\pi(\vec{x}'^{j,i,\delta})$ for all distinct pairs $i,j \in N, i > j$, i.e., $f_{ij}(\vec{x}') = 0$ for all $i,j \in N, i > j$. Hence $\vec{x}' \in \mathcal{P}r\mathcal{K}(v)$.

Now let us prove the converse. If $\vec{x}' \in \mathcal{P}r\mathcal{K}(v)$, then the maximal surplus is balanced $s_{ij}(\vec{x}',v) = s_{ji}(\vec{x}',v)$ for all $i,j \in N, i > j$, thus $f_{ij}(\vec{x}') = 0$ for all $i,j \in N, i > j$. But then, we have for at least one k that the equation (4.6) is equal to zero, hence we get $h_{\gamma_k}(\vec{x}') = 0$. In other words, $\vec{x}' \in M_{h_{\gamma_k}}$, which implies $\vec{x}' \in \cup_{k \in \mathcal{J}} \partial h^*_{\gamma_k}(\mathbf{0})$. □

5 Algorithm for Computing the Pre-Kernel

In the foregoing discussion we introduced an approach to characterize the pre-kernel by an arbitrary union of minimum sets or subdifferentials of conjugations at the null vector. This representation by convex sets was obtained by solving minimization problems of quadratic functions of type (4.5). But so-far our discussion was not constructive in providing a method to compute a pre-kernel element. The idea we present to find a point of the pre-kernel is based on an iterative procedure. We choose an arbitrary pre-imputation $\vec{\gamma}^{\,0} \in \mathcal{I}'(v)$ as a starting point to construct a quadratic objective function. Any selection of a starting point $\vec{\gamma}^{\,0}$ that satisfies the efficiency criterion will induce due to formula (3.10) a quadratic objective function of the form (4.5) corrected by a matrix Q_0 consisting only of integer values 2 in order to take efficiency into account. Let us denote this function by h_{γ^0}. If for all $i,j \in N,\; i > j$, it holds $f^0_{ij}(\vec{\gamma}^{\,0}) = 0$ and $f_0(\vec{\gamma}^{\,0}) = 0$, then $h_{\gamma^0}(\vec{\gamma}^{\,0}) = 0$ and we have found a pre-kernel element. For the case that for some distinct pairs i,j we have $f^0_{ij}(\vec{\gamma}^{\,0}) \neq 0$ and $f_0(\vec{\gamma}^{\,0}) = 0$, this induces $\beta^0 := h^0_{\gamma^0}(\vec{\gamma}^{\,0}) > 0$, and we have to solve the associated minimization problem. The vector $\vec{\gamma}^{\,1} = (-Q^0)^{-1}\,\vec{a}^{\,0}$ that solves $\beta^1 := \min_{\vec{x} \in \mathcal{I}(v)} h_{\gamma^0}(\vec{x})$ provides us with a unique solution, whenever the symmetric $(n \times n)$-matrix Q^0 is non-singular. The solution must be an element of the pre-imputation set $\mathcal{I}'(v)$, since if $f_0(\vec{\gamma}^{\,1}) \neq 0$, then we get $h_{\gamma^0}(\vec{\gamma}^{\,1}) > \beta^1$ and $\vec{\gamma}^{\,1}$ can not be a solution for $\min_{\vec{x} \in \mathcal{I}(v)} h_{\gamma^0}(\vec{x})$. In contrast, if the determinant of the matrix Q is zero, i.e. $\det |Q| = 0$, then the matrix Q is singular and it holds in our case that $\text{rank}(Q) = m < n$. The number of variables is greater than the number of equations and the solution is not anymore unique, then $Q\,\vec{x} = \vec{a}$ has an infinite number of solutions. In such a case, we select a particular solution $\vec{\gamma}^{\,1} = -Q^{MP\,0}\,\vec{a}^{\,0}$ from the minimum set to induce a new objective function. The point $\vec{\gamma}^{\,1}$ induces again for some distinct pairs i,j an inequality $f^1_{ij}(\vec{\gamma}^{\,1}) \neq 0$, otherwise we had already found by $\vec{\gamma}^{\,1}$ an element of the pre-kernel. Solving the minimization problem $\beta^2 := \min_{\vec{x} \in \mathcal{I}(v)} h_{\gamma^1}(\vec{x})$ we find the unique point $\vec{\gamma}^{\,2} = (-Q^1)^{-1}\,\vec{a}^{\,1}$ for a non-singular matrix Q^1. If $h_{\gamma^1}(\vec{\gamma}^{\,2}) = 0$ the algorithm stops, otherwise we continue in our search for a pre-kernel solution. By computing pre-imputation points in accordance with the described rule, we construct a non-increasing sequence $\{\beta^k\}_{k \geq 0}$ of real-valued minima attained by optimization problems which is non-increasing in the integers k for all $k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. Whenever such a non-increasing sequence $\{\beta^k\}_{k \geq 0}$ of real numbers is bounded from below by zero, the sequence will converge in the limit to zero and therefore to an objective function from which an element of the pre-kernel can be computed.

In the next step we introduce an algorithm that is based on the iterative procedure discussed above in order to generate a sequence of pre-imputation points that converges in the limit to a pre-kernel element. We can establish in Theorem 5.1 that such a sequence can be generated by Algorithm 5.1.

Algorithm 5.1. *Given a TU-Game $\langle N, v \rangle$ with indirect function π and objective function h of type* (3.11) *on the pre-imputation set $\mathcal{I}'(v)$, proceed as follows to generate a sequence of pre-imputation points.*

0. *Set $k = 0$ and select an arbitrary starting point $\vec{\gamma}^{\,0}$ from the pre-imputation set $\mathcal{I}'(v)$, otherwise go to Step 9.*

1. *Construct an objective function* h_{γ^0}*, otherwise go to Step 9.*

2. *If* $k = 0$ *go to Step 3, otherwise go to Step 4.*

3. *If the induced objective function* h_{γ^0} *at* $\vec{\gamma}^{\,0}$ *is*

   – *equal to zero, then go to Step 9.*

   – *greater than zero, then continue with Step 5.*

4. *Construct an objective function* h_{γ^k} *and continue with the next step, otherwise go to Step 9.*

5. *Solve the associated minimization problem* $\min_{\vec{x}\in\mathcal{I}'(v)} h_{\gamma^k}(\vec{x})$*, otherwise go to Step 9.*

6. *Increase the index* k *by 1 and label the solution of the minimization problem by* $\vec{\gamma}^{\,k+1}$*.*

7. *If the induced objective function* h_{γ^k} *at the solution* $\vec{\gamma}^{\,k+1}$ *is:*

   – *equal to zero, then go to Step 9.*

   – *greater than zero, then go to next step.*

8. *Replace the point* $\vec{\gamma}^{\,k}$ *by the new solution* $\vec{\gamma}^{\,k+1}$*, and return to Step 2.*

9. *Stop the algorithm.*

For a TU-Game $\langle N, v\rangle$ with indirect function π and objective function h of type (3.11) on the pre-imputation set $\mathcal{I}'(v)$, the Theorem 5.1 establishes monotonicity and stopping properties for Algorithm 5.1. The first part of Theorem 5.1 states that the algorithm generates in any iteration step a pre-imputation point. The algorithm never stops without specifying a pre-imputation point. The second part of the theorem shows that the sequence $\{\beta^k\}_{k\in\mathbb{N}_0}$ of minima attained from minimization problems generated by the algorithm is non-increasing in k for all $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and that this sequence converges in the limit to zero. This result formulates, in addition, the stopping rule of the Algorithm 5.1 to terminate the algorithm when the minimum attained for an optimization problem generated at an iteration step is equal to zero. The third part of Theorem 5.1 states that the sequence $\{\vec{\gamma}^{\,k}\}_{k\in\mathbb{N}_0}$ of pre-imputation points generated by the Algorithm 5.1 by solving in any iteration step the associated minimization problem converges in the limit to a pre-kernel element. In accordance with the stopping rule formulated in the second part of the theorem, this result indicates that the algorithm guarantees to terminate when a pre-kernel element is found.

Theorem 5.1. *Consider a TU-Game* $\langle N, v\rangle$ *with indirect function* π *and objective function* h *of type* (3.11) *on the pre-imputation set* $\mathcal{I}'(v)$*.*

1. *Algorithm 5.1 never stops in Step 0, Step 1, Step 4 and Step 5.*

2. *Algorithm 5.1 generates a sequence* $\{\beta^k\}_{k\in\mathbb{N}_0}$ *that is non-increasing in* $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$*, whereas* $\beta^{k+1} := \min_{\vec{x}\in\mathcal{I}(v)} h_{\gamma^k}(\vec{x})$ *and* $\beta^0 := h(\vec{\gamma}^{\,0})$*. Thus, there exists a limit point* β^{k^*} *at zero, s. t.* $\lim_{k\to\infty} \beta^k = \beta^{k^*} = 0$*.*

3. *Algorithm 5.1 generates a sequence* $\{\vec{\gamma}^{\,k}\}_{k\in\mathbb{N}_0}$ *of pre-imputation vectors that converges to a pre-kernel element. Hence, there exists a pre-imputation point* $\vec{\gamma}^{\,k^*}$ *s. t.* $\lim_{k\to\infty} \vec{\gamma}^{\,k} = \vec{\gamma}^{\,k^*} \in \mathcal{P}r\mathcal{K}(v)$*.*

Proof. Algorithm 5.1 does not stop at Step 0, because the pre-imputation set is never empty. Moreover, Algorithm 5.1 does not stop at Step 1 or Step 4, since by Proposition 4.1 any pre-imputation vector $\vec{\gamma}$ induces via the objective function h of type (3.11) a quadratic objective function h_γ of form (4.5). Finally, the Algorithm 5.1 does not stop at Step 5, since the conjugation h^*_γ is subdifferentiable at $\mathbf{0}$, we can deduce that the convex function h_γ attains a minimum (c.f. Theorem 27.1 (b) Rockafellar (1970, p. 264)).

We conduct this proof by induction. To this end set $k = 0$ and pick an arbitrary starting point $\vec{\gamma}^{\,0}$ from the pre-imputation set. Since $\mathcal{I}(v) \neq \emptyset$ by the first part of the theorem such an element can be selected. This implies that the value of the objective function h evaluated at $\vec{\gamma}^{\,0}$ is $h(\vec{\gamma}^{\,0})$. Set now $\beta^0 = h(\vec{\gamma}^{\,0})$. The vector $\vec{\gamma}^{\,0}$ induces by Proposition 4.1 a quadratic objective function h_{γ^0} with the property that $\beta^0 = h_{\gamma^0}(\vec{\gamma}^{\,0})$ holds, this is due to $s_{ij}(\vec{\gamma}^{\,0}, v) = e^v(S_{ij}, \vec{\gamma}^{\,0})$ for all pairs of i, j in N such that $i > j$, by taking an element $S_{ij} \in \mathcal{C}_{ij}(\vec{\gamma}^{\,0})$. Let us suppose without loss of generality that $\beta^0 > 0$ holds true. Write now $\beta^1 = \min_{\vec{x}\in\mathcal{I}(v)}\; h_{\gamma^0}(\vec{x})$ and assume that $\beta^1 > 0$. By the first part of the theorem the Algorithm 5.1 does not stop at Step 5, because this problem has a solution and it attains a minimum at a solution $\vec{\gamma}^{\,1}$. By Proposition 4.2 we can conclude that an optimal solution obtained by minimizing a quadratic convex function can be specified by an inverse matrix or a pseudo inverse matrix in connection with a coefficient vector. Thus, we get $\beta^0 \geq \beta^1$. Now assume by induction hypothesis that for integers $k' \geq 1$, it is satisfied that $\beta^{k-1} \geq \beta^k > 0$ for all $k = 1, \ldots, k'$. It suffices to show that $\beta^{k'} \geq \beta^{k'+1} > 0$ to conclude by induction. Notice that a vector $\vec{\gamma}^{\,k'}$ is a solution for the minimization problem $\beta^{k'} := \min_{\vec{x}\in\mathcal{I}(v)}\; h_{\gamma^{k'-1}}(\vec{x})$, if it solves the problem. This solution induces via the objective function h a quadratic objective function $h_{\gamma^{k'}}$ with the property that $\beta^{k'} = h(\vec{\gamma}^{\,k'}) = h_{\gamma^{k'-1}}(\vec{\gamma}^{\,k'}) = h_{\gamma^{k'}}(\vec{\gamma}^{\,k'})$. The associated minimization problem $\beta^{k'+1} = \min_{\vec{x}\in\mathcal{I}(v)}\; h_{\gamma^{k'}}(\vec{x})$ has now a solution denoted by $\vec{\gamma}^{\,k'+1}$. From this result we can establish that $h_{\gamma^{k'}}(\vec{\gamma}^{\,k'}) = \beta^{k'} \geq \beta^{k'+1} = h(\vec{\gamma}^{\,k'+1}) = h_{\gamma^{k'}}(\vec{\gamma}^{\,k'+1}) > 0$ is given. Thus, the sequence $\{\beta^k\}_{k\in\mathbb{N}_0}$ is non-increasing in k as desired. From Lemma 4.1, we can conclude that this descent sequence is bounded from below by zero. The proof of part (2) is completed by mentioning that a descent sequence that is bounded from below by zero converges to zero, hence we get $\lim_{k\to\infty}\; \beta^k = 0$. This argument indicates that it can be guaranteed that the Algorithm 5.1 terminates.

By part (2) we get a sequence $\{\beta^{k+1} := \min_{\vec{x}\in\mathcal{I}(v)}\; h_{\gamma^k}(\vec{x})\}_{k\geq 0}$ of minima attained from optimization problems that converges to $\beta^{k^*} = \min_{\vec{x}\in\mathcal{I}(v)}\; h_{\gamma^{k^*-1}}(\vec{x}) = 0$. Denote the solution of this minimization problem by $\vec{\gamma}^{\,k^*}$ which is in the solution set of $M_{h_{\gamma^{k^*}}} = \{\vec{x} \mid h_{\gamma^{k^*-1}}(\vec{x}) = 0\}$. By Proposition 4.1, we obtain a pre-kernel element from this optimization problem. Moreover, by part (2) of the theorem, we get a sequence $\{\vec{\gamma}^{\,k}\}_{k\in\mathbb{N}_0}$ of pre-imputation points with $\vec{\gamma}^{\,k} \in M_{h_{\gamma^k}}$ for all $k \in \mathbb{N}_0$. According to $\vec{\gamma}^{\,k} \in M_{h_{\gamma^k}}$ for all $k \in \mathbb{N}_0$, we can conclude that the sequence $\{\vec{\gamma}^{\,k} \in M_{h_{\gamma^k}}\}_{k\in\mathbb{N}_0}$ converges to $\vec{\gamma}^{\,k^*}$. Hence, the sequence converges to a pre-kernel element, that is, $\lim_{k\to\infty}\; \vec{\gamma}^{\,k} = \vec{\gamma}^{\,k^*} \in \mathcal{P}r\mathcal{K}(v)$. □

The Theorem 5.1 established alternatively that the Algorithm 5.1 determines a descent sequence of function values of minimum value functions H^k_γ on the parameter space $(Q, \vec{a}, \alpha)$ converging against zero. This is due from Lemma 4.1 from which we can conclude that the descent sequence is bounded from below by zero. The induced sequence of the associated solution vectors converges to a pre-kernel element. Hence, the limit point of the sequence of the corresponding solution vectors is a pre-kernel element. That implies that the Algorithm 5.1 guarantees to terminate definitely.

One advantage of the proposed algorithm compared to a convergent transfer scheme as proposed for instance by Stearns (1968) lies in its capability to change simultaneously all components of a pre-imputation rather than changing just its i-j-components in order to generate the next pre-imputation to approximate a pre-kernel solution. It should be obvious that in such a case the convergence can be very slow.

6 NUMERICAL EXAMPLE

In this section, we want to demonstrate by an numerical example the strength of the proposed algorithm in computing the pre-kernel. For this purpose, we rely on an example discussed in Kopelowitz (1967) and Stearns (1968). It is a six person weighted majority game based on the following parameters $[16; 2, 4, 4, 5, 6, 7]$. This game has a disconnected pre-kernel, which consists of the two points $\{(0, 0, 0, 0, 1/2, 1/2), (0, 1/5, 1/5, 1/5, 1/5, 1/5)\}$, whereas the last point is also the pre-nucleolus of the game. This game is zero-monotonic. For this class of TU-games the pre-kernel coincides with the kernel.

According to the fact that the game has a disconnected pre-kernel with just two elements, we need to apply the proposed iterative procedure for computing the pre-kernel at least twice. By doing so, we choose the following pre-imputations as starting points: $(1, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 0, 1)$.

Set $k = 0$ and let us start with the first pre-imputation $\vec{x}^{\,1} = \vec{\gamma}^{\,0} = (1, 0, 0, 0, 0, 0)$, then we will see that this induces the following quadratic objective function:

$$
\begin{aligned}
h^1_{\gamma^0}(\vec{x}) &= (1/2) \cdot \langle\, \vec{x}, Q^0\, \vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}^{\,0}\,\rangle + \alpha^0 \\
&= (1/2) \cdot \vec{x}^{\,T} \cdot \begin{pmatrix} 12 & 2 & 8 & -2 & 8 & -6 \\ 2 & 18 & -12 & 2 & 4 & 0 \\ 8 & -12 & 28 & 0 & 8 & -8 \\ -2 & 2 & 0 & 18 & -12 & 4 \\ 8 & 4 & 8 & -12 & 28 & -12 \\ -6 & 0 & -8 & 4 & -12 & 18 \end{pmatrix} \vec{x} + \vec{x}^{\,T} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} + 1.
\end{aligned}
$$

The reader may be aware that the matrix Q^0 must be corrected by a matrix Q_0 to take into account the efficiency property related to a pre-imputation. Thus the matrix Q^0 is obtained by the summing up $\sum_{\substack{i,j \in N \\ i>j}} Q_{ij}$ and Q_0. It must be mentioned at this point that each element of the matrix Q_0 equals 2, as the reader may check. Then we have all information incorporated which are necessary to compute a solution from the minimization problem: $\min_{\vec{x}\in\mathcal{I}(v)} h^1_{\gamma^0}(\vec{x})$. The vector $\vec{\gamma}^{\,1}$ that solves the minimization problem related to the objective function $h^1_{\gamma^0}$ is:

$$
\vec{\gamma}^{\,1} = (\frac{281}{2360}, \frac{343}{2950}, \frac{69}{590}, \frac{2003}{11800}, \frac{1081}{5900}, \frac{3389}{11800}).
$$

Furthermore, the value of the objective function $h^1_{\gamma^0}$ at $\vec{\gamma_1}$ is 5, and its minimum is attained at $h^1_{\gamma_0}(\vec{\gamma}^{\,1}) = 89/11800$. For $k = 1$, the solution vector $\vec{\gamma_1}$ of the minimization problem $\min_{\vec{x}\in\mathcal{I}(v)} h^1_{\gamma^0}(\vec{x})$ induces now the objective function:

$$
\begin{aligned}
h^1_{\gamma^1}(\vec{x}) &= (1/2) \cdot \langle\, \vec{x}, Q^1\, \vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}^{\,1}\,\rangle + \alpha^1 \\
&= (1/2) \cdot \vec{x}^{\,T} \cdot \begin{pmatrix} 24 & 6 & 14 & -16 & 20 & -16 \\ 6 & 16 & -8 & -2 & 6 & -2 \\ 14 & -8 & 26 & -8 & 14 & -12 \\ -16 & -2 & -8 & 26 & -16 & 14 \\ 20 & 6 & 14 & -16 & 28 & -20 \\ -16 & -2 & -12 & 14 & -20 & 26 \end{pmatrix} \vec{x} + \vec{x}^{\,T} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} + 1.
\end{aligned}
$$

As solution for the associated optimization problem we get :

$$
\vec{\gamma}^{\,2} = (\frac{382}{4087}, \frac{10}{67}, \frac{9}{61}, \frac{647}{4087}, \frac{738}{4087}, \frac{1094}{4087}),
$$

and $h^1_{\gamma^1}(\vec{\gamma}^{\,2}) = 13/4087$. Set now $k = 2$. The pre-imputation $\vec{\gamma}^{\,2}$ induces the final objective function:

$$h^1_{\gamma^2}(\vec{x}) = (1/2) \cdot \langle\, \vec{x}, Q^2\,\vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}^{\,2}\,\rangle + \alpha^2$$
$$= (1/2) \cdot \vec{x}^{\,T} \cdot \begin{pmatrix} 30 & 16 & 6 & -12 & 24 & -24 \\ 16 & 26 & -8 & -10 & 12 & -10 \\ 6 & -8 & 16 & -2 & 6 & -2 \\ -12 & -10 & -2 & 20 & -14 & 16 \\ 24 & 12 & 6 & -14 & 28 & -22 \\ -24 & -10 & -2 & 16 & -22 & 28 \end{pmatrix} \vec{x} + \vec{x}^{\,T} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} + 1,$$

which gives us the solution:

$$\vec{\gamma}^{\,3} = (0,1,1,1,1,1)/5,$$

and for the objective function, we receive the result that $h^1_{\gamma^2}(\vec{\gamma}^{\,3}) = 0$, hence we have found the first pre-kernel point. Moreover, we generated the descent real-valued sequence $\{5, 89/11800, 13/4087, 0\}$, since $5 > 89/11800 > 13/4087 > 0$. We see that we are able to find the nucleolus/pre-kernel point $(0,1,1,1,1,1)/5$ after three iterative steps.

Finally, let us consider how many iterative steps we need to compute the second and last pre-kernel point. For this purpose set $k = 0$ and let us take the second pre-imputation $\vec{x}^{\,2} = \vec{\gamma}^{\,0} = (0,0,0,0,0,1)$, this induces the following quadratic objective function:

$$h^2_{\gamma^0}(\vec{x}) = (1/2) \cdot \langle\, \vec{x}, Q^0\,\vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}^{\,0}\,\rangle + \alpha^0$$
$$= (1/2) \cdot \vec{x}^{\,T} \cdot \begin{pmatrix} 26 & 2 & 6 & -6 & 14 & -8 \\ 2 & 18 & 0 & -2 & 8 & 0 \\ 6 & 0 & 22 & -6 & 12 & -4 \\ -6 & -2 & -6 & 26 & -2 & 8 \\ 14 & 8 & 12 & -2 & 18 & -6 \\ -8 & 0 & -4 & 8 & -6 & 12 \end{pmatrix} \vec{x} + \vec{x}^{\,T} \cdot \begin{pmatrix} -6 \\ -6 \\ -6 \\ -6 \\ -10 \\ -2 \end{pmatrix} + 5.$$

Solving the associated minimization problem yields:

$$\vec{\gamma}^{\,1} = \left(\frac{323}{4801}, \frac{463}{4801}, \frac{335}{4801}, \frac{931}{4801}, \frac{2711}{4801}, \frac{1862}{4801}\right).$$

Furthermore, the value of the objective function $h^2_{\gamma^0}$ at $\vec{\gamma}_0$ is 5, and its minimum is attained at $h^2_{\gamma_0}(\vec{\gamma}^{\,1}) = 2432/4801$.

Let now $k = 1$. The pre-imputation $\vec{\gamma}^{\,1}$ induces the final objective function:

$$h^2_{\gamma^1}(\vec{x}) = (1/2) \cdot \langle\, \vec{x}, Q^1\,\vec{x}\,\rangle + \langle\, \vec{x}, \vec{a}^{\,1}\,\rangle + \alpha^1$$
$$= (1/2) \cdot \vec{x}^{\,T} \cdot \begin{pmatrix} 24 & 6 & 14 & -16 & 20 & -16 \\ 6 & 16 & -8 & -2 & 6 & -2 \\ 14 & -8 & 26 & -8 & 14 & -12 \\ -16 & -2 & -8 & 26 & -16 & 14 \\ 20 & 6 & 14 & -16 & 28 & -20 \\ -16 & -2 & -12 & 14 & -20 & 26 \end{pmatrix} \vec{x} + \vec{x}^{\,T} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} + 1,$$

which gives us the solution:

$$\vec{\gamma}^{\,2} = (0,0,0,0,1,1)/2,$$

and for the objective function, we receive the result that $h^2_{\gamma^2}(\vec{\gamma}^2) = 0$, hence we have found the second pre-kernel point. Similar to the first calculation, we generated the following finite descent real-valued sequence $\{5, 2432/4801, 0\}$, since $5 > 2432/4801 > 0$. In this case, we have just needed two iterative steps to find the second pre-kernel point $(0, 0, 0, 0, 1, 1)/2$. Thus, the proposed procedure enables us to compute a pre-kernel point by a reasonable amount of steps and time, which could be done without any computer help.

7 Concluding Remarks

We presented for the coalition structure related to the grand coalition a dual representation of the pre-kernel based on the indirect function of a cooperative game. We established that the pre-kernel can be alternatively described by an arbitrary union of closed and convex minimum sets derived by a collection of quadratic functions. Moreover, we provided an algorithm that led the partners –for instance of a common pool resource– to consider the pre-kernel as a reasonable division rule according to its simple computation process. We demonstrated that only some simple calculus from analysis and linear algebra is needed to find out a pre-kernel solution that can be accepted by unequal partners as a fair outcome. Finally, we are quite sure that the results we have discussed here can be generalized easily to more complex coalition structures. But this topic will be discussed elsewhere.

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