STRATEGY-PROOFNESS OF THE PROBABILISTIC SERIAL MECHANISM IN LARGE RANDOM ASSIGNMENT PROBLEMS

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Abstract. In the random assignment problem, the probabilistic serial mechanism (Bogomolnaia and Moulin 2001) is ordinally efficient and envy-free, but not strategy-proof. However, we show that agents have incentives to state their ordinal preferences truthfully when the market is sufficiently large. Given a fixed set of object types and an agent with a fixed expected utility function over these objects, if the number of copies of each object type is sufficiently large, then truthful reporting of ordinal preferences is a weakly dominant strategy for the agent (for any set of other participating agents and their possible preferences). The better efficiency and fairness properties of the probabilistic serial mechanism, together with the non-manipulability property we discover, support its implementation in many circumstances instead of the popular random serial dictatorship.

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1. Introduction

In an assignment problem, a number of indivisible objects that are collectively owned need to be assigned to a set of agents who can consume at most one object each. University housing allocation and student placement in public schools are examples of important assignment problems in real life.[1] The mechanism designer is faced with the problem of assigning the objects in an efficient and fair fashion, while eliciting the true preferences of the agents. The mechanism may need to meet other constraints as well. For example, fairness considerations preclude monetary transfers and motivate random assignments in many applications. Also, the assignment usually has to be based on agents' reports of ordinal preferences over objects rather than full cardinal preferences, as elicitation of cardinal preferences may be difficult.[2] Two main solutions to the random assignment problem have been considered in the literature: the random serial dictatorship mechanism (Abdulkadiroğlu and Sönmez 1998) and the probabilistic serial mechanism (Bogomolnaia and Moulin 2001). Although the random serial dictatorship mechanism is widely used in practical assignment problems,[3] our main result implies that the probabilistic serial mechanism has more desirable properties in reasonably large assignment problems.

Random serial dictatorship chooses each possible ordering of the agents with equal probability and, for each realization of the ordering, assigns the first agent his most preferred object, the next agent his most preferred object among the remaining ones, and so on. Random serial dictatorship is strategy-proof, that is, truthful reporting of ordinal preferences is a weakly dominant strategy for every agent (for any expected utility

[1]See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for application to house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003b) for student placement. For the latter application, Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discuss practical considerations in designing student placement mechanisms in New York City and Boston.

[2]The market-like mechanism of Hylland and Zeckhauser (1979) is one of the few solutions to the random assignment problem existent in the literature in which agents report their cardinal preferences over objects.

[3]Random serial dictatorship is used for housing allocation in many universities. In the context of school choice, it is used in the third round of the student placement mechanism in New York City for instance (Abdulkadiroğlu, Pathak, and Roth 2005).

function consistent with the ordinal preferences). Moreover, random serial dictatorship is ex-post efficient, that is, the assignment after the ordering lottery is resolved is Pareto efficient. Abdulkadiroğlu and Sönmez (1998) show that the random serial dictatorship mechanism is equivalent to the core from random endowments mechanism, and argue that the equivalence further justifies the use of random serial dictatorship in practice.

Despite its ex-post efficiency, random serial dictatorship may result in unambiguous efficiency loss ex ante. Bogomolnaia and Moulin (2001) provide an example in which the random serial dictatorship assignment is first-order stochastically dominated by another random assignment for each agent. Bogomolnaia and Moulin define a random assignment to be ordinally efficient if it is not first-order stochastically dominated for all agents by any other random assignment. Clearly, any ordinally efficient random assignment is ex-post efficient. Ordinal efficiency is probably the most compelling efficiency concept in the context of assignment mechanisms based solely on ordinal preferences.

Bogomolnaia and Moulin propose the probabilistic serial mechanism as an alternative to the random serial dictatorship mechanism. The basic idea is to regard each object as a continuum of "probability shares." Each agent "eats" the best available object with speed one at every point in time between 0 and 1. The probabilistic serial random assignment is defined as the profile of shares of objects eaten by agents by time 1. The random assignment prescribed by the probabilistic serial mechanism is ordinally efficient and envy-free (every agent prefers his random assignment to the one of any other agent) if all the agents report their ordinal preferences truthfully.

The desirable properties of the probabilistic serial mechanism come at some cost, however. The mechanism is not strategy-proof. In other words, in some circumstances an agent can receive a more desirable random assignment (with respect to his true expected utility function) by misstating his ordinal preferences. Ordinal efficiency and envy-freeness with respect to misstated preferences have little relevance since they do not imply ordinal efficiency and envy-freeness with respect to true preferences. Due to the incentive problem it has been unclear whether the probabilistic serial mechanism is a good solution to the random assignment problem.

We show that agents have incentives to state their ordinal preferences truthfully when the market is sufficiently large. More specifically, our main result shows that, given a fixed set of object types and an agent with a fixed expected utility function over these objects, if the number of copies of each object type is sufficiently large, then truthful reporting of ordinal preferences is a weakly dominant strategy for the agent (for any set of other participating agents and their possible preferences). The better efficiency and fairness properties of the probabilistic serial mechanism, together with the non-manipulability property we discover, support its use rather than the random serial dictatorship mechanism in many circumstances.

The lower bound on the size of the supply of each object type that is sufficient for truthful reporting of ordinal preferences to be a weakly dominant strategy equals 1.76322 times the ratio of the difference between the utility of the most preferred object and the utility of being unassigned an object, and the smallest difference between the utility of two objects, the more preferred of which is acceptable. We show that the bound cannot be improved by more than a factor of 1.76322.

In our model, the large market assumption means that there is a large number of copies of each object type. This model is very general and subsumes several interesting cases. For instance, the "replica economy" model that is often used to discuss asymptotic properties of markets is a special case, as the number of copies of each object type is large in an economy that is replicated a large number of times. Also, the assumption is natural in a number of real life applications. In the context of university housing allocation, rooms may be divided into several categories according to building and size, and all rooms of the same type can be treated as identical.[4] In student placement in public schools, there are typically a large number of identical seats at each school. Our result may apply to these markets. For example, in a school choice setting where a student finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant, a sufficient condition for truth-telling to be a dominant strategy for him in the probabilistic serial mechanism is that each school have at least 18 seats.

Related literature. Manea (2006a) shows that random serial dictatorship results in an ordinally inefficient assignment for most realizations of preference profiles in a large

[4]For example, the assignment of Harvard on-campus graduate housing is only based on the preferences of each student over eight types of rooms: two possible sizes, large and small, for each of four dorms.

market. The result complements our paper by suggesting that the efficiency gain of the probabilistic serial mechanism over the random serial dictatorship mechanism is realized in most large markets. Simulations based on real data also suggest that there exists an efficiency gain of the probabilistic serial mechanism over the random serial dictatorship mechanism in large markets. Using the data of student placement in public schools in New York City, Pathak (2006) compares the resulting random assignments for each student with respect to first order stochastic dominance in the probabilistic serial mechanism and the random serial dictatorship mechanism. He finds that about 50% of the students are better off in the former mechanism, and about 6% are better off in the latter mechanism (for the rest of the students, the random assignments corresponding to the two mechanisms are not comparable in the sense of first order stochastic dominance).

Incentive properties in large markets are studied in various areas of economics. An example is a pure exchange economy, where the classical result of Hurwicz (1972) implies that the Walrasian mechanism is not strategy-proof.[5] Roberts and Postlewaite (1976) show that, under some conditions, the Walrasian mechanism becomes increasingly difficult to manipulate as the market becomes large. Similarly, in the context of double auctions, Gresik and Satterthwaite (1989), Rustichini, Satterthwaite, and Williams (1994), and Cripps and Swinkels (2006) show that the equilibrium behavior converges to truth-telling as the number of traders increases. Two-sided matching is an area closely related to our model. In that context, Roth and Peranson (1999), Immorlica and Mahdian (2005) and Kojima and Pathak (2006) show that the deferred acceptance algorithm proposed by Gale and Shapley (1962) becomes increasingly hard to manipulate as the number of participants becomes large. Most of these results show either that the gain from false reporting of preferences converges to zero or that the equilibrium behavior converges to truth-telling, but do not claim that truth-telling is a dominant strategy in a sufficiently large market. Our result is stronger in this respect, as we show that truth-telling is an exactly dominant strategy in a (finitely) large market.

In contrast to the mechanisms mentioned above, some popular mechanisms may be highly manipulable even in an environment with a large supply of each object type. For

[5]The result of Hurwicz (1972) is generalized by Zhou (1991) and Serizawa (2002).

example, the housing assignment mechanism of Harvard College in the 1970s randomly assigned rooms in a house to students who indicated the house as their first choices. Students who indicated a house as their second or lower choices were assigned to the house only if there were still available rooms in the house. Hylland and Zeckhauser (1979) note that this mechanism is not strategy-proof. Indeed, some students may be able to manipulate the mechanism even in a large market.[6]

Finally, there is a growing literature on ordinal efficiency. Abdulkadiroğlu and Sönmez (2003a) give a characterization of ordinal efficiency based on the idea of dominated sets of assignments. McLennan (2002) proves that any ordinally efficient random assignment with respect to some ordinal preferences is welfare-maximizing with respect to some expected utility function consistent with the ordinal preferences. A short constructive proof is given by Manea (2006b). Kesten (2006) introduces the top trading cycles from equal division mechanism, and shows that it is equivalent to the probabilistic serial mechanism. The probabilistic serial mechanism is generalized to cases with indifferences in preferences and with existing property rights by Katta and Sethuraman (2006) and Yilmaz (2006), respectively. On the restrictive domain of the scheduling problem, Crès and Moulin (2001) show that the probabilistic serial mechanism is group strategy-proof and stochastically dominates the random serial dictatorship mechanism, and Bogomolnaia and Moulin (2002) give two characterizations of the probabilistic serial mechanism.

The remainder of the paper is organized as follows. Section 2 presents the model, Section 3 presents the main result, and Section 4 concludes. The proof of the main result is presented in the Appendix.

2. Model

A **random assignment problem** is a quadruple $\Gamma = (N, (\succ_i)_{i \in N}, O, (q_a)_{a \in O})$. N represents the set of **agents**, and O represents the set of **proper object types**; both N and O are finite. There are q_a copies of object a, where a positive integer q_a is called

[6]The mechanism of Harvard College described here is analogous to the so-called Boston mechanism studied by Abdulkadiroğlu and Sönmez (2003b) and Ergin and Sönmez (2006). Kojima and Pathak (2006) give an example of a large market where the Boston mechanism can be manipulated, and their example can be easily adapted to the housing allocation mechanism described here.

the **quota** of a. There exist an infinite number of copies of a **null object** $\emptyset$ (which is not included in O), $q_\emptyset = +\infty$. Each agent $i \in N$ has a **strict preference** $\succ_i$ over $\tilde{O} := O \cup \{\emptyset\}$. We write $a \succeq_i b$ if either $a \succ_i b$ or $a = b$ holds. When N and O are fixed, we write $\succ$ for $(\succ_i)_{i \in N}$, $\succ_{N'}$ for $(\succ_i)_{i \in N'}$ where $N' \subset N$, and q for $(q_a)_{a \in \tilde{O}}$.

A **deterministic assignment** for the problem Γ is a function α from N to $\tilde{O}$ such that at most q_a agents are assigned to a for each $a \in \tilde{O}$, with $\alpha(i)$ denoting the object that i receives at α. A **deterministic assignment matrix** is a matrix $X = [X_{ia}]_{i \in N, a \in \tilde{O}}$ with $X_{ia} \in \{0, 1\}$ for all i and a, $\sum_{a \in \tilde{O}} X_{ia} = 1$ for all i, and $\sum_{i \in N} X_{ia} \leq q_a$ for all a. For each deterministic assignment α, there exists a (one-to-one) **corresponding deterministic assignment matrix** X^α such that $X^\alpha_{ia} = 1$ if and only if $\alpha(i) = a$. Denote by $\mathcal{A}$ the set of all deterministic assignments.

A **lottery assignment** is a probability distribution w over $\mathcal{A}$, with $w(\alpha)$ denoting the probability of assignment α. A **random assignment** is a matrix $P = [P_{ia}]_{i \in N, a \in \tilde{O}}$, where $P_{ia} \geq 0$ for all i and a, $\sum_{a \in \tilde{O}} P_{ia} = 1$ for all i, and $\sum_{i \in N} P_{ia} \leq q_a$ for all a; P_{ia} stands for the probability that agent i receives object a. For each lottery assignment w, there exists a **corresponding random assignment** P^w, with P^w_{ia} equal to the probability that agent i is assigned object a under w, i.e., $P^w_{ia} = \sum_{\alpha \in \mathcal{A}, \alpha(i) = a} w(\alpha)$. We say that the lottery assignment w **induces** the random assignment P^w.

The following proposition is a generalization of the Birkhoff-von Neumann theorem (see Birkhoff (1946) and von Neumann (1953)), showing that the correspondence $w \to P^w$ from lottery assignments to random assignments is surjective on the set of all random assignments.

Proposition 1. *Every random assignment can be written as a convex combination of deterministic assignment matrices, hence any random assignment is induced by a lottery assignment.*[7]

Proof. Consider a random assignment $P = [P_{ia}]_{i \in N, a \in \tilde{O}}$. Let $P' = [P_{ia}]_{i \in N \cup N', a \in \tilde{O}}$ be a matrix with rows corresponding to the agents in $N \cup N'$, where N' is a set of n' fictitious agents not in N, with $P'_{ia} = P_{ia}$ for all $i \in N$ and $a \in \tilde{O}$, $P'_{ja} = (q_a - \sum_{i \in N} P_{ia})/n'$ for all

[7]The convex combination is not unique in general.

$j \in N'$ and $a \in O$, and $P'_{j\o} = 1 - \sum_{a \in O} P'_{ja}$ for all $j \in N'$.[8] For sufficiently large n', all entries of the matrix P' are non-negative. Each row of P' sums to 1, and column a of P' sums to q_a for all $a \in O$. Since all rows and columns have integer sums and each entry is non-negative, the procedure described by Hylland and Zeckhauser (1979) in the section "Conduct of the Lottery" may be adapted to the current environment to find a convex decomposition of P' in deterministic assignment matrices corresponding to an assignment problem with set of agents $N \cup N'$ and set of objects O. Obviously, the restriction of these deterministic assignment matrices for the problem $(N \cup N', (\succ_i)_{i \in N \cup N'}, O, (q_a)_{a \in O})$ to the agents in N, with the corresponding weights in the convex decomposition, yield a convex decomposition of P in deterministic assignment matrices for the problem $(N, (\succ_i)_{i \in N}, O, (q_a)_{a \in O})$.

□

By Proposition 1, for any random assignment P, there exists a lottery over deterministic assignments that induces P. Henceforth, we identify a lottery assignment with a random assignment, and use these terminologies interchangeably.

We assume that agents have **von Neumann-Morgenstern expected utility**. The **utility index** of agent i is a function $u_i : \tilde{O} \to \mathbb{R}$. Agent i's expected utility for the random assignment P is then given by $U_i(P) = \sum_{a \in \tilde{O}} P_{ia} u_i(a)$.[9] We say that u_i is **consistent** with $\succ_i$ when $u_i(a) > u_i(b)$ if and only if $a \succ_i b$.

A random assignment P **ordinally dominates** another random assignment P' **at** $\succ$ if for each agent i the lottery P_i first-order stochastically dominates the lottery P'_i,

$$\sum_{b \succeq_i a} P_{ib} \geq \sum_{b \succeq_i a} P'_{ib} \ \forall i \in N, \forall a \in \tilde{O}, \tag{2.1}$$

with strict inequality for some i, a. The random assignment P is **ordinally efficient at** $\succ$ if it is not ordinally dominated at $\succ$ by any other random assignment. If P ordinally dominates P' at $\succ$, then every agent i prefers P_i to P'_i according to any expected utility function with utility index consistent with $\succ_i$.

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[8] McLennan (2002) uses a similar construction.

[9] Without any risk of confusion, we let the domain of U_i be either the set of random assignment matrices or the set of random assignments for agent i.

Consider the binary relation $\triangleright(P,\succ)$ on $\tilde{O}$ defined by

$$a \triangleright (P,\succ) b \iff \exists i \in N, a \succ_i b \text{ and } \; P_{ib} > 0. \tag{2.2}$$

In a setting in which each object has quota 1 and there exist an equal number of agents and objects, Bogomolnaia and Moulin show that a random assignment P is ordinally efficient at $\succ$ if and only if $\triangleright(P,\succ)$ is acyclic. Their characterization extends straightforwardly to our setting. We say that P is **non-wasteful** if there exists no agent $i \in N$ and objects $a, b \in \tilde{O}$ such that $a \succ_i b$, $P_{ib} > 0$ and $\sum_{j \in N} P_{ja} < q_a$.

Proposition 2. *The random assignment P is ordinally efficient at $\succ$ if and only if the relation $\triangleright(P,\succ)$ is acyclic and P is non-wasteful.*

Now we introduce the **probabilistic serial** mechanism, which is an adaptation of the mechanism proposed by Bogomolnaia and Moulin to our setting. The idea is to regard each object as a divisible good of "probability shares." Each agent "eats" the best available object with speed one at every time $t \in [0,1]$ (object a is available at time t if less than q_a share of a has been eaten away by time t). The resulting profile of shares of objects eaten by agents by time 1 obviously corresponds to a random assignment matrix, which we call the **probabilistic serial random assignment**.

Formally, the **symmetric simultaneous eating algorithm**,[10] used to determine the probabilistic serial random assignment, is defined as follows. For any $a \in O' \subset \tilde{O}$, let $N(a, O') = \{i \in N | a \succeq_i b \text{ for every } b \in O'\}$ be the set of agents whose most preferred object in O' is a. For a preference profile $\succ$, the assignment under the probabilistic serial mechanism is defined by the following sequence of steps. Let $O^0 = \tilde{O}$, $t^0 = 0$, and $P^0_{ia} = 0$ for every $i \in N$ and $a \in \tilde{O}$. Given $O^0, t^0, [P^0_{ia}]_{i \in N, a \in \tilde{O}}, \ldots, O^{v-1}, t^{v-1}, [P^{v-1}_{ia}]_{i \in N, a \in \tilde{O}}$, for any $a \in O^{v-1}$ define

$$t^v(a) = \sup \left\{ t \in [0,1] | \sum_{i \in N} P^{v-1}_{ia} + |N(a, O^{v-1})|(t - t^{v-1}) < q_a \right\}. \tag{2.3}$$

[10] Bogomolnaia and Moulin (2001) consider a broader class of simultaneous eating algorithms, where eating speeds may vary across agents and time.

(For any set S, we denote by $|S|$ the cardinality of S.) Define

$$t^v = \min_{a \in O^{v-1}} t^v(a), \tag{2.4}$$

$$O^v = O^{v-1} \setminus \{a \in O^{v-1} | t(a) = t^v\}, \tag{2.5}$$

$$P^v_{ia} = \begin{cases} P^{v-1}_{ia} + t^v - t^{v-1}, & i \in N(a, O^{v-1}), \\ P^{v-1}_{ia}, & \text{otherwise.} \end{cases} \tag{2.6}$$

Since O is a finite set, there exists $\bar{v}$ such that $t^{\bar{v}} = 1$. We define $PS(\succ) := P^{\bar{v}}$ to be the probabilistic serial random assignment for the preference profile $\succ$.

Bogomolnaia and Moulin (2001) show that the random assignment resulting from the probabilistic serial mechanism is ordinally efficient in their simplified setting. Their proof can be adapted easily to our setting, using Proposition 2.

Proposition 3. *For all preference profiles $\succ$, the probabilistic serial random assignment $PS(\succ)$ is ordinally efficient at $\succ$.*

Bogomolnaia and Moulin (2001) also show that the probabilistic serial mechanism is **envy-free**, that is, every agent who reports her preference truthfully weakly prefers, in the sense of first order stochastic dominance, her own random assignment to a random assignment of any other agent; the proof extends to our setting as well. Ordinal efficiency and envy-freeness are not satisfied by another popular mechanism called **random serial dictatorship** (Abdulkadiroğlu and Sönmez 1998) or **random priority** (Bogomolnaia and Moulin 2001).

The high level of efficiency and fairness of the probabilistic serial mechanism comes at a cost, however. The mechanism is **not strategy-proof**, that is, an agent is sometimes made better off misstating his preferences (Bogomolnaia and Moulin (2001) show that, however, the probabilistic serial mechanism is **weakly strategy-proof**, an agent cannot misstate his preferences and obtain a random assignment that stochastically dominates the one corresponding to truth-telling). Actually, Bogomolnaia and Moulin (2001) prove that there exists no mechanism that satisfies strategy-proofness, ordinal efficiency and equal treatment of equals.[11] Moreover, the ordinal efficiency and envy-freeness of the probabilistic serial mechanism are based on the presumption that agents state their ordinal preferences truthfully. If agents behave strategically, then neither of the two desirable properties is guaranteed. Therefore, it is important to find out under what circumstances there exist incentives for truthful reporting of the ordinal preferences in the probabilistic serial mechanism.

3. Result

Theorem 1. Let $\succ_i$ be an arbitrary strict preference relation and u_i be a utility index consistent with $\succ_i$. There exists M such that, if $q_a \geq M$ for all $a \in O$, then for any ordinal preferences $\succ'_i$, any set of agents $N \ni i$ and any profile of strict preferences $\succ_{N\setminus\{i\}}$

$$U_i(PS(\succ_i, \succ_{N\setminus\{i\}})) \geq U_i(PS(\succ'_i, \succ_{N\setminus\{i\}})),$$

where U_i is the expected utility function with utility index u_i. One M that satisfies the claim is $M = xD/d$ where $x \approx 1.76322$,[12] $d = \min_{a \succ_i b, a \succeq_i \emptyset} u_i(a) - u_i(b), D = \max_{a \succeq_i b \succeq_i \emptyset} u_i(a) - u_i(b)$.

A formal proof of the theorem is given in the Appendix. For a sketch of the argument, we first note that manipulations of preferences in the symmetric simultaneous eating algorithm have two effects: (1) at each point in time in the symmetric simultaneous eating algorithm, given the same set of available objects, reporting false preferences may prevent the agent from eating his most preferred available object; (2) reporting false preferences can affect the schedule of availability of objects, that is, reporting an object that is highly desirable as less desirable may lengthen the time that the object is available, and thus change the eating behavior of other agents and the availability schedule for other objects. The first effect is always detrimental to the manipulating agent, while the second effect can be positive. We prove that the second effect becomes of smaller order than the first effect when the number of units of each object type becomes large. If over some time

[11] A mechanism satisfies equal treatment of equals if any two agents who report identical ordinal preferences receive identical random assignments. In particular, envy-freeness implies equal treatment of equals.

[12] x solves $x = \mathbf{e}^{1/x}$ ($\mathbf{e}$ is the base of the natural logarithm).

interval an agent eats object a in the case of misstated preferences and object b in the case of truthfully reported preferences, and the agent prefers a to b, then it can be shown that many agents are eating from a over that time interval in the case the agent misstates his preferences (in the case the agent states his preferences truthfully, a is not available over that time interval, since otherwise the agent would be eating from it; it follows that, even in the case the agent misstates his preferences, the share of a eaten before the beginning of that time interval is close to q_a, and since q_a is large, many agents should be eating a over that time interval). Then the time interval over which the manipulating agent is better off by misstating his preferences (eating a instead of b) is short. We show that the length of any such interval is of smaller order than the sum of length of intervals on which the consumption when the agent misstates his preferences is less preferred to the consumption when he reports his true preferences.

Let $\succ$ be the profile of true preferences $(\succ_i, \succ_{N\setminus\{i\}})$, and $\succ'$ be the profile of preferences $(\succ'_i, \succ_{N\setminus\{i\}})$ in which agent i reports $\succ'_i$ instead of $\succ_i$. Define λ to be the sum of the lengths of the time intervals on which agent i's consumption in the symmetric simultaneous eating algorithm under $\succ$ is $\succ_i$-preferred to the one under $\succ'$. Let $k = |\{a \in O | a \succ_i \emptyset\}|$ be the number of proper object types that are $\succ_i$-preferred to the null object.

In Appendix A, we find a rough lower bound on the quotas of all objects sufficient for truth-telling to be a dominant strategy for player i. Based on the observations above, we show that the sum of the lengths of the time intervals on which i benefits from the change in the availability schedule caused by misstating preferences is at most $\lambda((1 + 1/M)^k - 1)$, so the expected utility gain from misstating preferences over these intervals is at most $D\lambda((1+1/M)^k - 1)$. Since the expected utility loss over the intervals in which i's consumption under $\succ$ is $\succ_i$-preferred to the one under $\succ'$ is at least $d\lambda$,

$$U_i(PS(\succ)) - U_i(PS(\succ')) \geq d\lambda - D\lambda\left(\left(1+\frac{1}{M}\right)^k - 1\right). \tag{3.1}$$

We show that the right hand side of 3.1 is non-negative, and hence truth-telling is a dominant strategy for player i, if

$$M \geq (k+1)\frac{D}{d}. \tag{3.2}$$

Note that for this lower bound for M, the expected benefit from reporting false preferences is of order at most $D\lambda k/M$.

In Appendix B, we refine the lower bound. The argument is based on the observation that $\Lambda = \frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}$ has the property that the object eaten at any time t under $\succ$ is $\succeq_i$-preferred to the one eaten at time $t+\Lambda$ under $\succ'$. Then, we can evaluate the expected utility gain from misstating preferences by translating the eating schedule under $\succ$ by Λ with respect to the one under $\succ'$, with the only positive difference contribution corresponding to the difference in consumption over the interval $[0, \Lambda]$ when $\succ'$ is reported and consumption over the interval $[1-\Lambda, 1]$ when $\succ$ is reported. Thus a bound on the expected utility gain from misstating preferences of $D\Lambda$ is obtained. Therefore,

$$U_i(PS(\succ)) - U_i(PS(\succ')) \geq d\lambda - D\frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}. \tag{3.3}$$

A sufficient condition for the right hand side of 3.3 be nonnegative, and truth-telling be a dominant strategy for player i, is

$$M \geq x\frac{D}{d} \text{ with } x \approx 1.76322. \tag{3.4}$$

This computation yields a weaker sufficient condition for M, since the bound on the expected benefit from reporting false preferences is shown to be of order $D\lambda/M$ in Appendix B, while in Appendix A the bound we find is of order $D\lambda k/M$.

Remark 1. The bound we find in Appendix B (3.4) for truth-telling to be a dominant strategy for player i is linear in D/d. Note that $D/d \geq k$. However, if the utility difference between any two consecutively ranked objects does not vary a lot, then D/d is of order k. For example, in a school choice setting where a student finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant, a sufficient condition for truth-telling to be a dominant strategy for him in the probabilistic serial mechanism is that each school has at least 18 seats.

One important feature of the bound for M in 3.4 is that it is independent of the misstated ordinal preferences $\succ_i'$, the set of agents $N \setminus \{i\}$ and their profile of strict preferences $\succ_{N\setminus\{i\}}$. In particular, agent i may verify whether 3.4 holds only using his information about D/d. Therefore, whenever 3.4 holds, truth-telling is a best response for player i independently of how many other agents participate and what their reported preferences are. Even when M is not large enough to make truth-telling a dominant strategy for all agents, truth-telling may be a dominant strategy for some of the agents.

Remark 2. We present an example that serves three purposes. First, it illustrates some of the ideas of the proof. Second, it shows that the lower bound for M necessary for truth-telling is $D/d - 1$, hence the bound we find in the proof of Theorem 1 (3.4) cannot be improved by more than a factor of $x \approx 1.76322$ (we do not know whether there exists an example with the lower bound for M necessary for truth-telling closer to xD/d). Third, it shows that the conclusion of the theorem cannot be strengthened to imply the existence of M such that, in the probabilistic serial mechanism for all markets with $q_a \geq M$ for all $a \in O$, agent i has incentives to report his ordinal preferences truthfully irrespective of his expected utility function.

Consider an agent i in an environment with 2 types of proper objects, a and b, each having quota M. Let the true preferences of agent i be given by $a \succ_i b \succ_i \o$. Fix $D > d > 0$, and let u_i be a utility index such that $u_i(a) = D, u_i(b) = D - d, u_i(\o) = 0$. Note that u_i is consistent with i's true preferences.

Let $N = \{i\} \cup N' \cup N''$ be the set of agents, with N' containing M agents and N'' containing $M+1$ agents. Assume that the preferences of the agents are specified by

$$a \succ_j \o \succ_j b, \ \forall j \in N',$$

$$b \succ_j \o \succ_j a, \ \forall j \in N''.$$

Let $\succ_i'$ be a preference relation for agent i specified by $b \succ_i' a \succ_i' \o$. Suppose all other $2M+1$ agents report their preferences truthfully.

Consider the symmetric simultaneous eating algorithm. If i reports his preferences truthfully, then he eats object a in the time interval $[0, M/(M+1))$, and the null object in the time interval $[M/(M+1), 1]$. Therefore his assignment under the probabilistic serial mechanism is given by

$$(P_{ia}, P_{ib}, P_{i\o}) = \left(\frac{M}{M+1}, 0, \frac{1}{M+1} \right). \tag{3.5}$$

If i reports $\succ_i'$ instead of $\succ_i$, then he eats object b in the time interval $[0, M/(M+2))$, object a in $[M/(M+2), M(M+3)/(M+1)(M+2))$, and then the null object in $[M(M+$

$3)/(M+1)(M+2), 1]$. Hence his assignment under the probabilistic serial mechanism is given by

$$(P'_{ia}, P'_{ib}, P'_{i\o}) = \left(\frac{2M}{(M+1)(M+2)}, \frac{M}{M+2}, \frac{2}{(M+1)(M+2)} \right). \tag{3.6}$$

Figure 1 illustrates the eating schedules for agent i under the two preference profiles.

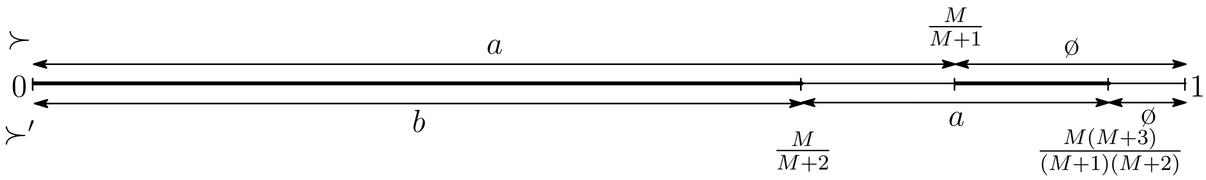

FIGURE 1. Eating schedules for agent i under $\succ$ and $\succ'$.

In the time interval $[0, M/(M+2))$, agent i eats a if he reports $\succ_i$ and b if he reports $\succ'_i$; agent i's expected utility loss from misstating his preferences corresponding to that time interval is $dM/(M+2)$. In the time interval $[M/(M+1), M(M+3)/(M+1)(M+2))$, i eats $\o$ if he reports $\succ_i$ and a if he reports $\succ'_i$; agent i's expected utility gain from misstating his preferences corresponding to that time interval is $DM/(M+1)(M+2)$. At any time except for these intervals, the object i eats if he reports $\succ_i$ is identical to the one he eats if he reports $\succ'_i$. In Figure 1, by misstating his preferences, agent i loses over the first thick interval, and gains over the second thick interval. (Note that the length of the latter is of order M times smaller than the length of the former.)

If U_i is the expected utility function corresponding to u_i,

$$U_i(P_i) - U_i(P'_i) = \frac{dM}{(M+1)(M+2)} \left(M + 1 - \frac{D}{d} \right). \tag{3.7}$$

Therefore, agent i has an incentive to report false preferences $\succ'_i$ when $M < D/d - 1$ (in particular, there exists no M such that, in the probabilistic serial mechanism for all markets with $q_a \geq M$ for all $a \in O$, agent i has incentives to report his ordinal preferences truthfully irrespective of his expected utility function). Conversely, if $M \geq D/d - 1$, truth-telling is a dominant strategy for i (note that the probabilistic serial assignment when i reports any preferences different from $\succ_i$ and $\succ'_i$ is first-order stochastically dominated by either P_i or P'_i).

Remark 3. The proof can be adapted (with straightforward changes to Claims 1 and 2 in the appendix) to show that the condition "$q_a \geq M$ for all $a \in O$" can be replaced with "$q_a \geq M$ for all $a \succ_i \emptyset$" in the statement of the theorem.

Theorem 1 has a couple of corollaries.

Corollary 1. *Suppose that the set O of possible proper object types and the set $\mathcal{U}$ of possible von Neumann-Morgenstern expected utility functions on lotteries over $\tilde{O}$ are fixed and finite (we maintain the assumption that ordinal preferences over $\tilde{O}$ consistent with the utility functions in $\mathcal{U}$ are strict). There exists M such that if $q_a \geq M$ for all $a \in O$, then for any set of agents N with utility functions in $\mathcal{U}$, truth-telling is a weakly dominant strategy for each agent in the probabilistic serial mechanism.*

Fix a problem $\Gamma = (N, (\succ_i)_{i \in N}, O, (q_a)_{a \in O})$ with utility index u_i consistent with $\succ_i$ for each i in N, and a positive integer M. The M-fold **replica economy** of (Γ, u) is a random assignment problem in which there are M "replicas" of each agent i, all having utility index u_i, and there are Mq_a copies of each object a in O.

Corollary 2. *For any random assignment problem $\Gamma = (N, (\succ_i)_{i \in N}, O, (q_a)_{a \in O})$ with utility indices $(u_i)_{i \in N}$, there exists $\underline{M}$ such that for any M-fold replica economy of (Γ, u) with $M \geq \underline{M}$, truth-telling is a weakly dominant strategy for each agent in the probabilistic serial mechanism.*

4. Conclusion

Truth-telling is a dominant strategy for an agent under the probabilistic serial mechanism when there is a large supply of each object type. This result gives support to the use of the mechanism in applications such as university housing and student placement in schools. A surprising feature of our result is that truth-telling is an exact dominant strategy as opposed to an "almost dominant strategy" common in the literature on asymptotic incentive compatibility. Fixing the set of object types, and one agent with his utility function, our conclusion holds irrespective of the set of other agents and their ordinal preferences.

The bound on the size of the supply of each object type we find in the proof of Theorem 1 cannot be improved by more than a factor of $x \approx 1.76322$. Whether the bound can be improved on at all is an open question. However, the bound we find is sufficiently low to make truth-telling a weakly dominant strategy in practical assignment problems.

Appendix A. Proof of Theorem 1

Let $\succ$ be the profile of true preferences $(\succ_i, \succ_{N\setminus\{i\}})$, and $\succ'$ be the profile of preferences $(\succ'_i, \succ_{N\setminus\{i\}})$ in which agent i states $\succ'_i$ instead of $\succ_i$.

Let an **eating function** e describe some eating schedule for each agent, $e_i : [0,1] \to \tilde{O}$ for all $i \in N$, with $e_i(t)$ representing the object agent i is eating at time t. We demand that e_i be right-continuous with respect to the discrete topology on $\tilde{O}$ (the topology in which all subsets are open), that is,

$$t < 1, e_i(t) = a \Rightarrow \exists \varepsilon > 0, e_i(t') = a,\ \forall t' \in [t, t+\varepsilon). \tag{A.1}$$

Let $n_a(t,e)$ be the number of agents eating from object a at time t, and $G_a(t,e)$ be the share of object a eaten away by time t under the eating function e, i.e.,

$$n_a(t,e) = \sum_{i\in N} \mathbf{1}_{e_i(t)=a}, \tag{A.2}$$

$$G_a(t,e) = \int_0^t n_a(s,e)ds, \tag{A.3}$$

where $\mathbf{1}$ is the indicator function. Note that $G_a(\cdot,e)$ is continuous.

Define $e^{\succ}$ as the eating function determined by the symmetric simultaneous eating algorithm when the agents report $\succ$. Formally, let $e_i^{\succ}(t) = a$ if $i \in N(a, O^{v-1})$ for $t \in [t^{v-1}, t^v)$, where $(O^v)_{v=0,\dots,\bar{v}}$ and $(t^v)_{v=0,\dots,\bar{v}}$ are defined by 2.4 and 2.5. Note that

$$PS(\succ)_{ia} = \int_0^1 \mathbf{1}_{e_i^{\succ}(t)=a} dt, \tag{A.4}$$

for each i and a.

Given $\succ$ and $\succ'$, let $\bar{e}$ be the eating function with $\bar{e}_i$ defined by

$$\bar{e}_i(t) = \begin{cases} e_i^{\succ}(t) & \text{if } e_i^{\succ}(t) = e_i^{\succ'}(t) \\ \emptyset & \text{otherwise} \end{cases}, \tag{A.5}$$

and, $\bar{e}_j$, for $j \neq i$, defined such that j starts eating from his most preferred object at time 0, and subsequently eats from his most preferred object at speed 1 among the ones still available. Note that the eating function $\bar{e}_i$ of i is fixed, and that $\bar{e}_j$ may be different from

$e_j^{\succ}$ or $e_j^{\succ'}$ for $j \neq i$ since available objects at various times may be different across $\bar{e}$, $e^{\succ}$ and $e^{\succ'}$ due to different eating behavior for i.

Let $\beta(t)$ and $\gamma(t)$ be the sums of the lengths of the time intervals, before time t, on which agent i's consumption in the symmetric simultaneous eating algorithm is $\succ_i$-preferred, and respectively $\succ_i$-less preferred, if the reported preferences change from $\succ$ to $\succ'$; let $\delta(t)$ be the sum of the lengths of the time intervals, before time t, on which agent i's consumption in the symmetric simultaneous eating algorithm is not identical if the reported preferences change from $\succ$ to $\succ'$,

$$\beta(t) = \int_0^t \mathbf{1}_{e_i^{\succ'}(s) \succ_i e_i^{\succ}(s)} ds \tag{A.6}$$

$$\gamma(t) = \int_0^t \mathbf{1}_{e_i^{\succ}(s) \succ_i e_i^{\succ'}(s)} ds \tag{A.7}$$

$$\delta(t) = \beta(t) + \gamma(t). \tag{A.8}$$

Note that $\beta(\cdot), \gamma(\cdot), \delta(\cdot)$ are continuous.

Lemma 1. For all $t \in [0,1]$ and $a \in O$,

$$0 \leq G_a(t, e^{\succ}) - G_a(t, \bar{e}) \tag{A.9}$$

$$-\delta(t) \leq G_{\emptyset}(t, e^{\succ}) - G_{\emptyset}(t, \bar{e}) \tag{A.10}$$

$$0 \leq G_a(t, e^{\succ'}) - G_a(t, \bar{e}) \tag{A.11}$$

$$-\delta(t) \leq G_{\emptyset}(t, e^{\succ'}) - G_{\emptyset}(t, \bar{e}). \tag{A.12}$$

Proof. By symmetry, we only need to prove the first two inequalities.

To prove the first inequality, we proceed by contradiction. Assume that there exist t and a such that $G_a(t, e^{\succ}) - G_a(t, \bar{e}) < 0$. Let

$$t_0 = \inf\{t \in [0,1] | \exists a \in O, G_a(t, e^{\succ}) - G_a(t, \bar{e}) < 0\}. \tag{A.13}$$

By continuity of $G_a(\cdot, e^{\succ}) - G_a(\cdot, \bar{e})$, it follows that $t_0 < 1$, and

$$0 \leq G_a(t_0, e^{\succ}) - G_a(t_0, \bar{e}), \ \forall a \in O. \tag{A.14}$$

(When $t_0 = 0$, this holds trivially.)

It follows that all objects that are not eaten away by time t_0 under $e^{\succ}$ are not eaten away by time t_0 under $\overline{e}$ either. So, the set of available objects at t_0 under $e^{\succ}$ is included in the set of available objects at t_0 under $\overline{e}$. Hence, if agent $j \in N$ is eating object $a \in O$ at t_0 under $\overline{e}$ and a is available at t_0 under $e^{\succ}$, then j needs to be eating a at t_0 under $e^{\succ}$, formally

$$\overline{e}_j(t_0) = a \;\&\; G_a(t_0, e^{\succ}) < q_a \Rightarrow e^{\succ}_j(t_0) = a, \forall j \in N. \tag{A.15}$$

(This holds for $j = i$ by the definition of $\overline{e}$.) Therefore,

$$G_a(t_0, e^{\succ}) < q_a \Rightarrow n_a(t_0, e^{\succ}) \geq n_a(t_0, \overline{e}). \tag{A.16}$$

Given the right-continuity of $e^{\succ}$ and $\overline{e}$, for sufficiently small $\varepsilon > 0$, we have that for all $t \in [t_0, t_0 + \varepsilon)$ and $a \in O$

$$G_a(t, e^{\succ}) = G_a(t_0, e^{\succ}) + n_a(t_0, e^{\succ})(t - t_0) \tag{A.17}$$

$$G_a(t, \overline{e}) = G_a(t_0, \overline{e}) + n_a(t_0, \overline{e})(t - t_0). \tag{A.18}$$

By A.14 and A.16, it follows that for all t in $[t_0, t_0 + \varepsilon)$ and $a \in O$ with $G_a(t_0, e^{\succ}) < q_a$

$$0 \leq G_a(t, e^{\succ}) - G_a(t, \overline{e}). \tag{A.19}$$

Note that if $G_a(t_0, e^{\succ}) = q_a$ the inequality holds trivially for all $t \geq t_0$.

By the definition of t_0, for all t in $[0, t_0]$ and a in O

$$0 \leq G_a(t, e^{\succ}) - G_a(t, \overline{e}). \tag{A.20}$$

Therefore, for all t in $[0, t_0 + \varepsilon)$ and a in O

$$0 \leq G_a(t, e^{\succ}) - G_a(t, \overline{e}), \tag{A.21}$$

which contradicts the definition of t_0.

To prove that $-\delta(t) \leq G_{\emptyset}(t, e^{\succ}) - G_{\emptyset}(t, \overline{e})$, note that

$$G_{\emptyset}(t, e^{\succ}) - G_{\emptyset}(t, \overline{e}) + \delta(t) = \int_0^t [n_{\emptyset}(s, e^{\succ}) - n_{\emptyset}(s, \overline{e}) + \mathbf{1}_{e^{\succ}_i(s) \neq e^{\succ'}_i(s)}] ds. \tag{A.22}$$

By an argument similar to the one above, using the fact that for all $a \in O$ and $t \in [0, 1]$

$$0 \leq G_a(t, e^{\succ}) - G_a(t, \overline{e}), \tag{A.23}$$

we have

(1) if $e^{\succ}(s) \neq \overline{e}(s)$ then $n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \overline{e}) - 1$

(2) if $e^{\succ}(s) = \overline{e}(s)$ then $n_{\o}(s, e^{\succ}) \geq n_{\o}(s, \overline{e})$,

so the integrand $n_{\o}(s, e^{\succ}) - n_{\o}(s, \overline{e}) + \mathbf{1}_{e_i^{\succ}(s) \neq e_i^{\succ'}(s)}$ is non-negative for all $s \in [0, t]$, completing the proof.

□

Lemma 2. For all $t \in [0, 1]$ and $a \in O$,

$$G_a(t, e^{\succ}) - G_a(t, \overline{e}) \leq \delta(t). \tag{A.24}$$

Proof. The inequality follows trivially from Lemma 1, noting that for each t in $[0, 1]$

$$\sum_{a \in \tilde{O}} G_a(t, e^{\succ}) - G_a(t, \overline{e}) = 0. \tag{A.25}$$

□

Lemma 3. For all $t \in [0, 1]$ and $a \in O$,

$$G_a(t, e^{\succ}) - G_a(t, e^{\succ'}) \leq \delta(t). \tag{A.26}$$

Proof. The inequality follows from Lemmata 1 and 2 (A.11 and A.24), writing

$$G_a(t, e^{\succ}) - G_a(t, e^{\succ'}) = [G_a(t, e^{\succ}) - G_a(t, \overline{e})] - [G_a(t, e^{\succ'}) - G_a(t, \overline{e})]. \tag{A.27}$$

□

Proof of the Theorem. If $e_i^{\succ}(t) \succeq_i e_i^{\succ'}(t)$ for all t in $[0, 1)$ the proof is immediate. So, assume $e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)$ for some t in $[0, 1)$.

Let

$$\{a_1, a_2, \ldots, a_{\bar{l}}\} = \{a \in O | \exists t \in [0, 1), a = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\}, \tag{A.28}$$

be the set of objects that are consumed at some time under $e_i^{\succ'}$, and are $\succ_i$-preferred to the consumption at that time under $e_i^{\succ}$. We label the set such that $a_1 \succ_i' a_2 \succ_i' \ldots \succ_i' a_{\bar{l}}$. For $l = 1, 2, \ldots, \bar{l}$, let

$$T_l = \inf\{t | a_l = e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)\} \tag{A.29}$$

be the first time when a_l is consumed under $e_i^{\succ'}$, and is $\succ_i$-preferred to the consumption at that time under $e_i^{\succ}$. Obviously, $0 < T_1 < T_2 < \ldots < T_{\bar{l}} < 1$.

Let $k = |\{a \in O | a \succ_i \o\}|$ be the number of proper object types that are $\succ_i$-preferred to the null object. Note that $\bar{l} \leq k$ since, for all l, a_l is $\succ_i$-preferred to some object that is eaten under $e^{\succ}$, hence is $\succ_i$-preferred to the null object, and that $\bar{l} \geq 1$ by the initial assumption that $e_i^{\succ'}(t) \succ_i e_i^{\succ}(t)$ for some $t \in [0, 1)$. Recall the definitions of $\beta(\cdot), \gamma(\cdot), \delta(\cdot)$ in A.6-A.8. Define

$$\lambda = \gamma(1), \tag{A.30}$$

as the sum of the lengths of the time intervals on which agent i's consumption in the symmetric simultaneous eating algorithm under $\succ$ is $\succ_i$-preferred to the one under $\succ'$.

Set $T_0 = 0, T_{\bar{l}+1} = 1$, as a technical notation convention.

Claim 1. If $q_a \geq M$ for all $a \in O$, then for all $l = 0, 1, \ldots, \bar{l}$,

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\lambda}{M} \left(1 + \frac{1}{M}\right)^{l-1}. \tag{A.31}$$

Proof of the claim. We prove the claim by induction on l for $l = 0, 1, \ldots, \bar{l}$.

For $l = 0$, the induction hypothesis holds trivially since $\beta(T_1) = 0$.

Let $l \geq 1$. Assume the induction hypothesis holds for $0, 1, \ldots, l-1$, and we prove that it holds for l.

By the induction hypothesis,

$$\delta(T_l) \leq \lambda + \sum_{g=1}^{l-1} \beta(T_{g+1}) - \beta(T_g) \leq \lambda + \frac{\lambda}{M} \sum_{g=1}^{l-1} \left(1 + \frac{1}{M}\right)^{g-1} = \lambda \left(1 + \frac{1}{M}\right)^{l-1}. \tag{A.32}$$

(Note that the inequality above holds for $l = 1$ as well.)

Since

$$a_l = e_i^{\succ'}(T_l) \succ_i e_i^{\succ}(T_l), \tag{A.33}$$

it follows that object a_l is not available at T_l under the eating function $e^{\succ}$, i.e., $G_{a_l}(T_l, e^{\succ}) = q_{a_l}$. By Lemma 3,

$$G_{a_l}(T_l, e^{\succ'}) \geq G_{a_l}(T_l, e^{\succ}) - \delta(T_l) \geq q_{a_l} - T_l > q_{a_l} - 1. \tag{A.34}$$

Note that $n_{a_l}(\cdot, e^{\succ'})$ is increasing on the time interval for which object a_l is available under $e^{\succ'}$, so

$$n_{a_l}(T_l, e^{\succ'}) > n_{a_l}(T_l, e^{\succ'})T_l \geq \int_0^{T_l} n_{a_l}(s, e^{\succ'})ds = G_{a_l}(T_l, e^{\succ'}) > q_{a_l} - 1. \tag{A.35}$$

Then $n_{a_l}(T_l, e^{\succ'}) \geq q_{a_l}$, because $n_{a_l}(T_l, e^{\succ'})$ is an integer. Since a_l is still available at T_l under $e^{\succ'}$, it follows that the number of agents eating from a_l, $n_{a_l}(s, e^{\succ'})$, is greater or equal to q_{a_l} for all times $s \geq T_l$ where a_l is still available under $e^{\succ'}$ (or, equivalently, $e_i^{\succ'}(s) = a_l$). Therefore

$$n_{a_l}(s, e^{\succ'}) \geq q_{a_l} \mathbf{1}_{e_i^{\succ'}(s)=a_l}, \forall s \in [T_l, T_{l+1}). \tag{A.36}$$

Then, by A.34,

$$\begin{aligned}
\delta(T_l) &\geq q_{a_l} - G_{a_l}(T_l, e^{\succ'}) && \text{(A.37)}\\
&= G_{a_l}(T_{l+1}, e^{\succ'}) - G_{a_l}(T_l, e^{\succ'}) && \text{(A.38)}\\
&= \int_{T_l}^{T_{l+1}} n_{a_l}(s, e^{\succ'}) ds && \text{(A.39)}\\
&\geq q_{a_l} \int_{T_l}^{T_{l+1}} \mathbf{1}_{e_i^{\succ'}(s)=a_l} ds && \text{(A.40)}\\
&= q_{a_l} (\beta(T_{l+1}) - \beta(T_l)), && \text{(A.41)}
\end{aligned}$$

where the last equality holds since, by the definition of the sequence $(a_l, T_l)_{l \in \{1,\ldots,\bar{l}\}}$, the times in $[T_l, T_{l+1})$ at which agent i's consumption in the symmetric simultaneous eating algorithm is $\succ_i$-preferred when the reported preferences change from $\succ$ to $\succ'$ are exactly the times at which agent i is eating object a_l.

Since $q_{a_l} \geq M$ by assumption, A.32 implies that

$$\beta(T_{l+1}) - \beta(T_l) \leq \frac{\lambda}{M} \left(1 + \frac{1}{M}\right)^{l-1}, \tag{A.42}$$

ending the proof of the induction step.

□

To finish the proof, note that if we set

$$\begin{aligned}
d &= \min_{a \succ_i b, a \succeq_i \emptyset} u_i(a) - u_i(b) && \text{(A.43)}\\
D &= \max_{a \succeq_i b \succeq_i \emptyset} u_i(a) - u_i(b), && \text{(A.44)}
\end{aligned}$$

then

$$U_i(PS(\succ)) - U_i(PS(\succ')) = \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s)) ds \geq d\gamma(1) - D\beta(1). \tag{A.45}$$

By definition, $\gamma(1) = \lambda$. Since $\bar{l} \leq k$, and $\beta(T_1) = 0$, adding the inequalities of Claim 1 for $l = 1, 2, \ldots, \bar{l}$,

$$\beta(1) \leq \sum_{g=0}^{\bar{l}} \frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{g-1} = \lambda\left(\left(1+\frac{1}{M}\right)^{\bar{l}} - 1\right) \leq \lambda\left(\left(1+\frac{1}{M}\right)^{k} - 1\right). \tag{A.46}$$

Therefore,

$$U_i(PS(\succ)) - U_i(PS(\succ')) \;\geq\; \lambda\left(d - D\left(\left(1+\frac{1}{M}\right)^{k} - 1\right)\right), \tag{A.47}$$

which is non-negative if

$$M \geq \frac{1}{\left(\frac{d}{D}+1\right)^{1/k} - 1}, \tag{A.48}$$

completing the proof.

□

Remark 4. We can linearize the bound in A.48 as a function of k and D/d. Using Taylor expansions of $(1+x)^{1/k} - 1$ at $x = 0$ we get the inequalities[13]

$$\frac{d}{D}\frac{1}{k} - \left(\frac{d}{D}\right)^2 \frac{1}{k}\left(1-\frac{1}{k}\right) \leq \left(\frac{d}{D}+1\right)^{1/k} - 1 \leq \frac{d}{D}\frac{1}{k}, \tag{A.49}$$

which offer bounds for the denominator in A.48.

Therefore, truth-telling is a dominant strategy for player i if

$$M \geq k\frac{D}{d}\frac{1}{1-\frac{d}{D}\left(1-\frac{1}{k}\right)}. \tag{A.50}$$

Noting that $d/D \leq 1/k$ it follows that $k+1 > \frac{k}{1-\frac{d}{D}\left(1-\frac{1}{k}\right)}$, hence a sufficient condition for truth-telling to be a dominant strategy for player i is

$$M \geq (k+1)\frac{D}{d}. \tag{A.51}$$

[13]These inequalities can be checked by taking first and second order derivatives of $(1+x)^{1/k} - 1 - \frac{1}{k}x$ and $(1+x)^{1/k} - 1 - \frac{1}{k}x + \frac{1}{k}(1-\frac{1}{k})x^2$ for $x \geq 0$.

Appendix B. Refinement of the Bound

Assume $q_a \geq M$ for all $a \in O$. We prove that

$$M \geq x\frac{D}{d} \text{ with } x \approx 1.76322 \tag{B.1}$$

is a sufficient condition for the conclusion of the theorem to hold.

Let

$$\Lambda = \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{k-1}. \tag{B.2}$$

Claim 2. For all $a \in O$, $t \leq T_{\bar{l}}$ and $t' = t + \Lambda$ with $t' \leq 1$, if $G_a(t, e^{\succ}) = q_a$ then $G_a(t', e^{\succ'}) = q_a$.

Proof of the claim. Let $a \in O$, $t \leq T_{\bar{l}}$ and $t' = t + \Lambda$, with $G_a(t, e^{\succ}) = q_a$.

By the proof of Claim 1 (A.32),

$$\delta(t) \leq \delta(T_{\bar{l}}) \leq M\Lambda. \tag{B.3}$$

By Lemma 3,

$$G_a(t, e^{\succ'}) \geq G_a(t, e^{\succ}) - \delta(t) \geq q_a - t > q_a - 1. \tag{B.4}$$

We prove that $G_a(t', e^{\succ'}) = q_a$ by contradiction. Assume that $G_a(t', e^{\succ'}) < q_a$. Note that $n_a(\cdot, e^{\succ'})$ is increasing on the time interval for which object a is available under $e^{\succ'}$, so

$$n_a(t, e^{\succ'}) > n_a(t, e^{\succ'})t \geq \int_0^t n_a(s, e^{\succ'})ds = G_a(t, e^{\succ'}) > q_a - 1. \tag{B.5}$$

Then $n_a(t, e^{\succ'}) \geq q_a$, because $n_a(t, e^{\succ'})$ is an integer. Since a is still available at t' under $e^{\succ'}$, it follows that

$$n_a(s, e^{\succ'}) \geq q_a, \forall s \in [t, t'). \tag{B.6}$$

Then, by B.3 and B.4, $G_a(t, e^{\succ'}) \geq G_a(t, e^{\succ}) - \delta(t) \geq G_a(t, e^{\succ}) - M\Lambda = q_a - M\Lambda > G_a(t', e^{\succ'}) - M\Lambda$. Therefore

$$M\Lambda > G_a(t', e^{\succ'}) - G_a(t, e^{\succ'}) \tag{B.7}$$
$$= \int_t^{t'} n_a(s, e^{\succ'})ds \tag{B.8}$$
$$\geq q_a(t' - t) \tag{B.9}$$
$$= q_a\Lambda, \tag{B.10}$$

a contradiction with $q_a \geq M$.

□

By the construction of the sequence (a_l, T_l), and by the fact that $G_{a_{\bar{l}}}(T_{\bar{l}} + \Lambda, e^{\succ'}) = q_{a_{\bar{l}}}$ when $T_{\bar{l}} + \Lambda \leq 1$, we get that $u_i(e_i^{\succ'}(s)) \leq u_i(e_i^{\succ}(s))$ for all $s \geq \min\{T_{\bar{l}} + \Lambda, 1\}$.

For technical purposes, extend the definition of $e_i^{\succ}$ such that $e_i^{\succ}(s) = e_i^{\succ}(0)$ for all $s \in [-\Lambda, 0)$. It follows from Claim 2, and the observation above, that $u_i(e_i^{\succ'}(s)) \leq u_i(e_i^{\succ}(s - \Lambda))$ for all $s \in [0, 1]$.

Then,

$$U_i(PS(\succ)) - U_i(PS(\succ')) = \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))ds \tag{B.11}$$
$$= \int_0^1 \max\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))\}ds \tag{B.12}$$
$$+ \int_0^1 \min\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ'}(s))\}ds \tag{B.13}$$
$$\geq d\lambda + \int_0^1 \min\{0, u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s - \Lambda))\}ds \tag{B.14}$$
$$= d\lambda + \int_0^1 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s - \Lambda))ds \tag{B.15}$$
$$= d\lambda + \int_0^1 u_i(e_i^{\succ}(s))ds - \int_{-\Lambda}^{1-\Lambda} u_i(e_i^{\succ}(s))ds \tag{B.16}$$
$$= d\lambda + \int_{1-\Lambda}^1 u_i(e_i^{\succ}(s))ds - \int_{-\Lambda}^0 u_i(e_i^{\succ}(s))ds \tag{B.17}$$
$$= d\lambda - \int_{-\Lambda}^0 u_i(e_i^{\succ}(s)) - u_i(e_i^{\succ}(s + 1))ds \tag{B.18}$$
$$\geq d\lambda - D\Lambda. \tag{B.19}$$

Therefore,

$$U_i(PS(\succ)) - U_i(PS(\succ')) \geq d\lambda - D\Lambda = \frac{d\lambda}{M}\left(M - \frac{D}{d}\left(1+\frac{1}{M}\right)^{k-1}\right). \tag{B.20}$$

Suppose $M \geq xD/d$, where x solves $x = \mathbf{e}^{1/x}$ ($\mathbf{e}$ is the base of the natural logarithm; $x \approx 1.76322$). Since $D/d \geq k$,

$$\left(1+\frac{1}{M}\right)^{k-1} < \left(1+\frac{1}{xk}\right)^{k} = \left(\left(1+\frac{1}{xk}\right)^{xk}\right)^{1/x} < \mathbf{e}^{1/x}. \tag{B.21}$$

It follows that

$$U_i(PS(\succ)) - U_i(PS(\succ')) \geq \frac{d\lambda}{M}\left(x\frac{D}{d} - \frac{D}{d}\mathbf{e}^{1/x}\right) = 0, \tag{B.22}$$

hence a sufficient condition for truth-telling to be a dominant strategy for player i is

$$M \geq x\frac{D}{d} \text{ with } x \approx 1.76322. \tag{B.23}$$

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