Social Learning with Partial Observations

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Abstract— **We study a model of social learning with partial observations from the past. Each individual receives a private signal about the correct action he should take and also observes the action of his immediate neighbor. We show that in this model the behavior of asymptotic learning is characterized in terms of two threshold values that evolve deterministically. Individual actions are fully determined by the value of their signal relative to these two thresholds. We prove that asymptotic learning from an ex ante viewpoint applies if and only if individual beliefs are unbounded. We also show that symmetry between the states implies that the minimum possible amount of asymptotic learning occurs.**

I. Introduction

Many important decision are taken by individuals under conditions of imperfect information. In such situations, it is natural for individuals to gather information in order to improve their decisions. A major source of information is the past actions of other individuals facing similar decision problems. This motivates the analysis of social learning problems, where a group of individuals are simultaneously learning from others and also taking important economic or social decisions. Examples of social learning problems include behavior in financial markets, where each trader may try to learn from the positions of other traders or from prices, consumer decisions in product markets, where purchases by other consumers are a key source of information, and political decision-making, where in voting or other political actions individuals typically learn from and condition on the behavior of others. A central question is therefore whether the equilibrium process of social learning will lead to the correct actions by groups.[1]

A large literature in game theory investigates the first question. A well-known result in this context, first derived by Banerjee [1] and Bikchandani et al. [3], establishes the possibility of a "pathological" result that features no learning and the possibility of incorrect actions by a large group of individuals. Consider *N* individuals ordered exogenously and choosing between two actions, say 0 and 1. Each individual receives a signal about which action is the right one and also observes the actions of all other agents that have moved before him. The signal received by each individual takes two possible values (one favoring 0 the other one favoring 1) and is identically and independently distributed across individuals. Banerjee [1] and Bikchandani et al. [3] show that the perfect Bayesian equilibrium of this game involves a particular type of "herding" in which following two consecutive actions in the same direction (for example, two individuals choosing 0), each subsequent individual ignores his own signal and follows the actions of these two individuals. Clearly, since two individuals choosing the action 0 is possible even when the right action is 1, this result illustrates a pathological form of non-learning and incorrect actions by individuals.

A more complete analysis of this model is provided by Smith and Sorensen [9], who analyze the case in which signals can also differ in their informativeness. Smith and Sorensen's main result can be summarized as follows. Let us refer to signals as *unbounded* if the likelihood ratio of a particular state can be arbitrarily large conditional on individual signals and as *bounded* otherwise. Smith and Sorensen show that with unbounded signals, there will be asymptotic learning, i.e., the probability of the correct action being chosen converges to 1.

This literature typically focuses on social learning environments in which individuals observe all previous actions. Consequently, the information set of individuals making decision later is necessarily finer than those moving earlier, which implies that Bayesian posteriors form *a martingale*. This property enables the use of the martingale convergence theorem and significantly simplifies the analysis. However, most relevant cases of social learning in practice do not feature this property. Often, each individual will have observed a different sample of actions than those who have acted before and will not necessarily have superior information relative to them. The existing literature, except for the more recent paper by Smith and Sorensen [8], has not studied the properties of equilibrium social learning in this more realistic environment. An investigation of the patterns of

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[1] A related and equally important question concerns what types of communication and observation structures will facilitate learning. For example, is learning more or less likely when individuals observe actions and communicate within their narrow communities? More generally, what is the impact of the topology of a social network on the patterns of learning? We study this question in our companion paper [7].

social learning in such an environment is not only important because of its greater realism, but also because it will enable us to address the second question posed above and study what types of social structures are more conducive to learning and information aggregation.

In this paper, we take a step in this direction by studying the simplest model of social learning without the martingale property. Each individual again receives a signal (with varying degree of informativeness) but only observes the action of the person who has moved before him. Despite the simplicity of this environment, existing results in the literature do not apply. Moreover, the mathematical structure of this simple case is very similar to the case in which each individual observes a uniformly random decision from the past and our result extend in a straightforward manner.

Our main results are as follows. First, we provide a recursive characterization of individual decisions in terms of two deterministic thresholds, such that the value of individual signals relative to these thresholds completely determines decisions. Second, as in Smith and Sorensen [9], unbounded signals ensure asymptotic learning. Third, when signals are bounded, there will never be asymptotic learning. Finally, we show that under a symmetry condition on the conditional signal distributions and with bounded signals, there will exist an equilibrium with the minimum amount of learning in the long-run. Under very mild conditions, this equilibrium is unique. In contrast, with asymmetry between the states, the amount of asymptotic learning can be quite high.

Our paper is related to the large and growing social learning literature (see [1], [3], [5], [4], [10]). Most closely related are the recent papers by Banerjee and Fudenberg [2] and Smith and Sorensen [8]. Banerjee and Fudenberg analyze a model of social learning in which individuals observe a random sample of past actions under the assumption that there is a continuum of agents, so that past actions reveal sufficient information about the underlying state. Smith and Sorensen study a related environment of social learning without the martingale property. While their method of analysis is different from ours, a number of our results are present in their work. In particular, Smith and Sorensen also show that unbounded signals will lead to social learning. However, our results on the dynamics of beliefs, the limiting distribution of probabilities and the role that asymmetry plays in asymptotic learning are novel.

The rest of the paper is organized as follows. In Section 2 we present the model, followed by an analysis of the properties of private beliefs in Section 3. In Section 4, we characterize the evolution of ex ante probabilities of taking the correct action. Section 5 presents our main results on asymptotic learning under unbounded signals and characterizes the convergence behavior of actions under bounded signals.

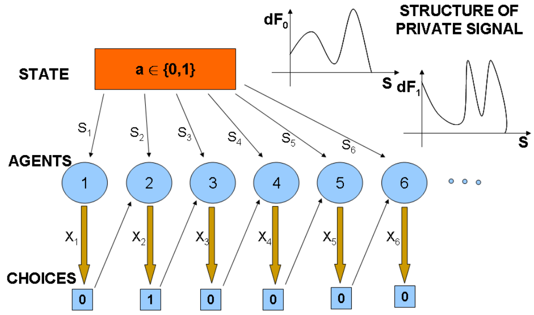

Fig. 1. Model of Social Learning with Limited Information.

II. The Model

The game consists of a countably infinite number of agents indexed by $n \in \mathbb{N}$, acting sequentially. Each agent n has a single action $x_n \in \{0,1\}$. The underlying state of the world is $a \in \{0,1\}$. If $x_n = a$, then the payoff of agent n is given by $u_n = 1$, and otherwise, $u_n = 0$. A priori, both states of the world are equally likely.

Let the information set of agent n be Ω_n. We assume that $\Omega_n = \{s_n, x_{n-1}\}$, where s_n is the private signal of the individual drawn independently from the conditional distribution F_a given the underlying state $a \in \{0,1\}$, and x_{n-1} is the action of the previous agent.

Our goal is to understand the limiting properties of a perfect Bayes-Nash equilibrium in this model. In particular, we want to determine the level of learning that is achieved by the agents as measured by their ex ante probability of choosing the best decision, i.e., $P(x_n = a)$.

Definition 1: (Asymptotic Learning) There is *asymptotic learning* if x_n converges to a in probability, i.e., $\lim_{n\to\infty} P(x_n = a) = 1$.

III. Private Beliefs

How the sequence of decisions $\{x_n\}$ evolves depends on inference based on individuals' signals regarding the underlying state. It is convenient to work with a transformation of these signals, which we refer to as *private beliefs* (see [9]).

Definition 2: (Private Belief) Agent n's private belief p_n is the probability that the state is equal to 1 conditional on his private signal s_n, i.e., $p_n = P(a = 1|s_n)$.

For a given signal s_n, by Bayes' rule, the private belief is

$$p_n = \frac{1}{1 + \frac{dF_0(s_n)}{dF_1(s_n)}}, \tag{1}$$

where dF_a reduces to the density of F_a if the distribution function has a density and the ratio in the denominator is the likelihood ratio.

Since p_n is a function of s_n only, the sequence of random variables $\{p_n\}$ is also independent and identically distributed. We will denote the cumulative distribution function for private beliefs given the true state a by G_a. That is,

$$G_a(x) = P(p_n \le x|a), \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

It can be seen that p_n contains all the useful information from the signal in estimating the state. Hence, $p_n = P(a = 1|p_n)$ and the following result follows.

Lemma 1: Given G_0 and G_1 defined as in Eq. (2), the following relation holds with probability 1,

$$\frac{dG_0(r)}{dG_1(r)} = \frac{1-r}{r}, \qquad \text{for all } r \in (0,1). \tag{3}$$

Moreover, given any cumulative distribution function G_1 such that $G_1(0) = 0$ and $G_1(1) = 1$, there exists a G_0 that satisfies Eq. (3) if and only if

$$\int_0^1 \frac{dG_1(r)}{r} = 2.$$

The proof of this lemma and all other omitted proofs are provided in [6]. Because the private beliefs contain all the useful information about the signals, we will directly work with private beliefs, or equivalently we suppose that each agent n knows only x_{n-1} and p_n when making his decision.

The following inequalities involving (G_0, G_1) will be used to provide bounds on the evolution of decision rules.

Lemma 2: Let (G_0, G_1) be a pair of distribution functions that satisfies Eq. (3). Then, for all $0 < z < p < 1$,

$$G_0(p) \ge \left(\frac{1-p}{p}\right) G_1(p) + \frac{p-z}{2} G_1(z),$$

and for all $0 < p < w < 1$,

$$1 - G_1(p) \ge (1 - G_0(p)) \left(\frac{p}{1-p}\right) + \frac{w-p}{2}(1 - G_1(w)).$$

Definition 3: (Bounded and Unbounded Private Beliefs) Let β and $1-\gamma$ be the infimum and the supremum of the support of the distribution function G_1, i.e.,

$$\beta = \inf_{x \in [0,1]} \{x : G_1(x) > 0\}. \tag{4}$$

$$\gamma = 1 - \sup_{x \in [0,1]} \{x : G_1(x) < 1\}. \tag{5}$$

Then, private beliefs are *unbounded* if $\beta = \gamma = 0$. The beliefs are *bounded* if both $\beta > 0$ and $\gamma > 0$.

We ignore the possibility that only one of β and γ is strictly positive to simplify the presentation.[2]

Unbounded private beliefs correspond to the likelihood ratio in Eq. (1) being unbounded, which implies that an agent can receive an arbitrarily strong signal about the underlying state. As in the existing work on the social learning literature, this feature will have important implications for the limiting behavior of the sequence $\{x_n\}$.

Throughout the paper, we adopt the following assumption.

Assumption 1: $\beta = \gamma$.

This assumption simplifies the exposition by imposing a natural symmetry on the distributions of private beliefs.

The next lemma provides a natural bound on the amount of learning at each step and will be used in the convergence analysis in the subsequent sections.

Lemma 3: Let β and γ be as defined in Equations 4 and 5 and let Assumption 1 hold, then

$$G_0(1/2) - G_1(1/2) \le 1 - 2\beta.$$

IV. Evolution of the Process

In this paper, we will characterize the limiting behavior of the agents by focusing on *ex ante* probabilities of correct decisions conditional on the true state a. These probabilities will be denoted

$$Y_n = P(x_n = 1|a = 1) \text{ and } N_n = P(x_n = 0|a = 0). \tag{6}$$

The unconditional probability of a correct decision is then

$$P(x_n = a) = \frac{Y_n + N_n}{2}, \tag{7}$$

and therefore asymptotic learning (from an ex ante point of view) is equivalent to the convergence of the sequence $\{(Y_n, N_n)\}$.[3]

Let us next define the thresholds

$$U_n = \frac{N_n}{1 - Y_n + N_n} \text{ and } L_n = \frac{1 - N_n}{1 - N_n + Y_n}, \tag{8}$$

which will fully characterize the decision rule as described by Lemma 4 below. Note that the sequence $\{(U_n, L_n)\}$ only depend on $\{(Y_n, N_n)\}$ and therefore are deterministic. This reflects the fact that each individual recognizes the amount of information that will be contained in the action of the previous agent, which determines his own decision thresholds. Individual actions are still stochastic since they are determined by whether the individual's private beliefs is below L_n, above U_n or in between.

Definition 4: Agent n's *strategy* σ_n is a mapping from his information set to his possible actions, i.e.,

$$\sigma_n : \Omega_n \to \{0, 1\}.$$

A *p*erfect Bayesian equilibrium of the game is a sequence of strategies for the players $\{\sigma_n^*\}$ such that for each n, σ_n^* maximizes the agent's expected utility given $\{\sigma_1^*, \ldots, \sigma_{n-1}^*, \sigma_{n+1}^*, \ldots\}$.

[2]Note that β and γ can be alternatively defined in terms of G_0 since the two distributions have the same support by Eq. (3).

[3]Note that since the amount of learning is captured by $P(x_n = a)$, asymptotic learning only requires that $\{x_n\}$ converges in probability.

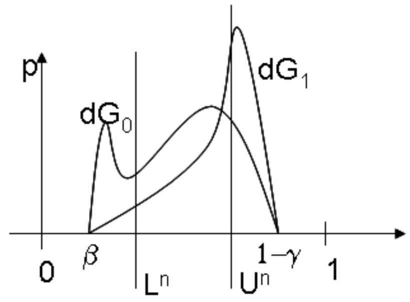

Fig. 2. Equilibrium Decision Rule Depicted on the Private Belief Interval.

Lemma 4: Let U_n and N_n be given by Eq. (8). Then, it is a perfect Bayesian equilibrium for agent n to select x_n according to the following rule:

$$x_n = \begin{cases} 0, & \text{if } p_n < L_n, \\ x_{n-1}, & \text{if } p_n \in [L_n, U_n], \\ 1, & \text{if } p_n > U_n. \end{cases}$$

Proof: To maximize his expected payoff, agent n will choose $x_n = 1$ only if

$$P(a=1|s_n, x_{n-1}) = P(a=1|x_{n-1}, p_n) \geq 1/2. \tag{9}$$

Using Bayes' Rule and the fact that both states are a priori equally likely,

$$P(a=1|x_{n-1}, p_n) = \frac{dP(x_{n-1}, p_n|a=1)}{\sum_{k=0}^{1} dP(x_{n-1}, p_n|a=k)}.$$

Given that x_{n-1} and p_n are independent conditionally on the state, we have that Eq. (9) holds if and only if

$$\frac{dG_1(p_n)}{dG_0(p_n)} \geq \frac{P(x_{n-1}|a=0)}{P(x_{n-1}|a=1)}.$$

Using Lemma 1, this condition is equivalent to

$$p_n P(x_{n-1}|a=1) \geq (1-p_n) P(x_{n-1}|a=0),$$

which can be rewritten to yield

$$p_n \geq \frac{P(x_{n-1}|a=0)}{\sum_{k=0}^{1} P(x_{n-1}|a=k)}.$$

By plugging in the two possible values of x_{n-1}, we obtain the desired decision rule. ■

Lemma 4 represents one particular tie-breaking rule, where agent n favors copying the choice of agent $n-1$ when p_n is equal to L_n or U_n and he is indifferent between two options. Any other choice of tie-breaking rule would also produce an equilibrium.

Lemma 5: Let Y_n, N_n, U_n and L_n be given by Eqs. (6) and (8). If the tie-breaking rule of Lemma 4 is adopted, then Y_n and N_n satisfy the following recursive relations:

$$N_{n+1} = G_0(L_n) + (G_0(U_n) - G_0(L_n)) N_n,$$

$$Y_{n+1} = 1 - G_1(U_n) + (G_1(U_n) - G_1(L_n)) Y_n.$$

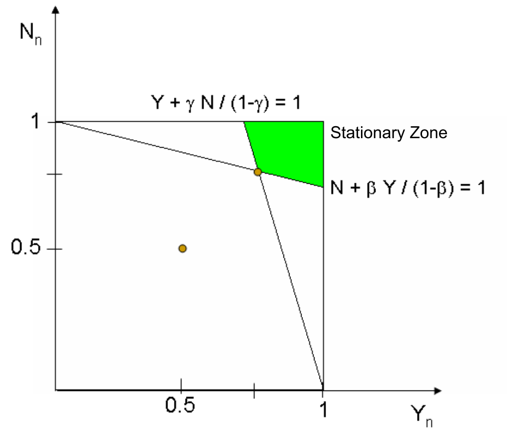

Fig. 3. Stationary Zone on (Y_n, N_n) Graph.

Lemma 6: If there exists an integer K such that

$$L_K \leq \beta \text{ and } U_K \geq 1-\beta,$$

then there exists a perfect Bayesian equilibrium where

$$Y_n = Y_K \text{ and } N_n = N_K \quad \text{for all } n \geq K,$$

where L_n and U_n are defined in Eq. (8) and β is defined in Eq. (4). Also, if there exists an integer K such that

$$L_K < \beta \text{ and } U_K > 1-\beta,$$

then the same holds for all equilibria.

Proof: Suppose such a K exists. Then, $G_0(L_K) = 1 - G_1(U_K) = 0$, which, by induction, using Lemma 5 implies that $Y_n = Y_K$ and $N_n = N_K$ for all $n \geq K$. In the case of a strict inequality, there is no issue of tie-breaking and all equilibria force stationarity. ■

This lemma defines a stationary zone such that once the sequence $\{(Y_n, N_n)\}$ enters this area, it remains constant. Using Eq. (8), it follows for any $\beta > 0$ that $L_n \geq \beta$ if and only if

$$N_n + \left(\frac{\beta}{1-\beta}\right) Y_n \leq 1. \tag{10}$$

Similarly, $U_n \leq 1-\beta$ if and only if

$$\left(\frac{\beta}{1-\beta}\right) N_n + Y_n \leq 1. \tag{11}$$

This region is the singleton (1,1) when beliefs are unbounded and is a non-degenerate quadrilateral as shown by the shaded area in Figure 3 when beliefs are bounded. Asymptotic learning is clearly equivalent to $\lim_{n\to\infty} \{(Y_n, N_n)\} = (1,1)$.

V. Convergence Analysis

The first useful property we can obtain about the sequence $\{x_n\}$ is what we refer to as "information monotonicity". Agents who act later will have higher probability of making the right choice. This is equivalent to the welfare improvement property of Smith and Sorensen [8].

Lemma 7: (Information Monotonicity) The sequence $P(x_n = a) = 2(Y_n + N_n)$ is nondecreasing.

Proof: The recursive relation in Lemma 5 yields

$$\begin{aligned} Y_{n+1} + N_{n+1} &= Y_n + N_n \\ &+ [(1-N_n)G_0(L_n) - Y_n G_1(L_n)] \\ &+ [(1-Y_n)(1-G_1(U_n)) - N_n(1-G_0(U_n))]. \end{aligned}$$

By Lemma 2, it follows that for any $z \in (0, L_n)$ and $w \in (U_n, 1)$, the two terms in the brackets are strictly positive, showing the desired result. ■

The next proposition is one of the main results of our paper and shows that the sequence $\{(Y_n, N_n)\}$ asymptotically approaches the stationary zone given by the shaded area in Figure 3.

Proposition 1: Let L_n and U_n be as defined in Eq. (8) and β as in Eq. (4). The sequences L_n and U_n satisfy

$$\limsup_{n\to\infty} L_n \le \beta, \quad \text{and} \quad \liminf_{n\to\infty} U_n \ge 1-\beta.$$

Proof: Let $L^* = \limsup_{n\to\infty} L_n$. Suppose $L^* > \beta$. Then, there exists a subsequence $\{L_N\}_{n\in\mathcal{N}}$ such that

$$L_n > \frac{L^* + \beta}{2}, \qquad \text{for all } n \in \mathcal{N}.$$

By Lemma 2, it can be seen that for every n in $\mathcal{N}$,

$$Y_{n+1} + N_{n+1} \ge Y_n + N_n + \frac{(1-N_n)(L_n - z)}{2} G_1(z),$$

for all $z \in (0, \frac{L^*+\beta}{2})$. Choose $z = \frac{L^*+2\beta}{3}$. It can be seen that

$$1 - N_n > \left(\frac{L^* + \beta}{2 - L^* - \beta}\right), \qquad \text{for all } n \in \mathcal{N}.$$

We also get that for this choice of z,

$$(L_n - z) \ge \frac{L^* - \beta}{6}, \qquad \text{for all } n \in \mathcal{N}.$$

Let C be defined as

$$C = \frac{1}{2}\left(\frac{L^* + \beta}{2 - L^* - \beta}\right)\left(\frac{L^* - \beta}{6}\right) G_1\left(\frac{L^* + 2\beta}{3}\right).$$

Note that C is a strictly positive constant and

$$Y_{n+1} + N_{n+1} \ge Y_n + N_n + C, \qquad \text{for all } n \in \mathcal{N},$$

which is impossible since $Y_n + N_n$ is a monotonically nondecreasing sequence bounded above by 2. Therefore, $L^* \le \beta$. A similar argument can be used to establish that

$$U^* = \liminf_{n\to\infty} U_n \ge 1 - \beta.$$

■

A. Asymptotic Learning

An immediate implication of Proposition 1 is that asymptotic learning occurs when the private beliefs are unbounded.

Proposition 2: Assume that private beliefs are unbounded. Then asymptotic learning occurs, i.e., $\lim_{n\to\infty} P(x_n = a) = 1$.

Proof: Since $\beta = 0$, Proposition 1 implies that $\lim_{n\to\infty} = 0$. Equivalently, $\lim_{n\to\infty} U_n = 1$. By Eq. (8), these imply that the sequence $\{(Y_n, N_n)\}$ converges to (1,1), showing the desired result. ■

Proposition 3: Let Assumption 1 hold and assume as well that the private beliefs are bounded. Then, $\lim_{n\to\infty} P(x_n = a) < 1$.

Proof: The proof is divided into two steps:

Step 1: First, we show that under the assumption that $\beta > 0$ (i.e., private beliefs are bounded), we have $Y_n + N_n < 2$ for all $n \ge 1$. We show this result by induction. We have $Y_0 + N_0 = 1$. Suppose that $N_n + Y_n < 2$ for some n. Then, by the evolution described in Lemma 5,

$$\begin{aligned} Y_{n+1} + N_{n+1} &= Y_n + N_n \\ &+ (1-N_n)G_0(L_n) - Y_n G_1(L_n) \\ &+ (1-Y_n)(1-G_1(U_n)) - N_n(1-G_0(U_n)). \end{aligned}$$

From this we obtain

$$\begin{aligned} Y_{n+1} + N_{n+1} \le Y_n + N_n &+ (1-N_n)G_0(L_n) \\ &+ (1-Y_n)(1-G_1(U_n)). \end{aligned}$$

Using the monotonicity of G_0 and G_1, we have

$$\begin{aligned} Y_{n+1} + N_{n+1} &\le Y_n + N_n \\ &+ (1-N_n)G_0(1/2) + (1-Y_n)(1-G_1(1/2)). \end{aligned}$$

Suppose first that $G_0(1/2) < 1$. Then

$$\begin{aligned} Y_{n+1} + N_{n+1} &< Y_n + N_n \\ &+ (1-N_n) + (1-Y_n)(1-G_1(1/2)) \\ &\le Y_n + N_n + (1-N_n) + (1-Y_n) = 2. \end{aligned}$$

If, on the other hand, $G_0(1/2) = 1$, then by Lemma 3, $G_1(1/2) \ge 2\beta > 0$, where the strict inequality is by the assumption that the private beliefs are bounded. Then,

$$\begin{aligned} Y_{n+1} + N_{n+1} &< Y_n + N_n \\ &+ (1-N_n)G_0(1/2) + (1-Y_n) \\ &= Y_n + N_n + (1-N_n) + (1-Y_n) = 2. \end{aligned}$$

Step 2: Since $Y_n + N_n < 2$ for all $n \ge 1$, the only way $Y_n + N_n$ could converge to 2 is if $(1,1)$ is a limit point of the set $\{(Y_n, N_n)\}_{n\ge1}$. We show by contradiction that this is not possible. Suppose $(1,1)$ is indeed a limit point of the set.

Then, $\forall\ \varepsilon > 0$, there exists some N such that, $Y_N > 1-\varepsilon$ and $N_N > 1-\varepsilon$. Let

$$\varepsilon = \frac{\beta}{4(1-\beta)}.$$

Then, L_N as defined in Eq. (8),

$$L_N \leq \frac{\varepsilon}{1-\varepsilon} = \frac{\beta}{4-5\beta} < \beta,$$

where the last inequality is true since $\beta \leq 1/2$. Equally, this value of ε implies that

$$U_N > 1-\beta.$$

By Lemma 6, the sequence (Y_n, N_n) enters the stationary zone and

$$(Y_k, N_k) = (Y_N, N_N), \qquad \text{for all } k \geq N.$$

Therefore, the set $\{(Y_n, N_n)\}_{n\geq 1}$ has finitely many points and $(1,1)$ is not a limit point. ■

Propositions 2 and 3 together show that asymptotic learning (from and ex-ante viewpoint) will occur if and only if the beliefs are unbounded.

B. Learning under Symmetry

When beliefs are bounded, Proposition 3 does not specify whether and where the sequence $\{(Y_n, N_n)\}$ will converge. We will next establish that under a symmetry assumption there exists an equilibrium with the minimum amount of asymptotic learning possible.

Assumption 2: (Symmetry) The states are symmetric if

$$G_0(r) = G_1(1-r) \text{ for all } r \in [0,1].$$

Assumption 3: G_0 and G_1 have densities.

Lemma 8: Let Assumption 2 hold. Then, there exists an equilibrium where $Y_n = N_n$ for all n. If Assumption 3 also holds, then this equilibrium is unique.

Lemma 9: Let β and N_n, Y_n be as defined in Eqs. (4) and (8). Assume symmetry holds. Then, there exists an equilibrium where for all $n \geq 1$, we have

$$N_n + Y_n \leq 2(1-\beta).$$

If Assumption 3 also holds, this equilibrium is unique.

The following lemma shows that the sequence $\{N_n + Y_n\}$ converges. The proof relies on using the upper bound on this sequence established in the preceding lemma and Proposition 1.

Lemma 10: Assume symmetry holds. Then, there exists an equilibrium where the sequence $\{N_n + Y_n\}$ converges to the limit $2(1-\beta)$, i.e.,

$$\lim_{n\to\infty} N_n + Y_n = 2(1-\beta).$$

Proof: By Proposition 1, we have

$$\limsup_{n\to\infty} L_n \leq \beta, \quad \text{and} \quad \liminf_{n\to\infty} U_n \geq 1-\beta.$$

This implies that for all $\varepsilon > 0$, there exist some $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$L_n \leq \beta - \varepsilon, \qquad \text{for all } n \geq K_1,$$

$$U_n \geq 1-\beta+\varepsilon, \qquad \text{for all } n \geq K_2.$$

Let $K_3 = \max\{K_1, K_2\}$. The preceding relations then immediately imply that for all $n \geq K_3$,

$$L_n \leq \beta + \varepsilon, \quad \text{and} \quad U_n \geq 1-\beta-\varepsilon.$$

Using the definition of L_n and U_n [cf. Eq. (8)], it follows from these relations that for all $n \geq K_3$,

$$N_n + \left(\frac{\beta+\varepsilon}{1-\beta-\varepsilon}\right) Y_n \geq 1,$$

$$Y_n + \left(\frac{\beta+\varepsilon}{1-\beta-\varepsilon}\right) N_n \geq 1.$$

Summing the preceding two relations yields

$$Y_n + N_n \geq 2(1-\beta-\varepsilon), \qquad \text{for all } n \geq K_3.$$

Combined with Lemma 9, we obtain

$$2(1-\beta-\varepsilon) \leq N_n + Y_n \leq 2(1-\beta), \qquad \text{for all } n \geq K_3.$$

Since ε was arbitrary, the preceding yields the desired convergence result, i.e., $\lim_{n\to\infty} N_n + Y_n = 2(1-\beta)$. ■

The next proposition contains the main convergence result of this subsection. In particular, we show that both sequences $\{N_n\}$ and $\{Y_n\}$ converge to the limit $(1-\beta)$.

Proposition 4: Assume that symmetry holds. Then, there exists an equilibrium where the sequences $\{N_n\}$ and $\{Y_n\}$ both converge to the limit $(1-\beta)$, i.e.,

$$\lim_{n\to\infty} N_n = \lim_{n\to\infty} Y_n = (1-\beta).$$

If Assumption 3 also holds, this equilibrium is unique.

Proof: The proof follows two steps:

Step 1: We first show that the sequence $\{L_n\}$ converges to the limit β, i.e, $\lim_{n\to\infty} L_n = \beta$. Proposition 1 establishes that $\limsup_{n\to\infty} L_n \leq \beta$. Therefore, it suffices to show that

$$\liminf_{n\to\infty} L_n \geq \beta.$$

Assume to arrive at a contradiction that $\liminf_{n\to\infty} L_n < \beta$. Let $\delta = 1/2(\beta - \liminf_{n\to\infty} L_n) > 0$. Then there exists a subsequence $\{L_n\}_{n\in\mathscr{N}}$ such that

$$L_n \leq \beta - \delta, \qquad \text{for all } n \in \mathscr{N}.$$

By the definition of L_n [cf. (8)], it follows that

$$1 \leq N_n + \left(\frac{\beta-\delta}{1-\beta+\delta}\right) Y_n, \qquad \text{for all } n \in \mathscr{N},$$

from which, in view of the fact that $\beta > \delta > 0$, we obtain that for all $n \in \mathcal{N}$,

$$1 \leq N_n + \left(\frac{\beta-\delta}{1-\beta+\delta}\right) Y_n \leq N_n + \left(\frac{\beta-\delta}{1-\beta}\right) Y_n.$$

Combined with Lemma 9, i.e., $N_n + Y_n \leq 2(1-\beta)$ for all $n \in \mathcal{N}$, this yields

$$\frac{N_n + Y_n}{1-\beta} - 1 \leq N_n + \left(\frac{\beta-\delta}{1-\beta}\right) Y_n,$$

or equivalently for all $n \in \mathcal{N}$,

$$N_n \left(\frac{\beta}{1-\beta}\right) + Y_n \leq 1 - \frac{\delta}{(1-\beta)} Y_n. \tag{12}$$

Since $N_n + Y_n$ converges to $2(1-\beta)$ (cf. Lemma 10), for every $\varepsilon > 0$ and for sufficiently large n, we have

$$Y_n \geq 2(1-\beta) - \varepsilon - N_n \geq 1 - 2\beta - \varepsilon,$$

where the second inequality follows by the fact that $N_n \leq 1$. Assume without loss of generality that $\beta < \frac{1}{2}$.[4] Then, ε can be taken arbitrarily close to 0, the preceding implies the existence of some $\alpha > 0$ such that $\frac{\delta}{1-\beta} Y_n \geq \alpha$ for all n sufficiently large. Hence, Eq. (12) implies that for all $n \in \mathcal{N}$ sufficiently large,

$$N_n \left(\frac{\beta}{1-\beta}\right) + Y_n \leq 1 - \alpha,$$

from which we can obtain

$$N_n \left(\frac{\beta}{1-\beta}\right) + \alpha N_n + Y_n \leq N_n \left(\frac{\beta}{1-\beta}\right) + \alpha + Y_n \leq 1.$$

Since the function $\frac{1-x}{x}$ is an unbounded increasing function in the $(0,1)$ interval, there exists some $w \in (\beta, 1)$ such that

$$\left(\frac{\beta}{1-\beta}\right) + \alpha = \left(\frac{\omega}{1-\omega}\right).$$

Combining the preceding two relations, we see that for all $n \in \mathcal{N}$ sufficiently large, we have

$$N_n \left(\frac{\omega}{1-\omega}\right) + Y_n \leq 1,$$

which using the definition of U_n [cf. (8)] can be rewritten as

$$U_n \leq 1 - \omega, \qquad \text{for all } n \in \mathcal{N} \text{ sufficiently large.}$$

Taking the limit along the subsequence $\mathcal{N}$, this implies that

$$\liminf_{n\to\infty} U_n \leq \limsup_{n\to\infty,\ n\in\mathcal{N}} U_n \leq 1 - \omega,$$

which in view of the fact that $w < \beta$ yields a contradiction to Proposition 1, thus showing that $\lim_{n\to\infty} L_n = 1 - \beta$.

Step 2: We now show that $\lim_{n\to\infty} N_n = \lim_{n\to\infty} Y_n = 1 - \beta$. From the definition of L_n [cf. (8)] and step 1,

$$\lim_{n\to\infty} \frac{1-N_n}{1-N_n+Y_n} = \beta,$$

[4] If $\beta = \frac{1}{2}$, the result holds trivially since no agent has any information about the state of the world.

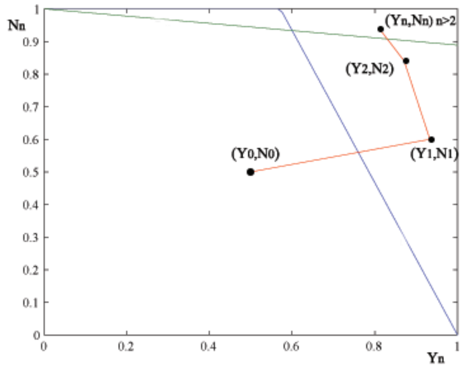

Fig. 4. Example Showing Asymmetry Could Lead to More Learning.

which implies that

$$\lim_{n\to\infty} (1-\beta) N_n + \beta Y_n = 1 - \beta.$$

We also have from Lemma 10 that

$$\lim_{n\to\infty} N_n + Y_n = 2(1-\beta).$$

Because $\beta < 0.5$, this pair of limits can only be satisfied if both Y_n and N_n converge. Furthermore, the limit points of both Y_n and N_n can only be $1-\beta$ to satisfy both limits. ■

If symmetry does not hold, then the sequence $\{Y_n + N_n\}$ might converge to a value greater than $2(1-\beta)$, i.e., not to the edge of region C in Figure **??**.

As an example of the behavior of asymptotic learning without symmetry, Figure 4 represents the dynamics of $\{(Y_n, N_n)\}$ for the following pair of distributions(G_0, G_1):

$$G_0(r) = \frac{18}{30}, \ r \in [0.1, 1-0.7),$$

$$G_1(r) = \frac{2}{30}, \ r \in [0.1, 1-0.7),$$

and both cumulative distributions having value 0 if $r < 0.1$ and value 1 for $r \geq 0.7$. In this example, private beliefs can take only two values, 0.1 and 0.7. The private belief of 0.1 implies a strong likelihood that 0 is the true state, while a belief of 0.7 implies a much weaker likelihood in favor of state 1. In this example, the sequence $\{(Y_n, N_n)\}$ converges to a point in the interior of the stationary zone as can be seen in Figure 4. As noted above, this limit point involves a greater amount of asymptotic learning than in the case with symmetric pair.

VI. Conclusions

In this paper, we presented an analysis of social learning when individuals only observe the action of their immediate neighbor. Despite the simplicity of this environment, the evolution of beliefs is substantially different than the typical models of social learning in the game theory literature. We characterized the behavior of asymptotic learning in terms of two threshold values that evolve deterministically. Individual actions are fully determined by the value of their signal relative to these two thresholds. We prove that asymptotic learning from an ex ante viewpoint applies if and only if individual beliefs are unbounded. We also show that for symmetric states bounded signals imply the minimum possible amount of asymptotic learning.

The tools introduced in this paper can be generalized to analyze social learning in environments in which individuals observe random samples of past actions and investigate how the topology of communication across agents affects information aggregation and the likelihood of asymptotic learning. This is an area we are investigating in [7].

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