"Knowing Whether" and Unawareness

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March 29, 2007

Extended Abstract

Unawareness is often interpreted as that one does not know an event, and he does not know that he does not know it, and so on *ad infinitum*. As one important aspect of bounded rationality, it plays an important role in economic life.

Let us start with the classic formulation. There is a finite state space Ω, an an arbitrary possibility correspondence $P : \Omega \to 2^{\Omega} \backslash \phi$. An event E is an nonempty subset of Ω. Knowledge operator is usually defined as $K(E) = \{\omega : P(\omega) \subseteq E\}$. The two basic properties of knowledge operator are:

$$\begin{array}{ll} \text{Necessitation} & K(\Omega) = \Omega, \\ \text{Monotonicity} & E \subseteq F \Rightarrow K(E) \subseteq K(F). \end{array}$$

Finally, it seems intuitive that we define unawareness as $U(E) = \cap_{i=1}^{\infty} (\neg K)^i (E)$,where $\neg K(E) = \Omega \backslash K(E)$.

The following example is taken from Dekel, Lipman and Rustichini (1998, hereafter DLR). The state space is $\Omega = \{a, b, c\}$, an (boundedly rational) agent's possibility correspondence is $P(a) = \{a\}$, $P(b) = \{b\}$ and $P(c) = \{a, b, c\}$. It is easy to check that at c, the agent does not know $\{a\}$, he does not know that he does not know, and so on. Hence it seems natural to say that at c, the agent is unaware of $\{a\}$. Indeed, according to the above definition, we have $U(\{a\}) = \{c\}$. However, DLR also argue that if an agent is unaware of an event, he cannot be plausibly aware of precisely which event he is unaware of. In the same example,

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$UU(\{a\}) = U(\{c\}) = \phi$, that is, at c, the agent is unaware of $\{a\}$, but is aware that he is unaware of $\{a\}$, which is completely unreasonable.

DLR consider the following three axioms on the knowledge operator and unawareness operator:

$$\begin{array}{ll} \text{Plausibility} & U(E) \subseteq \neg K(E) \cap \neg K \neg K(E) \\ \text{KU Introspection} & K(U(E)) = \phi \\ \text{AU introspection} & U(E) \subseteq UU(E). \end{array}$$

They show that standard state-space models of information and knowledge preclude unawareness. That is, any model that uses the state-space formulation must have either complete awareness or complete unawareness.

In this paper, we propose a different definition of unawareness, which is based on the "Knowing Whether" operator proposed by Hart, Heifetz and Samet (1996). Specifically, we define the knowledge operator as

$$K^W(E) = \{\omega : P(\omega) \subseteq E \text{ or } P(\omega) \subseteq \neg E\}.$$

Clearly, $K^W(E) = K(E) \cup K(\neg E)$. If $\omega \in K^W(E)$, then at ω, the agent knows whether the event E has happened or not, in other words, the agent knows the true value (0 or 1) of the event E. Also, we write "does not know whether" as $\neg K^W(E) = \Omega \backslash K^W(E)$. If $P(\omega) \cap E \neq \phi$ and $P(\omega) \cap (\neg E) \neq \phi$, then $\omega \in \neg K^W(E)$. Necessitation still holds, but monotonicity should be modified. Specifically,

$$\begin{array}{ll} \text{Necessitation} & K^W(\Omega) = \Omega, \\ \text{Monotonicity} & E \subseteq F \Rightarrow K_P^W(E) \subseteq K_P^W(F) \text{ and } K_N^W(E) \supseteq K_N^W(F). \end{array}$$

Finally, we define unawareness as

$$U^W(E) = \cap_{i=1}^{\infty} \left(\neg K^W\right)^i (E).$$

That is, an agent is unaware of an event if he does not know whether the event has occured, and he does not know whether he knows whether the event has occurred, and so on *ad infinitum*. Under this definition, DLR's axioms shall be modified as follows:

$$\begin{array}{ll} \text{Plausibility} & U^W(E) \subseteq \neg K^W(E) \cap \neg K^W \neg K^W(E) \\ \text{KU Introspection} & K^W\left(U^W(E)\right) \cap U^W(E) = \phi \\ \text{AU introspection} & U^W(E) \subseteq U^W U^W(E). \end{array}$$

Modified plausibility is automatically satisfied by the definition of unawareness. The meaning of KU introspection is that an agent will not know that he is unaware of something, but an agent may know the fact that he is not unaware of anything. With the new knowledge operator, the modification on KU introspection seems to be natural. AU introspection is the same as in DLR (1998), but with the new knowledge operator. The meaning is that the agent should be unaware of his unawareness. That is, the agent does not know whether he is unaware of an event, and he does not know whether he knows whether he is unaware of the event, and so on.

Now let us revisit the leading example of DLR. It is easy to see that $U^W(\{a\}) = \{c\}$. Moreover, $U^W\left(U^W(\{a\})\right) = \{c\}$, AU introspection is satisfied; $K^W\left(U^W(\{a\})\right) \cap U^W(\{a\}) = \phi$, the modified KU introspection is also satisfied. We also show that the symmetry axiom (Modica-Rustichini 1994), $U^W(E) = U^W(\neg E)$, is always satisfied.

We have examples showing that a non-partitional possibility correspondence does not necessarily accommodate unawareness. Nevertheless, we identify a necessary and sufficient condition on the possibility correspondence, under which there exists nontrivial unawareness. For that purpose, we first define reachability concerning the agent's information structure.

Definition 1 *For $\omega, \omega' \in \Omega$, we say that ω' is reachable from ω under $P(\cdot)$, written as $\omega \to \omega'$, if there exists a (finite) sequence $\{\omega_i\}_{i=1}^n$ such that $\omega_1 \in P(\omega)$, $\omega_{i+1} \in P(\omega_i)$, and $\omega' = \omega_n$.*

Our main result is that an agent has nontrivial unawareness when, according to his information structure, there is a pair of states such that one is reachable from another but not the other way around.

Theorem 1 *$U^W(E) \neq \phi$ under $P(\cdot)$ for some $E \subset \Omega$ with if and only if there exist $\omega, \omega' \in \Omega$ such that $\omega \to \omega'$ and $\omega' \nrightarrow \omega$.*

References

[1] DEKEL, E., B. LIPMAN, AND A. RUSTICHINI (1998), "Standard State-Space Models Preclude Unawareness," *Econometrica*, 66, 159-173.

[2] HART, S., A. HEIFETZ, AND D. SAMET (1996), ""Knowing Whether," "Knowing That," and The Cardinality of State Spaces," *Journal of Economic Theory*, 70, 249-256.

[3] MODICA, S., AND A. RUSTICHINI (1994), "Awareness and Partitional Information Structure," *Theory and Decision*, 37, 107-124.