Informative Cheap Talk Equilibria as Fixed Points*

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Abstract: We introduce a new fixed point method to analyze cheap talk games, in the tradition of Crawford and Sobel (1982). We illustrate it in a class of one-dimensional games, where the sender's bias may depend on the state of the world, and which contains Crawford and Sobel's model as a special case. The method yields new results on the structure of the equilibrium set. For games in which the sender has an *outward bias*, i.e. the sender is more extreme than the receiver whenever the state of the world is extreme, we prove that for any positive integer k, there is an equilibrium with exactly k pools, and at least one equilibrium with an infinite number of pools. We discuss the extent to which the fixed point method can be used to address other cheap talk signalling problems.

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1. Introduction

A stream of research examines how a privately informed agent, the "sender," can influence a decision maker, the "receiver," by supplying relevant unverifiable information. To influence the decision, all the sender can do is talk. Talking is free of costs, in the sense that messages do not enter the payoff of the players. This problem of cheap talk signalling is interesting when the sender and the receiver do not have the same preferences, i.e. when the sender is "biased."

The model of Crawford and Sobel (1982) is important in this literature. It is one of the first models to address the issue, and has served as a building block for most of the work in the area. We now have at hand an entire family of cheap talk signalling models that either enrich, build on, or apply to more specific settings, the model of Crawford and Sobel. In this paper, we introduce a new method to analyze models in this family. The key idea of the method is to look at cheap talk equilibria as the fixed points of a certain mapping. We thus label it the "fixed point method."

The method can be used to analyze a large class of cheap talk signalling games. In particular, it can help to analyze models that have raised technical difficulties, such as models where actions and types have more than one dimension.[1] The method also leads to a new natural way to address the problem of selecting among the many equilibria that typically arise in cheap talk signalling games.[2]

In this paper, we show how the fixed-point method works for a model in one dimension, which contains Crawford and Sobel's as a special case. Ours is more general, in that we allow the direction of the sender's bias to be either left or right, depending on the state of the world. In contrast, these authors require the sender's bias to be consistently in the same direction, across all states of the world. Thus, our model can be applied to a larger set of situations.

[1]See the pioneering work in this area, by Levy and Razin (2004), and Chakraborty and Harbaugh (2005). Models of cheap talk in multiple dimensions are not well understood. In a companion paper (2006b), we apply the fixed-point method to a class of multidimensional models, and obtain a description of the set of equilibria for this class.

[2]This remains a wide open question to this day, in spite of the recent advances by Kartik (2005) and Chen (2006). In a companion paper (2006a), we apply the fixed-point method to this problem, and obtain a criterion that selects a unique equilibrium in the class of games studied by Crawford and Sobel (1982).

In this unidimensional context, using our method has three advantages. First, it works exactly the same way, for games that satisfy Crawford and Sobel's consistent bias direction restriction, and for those who do not. Therefore the generalization is, in a sense, costless. Second, it yields a more detailed description of the rich structure of the equilibrium set, even for games that do satisfy all the assumptions of Crawford and Sobel. Third, the method requires few regularity assumptions, and some of our results hold even when the receiver's decision rule is not continuous in its information. All of these improvements are dividends of the fixed-point method.

In general, cheap talk signalling games can be described by a set of receiver's possible actions, a set of types that represent the sender's private information, a set of preferences for the sender, indexed by his type, and a decision rule under uncertainty for the receiver. An equilibrium outcome can be described by a partition of the sender type space in pools, and a list of actions indexed by the pools in the partition, satisfying two conditions. An interpretation is that sender types in a same pool send the same information to the receiver, therefore they induce the same action, but types in different pools send different information,therefore they induce possibly different actions. The first condition is that the action associated with a certain pool must be the decision prescribed by the receiver's rule when all he knows is that the type is in this pool. In other words, the receiver transforms the information he receives into actions in a way that is consistent with his decision rule. The second condition is that all types in any pool must like the action they induce at least as much as any other action in the list. This condition simply says that the sender types pool in an incentive-compatible fashion.

The fixed point method. We can map each pool partition into another pool partition in the following manner. For each pool in the initial partition, consider the action prescribed by the receiver's decision rule when all he knows is that the type is in this pool. This defines a list of actions. Next, sort sender types according to which action in the list they like the best. This yields a new pool partition of the type space. The equilibria of the game are exactly the fixed points of the mapping we just defined. Therefore, studying the equilibrium set amounts to study the set of fixed points of this mapping.[3]

A larger class of one dimensional models. The model we consider is more general that Crawford and Sobel's, in that we allow the direction of the sender's bias to be either left or right, depending on the state of the world. These authors require the sender's bias to be strictly in the same direction, across all states of the world. For example, all sender types could have a strict upward bias compared to the receiver. Or they could all have a strict downward bias. While this restriction is appropriate in many situations, it excludes a large class of problems. For example, the sender could have an outward bias. In this case, his preferred action is lower than the receiver's when his type is low, and higher than the receiver's when his type is high. He could also have an inward bias. In this case, his preferred action is higher than the receiver's when his type is low, and lower than the receiver's when his type is high. Our model contains upward, downward, outward and inward biases as special cases. More generally, we allow the direction of the bias to depend on the sender's type. We now provide examples of situations, where the sender has an outward or inward bias.

Outward and inward bias: a few examples. In our first example, the receiver is the government, and the sender is an expert, hired by the government to advise it on a one dimensional policy reform from a current status quo a^* to a new policy a. The expert's type represents the policy the expert believes the government should take. The government trusts the expert to indicate the direction of the change, i.e. whether a should be greater or lesser than a^*. The government takes into account factors that the expert will tend to ignore, such as the greater risks of facing popular resistance incurred when carrying out large changes. Thus, the government is more conservative than the expert, in the sense that it is reluctant to implement large policy changes. To fix ideas, let the type t be distributed in $[-1, 1]$, and the preferred policy of the government under complete information be $R(t) = a^* + \frac{t}{2}$. Instead, the expert would

[3]In unpublished work, Dimitrakas and Sarafidis (2005) use a version of the "fixed point method" outlined here, to study a variant of Crawford and Sobel's model. Their results and ours were obtained independently. For a discussion of their work, see the last section.

like the government to implement $S(t) = a^* + t$. In this example, the sender has an outward bias, since $S(-1) < R(-1) < R(1) < S(1)$.

In our second example, the receiver is a legislature with two members, and the sender is an expert, hired to advise it on a one dimensional policy a. The expert reports to the legislature, which then collectively choose the policy a. Specifically, the chosen policy is the outcome of a bargaining game among the two members of the legislature. To fix ideas, let the type t be any real number in $[0, 1]$, and let $S(t) = t$ be the policy the expert would like the government to implement. Let the preferred policy of one of member 1 under complete information be $R_1(t) = -\frac{3}{4} + \frac{3t}{2}$, and let the preferred policy of member 2 under complete information be $R_2(t) = \frac{1}{4} + \frac{3t}{2}$. Let the outcome of the legislative bargaining under complete information be $R(t) = \frac{R_1(t)+R_2(t)}{2} = -\frac{1}{4} + \frac{3t}{2}$. Here, the expert has an upward bias, with respect to member 1, and a downward bias, with respect to member 2. Indeed, we have for all $t \in [0, 1]$, $R_1(t) < S(t) < R_2(t)$. But when comparing the expert, and the legislature's rule $R(t)$, the sender has an inward bias, since $R(-1) < S(-1) < S(1) < R(1)$.

Other examples can be found in the literature. Stein (1989) uses a unidimensional model, where the sender is a central bank, and the receiver is a financial market. The equilibrium of this market determines an exchange rate. The central bank has a target exchange rate for today, but the market expect a reversal of the policy tomorrow. As a result, it is less reactive than the central bank would like it to be. Thus, the central bank has an outward bias, compared to the market. Melumad and Shibano (1991) also study cheap talk signalling, among other mechanisms, without Crawford and Sobel's restriction. Their main focus is on comparing equilibria with one and two pools, from the point of view of the expected utility of the sender and the receiver. In both cases, the authors restrict attention to the special case where the preferences of the sender are quadratic, and the decision rule of the receiver is linear. Our analysis applies to a much larger set of situations, as it does not rely on these assumptions.

Equilibrium sizes. A classic result for unidimensional cheap talk signalling, which holds also in our model, is that equilibrium pools must be intervals. Crawford and Sobel prove that, when the sender's bias is strictly upward (or strictly downward), the set of integers such that there are equilibria of size k, i.e. with exactly k intervals, is of the form $\{1, ..., K\}$ and there are no equilibria of infinite size. In contrast, we prove that when the bias is outward, there are equilibria of any finite size and at least one of infinite size. If we interpret the maximal equilibrium size as a measure of the sender's influence on the receiver, our results suggests that a sender with an outward bias enjoys a greater influence on the receiver than a sender with an strictly upward bias.

More generally, the following result holds for any game in the class we study. Either the set of equilibrium sizes is of the form $\{1, ..., K\}$, like in the strictly upward bias case, or it is $\mathbb{N} \cup \{\infty\}$, like in the outward bias case. In other words, if there is an equilibrium of size $k > 1$, then there exists an equilibrium of size $k - 1$. We also show that the latter is "nested" into the former, in the sense that the boundary points of the size k equilibrium define bounds within which a size $k - 1$ equilibrium necessarily exists.

Structure of the set of equilibria of a given size $k \geq 2$. We obtain new results on the structure of the equilibrium set. When the sender has an outward bias, the set of equilibria of a given size $k \geq 2$ is nonempty and has a complete lattice structure. In particular, it has a minimal element and a maximal element. Under the assumption that the highest sender type has an upward bias (this includes upward the upward and outward cases), the set of equilibria of a given size $k \geq 2$ may or may not be empty. If it is nonempty, this set is a semi-upper lattice. In particular, it has a maximal element. We then provide further results on this maximal equilibrium of size k. First, we provide a simple algorithm that converges monotonically to this equilibrium. We then provide comparative statics results on this equilibrium. Crawford and Sobel (1982) proved some results of this type. Ours are stronger, in that we do not assume the unicity of the equilibrium of size k to obtain them.

As we pointed out, the fixed-point method yields both a more precise description of the equilibrium set, and for a broader class of models, than Crawford and Sobel's work. However, the real contribution here is the introduction of the fixed-point method in this context. The method can be used to address other questions in the cheap talk signalling literature. Fixed point methods are pervasive in many areas of economic theory. We show that they are a powerful tool to analyze cheap talk signalling models as well.

The rest of the paper is organized as follows. Section 2 lays out the model. Section 3 studies the set of possible equilibrium sizes in general. Section 4 introduces a taxonomy of sender's biases, and specializes the results of section 3 to certain special cases. Section 5 provides further results on the structure of the equilibrium set. The case where the receiver maximizes a von Neuman Morgenstern utility function is studied in section 6. This section includes a comparative statics analysis. In section 7, we study the important uniform-quadratic case. Section 8 discusses the technical aspect of this paper and its articulation with other works. In section 9, we discuss the extent to which the fixed-point method can be used to address other cheap talk signalling problems.

2. The model

There are two players, the sender and the receiver. Only the sender has payoff-relevant private information. The interaction takes place in two stages. In the first stage, the communication stage, the sender learns his type, sends a message that is read by the receiver. Talking is "cheap", i.e. messages do not directly affect payoffs. In the second stage, the receiver takes an action.

Let $T \equiv [0, 1]$ be the sender's set of types, with typical element t. A *pool* is a nonempty subset of T. Let $\mathcal{T}$ be the collection of all pools. A sender strategy is described by a partition Π of T in pools. A typical pool I in the partition Π is a set of sender types that send identical signals or messages. The encoding of information, i.e. what messages are sent by each of the pools, is irrelevant.

Let $A \subseteq \mathbb{R}$ be a nonempty set of receiver's possible actions, with typical element a. The receiver reacts exogenously to the information received from the sender, and its reaction is a mapping $R : \mathcal{T} \to A$.

For any strategy Π, the *outcome function f for* Π maps sender types to the actions chosen by the receiver in reaction to the information they report under strategy Π. For any pool I, let $1_I : T \to \{0, 1\}$ be the characteristic function of I. The outcome function f for Π is

$$f(t) = \sum_{I \in \Pi} R(I) 1_I(t).$$

Two partitions Π and Π' are equivalent if they induce the same outcome. We now introduce a natural assumption on the receiver's reaction. Minimal-rationality requires that if some information is not taken into account in the decision of the receiver, then the suppression of this information does not affect this decision.

The receiver is **minimally-rational** if, for all family of disjoint pools $\mathcal{T}^* \subset \mathcal{T}$ and all $a \in A$ satisfying (for all $I \in \mathcal{T}$, $R(I) = a$), we have $R(\cup_{I \in \mathcal{T}^*} I) = a$.

This assumption has the following appealing consequence.

Lemma 1: *Suppose that R is minimally-rational. Let f be the outcome for some strategy Π, then the partition Π' in level curves of f also induces f. Thus Π' is equivalent to Π.*

Proof. Let $I' \in \Pi'$, and let $a \in A$ such that $I' = \{t \in T : f(t) = a\}$. Let Π^* be the (possibly infinite) sub-collection of Π consisting of sets I that have a nonempty intersection with I'. For all $I \in \Pi^*$, we have $R(I) = a$, which implies $I \subseteq I'$. Therefore I' equals the a union of the members of Π^*. By *minimal rationality* , this implies $R(I') = a$, the desired conclusion.■

A preference relation over A is a binary relation that is reflexive, transitive and complete. The sender has a preference relation $\succeq_t$ over A, which depend on his type t. For all $a, b \in A$, the proposition $a \succeq_t b$ means that the sender of type t weakly prefers action a to action b. The corresponding strict preference and indifference relations are denoted by $\succ_t$ and $\simeq_t$. Let $\succeq$ denote the family of preferences $\{\succeq_t\}_{t \in [0,1]}$.

An *equilibrium strategy* is a partition Π such that for all $I, I' \in \Pi$, for all $t \in I$, we have $R(I) \succeq_t R(I')$. Clearly, a strategy equivalent to an equilibrium strategy is also an equilibrium strategy. We say that an outcome f is an *equilibrium outcome* if it is induced by some equilibrium strategy. In particular, by minimal-rationality, f is an *equilibrium outcome* if its level-curves form an equilibrium partition. Our next assumption says that the sender's parameterized preferences shift in favor of higher actions as his type increases.

The family $\succeq$ satisfies the **single-crossing-property** if, for all $s, t \in T$ such that $s < t$, and for all $a, b \in A$ such that $a < b$, we have $b \succeq_s a \Rightarrow b \succ_t a$.

An interval partition is a partition Π whose elements are all intervals in T, (some of them possibly reduced to a point). Under the two above assumptions, any equilibrium partition is equivalent to some interval partition.

LEMMA 2: *Let R satisfy minimal-rationality and $\succeq$ satisfy the single-crossing property. Then any equilibrium partition Π is equivalent to an interval partition.*

Proof. Let Π be an equilibrium partition, let f be its outcome, and let Π' be the partition of level curves for f. From Lemma 1, Π and Π' are equivalent. By Theorem 2.8.1 in Topkis (1998) and the *single-crossing-property*, all elements in the partition Π' are intervals.$\blacksquare$

Thus we may restrict attention to interval equilibrium partitions. We now introduce additional restrictions on the sender's preferences and the receiver's behavior. We need the following definitions to present the remaining assumptions.

Let m, n be arbitrary positive integers. For all $x, y \in \mathbb{R}^m$, let $x \leq y$ if $x_i \leq y_i$ for all $i = 1, ..., m$. Furthermore, let $x < y$ if $x \leq y$ and $x \neq y$. Let $g : X \to Z$ be a mapping. We say that g is *nondecreasing* if, for all $x \leq y \in X$, we have $g(x) \leq g(y)$. We say that g is *increasing* if, for all $x < y \in X$, we have $g(x) < g(y)$.

A preference $\succeq_t$ is **single-peaked** if it is continuous and has a unique preferred action $S(t) \in A$ (the peak) and, among any two distinct actions on the same side of the peak, the one closest the peak is preferred. More precisely: *i)* There is an action $S(t) \in A$ such that for all $a, b \in A$ satisfying either $S(t) \geq a > b$ or $b > a \geq S(t)$, we have $a \succ_t b$. *ii)* For all $a \in A$, the set $\{(a, b) \in A : a \succeq_t b\}$ is closed.

Single-peakedness and the *single-crossing property* imply that $S(t)$ is nondecreasing. We say that the collection $\succeq$ is **type-continuous** if, for all $a, b \in A$, the set $\{t \in T : a \succeq_t b\}$ is closed.

Under *minimal-rationality* of R and the *single-crossing property* of $\succeq$, Lemma 2 says that any equilibrium partition is equivalent to an equilibrium partition where all pools are intervals. The last two assumptions will restrict the way the receiver reacts to interval pools. Abusing notations, for all $s \leq t \in T$, let $R(s, t) := R([s, t])$.

We say that R is **monotonic** if the function $(s,t) \mapsto R(s,t)$ is increasing. We say that R is **robust** if the receiver reaction to an interval pool that is not a singleton does not depend on whether the endpoints of the interval are included in the pool, i.e. for all $s < t \in [0,1]$, we have $R([s,t]) = R(]s,t[) = R(]s,t]) = R([s,t[)$.

An **admissible problem** is a pair $(R, \succeq)$ such that $\succeq$ is *single-peaked*, *type-continuous* and satisfies *the single-crossing property*, and R is *minimally-rational*, *monotonic* and *robust*. In the remainder of the paper, we restrict attention to admissible problems. For examples and applications, see Sections 6 and 7.

3. General results

In the entire section, let $(R, \succeq)$ be admissible. We characterize the set of equilibrium partitions for any admissible problem. We first introduce some notations and definitions, then examine the structure of the set of equilibria with finitely many intervals. Finally, we examine equilibria with infinitely many intervals.

3.1. Preliminaries

In last section, we introduced a partial order on real vectors, and defined two monotonicity properties for functions. We now need to extend this partial order to sets of real vectors, and define the corresponding monotonicity properties for correspondences.

Let m, n be arbitrary positive integers. For all two nonempty subsets $X, Y \subseteq \mathbb{R}^m$, let $X \leq Y$ if, for all $x \in X$, and all $y \in Y$, we have $x \leq y$. Similarly, let $X < Y$ if, for all $x \in X$, and all $y \in Y$, we have $x < y$. Let $X \subseteq \mathbb{R}^m$ and $Z \subseteq \mathbb{R}^n$. Let $G : X \twoheadrightarrow Z$ be a correspondence such that $G(x)$ is nonempty for all $x \in X$. We say that G is *nondecreasing* if, for all $x \leq y \in X$, we have $G(x) \leq G(y)$. We say that G is *increasing*[4] if for all $x < y \in X$, we have $G(x) < G(y)$.

We now introduce a correspondence which will play an important role in our results. Let $\tau : \{(a,b) \in A^2 : a \leq b\} \to [0,1] \cup \{-2, 2\}$ be such that, for all $a < b \in A$,

[4]An equivalent definition is that G is nondecreasing (increasing) if all selections from G are nondecreasing (increasing).

we have

$$
\begin{aligned}
\tau(a,b) &:= \{-2\} && \text{if } \; b \succ_0 a, \\
&:= \{2\} && \text{if } \; a \succ_1 b, \\
&:= \{t \in [0,1] : a \simeq_t b\} && \text{if } \; a \succeq_0 b \text{ and } b \succeq_1 a,
\end{aligned}
$$

and for all $a \in A$, we have

$$
\begin{aligned}
\tau(a,a) &:= \{-2\} && \text{if } \; S(0) > a, \\
&:= \{2\} && \text{if } \; S(1) < a, \\
&:= \{t \in [0,1] : S(t) = a\} && \text{if } \; S(0) \leq a \leq S(1).
\end{aligned}
$$

By the *single-crossing property* and *type-continuity*, the conditions $a \succeq_0 b$ and $b \succeq_1 a$ imply that the set $\{t \in [0,1] : a \simeq_t b\}$ is a singleton. Similarly, by *single-peakedness*, the *single-crossing property*, and *type-continuity*, the inequalities $S(0) \leq a \leq S(1)$ imply that the set $\{t \in [0,1] : S(t) = a\}$ is a nonempty closed interval. Therefore $\tau(a,b)$ is a nonempty closed interval for all $a \leq b$, and is a singleton when $a < b$. In addition, τ has the following monotonicity properties.

LEMMA 3: *τ is nondecreasing on its domain and increasing on $\tau^{-1}([0,1])$.*

Proof. Let us prove that τ is increasing on $\tau^{-1}([0,1])$. Let $a,b,c,d \in A$ such that $a \leq b$, $c \leq d$, $(a,b) \leq (c,d)$ and $(a,b) \neq (c,d)$. Suppose that $s \in \tau(a,b)$ and $t \in \tau(c,d)$ satisfy $s,t \in [0,1]$. We will prove that $s < t$. We distinguish three cases. *Case 1: $c < d$.* By *single-peakedness* of $\succeq_s$, we have $c \succ_s d$. Since $c \simeq_t d$, and by *the single-crossing property*, we have $s < t$. *Case 2: $a < b$.* By *single-peakedness* of $\succeq_t$, we have $b \succ_t a$. Since $b \simeq_s a$, and by *the single-crossing property*, we have $s < t$. *Case 3: $a = b < c = d$.* Since $S(s) = a$ and $S(t) = c$ and S is non-decreasing, therefore $s < t$. Thus τ is increasing on $\tau^{-1}([0,1])$.

Let us prove that τ is nondecreasing on its domain. For all $c \leq d \in A$ such that $\tau(c,d) = \{-2\}$, and for all $a \leq b \in A$ such that $(a,b) \leq (c,d)$, we have $\tau(a,b) = \{-2\}$, by *single-peakedness* of $\succeq_0$. Similarly, for all $a \leq b \in A$, such that $\tau(a,b) = \{2\}$, and all $c \leq d \in A$ such that $(a,b) \leq (c,d)$, we have $\tau(c,d) = \{2\}$, by *single-peakedness* of $\succeq_1$. Thus τ is nondecreasing on its domain.■

3.2. *Equilibria with finitely many intervals*

For all partition Π, let the *size of* Π be the number of distinct pools in the partition. An important class of interval equilibrium partitions are the ones that have a finite size.

For all $\kappa \geq 2$, let $W_\kappa \equiv \{x \in T^{\kappa+1} : x_0 \leq ... \leq x_\kappa\}$. For all $\kappa \geq 2$, let $\theta^\kappa : W_\kappa \twoheadrightarrow (T \cup \{-2, 2\})^{\kappa+1}$ be the correspondence such that for all $x \in W_\kappa$ we have $\theta^\kappa(x) := \theta_0(x) \times ... \times \theta_\kappa(x)$ with $\theta_0(x) = \{x_0\}$, $\theta_\kappa(x) = \{x_\kappa\}$ and

$$\theta_l(x) := \tau(R(x_{l-1}, x_l), R(x_l, x_{l+1})),$$

for all $l = 1, ..., \kappa - 1$. Since for all $x \in W_\kappa$, we have $x_0 \leq ... \leq x_\kappa$, and since R is nondecreasing, this correspondence is well-defined and nonempty valued.

Let $X_\kappa \equiv \{x \in W_\kappa : x_0 = 0 \text{ and } x_\kappa = 1\}$. We say that $x \in X_\kappa$ *represents an interval partition of size exactly* κ if there is a collection of nonempty intervals $\{I_l\}_{l=1,...,\kappa}$ such that $]x_{l-1}, x_l[$ is the interior of I_l for all $l = 1, ..., \kappa$. [5]

The following result characterizes the set of vectors $x \in X_\kappa$ that represent an interval equilibrium partition of size exactly κ. The result shows that this set coincides with the set of fixed-point of the mapping θ^κ. In addition, all vectors in this set satisfy $x_1 < ... < x_{\kappa-1}$.

Lemma 4: *Let $\kappa \geq 2$. For all $x \in X_\kappa$, the vector x defines an interval equilibrium partition of size exactly κ iff $x \in \theta^\kappa(x)$. When this is the case and $\kappa > 2$, we have $x_1 < ... < x_{\kappa-1}$.*

Proof. By Lemma 2 and *sender-continuity*, if x represents an equilibrium of size exactly κ, then x is a fixed-point of θ^κ.

Let $\kappa > 2$ and let x be a fixed-point of θ^κ in the set X_κ. To alleviate notations, for all relevant indices l, let $S_l \equiv S(x_l)$, let $a_l \equiv R(x_{l-1}, x_l)$, and let $I_l \equiv]x_{l-1}, x_l[$. We will prove that $S(x_1) < ... < S(x_{\kappa-1})$. Let $H \equiv \{h \in \{1, ..., \kappa - 2\} : S_h < S_{h+1}\}$.

[5] A necessary and sufficient condition for this to be the case is that for all $l \in \{1, ..., \kappa - 1\}$, we de not have $x_{l-1} = x_l = x_{l+1}$. Any interval strategy of size exactly κ is represented by a unique $x \in X_\kappa$. For example, the interval strategy $\{[0, 1/3],]1/3, 1/2[, [1/2, 1]\}$ is represented only by $x = (1/3, 1/2)$. Some vectors x do not represent any strategy. For example the list $x = (1/2, 1/2, 1/2)$.

Since $x \in X_\kappa$, and by *the single-crossing property,* we have $S_0 \leq S_1 \leq ... \leq S_\kappa$ and $S_0 < S_\kappa$. Therefore $H \neq \emptyset$. Let $h \in H$. Then in particular $x_h < x_{h+1}$. Suppose that $h > 1$. Then by *receiver-monotonicity,* we have $a_h < a_{h+1}$. Since x is a fixed-point, we have $a_{h-1} \simeq_{x_{h-1}} a_h$ and $a_h \simeq_{x_h} a_{h+1}$. By *single-peakedness* of the preferences $\succeq_{x_{h-1}}$ and $\succeq_{x_h}$, we have $S_{h-1} \leq a_h < S_h < a_{h+1}$. In particular, $h-1 \in H$. By induction, we obtain that $1, ..., h \in H$. By an identical reasoning, we obtain that $h, ..., \kappa - 2 \in H$, which proves the claim.

In particular, x represents a partition of size exactly κ. It only remains to prove that this partition is an equilibrium partition. For all $l = 1, ..., \kappa - 1$, we have $a_l \simeq_{x_l} a_{l+1}$. By *the single-crossing property*, this further implies that for all $t \in I_1 \cup ... \cup I_l$, we have $a_l \succeq_t a_{l+1}$ and that for all $t \in I_{l+1} \cup ... \cup I_\kappa$, we have $a_{l+1} \succeq_t a_l$. Thus for all $h, l \in \{0, ..., \kappa\}$, all $t \in I_h$, we have $a_h \succeq_t a_l$. Thus the $(I_1, ..., I_\kappa)$ form an equilibrium partition.■

A set of natural integers is **decreasing** if it is $\mathbb{N}$ or of the form $\{1, ..., K\}$. The following result describes the set of integers κ such that there are equilibria of size κ.

Theorem 1: *The set of integers κ such that there are equilibria of size κ is decreasing.*

The proof of Theorem 1 rests on Lemmas 5, 6 and 7. For all positive integer m, and all vectors $x, z \in \mathbb{R}^m$, we let $[x, z] := \{y \in \mathbb{R}^m : x \leq y \leq z\}$. Sets of this form are called **closed intervals**.

Lemma 5: *For all $\kappa \geq 2$, the mapping θ^κ is increasing. For all $x \in X_\kappa$, the set $\theta^\kappa(x)$ is a closed interval.*

Proof. By *receiver-monotonicity*, R is increasing. By Lemma 3, τ is increasing. Therefore θ^κ is increasing. Since R is a function and $\tau(a, b)$ is a closed interval for all $a \leq b$, therefore $\theta^\kappa(x)$ is a closed interval for all $x \in X_\kappa$. ■

To state the next Lemma, we need the following definitions. A **complete lattice** in $\mathbb{R}^m$ is a subset $L \subseteq \mathbb{R}^m$ such that, for any nonempty subset H of L, the set $\{x \in L : x \leq H\}$ is nonempty and has a greatest element in L, the infimum of H in L, denoted by $\inf_L H$; and the set $\{x \in L : x \geq H\}$ is nonempty and has a least element in L, the supremum of H in L, denoted by $\sup_L H$. In particular, a nonempty complete lattice L has a least element $\inf_L L$ in L and a greatest element $\sup_L L$ in L. A subset L' of a complete lattice L is a **subcomplete sublattice** of L if it is a complete lattice such that for all nonempty subset H of L', we have $\inf_{L'} H = \inf_L H$ and $\sup_{L'} H = \sup_L H$.

For example, it is easy to verify that W_κ is a nonempty complete lattice. We have $\inf_{W_\kappa} W_\kappa = (0, \ldots, 0)$ and $\sup_{W_\kappa} W_\kappa = (1, \ldots, 1)$. Also, let 0_κ and 1_κ be the vectors $(0, \ldots, 0, 1)$ and $(0, 1, \ldots, 1)$ in X_κ. It is easy to verify that X_κ is a nonempty complete lattice. We have $\inf_{X_\kappa} X_\kappa = 0_\kappa$ and $\sup_{X_\kappa} X_\kappa = 1_\kappa$. Furthermore, for any positive integer m, and all vectors $x, z \in \mathbb{R}^m$, the closed interval $[x, z]$ is a complete lattice. In particular, for all positive integer κ, and all $x \in X_\kappa$, the set $\theta^\kappa(x)$ is a complete lattice. Finally, if L is a complete lattice and $x, z \in L$, then $[x, y] \cap L$ is subcomplete sublattice of L.

Lemma 6: *Let $\kappa \geq 2$. Suppose that there is a nonempty complete lattice $L \subseteq X_\kappa$ such that for all $x \in L$, we have $\theta^\kappa(x) \subseteq L$. Then the set of fixed-points of θ^κ in L is a nonempty complete lattice.*

Proof. By Lemma 5, θ^κ is increasing. Since $\theta^\kappa(x) \subseteq L$, is a closed interval for all $x \in X_\kappa$, the inclusion $\theta^\kappa(x) \subseteq L$ implies that this set is a subcomplete sublattice of L for all $x \in X_\kappa$. The result then follows from Zhou's (1994) extension of Tarski's (1955) fixed-point theorem to correspondences. Note that θ^κ satisfies a stronger monotonicity condition than the one required for Zhou's result.■

For all vector $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$, let $x_{-j} \equiv (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$.

Lemma 7: *Let $\kappa \geq 3$. Suppose that $x \in X_\kappa$ is a fixed-point of θ^κ. Then the set $L \equiv \left[x_{-(\kappa-1)}, x_{-1}\right]$ is a subset of X_κ, it is a nonempty complete lattice, and for all $y \in L$, we have $\theta^{\kappa-1}(y) \subseteq L$. In addition, $\theta^{\kappa-1}$ admits a fixed-point in L.*

Proof. First, it is clear that $L \subseteq X_\kappa$. Second, L is a closed interval, therefore a nonempty complete lattice. Third, we show that $\theta^{\kappa-1}(x_{-1}) \leq x_{-1}$. Since $x \in \theta^\kappa(x)$, then $x_{-0} \in \theta^{\kappa-1}(x_{-0})$. We have $x_{-1} \leq x_{-0}$, thus by monotonicity of $\theta^{\kappa-1}$, we have $\theta^{\kappa-1}(x_{-1}) \leq \theta^{\kappa-1}(x_{-0})$. Therefore $\theta^{\kappa-1}(x_{-1}) \leq x_{-0}$. But since the first coordinate of $\theta^{\kappa-1}(x_{-1})$ is $\{0\}$, and x_{-1} only differs from x_{-0} by its first coordinate, which precisely equals 0, therefore $\theta^{\kappa-1}(x_{-1}) \leq x_{-1}$. Fourth, by an identical reasoning, we can prove that $x_{-(\kappa-1)} \leq \theta^{\kappa-1}(x_{-(\kappa-1)})$. From these last two inequalities and since $\theta^{\kappa-1}$ is increasing, we conclude that for all $y \in L$, we have $\theta^{\kappa-1}(y) \subseteq L$, the desired conclusion. Lemma 6 ensures then that $\theta^{\kappa-1}$ has a fixed-point in L.■

Proof of Theorem 1. It is immediate, from Lemmas 4 and 7.■

3.3. Equilibria with infinitely many intervals

The following lemma is useful.

LEMMA 8: *$S(\cdot)$ is continuous in t.*

Proof. Let $t \in [0,1]$ and $\varepsilon > 0$. Let $a \equiv S(t)$. By *sender-continuity*, the set $O \equiv \{s \in [0,1] : a \succ_s a - \varepsilon \text{ and } a \succ_s a + \varepsilon\}$ is open in $[0,1]$. Therefore it is a neighborhood of t in $[0,1]$. For all $s \in O$, by *single-peakedness* of $\succeq_s$, we have $S(s) \in (a - \varepsilon, a + \varepsilon)$. Therefore $S(\cdot)$ is continuous at t.■

We say that the receiver reaction R is **continuous** if the mapping $(s,t) \mapsto R(s,t)$ is continuous in the usual sense. The following result says that if R is continuous, then exactly one of the following alternatives is true. Either there are finite size equilibria of any positive integer size and there is at least one equilibrium of infinite size, or the set of finite equilibrium sizes is a bounded decreasing set, and there are no equilibria of infinite size.

THEOREM 2: *Let R be continuous. The set of integers κ such that there are equilibria of size κ is $\mathbb{N}$ iff there is at least one equilibrium of infinite size.*

Proof. The continuity of R is needed only to prove the *only if* implication. We first prove the *only if* implication. (Claims 1 to 6). Let Π^κ be a sequence of equilibria such that for all $\kappa = 1, 2, ...$, the equilibrium Π^κ is of size exactly κ. For all $\kappa \geq 0$, let $i_\kappa : [0,1] \to [0,1]$ and $s_\kappa : [0,1] \to [0,1]$ such that for all $t \in [0,1]$, the real numbers $i_\kappa(t)$ and $s_\kappa(t)$ are respectively the infimum and the supremum of the pool containing t in the partition Π^κ. Clearly, these functions are both nondecreasing and satisfy for all κ and all $t \in [0,1]$, the inequalities $i_\kappa(t) \leq t \leq s_\kappa(t)$.

Claim 1: There is a subsequence $\{n\}$ and (unique) nondecreasing functions $i(\cdot)$ and $s(\cdot)$ such that $i_n(\cdot)$ converges to $i(\cdot)$ and $s_n(\cdot)$ converges to $s(\cdot)$. Moreover, for all $t \in [0,1]$, we have $i(t) \leq t \leq s(t)$.

Proof: The functions i_κ and s_κ are all nondecreasing and uniformly bounded on $[0,1]$. Helly's selection Theorem guarantees that a sequence of nondecreasing uniformly bounded functions on $[0,1]$, has a subsequence which converges to a nondecreasing function. Let $\{m\}$ denote a sequence and let $i : [0,1] \to [0,1]$ be a nondecreasing function such that $i_m(\cdot)$ converges to $i(\cdot)$. Next, let $\{n\}$ be a subsequence from $\{m\}$ and let $s : [0,1] \to [0,1]$ be a nondecreasing function such that $s_n(\cdot)$ converges to $s(\cdot)$. The last inequalities are obvious.$\|$

Let Π^* be a (possibly infinite) partition of $[0,1]$ into level curves of $i(\cdot) + s(\cdot)$. Since $i(\cdot) + s(\cdot)$ is nondecreasing, each pool in Π^* is an interval, possibly a singleton.

Claim 2: The functions $i(\cdot)$ and $s(\cdot)$ are constant on any pool of Π^*.

Proof: Let t, t' be in the same pool of the partition Π^*. Then $i(t) + s(t) = i(t') + s(t')$ holds. Since both $i(\cdot)$ and $s(\cdot)$ are nondecreasing, the equality implies that $i(t) = i(t')$ and $s(t) = s(t')$. $\|$

For all $t \in [0,1]$, let $I(t)$ be the interval that contains t in the partition Π^*.

Claim 3: For all $t \in [0,1]$, we have $\inf[I(t)] = i(t)$ and $\sup[I(t)] = s(t)$.

Proof: By Claim 2, for all $t \in [0,1]$ and all $t' \in I(t)$, we have $i(t) = i(t')$. Since $i(t') \leq t'$, we obtain $i(t) \leq t'$, for all $t' \in I(t)$. Therefore $i(t) \leq \inf[I(t)]$ for all for all $t \in [0,1]$. An identical reasoning proves $\sup[I(t)] \leq s(t)$ for all $t \in [0,1]$. Thus for all $t \in [0,1]$, we have $i(t) \leq \inf[I(t)] \leq t \leq \sup[I(t)] \leq s(t)$. For any type t satisfying $i(t) = s(t)$, all these inequalities hold as equalities and there is nothing more to prove.

For the other types, we still need to prove that $]i(t), s(t)[\subseteq I(t)$. Let then t be such that $i(t) \neq s(t)$. Let t' be such that $i(t) < t' < s(t)$. Let u and v be types such that $i(t) < u < t'$ and $t' < v < s(t)$. Since $\lim_{n\infty} i_n(t) = i(t)$ and $\lim_{n\infty} s_n(t) = s(t)$, there is a positive integer n^* such that for all $n \geq n^*$, we have $i_n(t) \leq u$ and $s_n(t) \geq v$. Thus for all $n \geq n^*$, we have $i_n(t') = i_n(t)$ and $s_n(t') = s_n(t)$. Taking the limit as n goes to infinity, we obtain $i(t') = i(t)$ and $s(t') = s(t)$. Therefore $t' \in I(t)$, for all $t' \in]i(t), s(t)[$. Therefore $]i(t), s(t)[\subseteq I^*(t)$. This and the inequalities we obtained in the last paragraph yield the desired conclusion.$\|$

Claim 4: Π^* is an equilibrium.

Proof: For all $t \in [0,1]$, we have $\lim_{n\infty} R(i_n(t), s_n(t)) = R(i(t), s(t))$, by continuity of R. Since Π^n is an equilibrium, for all $t, t' \in [0,1]$, we have $R(i_n(t), s_n(t)) \succeq_t R(i_n(t'), s_n(t'))$. This relation and the continuity of both R and the preference $\succeq_t$ imply that for all $t, t' \in [0,1]$, we have $R(i(t), s(t)) \succeq_t R(i(t'), s(t'))$. Therefore Π^* is an equilibrium.$\|$

It only remains to show that Π^* has an infinity of intervals.

Claim 5: There is $t^* \in [0,1]$ such that $i(t^*) = t^* = s(t^*)$. The partition Π^* has an infinity of intervals.

Proof: For all $n \geq 2$, there exists $u_n, v_n \in [0,1]$ such that $s_n(v_n) - i_n(u_n) \leq 2/(n-1)$ and $s_n(u_n) = i_n(v_n)$. Let $\{q\}$ be a subsequence such that $s_q(u_q)$ converges to $t^* \in [0,1]$. Then the sequences $i_q(u_q)$ and $s_q(v_q)$ both also converge to t^*. Since R is continuous, we have $\lim_{q\infty} R(i_q(u_q), s_q(u_q)) = R(t^*, t^*)$. By *type-continuity* and Lemma 8, the function $S(t)$ is continuous and thus $\lim_{q\infty} S(s_q(u_q)) = S(t^*)$. For all q, by *single-peakedness* of the preference $\succeq_{s_q(u_q)}$, we have

$$R(i_q(u_q), s_q(u_q)) \leq S(s_q(u_q)) \leq R(i_q(v_q), s_q(v_q)).$$

In the limit where q goes to infinity, we obtain $R(t^*, t^*) = S(t^*)$. Since Π^q is an equilibrium, we have

$$R(i_q(t^*), s_q(t^*)) \succeq_{t^*} R(i_q(u_q), s_q(v_q)).$$

By continuity of R and *type-continuity*, we can take the limit as q goes to infinity, which yields $R(i(t^*), s(t^*)) \succeq_{t^*} R(t^*, t^*)$. Since $R(t^*, t^*) = S(t^*)$, we obtain $R(i(t^*), s(t^*)) = R(t^*, t^*)$. Since R is increasing and $i(t^*) \leq t^* \leq s(t^*)$, then either we have $i(t^*) = t^* = s(t^*)$ or we have $i(t^*) < t^* < s(t^*)$. Suppose, by contradiction, that the second case holds, i.e. the inequalities are strict. Let u be a type such that $i(t^*) < u < t^*$ and let v be a type such that $t^* < v < s(t^*)$. Then there is a positive integer q° such that for all $q \geq q^\circ$, we have $u_q \in]u, v[$, $i_q(t^*) < u$ and $s_q(t^*) > v$. Thus for all $q \geq q^\circ$, we have $i_q(u_q) = i_q(t^*) < u$ and $s_q(u_q) = s_q(t^*) > v$. Thus for all $q \geq q^\circ$, we have $s_q(v_q) - i_q(u_q) > v - u > 0$, which contradicts that $s_q(v_q) - i_q(u_q)$ converges to 0. Therefore $i(t^*) = t^* = s(t^*)$.

For all $t < t^*$, we have $i(t) < t^*$ and $s(t) \leq t^*$. Since R is increasing, we have $R(i(t), s(t)) < R(t^*, t^*) = S(t^*)$. Therefore $S(t^*) \succ_{t^*} R(i(t), s(t))$ and thus $s(t) < t^*$. Therefore, if $t^* > 0$, the partition Π^* has infinitely many intervals in a left-neighborhood of t^*. Similarly, if $t^* < 1$, the partition Π^* has infinitely many intervals in a right-neighborhood of t^*. $\|$

We now prove the *if* implication.

Claim 6: If there is an equilibrium of infinite size, then for all $\kappa \geq 1$, there are two vectors $y, z \in X_\kappa$, such that for all $x \in [y, z]$ we have $\theta^\kappa(x) \subseteq [y, z]$.

Proof: If $R(0,0) = S(0)$, let $y := 0$. If $R(0,0) \neq S(0)$, then there are $t_1, ..., t_\kappa$ such that $i(t_1) = 0$ and for all $h = 1, ..., \kappa - 1$, we have $s(t_h) = i(t_{h+1})$. For all $h = 1, ..., \kappa - 1$, let $y_h := s(t_h)$ and $y := (y_1, ..., y_{\kappa-1})$. Similarly, if $R(1,1) = S(1)$, let $z := 1$. If $R(1,1) \neq S(1)$, then there are $t'_1, ..., t'_\kappa$ such that $s(t'_1) = 1$ and for all $h = 1, ..., \kappa - 1$, we have $i(t'_h) = s\left(t'_{h+1}\right)$. For all $h = 1, ..., \kappa - 1$, let $z_h := i\left(t'_{\kappa-h}\right)$ and $z := (z_1, ..., z_{\kappa-1})$. It is easily verified that y and z satisfy the desired condition. $\|$

Claim 6 and Lemma 7 imply that θ^κ has a fixed point in $[y, z]$. By Lemma 4, this vector represents an equilibrium of size exactly κ. This ends the proof of the Theorem.■

4. A taxonomy of biases

We introduce here a taxonomy of categories of admissible problems, according to the nature of the bias of the sender versus the receiver. We then refine the results of Section 3 within some of these categories.

Abusing notations, let $R(t) \equiv R(t, t)$. This is the reaction of the receiver to the belief that the type of the sender is certainly t.

One important case occurs when one of the functions $R(t)$ or $S(t)$ dominates the other by at least some positive constant.

The sender has a **strictly upward bias** if there is $\varepsilon > 0$ such that either for all $t \in [0,1]$, we have $R(t) + \varepsilon < S(t)$, and it has a **strictly downward bias** if or for all $t \in [0,1]$, we have $S(t) < R(t) - \varepsilon$. The sender has a **strictly consistent bias** if it has a strict bias, either upward or downward.

Crawford and Sobel's main result is obtained under assumptions that imply that the sender has a strictly consistent bias.[6] The following result generalizes Theorem 1 in Crawford and Sobel (1982) to problems where R is not necessarily continuous, and the sender has a strictly consistent bias.

THEOREM 3: *Let $(R, \succ)$ be such that the sender has a strictly consistent bias. The set of integers κ such that there are equilibria of size κ is decreasing and bounded. No equilibrium has an infinite size.*

Proof. When the sender has a *strictly consistent bias*, there is $\varepsilon > 0$ such that if u and v are actions induced in equilibrium, they satisfy $|u - v| > \varepsilon$ (see Crawford and Sobel 1982, Lemma 1, for a detailed proof). Therefore the set of actions induced in equilibrium is finite. Let κ be a positive integer such that there is an equilibrium of size κ. Consider one such equilibrium and let a_1 and a_κ be the most extreme actions induced in equilibrium. Then $\epsilon(\kappa - 1) \leq a_\kappa - a_1 \leq R(1) - R(0)$. Therefore $\kappa \leq (R(1) - R(0))/\epsilon + 1$. The Theorem is an immediate consequence of this fact and Theorems 1 and 2.■

Another important case occurs when the locus of the sender's preferred actions contains the locus of the receiver's optimal actions. In other worlds, the sender is weakly *more* responsive to the state of the world than the receiver in extreme situations. This condition is incompatible with a *strictly consistent bias.*

Outward bias. The sender has an outward bias if $[R(0), R(1)] \subseteq [S(0), S(1)]$.

For all $\kappa \geq 1$, let X^*_κ be the set of lists $x \in X_\kappa$ such that the vector x represents an interval equilibrium partition with (exactly) κ intervals.

THEOREM 4: *Let $(R, \succeq)$ satisfy outward bias. Then the set $X^*(\kappa)$ is a nonempty complete lattice. If, in addition, the function R is continuous, then at least one equilibrium has an infinite size.*

Proof. For $\kappa > 1$, the set $L \equiv X_\kappa$ is a nonempty complete lattice, and under *outward bias*, it satisfies the conditions of Lemma 6. The first claim in Theorem

[6] In their main result, Theorem 1, Crawford and Sobel (1982) assume that for all $t \in [0, 1]$, we have $R(t) \neq S(t)$. Under sender-continuity and the continuity of R (both of them are implied by their assumptions), this condition is equivalent to strictly consistent bias, as these authors show in their Lemma 1.

4 follows immediately from Lemma 6. The second claim follows immediately from Lemma 6 and Theorem 2.■

For completeness, we say that the sender has an *inward bias* when the locus of sender's preferred actions is strictly included in the locus of receiver's optimal actions. In other worlds, the sender is strictly *less* responsive to the state of the world than the receiver in extreme situations, i.e. $[S(0), S(1)] \subsetneq [R(0), R(1)]$. We do not have a more precise result than Theorem 1 for this case. We study an example in Section 7.

The three conditions of *strictly consistent bias, outward bias* and *inward bias* are mutually exclusive. But there are admissible problems that do not belong to any of the three cases. Such problems are such that $S(0) - R(0)$ and $S(1) - R(1)$ have strictly the same sign but the graphs of $R(\cdot)$ and $S(\cdot)$ are not bounded away from each other (e.g. they cross).

5. On equilibria with the same number of intervals

We present here additional results on the structure of the set of equilibria with a given number κ of intervals, for a class of problems that includes both the *strictly upward bias* and the *outward bias* cases. The sender has an **upward bias at 1** if $R(1) \leq S(1)$. A symmetric situation also of interest occurs when the sender has a **downward bias at 0**, i.e. if $R(1) \leq S(1)$. Symmetric results can be obtained in this case, so we will restrict attention to situations where the sender has an upward bias at 1.

To state the next Lemma, we need the following definitions. A **complete upper-semilattice** in $\mathbb{R}^m$ is a subset $L \subseteq \mathbb{R}^m$ such that, for any nonempty subset H of L, the set $\{x \in L : x \geq H\}$ is nonempty and has a least element, the supremum of H in L, denoted by $\sup_L H$. In particular, a nonempty complete upper-semilattice L has a greatest element $\sup_L L$ in L.

Lemma 9: *Let $(R, \succeq)$ be such that the sender has an upward bias at 1. Then for all κ such that $X_\kappa^* \neq \emptyset$, the set X_κ^* is a complete upper-semi lattice. In particular, it has a greatest element.*

Proof. Let the sender have an *upward bias at 1*, and let κ be such that $X^*_\kappa \neq \emptyset$. Since the sender has an upward bias at 1, we have $\theta^\kappa(x) \leq 1_\kappa$ for all $x \in X_\kappa$. Let Y be an arbitrary nonempty set of fixed-points of θ^κ, i.e. $Y \subseteq X^*(\kappa)$. Let $\hat{y} := \sup_{X_\kappa}(Y)$. This supremum exists, since X_κ is a complete lattice. Since all elements in Y are fixed-points of θ^κ, then for all $y \in Y$, we have $y \leq \sup(\theta^\kappa(y)) \leq \sup(\theta^\kappa(\hat{y}))$. Therefore $\hat{y} \leq \sup(\theta^\kappa(\hat{y}))$. Let $U := [\hat{y}, 1_\kappa]$. For all $u \in U$, we have $\hat{y} \leq \sup(\theta^\kappa(\hat{y})) \leq \sup(\theta^\kappa(u)) \leq 1_\kappa$. Thus for all $u \in U$, we have $\theta^\kappa(x) \cap U \neq \emptyset$. Let $Z(x) := \theta^\kappa(x) \cap U$. Consider the correspondence $Z : U \twoheadrightarrow U$. The set U is a closed interval, therefore it is a nonempty complete lattice. For all $x \in U$, the set $Z(x)$ is also a closed interval included in U, therefore it is a nonempty subcomplete sublattice of U. Since Z is increasing, we can apply Zhou's (1994) extension of Tarski's (1955) fixed-point Theorem to correspondences. Therefore the set of fixed-points of Z in U is a nonempty complete lattice. Let $\overline{y}$ be the least fixed-point of Z in U. The vector $\overline{y}$ has the following properties. *i)* It is a fixed-point of θ^κ on X_κ, i.e. $\overline{y} \in X^*_\kappa$. *ii)* Since $\overline{y} \in U$, then $\overline{y}$ is an upper-bound of Y. *iii)* Any upper-bound u of Y in X^*_κ is a fixed-point of Z in U, and therefore $\overline{y} \leq u$. Therefore $\overline{y}$ is the supremum of Y in X^*_κ, the desired conclusion.■

Under the conditions of Lemma 9, the set of vectors that represent equilibria with κ intervals has a greatest element, whenever this set is nonempty. Let the **greatest equilibrium with κ intervals** be the equilibrium represented by the greatest element of X^*_κ. The following result shows that the greatest equilibrium with κ intervals is nested within the greatest equilibrium with $\kappa + 1$ intervals, whenever the latter exists.

LEMMA 10: *Let $\kappa \geq 3$. Suppose that the sender has an upward bias at 1. Suppose that $X^*_{\kappa+1} \neq \emptyset$ (and therefore also $X^*_\kappa \neq \emptyset$). Let $\overline{x}$ be the greatest element in X^*_κ, and let $\overline{y}$ be the greatest element in $X^*_{\kappa+1}$. Then $\overline{y}_{-(\kappa-1)} \leq \overline{x} \leq \overline{y}_{-1}$.*

Proof. First, Lemma 7 ensures that there exists some $x \in X^*_\kappa$ such that $\overline{y}_{-(\kappa-1)} \leq x \leq \overline{y}_{-1}$. Since $x \leq \overline{x}$, it follows that $\overline{y}_{-(\kappa-1)} \leq \overline{x}$. Second, let $y^* \in X^*_{\kappa+1}$, let $y^\circ \equiv (0, \overline{x}_0, \ldots, \overline{x}_\kappa) \in X_{\kappa+1}$, and let $L := [y^*, 1_\kappa] \cap [y^\circ, 1_\kappa] \cap X_{\kappa+1}$. This set is such that for all $y \in L$, we have $\theta^{\kappa+1}(y) \in L$, and it is a nonempty complete lattice.

Therefore it contains a fixed point of $\theta^{\kappa+1}$. Thus there exists $y \in X^*_{\kappa+1}$ such that $\overline{x} \leq y_{-1}$. Since $y \leq \overline{y}$, we then have $\overline{x} \leq \overline{y}_{-1}$.■

We now present a comparative statics result on the greatest equilibrium with κ intervals, for two distinct admissible problems where the sender has an upward bias at 1.

COROLLARY 1: *Let $(R_1, \succeq^1)$ and $(R_2, \succeq^2)$ be two admissible problems. Let $\kappa \geq 2$. Suppose that the sender has an upward bias at 1 in both of these problems. Suppose that for all $x \in X_\kappa$, we have $\inf[\theta_1^\kappa(x)] \leq \inf[\theta_2^\kappa(x)]$ and $\sup[\theta_1^\kappa(x)] \leq \sup[\theta_2^\kappa(x)]$. Suppose further that problem 1 has an equilibrium with exactly κ intervals. Then problem 2 also has an equilibrium with exactly κ intervals. Let $\overline{x}^1$ and $\overline{x}^2$ be the respective greatest such equilibria for problem 1 and 2. Then $\overline{x}^1 \leq \overline{x}^2$. If, in addition, for all $x \in X_\kappa$, we have $\theta_1^\kappa(x) < \theta_2^\kappa(x)$, then $\overline{x}^1 < \overline{x}^2$.*

Proof. This result follows directly from Lemma 9 in this paper, and Theorem 2.5.2 by Topkis (1998), which extends Milgrom and Roberts' (1994) Theorem 3 to correspondences. ■

In practice, the following conditions on the primitives $(R_1, \succeq^1)$ and $(R_2, \succeq^2)$ imply that $\theta_1^\kappa(\cdot) \leq \theta_2^\kappa(\cdot)$ (which is stronger than the joint inequalities $\inf[\theta_1^\kappa(x)] \leq \inf[\theta_2^\kappa(x)]$ and $\sup[\theta_1^\kappa(x)] \leq \sup[\theta_2^\kappa(x)]$).

- *Sender 2 is more leftist than Sender 1; receivers are identical.*
  For all $t \in [0,1]$, all $a < b \in A$, we have $[a \succeq_t^1 b] \Rightarrow [a \succeq_t^2 b]$.
- *Receiver 2 is more rightist than Receiver 1; senders are identical.*
  For all $s \leq t \in [0,1]$, we have $R_1(s,t) \leq R_2(s,t)$.

Corollary 1 plays an important role in Section 6.2. There, we will show that comparative statics results on welfare due to Crawford and Sobel (1982) hold under broader conditions than what they assume.

We obtained the existence of a greatest equilibrium with κ intervals, as the greatest element of the set of fixed points of the correspondence θ^κ. It is easy to show that this equilibrium is also the greatest fixed point of the function $x \mapsto \sup[\theta^\kappa(x)]$ in

X_κ (as an application of Corollary 1, for example). If R is continuous, the following algorithm converges to this equilibrium. Let $\{x_n\}$ be the sequence of elements of X_κ such that $x_0 = 1_\kappa$ and for all $n \geq 0$, we have $x_{n+1} = \sup[\theta^\kappa(x_n)]$.

THEOREM 5: *Let R be continuous. Let $\kappa \geq 2$. If the sender has an upward bias at 1, and $X^*(\kappa) \neq \emptyset$, then $\{x_n\}$ converges to the greatest element of $X^*(\kappa)$.*

Proof. By Lemma 9, the set $X^*(\kappa)$ has a greatest element $\sup(X_\kappa^*)$. Since the sender has an upward bias at 1, we have $x_1 \leq x_0$. Since $x \mapsto \sup[\theta^\kappa(x)]$ is increasing, this implies that the sequence $\{x_n\}$ is nonincreasing. Since $\{x_n\}$ is bounded below by $\sup(X_\kappa^*)$, therefore it converges to a limit $x' \in X_\kappa$. Since R is continuous and by *type-continuity*, the function $x \mapsto \sup[\theta^\kappa(x)]$ is continuous. Therefore $x' \in X_\kappa^*$ and in addition $\sup(X_\kappa^*) \leq x'$ i.e. $x' = \sup(X_\kappa^*)$.■

This result can be used in practice as an algorithm to compute equilibria numerically.

6. Welfare comparisons for the receiver

In this section, we compare the receiver's welfare across the different equilibria of the same game, and across games, when the preferences of the sender vary. We suppose that the receiver has a Bernoulli utility function $U^a : A \times [0,1] \to \mathbb{R}$ and a non-atomic prior represented by the density $f(t)$. Its reaction function R maximizes its expected utility with respect to the prior f. Thus for any interval pool $[\underline{t}, \overline{t}]$, we have

$$R(\underline{t}, \overline{t}) \equiv \arg\max_a \int_{\underline{t}}^{\overline{t}} U^a(a,t) f(t)\, dt.$$

We suppose that the preferences represented by the utility function U^a satisfy *single-peakedness*, the *single-crossing property*, and that U^a is continuously differentiable in (a,t). For all $\kappa \geq 2$, and all $y \in X_\kappa$, let $E(y)$ be the expected indirect utility of the receiver, when he believes that the sender plays the strategy represented by the vector y and responds optimally. We have

$$E(y) = \sum_{h=0}^{\kappa-1} \int_{y_h}^{y_{h+1}} U^a(R(y_h, y_{h+1}), s) f(s)\, ds.$$

We will show that $E(y)$ is nondecreasing in y within a certain region of X_κ, which we introduce next.

Consider the auxiliary game where both the sender and the receiver have the same preferences, represented by the utility function U^a. Let $\tilde{\theta}^\kappa$ be the corresponding mapping of order κ, for all $\kappa \geq 2$. For all $\kappa \geq 2$, let Z_κ be the set of vectors in W_κ satisfying $z \leq \tilde{\theta}^\kappa(z)$. The following result plays an important role in the analysis.

LEMMA 11: *Let $\kappa \geq 2$. Then Z_κ is a subcomplete upper-subhemilattice of W_κ.*

Proof. Let Z be an arbitrary nonempty subset of Z_κ, and let z be an arbitrary element of Z. Since $\tilde{\theta}^\kappa$ is nondecreasing, then $\tilde{\theta}^\kappa(z) \leq \tilde{\theta}^\kappa(\sup_{W_\kappa}(Z))$. Since $z \in Z_\kappa$, then $z \leq \tilde{\theta}^\kappa(z)$. Therefore $z \leq \tilde{\theta}^\kappa(\sup_{W_\kappa}(Z))$. Since this holds for all $z \in Z$, therefore $\sup_{W_\kappa}(Z) \leq \tilde{\theta}^\kappa(\sup_{W_\kappa}(Z))$. Therefore $\sup_{W_\kappa}(Z) \in Z_\kappa$.■

The following condition ensure that the set Z_κ is connected in a particular way.

Condition (N) let $x, x' \in W_\kappa$ satisfying $\tilde{\theta}^\kappa(x) = x$ and $\tilde{\theta}^\kappa(x') = x'$. Then if $x_0 = x'_0$ and $x_1 < x'_1$, then $x_h < x'_h$, for all $h = 2, \ldots, \kappa$.

Crawford and Sobel (1982) introduce a similar, but substantially stronger condition (M). Condition (N) only restricts the receiver's preferences, while condition (M) is a joint restriction on the preferences of the receiver and the sender. Unlike condition (M), condition (N) does not imply that there is at most one equilibrium of any given size κ.

LEMMA 12: *Let $\kappa \geq 2$, and let (N) hold. Let $x, x' \in W_\kappa$ satisfy $\tilde{\theta}^\kappa(x) = x$, $\tilde{\theta}^\kappa(x') \geq x'$ and $(x_0, x_\kappa) = (x'_0, x'_\kappa)$. Then $x' \leq x$.*

Proof. By Lemma 11, the set $Z \equiv \{z \in Z_\kappa : z_0 = x_0 \text{ and } z_\kappa = x_\kappa\}$ is a subcomplete upper-subhemilattice of W_κ. Let x^* be the greatest element of Z. Let $x^{**} \equiv \tilde{\theta}^\kappa(x^*)$. We have $\tilde{\theta}^\kappa(x_0, \ldots, x_0, x_\kappa) \geq (x_0, \ldots, x_0, x_\kappa)$ and $\tilde{\theta}^\kappa(x_0, x_\kappa, \ldots, x_\kappa) \leq (x_0, x_\kappa, \ldots, x_\kappa)$. Therefore $x^{**} \in X_\kappa$. The monotonicity of $\tilde{\theta}^\kappa$ and $x^* \leq x^{**}$ further imply that $x^{**} \in Z_\kappa$, therefore $x^{**} \in Z$. Therefore $x^{**} \leq x^*$, i.e. $x^{**} = x^*$. By condition (N), we have $x = x^*$. Since $x' \in Z$, this implies $x' \leq x$.■

LEMMA 13: *Let $\kappa \geq 2$, and let (N) hold. Let $y' \leq y'' \in Z_\kappa$. For all $t \in [0,1]$, let $g(t) \equiv ty' + (1-t)y''$. Let $y : [0,1] \to Z_\kappa$, such that for all $t \in [0,1]$, we have $y(t) \equiv \sup(Z_\kappa \cap [0_\kappa, g(t)])$. Then the path $y(t)$ satisfies $y(0) = y'$ and $y(1) = y''$, and is increasing and continuous.*

Proof. By Lemma 11, the set Z_κ is a subcomplete upper subhemilattice of X_κ. Therefore the set $Z_\kappa \cap [0_\kappa, g(t)]$ is also a subcomplete upper subhemilattice of X_κ, which contains y'. It follows that for all $t \in [0,1]$, we have $y(t) \in Z_\kappa$. For all $t \leq t' \in [0,1]$, we have $Z_\kappa \cap [0_\kappa, g(t)] \subseteq Z_\kappa \cap [0_\kappa, g(t')]$. Therefore $y(t)$ is nondecreasing.

It only remains to prove that $y(t)$ is continuous everywhere on $[0,1]$. Since $y(t)$ is nondecreasing, then for all $t \in]0,1]$, the limit $y(t^-) := \lim_{s \to t^-} y(s)$ exists, and we have $y(t^-) \leq y(t)$. Similarly, for all $t \in [0,1[$, the limit $y(t^+) := \lim_{s \to t^+} y(s)$ exist, and we have $y(t) \leq y(t^+)$. By continuity of $\tilde{\theta}^\kappa$, we have $y(t^-), y(t^+) \in Z_\kappa \cap [0_\kappa, g(t)]$. Since $y(t)$ is the greatest element of this set, then in fact $y(t) = y(t^+)$, for all $t \in [0,1[$.

To prove that $y(t)$ is continuous everywhere on $[0,1]$, it only remains to establish that $y(t^-) = y(t)$ also holds. Suppose, by contradiction, that this is not true, so that $y(t^-) < y(t)$. Then there are indices k, l such that $0 < k \leq l < \kappa$ and satisfying $y_{k-1}(t^-) = y_{k-1}(t)$, $y_{l+1}(t^-) = y_{l+1}(t)$, and for all $h \in \{k, \ldots, l\}$, we have $y_h(t^-) < y_h(t)$. Let h be an arbitrary index such that $k \leq h \leq l$. Since $y_h(t) \leq g_h(t)$, therefore we also have $y_h(t^-) < g_h(t)$. For all $\epsilon > 0$ small enough, we have $y_h(t-\epsilon) < g_h(t-\epsilon)$. The only other constraint that restricts $y_h(t-\epsilon)$ must then bind. Therefore $\tilde{\theta}^\kappa_h(y(t-\epsilon)) = y_h(t-\epsilon)$. By continuity of $\tilde{\theta}$, it follows that $\tilde{\theta}^\kappa_h(y(t^-)) = y_h(t^-)$ holds, for all h such that $k \leq h \leq l$. Let $x \equiv (y_{k-1}(t^-), \ldots, y_{l+1}(t^-))$ and $x' \equiv (y_{k-1}(t), \ldots, y_{l+1}(t))$. We have $\tilde{\theta}^{l-k+2}(x) = x$ and $\tilde{\theta}^{l-k+2}(x') \geq x'$. By Lemma 12, we conclude that $x' \leq x$, a contradiction.■

LEMMA 14: *Let $\kappa \geq 2$, and let (N) hold. Then $E(y)$ is increasing on $Z_\kappa \cap X_\kappa$.*

Proof. Let $y' \leq y'' \in Z_\kappa$. By Lemma 12, the following object exists. Let $y(t)$ be a continuous increasing path such that $y(0) = y'$ and $y(1) = y''$ and $y(t) \in Z_\kappa$. For all $t \in [0,1]$, let $W(t) := E(y(t))$. We will show that $W(0) \leq W(1)$. For all $t \in [0,1)$, let

$$Dy(t) := \liminf_{h \to 0^+} \frac{y(t+h) - y(t)}{h}, \text{ and } DW(t) := \liminf_{h \to 0^+} \frac{W(t+h) - W(t)}{h}.$$

Since $E(y)$ is everywhere continuously differentiable, and $y(t)$ is continuous, we have

$$DW(t) = \sum_{k=1}^{\kappa-1} \frac{dE}{dy_k}(y(t))Dy_k(t).$$

By the envelope theorem,

$$\frac{dE}{dy_k}(y(t)) = \left[U^a\left(R\left(y_{h-1}, y_h\right), y_h\right) - U^a\left(R\left(y_h, y_{h+1}\right), y_h\right)\right] f\left(y_h\right).$$

Since $y(t) \in Z_\kappa$, then $\frac{dE}{dy_k}(y(t)) \geq 0$. Since $y(t)$ is nondecreasing, then $Dy_k(t) \geq 0$. Therefore we obtain $DW(t) \geq 0$ for all $t \in [0,1)$. Since $y(t)$ is continuous on $[0,1]$, then $W(0) \leq W(1)$, the desired conclusion.■

The next results are consequences of the previous lemma. They apply to situations where the sender has a particular form of strictly upward bias. Given two preferences $\succeq$ and $\succeq'$, we say that the preference $\succeq'$ has a **pairwise strictly upward bias** with respect to $\succeq$, if for all $t \in [0,1]$, all two actions $a < b \in A$, we have $b \succeq_t a \Rightarrow b \succ'_t a$.

THEOREM 6: *Let $\kappa \geq 2$, and let condition (N) hold. Suppose that the sender has a pairwise strictly upward bias with respect to the receiver. Let y' and y'' be two equilibria of size κ such that $y' \leq y''$. Then the receiver's expected payoff is greater at y'' than at y'.*

Proof. We have $y', y'' \in Z_\kappa \cap X_\kappa$. The Theorem then follows from Lemma 14.■

THEOREM 7: *Let $\kappa \geq 1$, and let (N) hold. Suppose further that the sender has a pairwise strictly upward bias with respect to the receiver. Let $\overline{x}$ represent the greatest equilibrium of size κ and let $\overline{y}$ represent the greatest equilibrium of size $\kappa+1$. The receiver's expected payoff is then greater at $\overline{y}$ than it is at $\overline{x}$.*

Proof. Let $z \in X_{\kappa+1}$ such that $z \equiv (0, \overline{x})$. We have $\overline{y}, z \in Z_{\kappa+1} \cap X_{\kappa+1}$. By Lemma 10, we have $z < \overline{y}$. The Theorem then follows from Lemma 14.■

Our last result compares the receiver's indirect utility at the greatest equilibrium of a given size κ, when informed by two different senders. The result shows that if sender 2 has a strictly pairwise upward bias with respect to sender 1, and sender 1 has a pairwise strictly upward bias with respect to the receiver, then the receiver's indirect utility is higher when informed by sender 1, than when informed by sender 2.

THEOREM 8: *Consider two sender preferences $\succeq^1$ and $\succeq^2$. Let $\kappa \geq 2$, and let condition (N) hold. Suppose that $\succeq^2$ has a pairwise strictly upward bias with respect to $\succeq^1$, and that $\succeq^1$ has a pairwise strictly upward bias with respect to U^a. Then the receiver's expected payoff at the greatest equilibrium of size κ is higher against $\succeq^1$ than against $\succeq^2$.*

Proof. Let y' and y'' represent the greatest equilibrium of size κ against $\succeq^1$, and against $\succeq^2$. By Corollary 1, we have $y'' < y'$. We also have $y', y'' \in Z_\kappa \cap X_\kappa$. The Theorem then follows from Lemma 14.■

A final remark on the comparison made in Theorem 8 is on order. The result is obtain by constructing a continuous path between the equilibrium of the first game and the equilibrium of the second game. Along the path, the indirect utility of the receiver decreases. An alternative strategy would be to consider a continuous path $\succeq^v$ from the preference $\succeq^1$ to the preference $\succeq^2$, indexed by $v \in [1,2]$, and such that for all $v < v'$, the preference $\succeq^{v'}$ has a strictly pairwise upward bias with respect to $\succeq^v$. We can then consider the maximal equilibrium of size κ for each of the games $(R, \succeq^v)$, which defines a path $\overline{x}^v \in X_\kappa$. By Corollary 1, the path $\overline{x}^v$ is decreasing. If the path $\overline{x}^v$ is also continuous, then we can show that the indirect utility of the receiver is decreasing along the path, along the lines of lemma 14, and the conclusion of Theorem 8 holds. Therefore, to obtain this welfare comparison, it suffices to prove the continuity of the path $\overline{x}^v$, which may or may not hold, independently of whether condition (N) is satisfied. Crawford and Sobel's condition (M) implies this continuity.

7. The uniform-quadratic example

We now consider the special case where the prior distribution is uniform and utilities are quadratic. Let F be the uniform distribution over $T = [0, 1]$, so that

$F(t) = t$. Let $d > 0$. Let

$$U^r(a,t) = -(a-t)^2 \text{ and } U^s(a,t) = -(a-b-dt)^2.$$

Straightforward calculations yield $R(s,t) = \frac{s+t}{2}$. It is immediate that the conditions listed at the beginning of this section are satisfied, so that the problem is admissible. Table 1 shows that nature of the sender's bias for different values of the parameters b and d.

	$b + d \leq 1$	$b + d \geq 1$
$b \leq 0$	Downward	Outward
$b \geq 0$	Inward	Upward

Table 1: Nature of the sender's bias for different values of b and d.

Crawford and Sobel (1982) studied in detail the case where $b > 0$ and $d = 1$ as an example of *strictly upward bias* and gave an explicit solution. We give here an explicit solution for all values of the parameters, using the same methodology.

In equilibrium, a cutoff type x_h must be indifferent between inducing the receiver's reaction to information the interval $[x_{h-1}, x_h]$ and the receiver's reaction to the information $[x_h, x_{h+1}]$. This implies the arbitrage condition

$$b + dx_h - \frac{x_{h-1} + x_h}{2} = \frac{x_h + x_{h+1}}{2} - (b + dx_h),$$

which can be rewritten as

$$(A_h) \qquad x_{h+1} + (2-4d)\, x_h + x_{h-1} - 4b = 0.$$

The vector $x = (x_0, ..., x_\kappa)$ represents an equilibrium with exactly κ intervals iff it is nondecreasing $x_0 \leq ... \leq x_\kappa$, solves the system $A_1, ..., A_{\kappa-1}$ and satisfies the boundary conditions $x_0 = 0$ and $x_\kappa = 1$ (problem A). We now solve problem A for all values of the parameters.

The discriminant of the equation

$$(*) \qquad \omega^2 + (2-4d)\,\omega + 1 = 0.$$

is $16d(d-1)$. It is null iff $d=1$, positive iff $d>1$ and negative iff $d<1$.

Case 1: $d=1$.

Crawford and Sobel (1982) show that in this case, a vector $(x_0,...,x_\kappa)$ is a solution of A iff it is nondecreasing, and for all $h=0,...,\kappa$, we have

$$x_h = 2bh^2 + (\frac{1-2b\kappa^2}{\kappa})h.$$

The vector defined by the formula above is nondecreasing iff

$$\kappa \leq \left\lfloor \frac{1+\sqrt{1+2/|b|}}{2} \right\rfloor.$$

Therefore there is exactly one equilibrium with κ intervals, for each positive integer κ satisfying this last inequality (i.e. for a bounded and decreasing set of integers), and it is described by the vector x defined above.

Case 2: $d>1$.

Let $\lambda<\theta$ be the solutions of $(*)$. We have $0<\lambda<1<\theta$. Let $x^* \equiv \frac{b}{d-1}$. A vector $(x_0,...,x_\kappa)$ is a solution of A iff it is nondecreasing, and for all $h=0,...,\kappa$, we have

$$x_h = x^* + a_\kappa \lambda^h + b_\kappa \theta^h. \tag{1}$$

The boundary conditions $x_0=1$ and $x_\kappa=1$ determine the constants

$$a_\kappa = -\frac{1+x^*(\theta^\kappa-1)}{\theta^\kappa-\lambda^\kappa} \text{ and } b_\kappa = \frac{1-x^*(1-\lambda^\kappa)}{\theta^\kappa-\lambda^\kappa}. \tag{2}$$

We now examine under what conditions the vector x is nondecreasing, i.e. defines an equilibrium with κ intervals. We distinguish three cases.

<u>Outward bias: $0 \leq x^* \leq 1$.</u> In this case, the vector x defined by the formula above is nondecreasing, for all $\kappa \in \mathbb{N}$, since $a_\kappa<0$ and $b_\kappa>0$. Therefore there is a unique equilibrium with κ intervals, for all $\kappa \in \mathbb{N}$, and it is described by the formula above. There is also a unique equilibrium with an infinity of intervals. It is described by the

sequence $\{x_h^\infty\}_{h\in\mathbb{Z}}$ such that $x_0^\infty := x^*$ and for all $h > 1$, we have $x_h^\infty = x^*(1-\lambda^{h-1})$ and $x_{-h}^\infty = x^* + (1-x^*)\,\theta^{-h+1}$.

Strong downward bias: $x^* < 0$. For all $\kappa > 0$, we have $b_\kappa > 0$. A necessary and sufficient condition for x to be nondecreasing is that

$$\frac{a_\kappa}{b_\kappa} \le \frac{\theta - 1}{1-\lambda}$$

i.e.

$$b_\kappa \ge -\frac{(1-\lambda)x^*}{\theta - \lambda}.$$

This inequality is compatible with (2) only within a bounded decreasing set of positive integers. For all κ in this set, there is a unique equilibrium with κ intervals. It is defined by (1).

Strong upward bias: $x^* > 1$. For all $\kappa > 0$, we have $a_\kappa < 0$. A necessary and sufficient condition for x to be nondecreasing is that

$$\frac{b_\kappa}{a_\kappa} \le \frac{1-\lambda}{\theta - 1}$$

i.e.

$$a_\kappa \le -\frac{(\theta-1)x^*}{\theta - \lambda}.$$

This inequality is compatible with (2) only within a bounded decreasing set of positive integers. For all κ in this set, there is a unique equilibrium with κ intervals. It is defined by (1).

Case 3: $d < 1$.

Let $x^* := \frac{b}{1-d}$. If $0 \le x^* \le 1$, the sender has an inward bias. Otherwise he has either a *strictly upward bias*, or a *strictly downward bias*. Let $z = e^{\pm i\rho}$ be the complex solutions of $(*)$. A solution x for problem A satisfies

$$x_h = x^* + A_\kappa \sin\left(\rho h + \varphi_\kappa\right).$$

The constants φ_κ and A_κ are jointly determined by the boundary conditions

$$\begin{aligned} x^* + A_\kappa \sin(\varphi_\kappa) &= 0 \\ x^* + A_\kappa \sin(\rho\kappa + \varphi_\kappa) &= 1. \end{aligned}$$

The vector x is a nondecreasing solution iff φ_κ satisfies

$$\frac{\sin(\rho\kappa + \varphi_\kappa)}{\sin(\varphi_\kappa)} = -\frac{1 - x^*}{x^*} \text{ and } \varphi_\kappa \in \left[-\frac{\pi+\rho}{2}, \frac{\pi + 1 - 2\rho\kappa}{2}\right]$$

It is easy to verify that the set

$$\left\{\frac{\sin(\rho\kappa + \varphi)}{\sin(\varphi)} : \varphi \in \left[-\frac{\pi+\rho}{2}, \frac{\pi + \rho - 2\rho\kappa}{2}\right]\right\}$$

is strictly decreasing in κ and empty for $\kappa > \pi/\rho + 1$. Therefore A has a nondecreasing solution only within a bounded and decreasing set of positive integers. There is one equilibrium with κ intervals for all κ in this set, and there are no equilibria of infinite size.

8. Related literature

8.1. *Fixed points versus difference equations*

In last section, we use a difference equation technique to obtain explicit expressions of the equilibria, in the uniform-quadratic example. More generally, the method is well suited to compute equilibria in examples. In contrast, in the general model, difference equations do not capture essential features of the structure of the problem, and are an inappropriate tool to describe the structure of the set of equilibria, except for the special case, where the sender's bias is strictly upward (or strictly downward).

Crawford and Sobel (1982) are able to prove Theorem 3 in this paper (Theorem 1 in theirs) using a difference equations approach. We now explain why this approach does not work to prove Theorem 1 in this paper, when the sender's bias is neither upward nor downward.

To explain this, let us recall the reasoning in Crawford and Sobel (1982). Any interval partition of the space may be described by a sequence of boundary types: $t_0, \ldots, t_\kappa$. For such a sequence to represent an equilibrium partition, it must satisfy two additional conditions. First an equilibrium sequence must satisfy a recursive equation obtained from the equilibrium "arbitrage conditions." This equation links together any three consecutive terms t_{k-1}, t_k and t_{k+1} for all $k = 1, \ldots, \kappa - 1$. Second, an equilibrium sequence must satisfy the boundary conditions $t_0 = 0$ and $t_1 = 1$. Now, suppose that the sender's bias is upward. Then one can show that any sequence that satisfies the two aforementioned condition is necessarily monotone nondecreasing, i.e. such that $t_0 \leq \ldots \leq t_k \ldots \leq t_k$. This in turn implies that any such sequence describes a type space partition, and even an equilibrium partition. The conclusion of Theorem 1 in this paper follows easily from there. Suppose that $t_0, \ldots, t_\kappa$ is an equilibrium sequence of size κ, and thus satisfies the two conditions. Consider now the subsequence made up of the first κ terms of this equilibrium sequence, i.e. $t_0, \ldots, t_{\kappa-1}$. We can now continuously deform this subsequence into a sequence $t'_0, \ldots, t'_{\kappa-1}$ that satisfies the two conditions, and therefore is an equilibrium sequence of size $\kappa - 1$, the desired conclusion. This is done by sliding t_1 upwards, and by sliding the remaining terms $t_2, \ldots, t_{\kappa-1}$, in such a way that the recursive equation is satisfied at each point of the transformation. The transformation ends when $t_{\kappa-1}$ hits 1, which must occur by continuity.

When the sender has a downward bias, one can similarly transform the subsequence made up of the last κ terms of this equilibrium sequence into a size κ equilibrium, and the same result obtains.

Unfortunately, when the bias is neither upward nor downward, the whole edifice collapses. But, where? In the general case consider in Theorem 1, a sequence satisfying the two above conditions may not be monotone nondecreasing. As a result, at the end of transformation previously introduced, we may end with a sequence that is not monotone nondecreasing, and therefore does not describe a partition, let alone an equilibrium partition.

As a consequence, we are confident that difference equations are useful only to study the upward or downward bias cases, or to solve examples. All results in this paper, except for Theorem 3, and the results in sections 6 and 7, are dividends of the fixed-point methods. This is the case, in particular, for the results on the outward bias case, the lattice and semi-lattice structures, the algorithm, and monotone comparative statics. Our results on welfare comparisons for the receiver in section 6, are also stronger than their counterparts in Crawford and Sobel (1982). The generalization is again a byproduct of the fixed point method.

The method we use is particularly effective in the unidimensional model thanks to two combined factors. The first one is the lattice structure that appears in the model. The second is the monotonicity of the equilibrium mapping. Together, these features allow us using prowerful results such as Tarski's fixed point theorem, and monotone comparative statics. Tarski's theorem not only yields existence of equilibria, but also results on the structure of the set of equilibria. For instance, when the sender's highest type has an upward bias, we prove that the set of equilibria of a given size k has a maximal element, if it is not empty. This enable us to further study this particular equilibrium, without assuming that it is the only one of this size. In the strictly upward case, we provide robust comparative statics without assumptions on equilibrium unicity. Crawford and Sobel (1982) obtained comparative statics results on equilibria and welfare which relied on a strong condition (condition (M)). This assumption implied in particular the unicity of an equilibrium of a given size. We obtain these same results under a much weaker condition (condition (N)), using our results on the structure of the set of equilibria.

8.2. Fixed points in Bayesian games

From the technical point of view, this paper is connected to an old tradition in economics and game theory of studying equilibria as fixed points of a certain mapping. In particular, our work is related to the theory of supermodular games, in that equilibria are the fixed-points of an increasing mapping, and we borrow many technical tools from this literature. Our work is also related to a literature on monotone pure strategies equilibria in Bayesian games.[7] It is also related to Athey's (2001) work on Bayesian games, with finite action sets and a unidimensional continuum of types, for each player. Her objective is to prove that, under certain monotonicity and regularity conditions, any such game has an equilibrium in pure monotone-in-type

[7]See Athey (2001), McAdams (2003) and Van Zandt and Vives (2006).

strategies. This is a key difference with our work, since in our setting, the existence of such equilibria is trivial. Athey represents strategies by the means of a vector of jump points, as we do. She defines a mapping from this set to itself and applies a fixed point theorem, as we do. The mapping we define is non-decreasing, allowing us to use Tarski's fixed-point theorem. Athey's mapping is not monotone, which leads her to invoke instead Kakutani's fixed point theorem. Finally, Athey uses the fixed-point theorem once and obtains the existence of at least one monotone equilibrium. In contrast, we use the fixed-point argument either as an induction step to prove existence of equilibria of inferior sizes (Theorem 1) or directly for all possible equilibrium sizes, i.e. an infinite number of times, to obtain an (at least countable) infinity of equilibria (Theorem 4).

9. Other applications of the method

There are many other possible applications of the fixed point method to cheap talk signalling models, well beyond the scope of this paper.

One important open question in the literature on cheap talk is the absence of a well-established theory of how to select equilibria. This is especially true in the unidimensional framework we study here.[8] In a companion paper, we propose to select the equilibria that are asymptotically stable fixed-points of the equilibrium mapping. Within the class studied by Crawford and Sobel (1982), we prove that a unique equilibrium satisfies this criterion. It is the maximal element of the set of equilibria of maximal size, the one we sometimes label the "most informative equilibrium."

We now turn our attention beyond the unidimensional framework. How useful is the fixed point method in more sophisticated models? In most, if not all, cheap talk models, it is possible to describe the equilibria of the game as the fixed points of an equilibrium mapping such as the one we introduced here. But for this strategy to yield any result, one needs at least one of the two following conditions to hold. Either the mapping should be nondecreasing, or the mapping's domain should contain the

[8]For recent advances on selection in this model that do not rely on the fixed-point method, see Kartik (2005) and Chen (2006).

mapping's image. In this paper, the second condition only holds when the sender's bias is outward. All of our results that do not take this as an assumption crucially rely on the monotonicity of the equilibrium mapping.

Unfortunately, in more sophisticated models, the equilibrium mapping is not likely to be monotone. However, even in this case, it may still be possible to use the fixed-point method provided that the domain of the equilibrium mapping contains its image. As a consequence, neither Tarski's fixed-point theorem nor any of its variants has any bite. Nevertheless, under certain conditions, Kakutani's, or even Brouwer's, fixed-point theorems can be used.

Levy and Razin (2004) and Chakraborty and Harbaugh (2005, 2006) introduced a multidimensional version of Crawford and Sobel's (1982) game. Unfortunately, this model raises serious technical difficulties. These authors provide partial results on the equilibrium set, but not a detailed description of the equilibrium set. In a companion paper (2006b), we apply the fixed-point method to this multidimensional model. We define the equilibrium mapping in this context, which maps pavements of the multi-dimensional type space to pavements of the same space. As one expects, this mapping is not monotone. However, in the special case where the sender has an outward bias, the domain of the equilibrium mapping contains its image. In this context, the assumption says that the support of the sender's preferred action contains the support of the receiver's preferred action. In this case, we prove, via Kakutani's fixed-point theorem, a result similar to this paper's Theorem 4. This yields a detailed description of the equilibrium set. We prove that this game has infinitely many equilibria, at least one of each finite size, and at least one of infinite size. Whether the method can be somehow adapted to the case where the sender's bias is not outward, to perhaps obtain a result analogous to this paper's Theorem 1, is an open question.

A simpler model where the equilibrium mapping is not monotone is the model of unidimensional cheap talk with an "uncertain bias," studied by Morgan and Stocken (2003), Li (2005) and Dimitrakas and Sarafidis (2005). In this model, the sender's privately known type has two dimensions and actions have one dimension. One of the type dimensions is relevant to both the receiver's and the sender's preferred decision, and the other type dimension is only relevant to the sender's preferred decision, and is therefore interpreted as an uncertain sender's bias. These authors all restrict attention to the case where both the sender and the receiver have quadratic preferences. Despite the equilibrium mapping not being monotone, Dimitrakas and Sarafidis (2005) successfully apply the fixed-point method. To this end, they restrict attention to the case where the support of the marginal distribution of the sender's bias is of the form $[0, b]$, with $b > 0$. This assumption precisely ensures that the equilibrium mapping's domain contains its image. As a result, a result similar to this paper's Theorem 4 holds, via Brouwer's fixed point theorem. This yields a detailed description of the equilibrium set. They prove that there are infinitely many equilibria, at least one of each finite size, and at least one of infinite size. Also here, whether the method can be adapted to other marginal distributions of the bias,[9], to perhaps obtain results analogous to this paper's Theorem 1 or Theorem 4, is an open question.

As an unfortunate consequence of the non monotonicity of the equilibrium mapping, both in Gordon (2006b) and Dimitrakas and Sarafidis (2005), the results in sections 5 and 6 in this paper do not extend to these frameworks. The rich structure we identify in this paper's model does not seem exist in these other models.

In the model we studied in this paper, as in all the models previously discussed in this section, and as in most cheap talk models, messages are interchangeable, in the sense that the equilibrium allocation of messages to pools is irrelevant. In our exposition of the method in this paper, we strongly rely on this property of the model. However, there are cheap talk models where messages are not irrelevant. One recent example is a model of noisy cheap talk signalling by Board and Blume (2006). In their framework, the presence of noise gives some messages an endogenous meaning in equilibrium. Therefore, as these authors point out,[10] the fixed point method, as presented here, cannot be applied to their model. Whether the method could be adapted to settings such as theirs, or even to costly signalling games, is an open question.

[9]For example, Morgan and Stocken (2003) consider marginal distributions of the bias with a support of the form $\{0, b\}$, where $b > 0$. Li (2005) consider marginal distributions of the bias with a support of the form $\{-b, +b\}$, where $b > 0$.

[10]Board and Blume (2006), p. 12.

The fixed point method improves our understanding of cheap talk signalling in general. The task of applying the method to these other questions is beyond the reach of this paper. We leave it for future research.

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