Monopoly and oligopoly supply of a good with dynamic network externalities

Robert Driskill
Department of Economics
Vanderbilt University
Nashville, TN
robert.driskill@vanderbilt.edu

January 23, 2007

Contents

Abstract

This paper models a good for which there are dynamic network externalities, and investigates the properties of the Markov Perfect equilibrium (MPE) that arises when there is monopoly or oligopoly supply. The framework is a continuous-time overlapping-generations model with constant probability of death in which every member of a cohort born at some time t must make a once-and-for-all decision as to whether to purchase a good which enhances such a member's income. (One interpretation of such a good would be education; the key idea, though, is to capture the notion that the size of the network has a positive externality, which has application beyond education). Each cohort is heterogeneous in regards to the effect of this good on an individual's earnings. Furthermore, the enhancement of earnings at every moment depends on how many other people at that time also possess the good. Hence, each member of a cohort faces the problem of forecasting how many people in the future will purchase the good. Key results are that positive network externalities may lead to steady-state price less than marginal cost, disadvantageous market power, and multiple equilibria (only one of which is the limit of a finite-horizon solution).

1 Introduction

This paper models a good for which there are dynamic network externalities, and investigates the properties of the Markov Perfect equilibrium (MPE) that arises when there is monopoly or oligopoly supply. The framework is a continuous-time overlapping-generations model with constant probability of death in which every member of a cohort born at some time t must make a once-and-for-all decision as to whether to purchase a good which enhances such a member's income. One interpretation of such a good would be education; the key idea, though, captures the notion that the size of the network has an externality, which has application beyond education. Each cohort is heterogeneous in regards to the effect of this good on an individual's earnings. Furthermore, the enhancement of earnings at every moment depends on how many other people at that time also possess the good. Hence, each member of a cohort faces the problem of forecasting how many people in the future will purchase the good. Such a forecast is predicated on the assumption that future generations (like present and past generations) attempt to maximize the present discounted value of lifetime resources. All consumers also understand that suppliers follow time-consistent strategies, i.e., cannot commit to an entire path of output.

With this structure of dynamic demand, we investigate the MPE for both monopoly and oligopoly supply, and contrast this with the equilibrium outcomes when firms are perfect competitors and when firms are assumed able to commit to paths of output through time. Our key results are:

1. With increasing marginal costs of production and sufficiently large positive network externalities, there exist parameter values such that steady-state

level of monopoly output is greater than the steady-state level of perfectly competitive output. When this occurs, steady-state price is less than marginal cost. This is similar to the finding in Driskill and McCafferty (2001) for addictive goods.

2. With sufficiently large positive network externalities, market power may be disadvantageous in the sense that industry profits are not maximized by monopoly.

3. For the infinite-horizon model, there exist parameter values for which there are two stable MPE. We prove that only one of them is the limit of a finite-horizon equilibrium as the horizon tends to infinity.

2 The model.

The basic building blocks of this model are specifications of consumer and firm behavior. The key assumption about consumers is that they know whether firms can commit to paths of output or whether they cannot and instead choose time-consistent, i.e., Markov, strategies. This distinction, familiar from the durable goods monopoly problem, allows us to highlight the importance of the assumption of time-consistent behavior by firms in this dynamic analysis. The more compelling assumption to us is that of a Markov strategy space for firms, but the assumption of commitment ability provides a useful benchmark.

2.1 Demand

Consider an overlapping generations model in which, at every moment, a generation of size ρ is born. Each member faces a constant probability of death also equal to ρ. At the moment of birth, individuals must make an irrevocable choice of whether or not to purchase a good at price p which affects their lifetime earnings. If they choose to not purchase the good, they receive through the rest of their life a constant flow of income R_a. If they choose to purchase the good, they receive a flow income R_m that depends on the total number of people alive who have also purchased the good. Let u denote the size of the group of members of a generation that choose to purchase the good and z denote the size of the total group of people who are alive and have also purchased the good. The evolution of the stock of people alive who have bought the good is thus governed by the differential equation

$$\dot{z} = u - \rho z \tag{1}$$

where $\dot{z} \equiv \frac{dz}{dt}$. That is, increases in the stock equal new purchasers minus the deaths of previous purchasers.

Members of each generation are heterogeneous in how their lifetime earnings are affected by the purchase of the good. Perfect annuity markets exist, there are no bequest motives, and everyone can borrow and lend at the fixed interest rate r, which is also each individual's rate of time preference.

A member of generation x, i.e., someone born at time $t = x$, maximizes expected utility:

$$\max_{c} \int_{t=x}^{t=\infty} e^{-\rho(t-x)} U(c) e^{-r(t-x)} dt \tag{2}$$

subject to his or her budget constraint. If the individual does not purchase the good, the budget constraint is:

$$\dot{w} = (r + \rho)w + R_a - c_a \tag{3}$$

where r is the constant rate of interest at which people can borrow and lend, w is non-human wealth and c_a is consumption. If the individual does purchase the good, the budget constraint is

$$\dot{w} = (r + \rho)w + R_m - c_m - p \tag{4}$$

The optimal program requires consumption to be constant. Hence, the present discounted value of this constant rate of consumption equals the present discounted value of lifetime resources. If one does not purchase the good, the present discounted value of lifetime resources is

$$\int_{t=x}^{t=\infty} R_a e^{-(r+\rho)(t-x)} dt = \frac{R_a}{r + \rho} \tag{5}$$

where, as noted, R_a is a positive constant.

If one purchases the good, the present discounted value of lifetime resources is

$$\int_{t=x}^{t=\infty} R_m(z) e^{-(r+\rho)(t-x)} dt - p. \tag{6}$$

Because consumption is constant, each member of generation x chooses whether or not to buy the good based on which choice gives him or her the highest present value of lifetime resources. We assume that there is an externality associated with the number of people alive who have purchased the good. If there is a positive externality, an individual's earnings are an increasing function of z. This might be the case if the good in question were basic education, for example: the more people there are that can read and write, the more valuable it is to any individual to be able to read and write. If there is a negative externality, then an individual's earnings are a decreasing function of z. This might occur if the good in question has "congestion effects."

To capture this externality, we assume that for individual i of generation x,

$$R_m^{i,x}(z) = \hat{R}_m^{i,x} + \widehat{A} z(t), \tag{7}$$

where $\widehat{A}$ is a constant (either positive or negative) and, for each individual, $\hat{R}_m^{i,x}$ is a positive constant. Assume, though, that individuals in any generation are heterogeneous in terms of $\hat{R}_m^{i,x}$ and that the distribution of $\hat{R}_m^{i,x}$ over members of a generation is uniform with highest value $\bar{R}_m$ and lowest value $\underline{R}_m$. Without loss of generality, think of the members of any generation as being sequenced in order of decreasing values of $\hat{R}_m^{i,x}$. The relationship between $\hat{R}_m^{i,x}$ for the last member of a generation that purchases the good and the size of the group of members of a generation that purchase the good is thus:

$$\hat{R}_m^{i,x} = \bar{R}_m - \widehat{a}u,\ \widehat{a} \equiv \frac{\overline{R}_m - \underline{R}_m}{\rho}. \tag{8}$$

For any generation x, an (interior) equilibrium distribution of its members between those that purchase the good and those that don't is determined by equality of lifetime resources for the marginal member:

$$\int_{t=x}^{t=\infty} [\hat{R}_m^{i,x} + \widehat{A}z(t)]e^{-(r+\rho)(t-x)}dt - p = \frac{R_a}{r+\rho}. \tag{9}$$

Using (7) and integrating and rearranging yields:

$$p = R - \alpha u(x) + A\int_{t=x}^{t=\infty} z(t)e^{-(r+\rho)(t-x)}dt; \tag{10}$$

where for notational convenience we have made the following definitions of new variables

$$R \equiv \frac{\overline{R}_m - R_a}{r+\rho};\ \alpha \equiv \frac{\widehat{a}}{r+\rho};\ A \equiv \frac{\widehat{A}}{r+\rho} \tag{11}$$

The forecasting problem faced by members of generation x is thus determination of the occupational choices of all future generations.

The solution to this forecasting problem depends upon which of two assumptions are made about firm behavior. The first assumption is that firms can commit to a *path* of output through time. This assumption, which means that firms do **not** follow time-consistent strategies, implies that the demand curve that constrains firms is simply equation (10).

The second assumption is that firms choose Markov strategies. We assume that individuals know the structure of the model, which means that along with knowledge of parameters and functional forms for the various structural equations in the model, they know that firms use these time-consistent Markov strategies in their attempts to maximize the present discounted value of profits. This knowledge, along with the equation of motion for z, allow people to infer that future values of the equilibrium price and equilibrium output of the good are a linear function of the state variable z:

$$p(t) = H + hz(t) \tag{12}$$

$$u(t) = \gamma_0 + \gamma z(t). \tag{13}$$

where $H,\ h,\ \gamma_0$, and γ are as-yet-to-be-determined coefficients that are functions of the underlying structural parameters of the model. Using (13) along with (1), we can rewrite (10) as:

$$p(x) = \chi_0 - \alpha u + \chi z \tag{14}$$

where

$$\chi \equiv \frac{A}{r+2\rho-\gamma} \tag{15}$$

and

$$\chi_0 \equiv \frac{\chi\gamma_0 + R}{r+\rho} \tag{16}$$

Under the assumption that firms choose Markov strategies, equation (14) is the downward-sloping instantaneous demand curve that constrains firms at any moment in time. This demand curve shifts through time as the value of z, the stock of people alive who have purchased the good, changes. The direction of this shift is determined by the sign of χ, which in equilibrium is a function of all the structural parameters in the model. We will eventually show that $\chi \gtreqless 0$ if and only if $A \gtreqless 0$. That is, we will show that if there is a positive externality $(A > 0)$, then the demand curve shifts out as z increases. If there is a negative externality $(A < 0)$, then the demand curves shifts in as z increases.

Equations (15) and (16) describe a pair of relations between (γ, χ) and (γ_0, χ_0). Firm behavior will provide another pair of relationships for these four variables. These four relationships then determine the four equilibrium values of these variables.

What is useful for our purposes is that the demand curve (14) and the relationships (15) and (16) are isomorphic to those derived from the model of addictive behavior in Driskill and McCafferty (2001). Thus, we can graft onto this demand model the oligopolistic supply model in Driskill and McCafferty (2001), and exploit the theorems proved there.

2.2 Firm behavior

Now consider firms. We first specify a cost structure, and then investigate optimal behavior under the aforementioned two assumptions about firm strategies: one in which firms can commit to a path of output through time, and a second in which firms choose time-consistent Markov strategies.

2.2.1 Cost structure

We follow Driskill and McCafferty (2001) in that we allow for the possibility of multiplant firms and rising marginal costs of production. In particular, there are n identical firms indexed by i, each of which owns l/n of M identical production units, each unit indexed by m and having instantaneous cost of production given by

$$C_m = F_m + c_0 u_m + \frac{\hat{c}}{2} u_m^2; m = 1, 2, ..., M, \tag{17}$$

where F_m denotes plant fixed costs, $u_m$is plant output, and c_0 and $\hat{c}$ are non-negative constants. Assuming firms minimize costs by producing equal amounts at each plant, firm cost of production will be given by

$$C_i = c_0 u_i + \frac{nc}{2} u_i^2 + F_i, c = \frac{\hat{c}}{M}, i = 1, 2, ..., n \tag{18}$$

where u_i is firm output and $F_i$are firm fixed costs. This assumption about cost functions leaves the *industry* cost function invariant to the number of firms within the industry.[1] Note that for simplicity we ignore integer constraints and take the number of plants and number of firms as exogenous[2].

2.2.2 The commitment equilibrium

Firms take as given other firms' strategies and the optimal behavior of consumers as embodied in equation (10). It is useful for this problem to define

$$\lambda_c \equiv A \int_{t=x}^{t=\infty} z(t) e^{-(r+\rho)(t-x)} dt$$

and re-write (10) as:

$$p = \lambda_c + R - \alpha u; \tag{(10.i)}$$

$$\dot{\lambda}_c = \lambda_c (r + \rho) + Az \tag{(10.ii)}$$

The i^{th} firm's problem is to choose a strategy $u_i(t)$ so as to maximize the present discounted value of profits:

$$\max_{u_i \in S_i} \Pi_i \equiv \int_0^{\infty} \{p(\lambda_c, u, z) u_i - (c_0) u_i - \frac{nc}{2} u_i^2\} e^{-rt} dt$$

$$s.t. \ \ \dot{z} = u - \rho z$$

and

$$p = \lambda_c + R - \alpha u$$

$$\dot{\lambda}_c = \lambda_c (r + \rho) + Az$$

where $u \equiv \sum_{i=1}^{n} u_i$ and S_i, the firm's strategy space, is defined as $S_i = \{u_i(t)$ such that $u_i(t)$ is continuous and differentiable$\}$.

Define the current-value Hamiltonian as:

$$\begin{aligned} H_i &= \{\lambda_c + R - \alpha u\} u_i - (c_0) u_i - \frac{nc}{2} u_i^2 + \theta_i [u - \rho z] \\ &\quad + \phi_i [\lambda_c (r + \rho) + Az] \end{aligned}$$

First-order conditions are:

$$\lambda + R - \alpha u - \alpha u_i - (c_0) - ncu_i + \theta_i - \delta \phi_i = 0$$

$$\dot{\theta}_i = (r + \rho) \theta_i + A \phi_i$$

[1] This feature means that a merger, for example, that would simply change the number of firms but not the number of plants, would not change the industry cost function.

[2] Endogenization of the number of plants and firms would be desirable, but is simply beyond our modeling abilities. One way to think of our specification is as an intermediate-length "run" where number and scale of plants and number of firms is fixed; only in the "long run" are these choice variables.

$$\dot{\phi}_i = -\rho\phi_i - u_i.$$

Because our interest in this commitment equilibrium is primarily as a benchmark, we leave out the details of the dynamic analysis and analyze the steady state. The steady-state equilibrium level of output of the symmetric equilibrium, denoted as $\overline{u}_C(n)$, where the subscript "C" is mnemonic for commitment and the parenthetical "n" signifies that the value depends on the number of firms, is computed as

$$\overline{u}_C(n) = \frac{R - c_0}{\alpha + c - \frac{A}{\rho(r+\rho)} + \frac{1}{n}\left\{\alpha - \frac{A}{\rho(r+\rho)}\right\}} \tag{19}$$

This will provide a benchmark for comparison that will aid in understanding the results from the more appealing assumption of time-consistent firm behavior.

2.2.3 The competitive benchmark

As a focus of comparison, consider the competitive equilibrium, found by assuming price equals marginal cost. The steady state of this equilibrium, denoted as $\overline{u}_\infty$, (where the subscript "∞" is used to emphasize that price equals marginal cost also results from letting $n \to \infty$) is readily computed as:

$$\overline{u}_\infty(n) = \frac{R - c_0}{\alpha + c - \frac{A}{\rho(r+\rho)}} \tag{20}$$

2.2.4 Comparison between perfect competition and commitment

Note that the difference between the two steady-state levels of output is the following term in the denominator of $\overline{u}_C(n)$:

$$\frac{1}{n}\left\{\alpha - \frac{A}{\rho(r+\rho)}\right\}. \tag{21}$$

Hence, steady-state output in the commitment equilibrium is less than steady-state output in the perfect competition equilibrium whenever $A < \alpha\rho(r+\rho)$, and is greater if $A > \alpha\rho(r+\rho)$. Heuristically, we can understand this result by noting that if $A \leq 0$, monopolistic or oligopolistic firms would restrict output relative to perfect competition. For $A > 0$, the case of positive externalities, a tension arises between the impulse of firms with market power to restrict output in the face of downward-sloping demand and the impulse to "build" demand by expanding output. The value of A at which these two forces just counterbalance each other is $\alpha\rho(r+\rho)$. For positive network externalities of greater magnitude than this, the "build demand" force is sufficiently large that steady-state output exceeds the competitive level.

2.2.5 Firm behavior with Markov strategies

Continue to assume that firms take as given other firms' strategies, but now they are constrained by the demand curve (14). The i^{th} firm's problem is to choose a strategy $u_i(t,z)$ from a strategy space S_i so as to maximize the present discounted value of profits:

$$\max_{u_i \in \rho} \Pi_i \equiv \int_0^\infty \{(p(u,z))u_i - c_0 u_i - \frac{nc}{2}u_i^2\} e^{-rt} dt \tag{22}$$

$$s.t.\ \ \dot{z} = u_i + \sum_{j \neq i} u_j(z,t) - \rho z \tag{23}$$

$$p = x_0 - \alpha u + xz \tag{14}$$

where $u \equiv \sum_{i=1}^{n} u_i$ and S_i, the firm's strategy space, is defined as $S_i = \{u_i(z,t)$ such that $u_i(z,t)$ is continuous and differentiable in $(z,t)\}$.

First-order conditions are:

$$p + \frac{\partial p}{\partial u_i} u_i - c_0 - ncu_i + \lambda_i = p - \alpha u_i - c_0 - ncu_i + \lambda_i = 0 \tag{24}$$

$$\dot{\lambda}_i = \lambda_i (r + \rho - \sum_{j \neq i} u_j(z,t)) - \chi u_i + \alpha \sum_{j \neq i} u_j(z,t) \tag{25}$$

$$\lim_{T \to \infty} \lambda_i(T) e^{-rT} = 0 \tag{26}$$

where λ_i is the firm's current-value costate variable that, as is well known, measures the marginal increase in the maximized value of (22) from an infinitesimal increase in z.

Observe from (24) that the oligopolist's first-order condition differs from a one-shot oligopolist's by the value of the costate variable, λ_i. That is, a one-shot oligopolist would choose output such that instantaneous marginal revenue would equal instantaneous marginal cost:

$$\overbrace{p - \alpha u_i}^{MR} - \overbrace{(c_0 + ncu_i)}^{MC} = 0$$

If $\lambda_i > 0$, output is greater than the one-shot level, and if $\lambda_i < 0$, output is less than the one-shot level. If this were to be the case, firms would invest (de-invest) in building (reducing) demand by producing more (less) today than would be produced under the myopic rule of (instantaneous) marginal revenue equals (instantaneous) marginal cost. Also observe that if it were the case that $\lambda_i + \alpha u_i > 0$, then output would be so high that marginal cost would be above price. What we will show is that for $A > 0$, at any moment $\lambda_i > 0$. We also show that for $A > 0$ and sufficiently large, there are some structural parameter values such that $\lambda_i + \alpha u_i > 0$ in the steady state.

Time-differentiating (24), equating this to (25), and substituting (24) for λ_i in the resulting expression yields a linear relationship between u_i and z. This linear relationship is the firm's strategy: $u_i = K + kz$. Assuming a symmetric equilibrium, aggregation of this relationship yields a linear relationship between u and z. Noting that in equilibrium this must equal (13), equating coefficients yields the following pair of relationships between (χ, γ) and (χ_0, γ_0), respectively, derived from firm optimization:

$$\chi = \frac{\gamma\{\alpha(1+\frac{1}{n})+c\}\{r+2\rho\} - \gamma^2(2\alpha+(2-\frac{1}{n})c)}{\{r+2\rho-2\gamma(1-\frac{1}{n})\}} \equiv \phi(\gamma) \tag{27}$$

$$\begin{aligned} -\{r+\rho-(1-\frac{1}{n})\gamma\}\{c_{0-}\chi_0\} \quad = & \\ & \gamma_0[\{\alpha(1+\frac{1}{n}+c\}\{r+\rho-(2-\frac{1}{n})\gamma\} \\ & +\chi(1-\frac{1}{n})+\alpha\gamma(\frac{n-1}{n^2})] \end{aligned} \tag{28}$$

where $\gamma = nk$ and $\gamma_0 = nK$.

2.3 The Markov perfect equilibrium

Definition: A Markov Nash equilibrium for the above game is:

1. A decision rule $u^* = d(p, z, t)$ that satisfies the consumer's dynamic optimization problem $\forall z, t$;
2. An n-tuple of Markov strategies $\{u_1^*(t,z), u_2^*(t,z), ..., u_n^*(t,z)\} \in S_1 \times S_2 \times ... \times S_n$ such that for every possible initial condition $\{z_0, t_0\}$:

$$J^i(u_i^*, u_j^*) \geq J^j(u_i, u_j^*)$$

   for every $u_i \in S_i,\ i,j = 1,2,...,n;\ i \neq j$;
3. A market-clearing condition that requires $\forall z, t$,

$$u^* = \sum_{\iota} u_i^*(z,t)$$

Note that the market-clearing condition implies that there exists an equilibrium price function $p = (z, t)$, implicitly defined by equating $d(p, z, t)$ to $\sum_i u_i^*(z,t)$.

3 Solution

As noted, even though the structure of the demand side of this model differs from the addictive structure found in Driskill and McCafferty, it leads to an instantaneous demand curve that is isomorphic in form to that of Driskill and McCafferty (2001). The supply side is identical to the supply side of Driskill and McCafferty (2001). Thus, we can exploit the theorems from Driskill and MCafferty (2001), albeit the interpretations will be different.

First, let us note that the value of γ that solves $\phi(\gamma) = \psi(\gamma)$ can be used recursively to solve for χ, χ_0, and γ_0. The features of the equilibrium value of γ are described in the following proposition.

Proposition 1 *If*

$$A < \frac{\rho(r+\rho)}{\left[r + \left(\frac{2\rho}{r+\rho}\right)\right]} \left\{ \alpha \left[r\left(1+\frac{1}{n}\right) + \frac{2\rho}{n}\right] + c\left(r + \frac{\rho}{n}\right)\right\}$$

then there exists a unique solution $\gamma^* < \rho$ *to the equation* $\phi(\gamma) = \psi(\gamma)$. *Furthermore, if*

$$0 < A < \frac{\rho(r+\rho)}{\left[r + \left(\frac{2\rho}{r+\rho}\right)\right]} \left\{ \alpha \left[r\left(1+\frac{1}{n}\right) + \frac{2\rho}{n}\right] + c\left(r + \frac{\rho}{n}\right)\right\},$$

then

$$\begin{aligned} 0 &< \gamma^* < \rho; \\ \chi &> 0; \\ h &> 0. \end{aligned}$$

If

$$A = 0$$

then

$$\chi = \gamma = h = 0.$$

If

$$A < 0$$

then

$$\begin{aligned} \gamma &< 0; \\ \chi &< 0; \\ h &< 0. \end{aligned}$$

Proof. See Driskill and McCafferty (2001), appendix A. ■

Remark 2 *We look for restrictions in parameter values to insure existence of values of* $\gamma^* < \rho$ *because we are interested in stable equilibria. It turns out that there are positive values of* A *that don't satisfy the above restriction but for which there exist two solutions, both of which are less than* ρ. *We will address this issue later and show that only one of these values is consistent with the equilibrium of a finite-horizon analogue of this model. The interesting feature of the upper bound restriction on* A *is that, for any value of* A, *there exists a large enough value of* c, *the marginal cost parameter, to ensure existence of a unique, stable equilibrium.*

With these solutions in hand, the key features of the equilibrium for this model are described in the following proposition.

Proposition 3 *Given the equilibrium solutions* $(\gamma_0^*, \gamma^*, \chi_0^*, \chi^*)$, *the perfect Markov equilibrium of the preceding game is fully described by the following equations that characterize firm strategies, consumer decision rules, and equilibrium price and output functions:*

$$u_i^* = \frac{\gamma_0}{n} + \frac{\gamma}{n} z \qquad \text{(strategies)}$$

where γ *solves*

$$\phi(\gamma) = \psi(\gamma);$$

$$p = \chi_0 - \alpha u + \chi z \qquad \text{(demand)}$$

where $\chi = h + \alpha\gamma = \psi(\gamma) = \phi(\gamma)$;

$$u_{MPE}^* = \overline{u}_{MPE} - \gamma\overline{z} + \gamma z \qquad \text{(output)}$$

$$p = \overline{p} - h\overline{z} + hz \qquad \text{(price)}$$

where an overbar denotes a steady state value and

$$\begin{aligned}
\overline{u}_{MPE} &= \frac{R - c_0}{\alpha + c - \frac{A}{\rho(r+\rho)} + \frac{1}{n}\left[\frac{\alpha(r+\rho)-\chi}{r+\rho-\left(\frac{1}{1+n}\right)\gamma}\right]}; \\
\overline{p} &= R + \overline{u}_{MPE}\left[\frac{A}{\rho(r+\rho)} - \alpha\right]; \\
\overline{z} &= \frac{\overline{u}_{MPE}}{\rho}.
\end{aligned}$$

Proof. See Driskill and McCafferty (2001), appendix A. ■

4 Comparisons

4.1 The competitive solution

Comparison between the steady-state level of output under perfect competition and the steady-state solution in the preceding proposition shows that whether or not the competitive level of output is greater or less than the oligopolistic level of output depends on the sign of $\alpha(\rho+r)-\chi$. The variable χ, of course, is endogenous. Insight into the comparison comes from considering the first-order condition for the firm, equation (24), which we re-write here in a useful form:

$$p - \overbrace{(c_0 + ncu_i)}^{MC} = -(\lambda_i - \alpha u_i) \qquad (29)$$

At any instant, $(\lambda_i - \alpha u_i)$ thus measures the gap between the competitive condition of price equal to marginal cost. Now, from the definitions of steady-state variables, we can find that

$$-\left(\overline{\lambda}_i - \alpha \overline{u}_i\right) = \frac{\alpha(\rho + r) - \chi}{r + \rho - (n-1)k}$$

Because in a stable equilibrium, $r + \rho - (n-1)k > 0$, this implies that $p \gtreqless MC$ as $\alpha(\rho + r) \gtreqless \chi$.

What χ measures is the value of building (or depleting) future demand. Big positive values of χ infer that changes in z lead to big shifts in the demand curve. Of course, χ is endogenous, and the question is what parameter values give rise to different values of χ. The following propositions describe these relationships for the case of monopoly. The first proposition deals with results for which the parameter A is restricted by an upper bound $\alpha(r + \rho)^2$. The second proposition deals with results for which the parameter A lies within the interval $(\alpha(r+\rho)^2, \alpha(r+\rho)^2 + \alpha\rho(r+\rho)$, while the third proposition deals with results for which $A \geq \alpha(r + \rho)^2 + \alpha\rho(r + \rho)$.

Proposition 4 *Assume the conditions for existence and uniqueness are satisfied. Then, if $A \leq \alpha(r + \rho)^2$, the competitive steady-state level of output is greater than the monopoly steady state value of output.*

Remark 5 *For $A < 0$, the solution to the model is isomorphic to a model of durable goods with adjustment costs (corresponding to the parameter α in this model), depreciation (corresponding to the parameter ρ in this model) and increasing marginal cost (corresponding to the parameter c in this model). As shown in Karp (1993) and Driskill (1997, 2001), in such a model the steady-state competitive level of output is greater than the oligopoly level. What this proposition indicates is that the implications of time-consistent behavior also make this the case even with $A > 0$.*

Proposition 6 *Assume the conditions for existence and uniqueness are satisfied. Then, if $\alpha(r+\rho)^2 < A < \alpha(r+\rho)^2+\alpha\rho(r+\rho)$, the competitive steady-state level of output may be greater than, less than, or equal to the monopoly steady-state level of output, depending on the value of c.*

Remark 7 *The behavior of the steady-state level of output is not monotonic in c. A sufficiently large value of c is necessary to insure existence of a unique equilibrium, but beyond that larger values of c reduce $\overline{u}_{MPE}$.*

Proposition 8 *For $A \geq \alpha(r + \rho)^2 + \alpha\rho(r + \rho)$ and for c sufficiently large to satisfy the condition for a unique stable equilibrium, the competitive steady-state level of output is less than the monopoly steady-state level of output, depending on the value of c.*

For the case of increasing marginal cost (of sufficient magnitude), the last two propositions tell us that monopoly output could be greater than competitive output. For such cases, a further implication is that, in the steady state, price is below marginal cost:

Proposition 9 *If* $\overline{u}_{MPE} > \overline{u}_{\infty}$*, then* $p < MC$*.*

Proof. Steady-state price is

$$\overline{p} = R + \overline{u}_{MPE}(n)\left[-\alpha + \frac{A}{\rho(r+\rho)}\right].$$

Thus, $\overline{p}$ minus marginal cost is

$$R - c_0 + \overline{u}_{MPE}(n)\left[-\alpha - c + \frac{A}{\rho(r+\rho)}\right].$$

For this equation to be negative, it must be that

$$\overline{u}_{MPE}(n) > \frac{R - c_0}{\alpha + c - \frac{A}{\rho(r+\rho)}} = \overline{u}_{\infty} \tag{30}$$

■

Remark 10 *Of course, there must be restrictions satisfied such that profits in such a case are sufficient to cover fixed costs.*

4.1.1 Comparison with the commitment equilibria

Insight into the above results can be developed by a comparison between the commitment and Markov equilibria. For positive externalities, we focus on results for monopoly, which are analytically tractable, and then provide a conjecture for the more general case.

By comparison of $\overline{u}_{MPE}$ and $\overline{u}_C$, we see that: $\overline{u}_C(n) \lesseqgtr \overline{u}_{MPE}(n)$ as $\left\{\alpha - \frac{A}{s(r+s)}\right\} \gtreqless \left\{\frac{\alpha(r+s)-\chi}{r+s-\left(1-\frac{1}{n}\right)\gamma}\right\}$. For the case of negative externalities, we have the following proposition that says that $\overline{u}_{MPE} < \overline{u}_C$.

Proposition 11 *If* $A < 0$*, then* $\overline{u}_C < \overline{u}_{MPE}$*.*

Proof. From the consumer's problem, we know that

$$\chi = \frac{A}{r + 2\rho - \gamma}.$$

Hence, $A = \chi\rho + \chi(r + \rho - \gamma)$. Because $\gamma < \rho$ and $\chi < 0$ if $A < 0$, this implies that A is the sum of two negative numbers, $\chi\rho$ and $\chi(r + \rho - \gamma)$, so $A < \chi\rho$. Consequently, $\frac{A}{\rho(r+\rho)} < \frac{\chi}{r+\rho}$, which implies $\left\{-\frac{A}{\rho(r+\rho)}\right\} > \left\{-\frac{\chi}{r+\rho}\right\}$. Thus,

$$\left\{\alpha - \frac{A}{\rho(r+\rho)}\right\} > \left\{\alpha - \frac{\chi}{r+\rho}\right\}$$

Now, $A < 0 \Rightarrow \gamma < 0$, so

$$\left\{\alpha - \frac{\chi}{r+\rho}\right\} > \left\{\frac{\alpha - \frac{\chi}{(r+\rho)}}{1 - \frac{(1-\frac{1}{n})\gamma}{r+\rho}}\right\}$$

That is, we are multiplying $\left\{\alpha - \frac{\chi}{r+\rho}\right\}$ by a number that is less than or equal to one (1), namely $\frac{1}{1-\frac{(1-\frac{1}{n})\gamma}{r+\rho}}$ Hence,

$$\left\{\alpha - \frac{\chi}{r+\rho}\right\} > \frac{1}{n}\left\{\frac{\alpha - \frac{\chi}{(r+\rho)}}{1 - \frac{(1-\frac{1}{n})\gamma}{r+\rho}}\right\} = \frac{1}{n}\left\{\frac{\alpha(r+\rho) - \chi}{r+\rho-(1-\frac{1}{n})\gamma}\right\}$$

where again we multiply $\left\{\frac{\alpha - \frac{\chi}{(r+\rho)}}{1 - \frac{(1-\frac{1}{n})\gamma}{r+\rho}}\right\}$ by a number less than or equal to one (1), namely $\frac{1}{n}$. Hence,

$$\left\{\alpha - \frac{A}{\rho(r+\rho)}\right\} > \frac{1}{n}\left\{\frac{\alpha(r+\rho) - \chi}{r+\rho-(1-\frac{1}{n})\gamma}\right\}.$$

■

For the case of positive externalities ($A > 0$) we focus on the case of monopoly. In this case, we also find that the steady-state commitment equilibrium level of output is greater than the steady-state MPE level of output:

Proposition 12 *If $A > 0$, then $\overline{u}_C(1) < \overline{u}_{MPE}(1)$.*

Proof. Again, $A = \chi\rho + \chi(r+\rho-\gamma)$. Because if $A > 0$, then $\gamma < \rho$ and $\chi > 0$, this implies that A is the sum of two positive numbers. Hence, $A > \chi\rho$. Consequently, $\frac{A}{\rho(r+\rho)} > \frac{\chi}{r+\rho}$, which implies $\left\{-\frac{A}{\rho(r+\rho)}\right\} < \left\{-\frac{\chi}{r+\rho}\right\}$. Thus,

$$\left\{\alpha - \frac{A}{\rho(r+\rho)}\right\} < \left\{\alpha - \frac{\chi}{r+\rho}\right\}$$

Comparison of the expression for $\overline{u}_{MPE}(1)$ and $\overline{u}_C(1)$ shows that this means $\overline{u}_{MPE}(1) < \overline{u}_C(1)$. ■

5 Disadvantageous market power

The above comparisons also suggest that there may be disadvantageous market power. Again we can exploit the isomorphism between Driskill and McCafferty (2001) and report the following results on what Karp (1996) has dubbed "disadvantageous market power."

Proposition 13 *Assume the conditions for a unique stable equilibrium are satisfied. Then for* $A \leq \alpha(r+\rho)$*, there exists* $n > 1$ *such that steady-state industry profits are maximized. For* $\alpha(r+\rho) < A < \alpha(r+\rho) + \alpha(r+\rho)^2$*, steady state industry profits are greater under perfect competition than under monopoly.*

Proof. See Driskill and McCafferty (2001). ■

6 Multiple stable equilibria and a selection criterion

When $A > 0$, the model can be characterized as one with increasing returns. As might be expected in a model with increasing returns, there are parameter values for which there exist two stable equilibria. We demonstrate this for the tractable case of duopoly, and show that only one of the two stable equilibria is the limit of the solution to the finite-horizon version of the model.

For the case of duopoly, the equation $\phi(\gamma) = \psi(\gamma)$ reduces to the following quadratic equation:

$$\gamma^2 - \frac{3}{4}(r+2\rho)\gamma + \widetilde{A} = 0, \ \widetilde{A} \equiv \frac{A}{2\alpha}. \tag{31}$$

For the roots to this equation to be real, we need the following restriction on the parameters of the model:

$$\widetilde{A} < \frac{9}{64}(r+2\rho)^2 \tag{32}$$

We assume throughout that this is satisfied. Denote the smaller root of (31) as γ_1 and the larger as γ_2. If $\widetilde{A} \leq 0$, $\gamma_1 \leq 0$ and $\gamma_2 > \rho$. If $\widetilde{A} > 0$, two possibilities emerge. First, if $0 < \widetilde{A} < \frac{3}{4}r\rho + \frac{1}{2}r\rho^2$, then $0 < \gamma_1 < \rho < \gamma_2$.[3] On the other hand, if $\frac{3}{4}r\rho + \frac{1}{2}r\rho^2 < \widetilde{A} < \frac{9}{64}(r+2\rho)^2$ and $2\rho > 3r$, then $0 < \gamma_1 < \gamma_2 < \rho$. For example, if $\rho = 2$, $r = 1$, and $\widetilde{A} = \frac{224.75}{64}$, then $\gamma_1 = \frac{29}{16}$ and $\gamma_2 = \frac{31}{16}$.

Because most of the literature has viewed as less problematic the case of multiple equilibria where only one of the equilibria is stable, our primary interest is in situations in which multiple stable equilibria exist, that is, in situations in which parameter values are such that $0 < \gamma_1 < \gamma_2 < \rho$. Note, though, that our results apply to the case where one stable and one unstable equilibria exist, that is, for the case where parameters are such that $\gamma_1 < \rho < \gamma_2$.

Assume a horizon of length T. The first-order conditions for the consumer differ from those in the infinite-horizon case only in that, in the finite-horizon case, $\lambda_c(T) = 0$. This means that at T consumers act myopically and choose consumption so as to equate price to instantaneous marginal utility:

$$p(T) = R - \alpha u(T) \tag{33}$$

[3] Note that $\frac{3}{4}r\rho + \frac{1}{2}r\rho^2 < \frac{9}{64}(r+2\rho)^2$.

The other difference between the finite and infinite-horizon game is that the equilibrium output function in the finite-horizon game has time-varying parameters:

$$u^*(t) = \gamma_0(t) + \gamma(t)z(t). \tag{34}$$

Following the same steps as in the infinite-horizon case, the consumer's first-order conditions can be manipulated to yield the following instantaneous demand curve:

$$p(t) = \chi_0(t) - \alpha u(t) + \chi(t)z(t) \tag{35}$$

where the constraints on $\chi_0(t),\ \chi(t),\ \gamma_0(t)$, and $\gamma(t)$ obey the following differential equations:

$$\dot{\chi} = \chi(r + 2s - \gamma) - A \tag{36}$$

$$\dot{\chi}_0 = \chi_0(r + s) - R - \gamma_0\chi \tag{37}$$

Note that (33) and (35) together imply the following terminal conditions on $\{\chi_0(T), \chi(T)\}$:

$$\chi_0(T) = \alpha_0 \tag{38}$$

$$\chi(T) = 0 \tag{39}$$

Turning to firm behavior, the only difference between the first-order conditions for firms in the finite-horizon and infinite-horizon case is that, in the finite-horizon case, $\lambda_i(T) = 0$. This means that at T, firms act as one-shot profit maximizers and choose output such that instantaneous marginal revenue equals instantaneous marginal cost. The other key difference is that firm strategies have time-varying parameters:

$$u_i = k_0(t) + k(t)z(t),\ i = 1, 2.$$

Repeated time-differentiation and substitution of the first-order conditions along with use of the equilibrium price and output functions yields the following differential equations that must be obeyed by the parameters $\{\chi_0(t), \chi(t), \gamma_0(t), \gamma(t)\}$:

$$\dot{\gamma} = \frac{-\gamma^2 + \gamma(r + 2\rho)\left(\frac{3}{4}\right) - A}{\left(\frac{3}{2}\right)\alpha} \tag{40}$$

$$\dot{\gamma}_0 = \frac{\chi_0\left(\frac{1}{2}\right)\gamma}{\left(\frac{3}{2}\right)\alpha} + \frac{\gamma_0[\left(\frac{3}{2}\alpha\right)(r + 2\rho - \left(\frac{3}{2}\gamma\right)) + \left(\frac{1}{4}\alpha\gamma\right) - \left(\frac{1}{2}c\gamma\right) - (r + \rho)(\alpha_0 - c)]}{\left(\frac{3}{2}\right)\alpha} \tag{41}$$

Equations (38) and (39) constitute a system of differential equations in the four parameters $\chi_0(t),\ \chi(t),\ \gamma_0(t)$, and $\gamma(t)$. The system can be solved recursively starting with (38), which only involves γ, and then moving to (39), and thence to (36) and (37). By construction,the critical values of these parameters are the values of the parameters from the autonomous infinite-horizon problem. Terminal conditions are derived from the first-order conditions at T:

$$\gamma(T) = 0; \tag{42}$$

$$\gamma_0(T) = \frac{(R-c)}{\left(\frac{3}{2}\alpha\right)} \tag{43}$$

We can now state the following proposition:

Proposition 14 *Let $\widetilde{z}, \widetilde{u}$, and $\widetilde{p}$ denote the equilibrium stock of consumption capital, equilibrium industry output (and consumption), and equilibrium price, respectively, in the finite-horizon game. For any z_0, $t > 0$ and any arbitrarily small $\varepsilon > 0$, there exists a horizon length $T > t$ sufficiently large such that $|\widetilde{z}(t) - z_1(t)| < \varepsilon, |\widetilde{u}(t) - u_1^*(t)| < \varepsilon$, and $|\widetilde{p}(t) - p_1^*(t)| < \varepsilon$.*

Proof. We need to show that the parameters $\chi_0(t)$, $\chi(t)$, $\gamma_0(t)$, and $\gamma(t)$ spend more and more time "close" to the critical values $\chi_{0,1}^*$, χ_1^*, $\gamma_{0,1}^*$, and γ_1^*. Because of the recursive nature of the problem, it will suffice to show that $\gamma(t) \to \gamma_1^*$ as $T \to \infty$. Consider (28). Denote the r.h.s. as $\theta(\gamma)$. The relevant properties of θ are:

$$\begin{aligned}
\theta(0) &= -\frac{2A}{3\alpha} = \theta(\gamma(T)) < 0; \\
\theta' &= \frac{-2\gamma + \frac{3}{4}(r+2\rho)}{\frac{3\alpha}{2}}; \\
\theta'' &= \frac{-4}{3\alpha} < 0; \\
\theta'(\gamma(T)) &= \frac{1}{2}(r+2\rho) > 0.
\end{aligned}$$

That is, $\theta(\gamma)$ is a concave function with a value of zero at γ_1^* and γ_2^*. Furthermore,because $\theta(\gamma(T)) < 0$ and $\theta'(\gamma(T))$, it must be that $\gamma(T) < \gamma_1^*$. Hence, $\forall t \in (0,T), \dot{\gamma} < 0$. Hence, as $T \to \infty, \gamma(t) \to \gamma_1^*$. ■

The above relationship between $\theta(\gamma(T))$, $\theta(\gamma)$, and γ is illustrated in Figure 1:

$-2x^2 + 2x - .5$

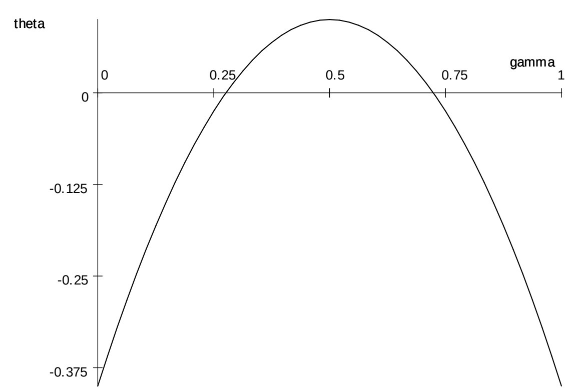

Figure 1

7 Conclusion

With less than perfect competition, the presence of strong positive dynamic network externalities may lead to networks so large that steady-state price is less than marginal cost. Furthermore, the presence of positive dynamic network externalities also creates disadvantageous market power: steady-state industry profits will be lower with fewer firms in the industry.

Finally, positive dynamic externalities may produce multiple stable equilibria. In such a case, though, we prove that only the smallest equilibria is the limit of a finite-horizon solution.

8 References

Driskill, R., "Durable Goods Monopoly with Increasing Costs and Depreciation," *Economica*, 64 (February 1997), 137-54.

Driskill, R., "Durable Goods Oligopoly," *International Journal of Industrial Organization*, Vol. 19, (2001), 391-413.

Driskill, R. and S. McCafferty, "Monopoly and Oligopoly Provision of Addictive Goods", *International Economic Review*, Vol. 42, no. 1, (February 2001), 43-72.

Fershtman, C. and M. Kamien, "Price Adjustment Speed and Dynamic Duopolistic Competition," *Econometrica* 55, (1987), 1140-51.

Fershtman, C. and M. Kamien, "Turnpike Properties in a Finite-Horizon Differential Game: Dynamic Duopoly with Sticky Prices," *International Economic Review*, 31, (1990), 49-60.

Karp, L., 1996a, "Depreciation Erodes the Coase Conjecture," *European Economic Review* 40, 473-490.

Karp, L., 1996b, "Monopoly Power Can Be Disadvantageous in the Extraction of a Durable Nonrenewable Resource," *International Economic Review* 37(4), (November), 825-849.

Katz, M. and C. Shapiro, 1985, "Network externalities, competition and compatability," *American Economic Review* Vol. 75, No. 3 (June), p. 424-440.

Mason, R., "Network externalities and the Coase conjecture," *European Economic Review* 44 (2000)

Matsuyama, K., 1991, "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium," *Quarterly Journal of Economics* 104, 617-650.