Cooperation in the Repeated Prisoner's Dilemma Game with Local Interaction and Local Communication*

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March 23, 2007

Abstract

The paper considers the repeated prisoner's dilemma game under a network where each agent interacts with his neighbors and he cannot observe the actions of others who are not directly connected to him. In this setting, when agents are sufficiently patient and the loss from being cheated is small enough, a trigger strategy that observing a deviation causes a permanent punishment cannot be a sequential equilibrium. Also, although the modification of the trigger strategy, following Ellison (1994), can be a sequential equilibrium supporting cooperation, it is not stable to mistakes in the sense that a mistake to play defection causes that all agents play defection forever. In this paper, we allow agents to communicate with their neighbors and construct a sequential equilibrium which supports cooperation and is stable to mistakes when the discount factor is high enough. Here, the role of local communication is to enable agent to resolve the discrepancy of his neighbors' beliefs on the punishment periods.

JEL Classification: C72, C73
Keywords: Repeated prisoner's dilemma game, Local interaction, Local communication, Sequential equilibrium, Network.

1 Introduction

In this paper, we consider a society where each agent locally interacts and communicates with others. The environment has the following features. Each agent is directly or indirectly connected with the other agents and his payoff depends only on the actions of himself and other agents who are directly connected to him. Each agent cannot take

*I would like to thank Kalyan Chatterjee for his guidance and encouragement. I also would like to thank Edward Green, James Jordan, and Anthony Kwasnica for helpful comments.
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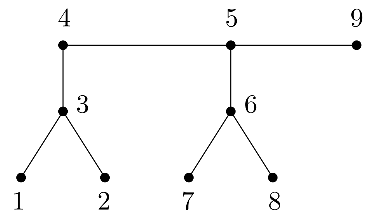


Figure 1: Car dealers in a village

different actions against different neighbors. Furthermore, information process is local so that an agent does not observe the action of other agents who are not directly connected to him. An example of this situation is producing local public goods such that the benefit of a public good is shared by one's neighbors but the cost is private. Another example is local competition and collusion of firms, which is discussed in Salop (1979) and Syverson (2004).

For a concrete example, consider a village whose map is described in Figure 1. In this village, there are nine car dealers, labeled 1 through 9. Each consumer buys a car from a dealer who offers the lowest price among the dealers which are close to him. For example, a consumer who lives between 3 and 4 buys from 3 if 3's price is lower than 4's price, and buys from 4 otherwise. In this setting, each car dealer plays the prisoner's dilemma game against his neighbored dealers. That is, he may either cooperate with his neighbors by choosing a high price, or defect by choosing a low price. Notice that competition among dealers is local, because each of them competes only with his neighbored dealers. Also, since car dealers cannot discriminate consumers, each dealer has to choose the same action against all his neighbors.

We assume that each dealer does not observe the price of others which are not adjacent to him. This may happen when there is a cost to observe the prices which other dealers offer. If there is a cost to monitor the others' actions, each dealer may not want to pay the cost to see the prices which is irrelevant to his profit.[1] Also, we can imagine that, since the price is a private offer to consumers, each car dealer cannot observe the others' prices. In the prisoner's dilemma game, each agent can recognize his opponent action from his realized payoff, even if he cannot observe his opponent's actions directly. If this is the case, then each dealer cannot recognize the prices of the other dealers which do not affect his profit directly.

In the paper, we are interested in an infinitely repeated game where each agent interacts with his neighbors by playing a prisoner's dilemma game. Can efficient outcome be supported by an equilibrium? To answer this question, we may consider a trigger strategy

[1]Monitoring cost is considered in Ben-Porath and Kahneman (2003) and Miyagawa et al. (2004).

such that each agent chooses defection if and only if he observes defection in his past history.[2] If the loss from opponent's defection is sufficiently high, the trigger strategy is a sequential equilibrium for sufficiently patient agents. If the loss is small enough, then the trigger strategy cannot be a sequential equilibrium. The intuitive reason is that an agent, when he observes a defection by his neighbor, is reluctant to punish the defector and bear the loss for fear of losing the future gain from cooperating with other neighbors. Indeed, when the loss is small, the trigger strategy can be a sequential equilibrium for the agents who are not sufficiently patient. Ellison (1994) provides an idea of constructing a sequential equilibrium for sufficiently patient agents by modifying the trigger strategy. The idea is to dilute the game into a certain number of replica games so that agents are not sufficiently patient in a replica game. After diluting the game, agents play the trigger strategy in each replica game and ignore observations in other replica games.[3]

Though it is not difficult to construct a sequential equilibrium which supports cooperation, the modification of trigger strategy has an undesirable feature. That is, it is not stable to mistakes in the sense that if an agent chooses defection by a mistake in a replica game, playing defection spreads over the network and cooperation is never recovered. Cho (2007) shows that, if there is a small possibility of mistakes, the equilibrium which is stable to mistakes can result in the more efficient outcome than the trigger strategy equilibrium, though both the equilibria give the same payoffs in the limit as the possibility of mistakes vanishes.

The usual way of handling this is to have punishment of fixed finite length. That is, if an agent observes his neighbor playing defection, then he plays defection to punish his neighbor in finite periods. However, the local observability may cause a discrepancy between the expectations of agents on the period when their neighbor ends a defection phase, which is a span of periods when he plays defection. If there is such a discrepancy in some history, then the agent whose neighbors have different beliefs on his action may not be able to satisfy the expectations of all his neighbors in any period. This may cause an infinite repetition of defection phases, and so cooperation is never recovered. Furthermore, the strategy described above is not a sequential equilibrium.

In Cho (2007), we resolve this discrepancy by introducing a public randomization, an idea from Ellison (1994). Since the realization of randomization is publicly known to all agents in the society, agents can reestablish cooperation in the same period when a specific event happens. However, to reestablish cooperation, we need a consensus of the whole society on what the specific event is, or when they turn back to cooperation.

[2]Since defection spreads over the network, the equilibrium with this strategy is sometimes called a *contagious equilibrium*.

[3]For details, see Section 6.

Since it seems difficult to achieve a global agreement in a society with large population, introducing a public randomization is not wholly satisfactory.

In this paper, we resolve the discrepancy of expectations by allowing agents to communicate locally and construct a sequential equilibrium which is stable to mistakes and supports cooperation. In the model we analyze here, each agent can communicate with his neighbors by sending a message without cost. The message does not affect the payoffs directly. Indeed, the role of local communication is to enable agent to inform his neighbors of the period when he starts a new defection phase. Thus, all his neighbors have the same expectations on the period when he will turn back to cooperation after playing defection in finite periods. The length of defection phase is related to the severity of punishment and determined for the strategy to be an equilibrium.

The sequential equilibrium we construct in this paper has the following features. In each period, each agent forms an expectation on his neighbors' actions based on past history. That is, if there is a surprise between linked agents in the previous period, then each agent expects that the other starts a new defection phase of finite periods. For an agent i, if his neighbors' expectations on his action agree, then he follows their expectations. If some of neighbors expect that agent i starts a new defection phase, agent i starts a new defection phase and informs all his neighbors of it. Under this strategy, cooperation will be recovered from any history in contiguous periods among the directly connected agents. That is, an agent recovers cooperation in some period, then all his neighbors play cooperation in the next period, and so on.

The related literatures for this paper is about the relationship between efficiency and equilibrium. It is well known as the Folk Theorem that, in repeated games, an efficient and individually rational outcome can be obtained as an equilibrium. The earliest work on this issue is Friedman (1971) who showed that, in a infinitely repeated game, any outcome that Pareto dominates a Nash equilibrium in a stage game can be supported in a perfect equilibrium for sufficiently patient agents. Aumann and Shapley (1976) and Fudenberg and Maskin (1986) extend this result to the feasible and individually rational outcome. These studies consider infinitely repeated games. Benoit and Krishna (1985) explore finitely repeated games and get similar results to Fudenberg and Maskin (1986) when stage games are repeated in a sufficiently large number of periods.

All the above studies assume that monitoring is perfect, so that if an agent deviates form supposed actions, all other agents can punish him immediately.[4] However, it is possible that agents do not have complete information on the past actions but receive random signals. Green and Porter (1984), Fudenberg et al. (1994), Mailath and Morris

[4]Fudenberg and Maskin (1986) consider the incomplete information games as well as complete information games.

(2002) and Kandori (2003), explore the situation where agents cannot observe the actions of other agents but can observe a public random signal. Sekiguchi (1997), Kandori and Matsushima (1998), Bhaskar and Obara (2002), Ely and Välimäki (2002), Horner and Olszewski (2006), and Obara (2007) study the situation where each agent can observe a signal which is private information and whose distribution depends on the past actions. Almost of these studies verify that almost efficiency can be obtained as an equilibrium when signals have enough information on agents' actions. Kandori and Matsushima (1998), Kandori (2003), and Obara (2007) consider also the role of communication, which means each agent can send a message to all agents.

There can be other environments where monitoring is not perfect so that immediate punishment is not possible. Kandori (1992b) and Ellison (1994) explore anonymous random matching model in which agents are matched randomly in each period and agents cannot observe the actions taken by agents in other matchings. Kandori (1992b) showed that a contagious strategy can be a sequential equilibrium which supports an efficient outcome. The contagious equilibrium is not stable to mistakes. Ellison (1994) considers a repeated prisoner's dilemma game with public random device and shows that there is a sequential equilibrium which supports cooperation and is stable to mistakes.

The overlapping generation environment is another one where immediate punishment is not possible since old generation will die in the next period. With the overlapping generation model, Kandori (1992a) shows that almost efficiency can be obtained as an equilibrium if overlapping periods are sufficiently long. Bhaskar (1998) considers the prisoner's dilemma game between young and old generation and shows that efficient payoffs can be obtained as an equilibrium in mix strategies.

The literatures which share the environment with this paper are Ben-Porath and Kahneman (1996) and Xue (2004) in the sense that, under a network, an agent can observe the action of other agents who are directly connected to him. Ben-Porath and Kahneman (1996) allow agents to send a message about their observation to all agents, and show that if each agent has at least two neighbors, efficient outcome can be supported as an equilibrium. Xue (2004) considers a repeated prisoner's dilemma games under a line-shaped network, and construct a sequential equilibrium in which cooperation is supported and recovered from any history. Although the equilibrium strategy in Xue (2004) is interesting, it has an undesirable feature that it is complicated and difficult to implement.

The remainder of the paper is organized as follows. In Section 2, we explain the environment and solution concept. In Section 3, we construct a strategy σ^* in which cooperation is recovered in finite periods from any history. In Section 4, we construct a belief system which is consistent with σ^*. In Section 5, we show that the strategy σ^* is a sequential equilibrium with the belief system. Some discussions follow in Section 6, and we conclude in Section 7.

2 The Model

There is a finite set $N = \{1, \ldots, n\}$ of agents who live in infinite periods. Agents are connected by an undirected *network* G, which is a collection of *links* $ij \equiv \{i, j\} \subset N$. We assume that G is *minimally connected*. That is, G satisfies that, for all $i \in N$ and $j \in N$ with $i \neq j$, there is a unique subset $\{i_1, i_2, \ldots, i_L\}$ of N satisfying $i_1 = i$ and $i_L = j$ and $i_l i_{l+1} \in G$ for $l = 1, \ldots, L-1$. We call such a subset $\{i_1, i_2, \ldots, i_L\}$ a *chain between* i *and* j and write as $i \leftrightarrow j$.[5] For each agent i, we define a *distance of* j *from* i, denoted $d(j; i)$, by the number of links which consist of agents in $i \leftrightarrow j$.

If $ij \in G$, then agent j (*resp.* agent i) is said to be a *neighbor* of i (*resp.* j).[6] For each agent i, let G_i denote the set of agent i's neighbors. That is, $G_i = \{j \in N : ij \in G\}$ and let $\overline{G}_i = G_i \cup \{i\}$. Since G is undirected, $ij \in G$ is equivalent to $ji \in G$, and $j \in G_i$ if and only if $i \in G_j$. Agent i is an *end agent*, if he has only one neighbor. Thus, if agent i is an end agent, then G_i is a singleton set. Note that since G is minimally connected, there are at most $n-1$ end agents which is obtained in a star-shaped network, and at least two end agents which is obtained in a line-shaped network.

In each period $t \in \{1, \ldots\}$, agent i plays a *prisoner's dilemma game with communication* against his neighbors. That is, in each period t, agent i chooses $\tilde{a}_i^t \in \{C, D\}$ which generates the payoffs of a prisoner's dilemma game. C and D represent the *cooperation* and *defection*, respectively. In addition, agent i can communicate with his neighbors by sending a message $\tilde{m}_i^t \in \{0, 1\}$ which does not affect the payoffs directly. Then, we can let $a_i^t \in \{C_0, C_1, D_0, D_1\}$ be agent i's action in period t, where $a_i^t = C_{\tilde{m}}$ (*resp.* $a_i^t = D_{\tilde{m}}$) means that agent i plays C (*resp.* plays D) and sends a message $\tilde{m} \in \{0, 1\}$ in period t. [7] The payoffs of prisoner's dilemma game with communication between i and j are given as in Table 1. Here, $g > 0$ and $l > 0$. We assume that l and g are so small that, for all $i \in N$,

$$l(|G_i| - 1) < 1 \text{ and } g(|G_i| - 1) < 1.^8 \tag{1}$$

[5] In some papers such as Jackson and Wolinsky (1996) and Bala and Goyal (2000), a subset $\{i_1 i_2, i_2 i_3, \ldots, i_{L-1} i_L\}$ of G where $\{i_1, i_2, \ldots, i_L\}$ is a chain between i and j is called a *path* in G connecting i and j. We avoid this definition to escape from the confusion with σ-*path* we will define later.

[6] In some papers such as Hojman and Szeidl (2006), a neighbor refers to an indirectly connected agent as well as a directly connected agent.

[7] Indeed, considering three actions (C, D_0, and D_1) for each agent in each period is enough to construct a sequential equilibrium which supports cooperation and in which cooperation is recovered in finite periods from any history. That means, an agent are allowed to send a message only when he plays defection. In this paper, we consider four actions (C_0, C_1, D_0, and D_1) to make choice on messages independent of choice on cooperation and defection.

[8] For a set A, $|A|$ denotes the number of elements in A.

$i \setminus j$	C_0, C_1	D_0, D_1
C_0, C_1	$1, 1$	$-l, 1+g$
D_0, D_1	$1+g, -l$	$0, 0$

Table 1: Payoffs in prisoner's dilemma game with communication

Note that (1) implies $g - l < 1$ which guarantees that all agents playing C is the efficient outcome. Each agent has to take the same action against his neighbors. We simplify the notation by letting $\boldsymbol{a}_i = (a_i^t)_{t=1}^{\infty}$, $a_K^t = (a_j^t)_{j \in K}$, and $\boldsymbol{a}_K = (a_K^t)_{t=1}^{\infty}$ for $K \subset N$.

Let $w(a, a')$ be agent i's payoff in prisoner's dilemma game with communication against $j \in G_i$ when i plays a and j plays a'. That is, $w(C, C) = 1$, $w(C, D) = -l$, $w(D, C) = 1+g$ and $w(D, D) = 0$ where $C \in \{C_0, C_1\}$ and $D \in \{D_0, D_1\}$. The stage game payoff of agent i in period t, when agents play a_N^t, is the sum of his payoffs in prisoner's dilemma games with communication against his neighbors:[9]

$$u_i(a_N^t) = \sum_{j \in G_i} w(a_i^t, a_j^t).$$

The payoff of agent i in the repeated prisoner's dilemma game with communication, when $\boldsymbol{a}_N$ is played, is the average of discounted stage game payoffs:

$$U_i(\boldsymbol{a}_N) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_N^t) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \sum_{j \in G_i} w(a_i^t, a_j^t),$$

where $\delta \in (0, 1)$ is a common discount factor. Note that the stage game payoff of agent i in period t depends only on actions $a_{\overline{G}_i}^t$ of himself and his neighbors and so the payoff of the repeated game depends only on $\boldsymbol{a}_{\overline{G}_i}$.

A *history* h^t in period t is a profile of actions played before period t. That is, $h^1 = \varnothing$ and $h^t = (a_N^s)_{s=1}^{t-1}$ for $t \geq 2$. Let H^t be the set of all histories h^t in period t. In our model, each agent i can observe only the actions played by himself and his neighbors. Thus, for $t \geq 2$, histories $h^t = (a_N^s)_{s=1}^{t-1}$ and $\hat{h}^t = (\hat{a}_N^s)_{s=1}^{t-1}$ are in the same information set o_i^t of agent i if and only if $(a_{\overline{G}_i}^s)_{s=1}^{t-1} = (\hat{a}_{\overline{G}_i}^s)_{s=1}^{t-1}$.[10] With a slight abuse of notation, we write $o_i^1 = \varnothing$

[9] We may want to normalize agent i's stage game payoff by letting

$$u_i(a_N^t) = \frac{1}{|G_i|} \sum_{j \in G_i} w(a_i^t, a_j^t).$$

This normalization does not affect the result.

[10] In Kandori and Matsushima (1998) and Xue (2004), a *(joint or global) history* refers to a history h^t and a *private history* of agent i refers to an information set o_i^t.

and $o_i^t = (a_{\overline{G}_i}^s)_{s=1}^{t-1}$ for $t \geq 2$ for an information set o_i^t of agent i in period t. Since agents are finite and actions in each stage game are finite, each information set o_i^t has finite histories. We write $o_i^t(h^t)$ for the information set of agent i which history h^t belongs to. Let O_i^t be the set of agent i's information sets in period t.

We restrict our attention to pure strategies. A *strategy* of agent i is a function $\sigma_i : \bigcup_{t=1}^{\infty} O_i^t \to \{C_0, C_1, D_0, D_1\}$. Under σ_i, agent i chooses action $\sigma_i(o_i^t)$ in period t when he observes o_i^t. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$. Let Σ_i be the set of all strategies of agent i and $\Sigma = \times_{i \in N} \Sigma_i$. Given a strategy σ, a *σ-path conditioning on* h^t, denoted $\boldsymbol{\alpha}_N(\sigma; h^t) = ((\alpha_i^s(\sigma; h^t))_{s=1}^{\infty})_{i \in N}$, is the string of actions which agents actually play under the strategy σ, given that h^t is reached. Formally, $\boldsymbol{\alpha}_N(\sigma; h^t)$ is defined as follows. Let $h^t = (a_N^s)_{s=1}^{t-1}$ with $t \geq 2$ (*or,* $h^t = \varnothing$ for $t = 1$) be a history and σ be a strategy. Consider an agent i. Then, $\alpha_i^s(\sigma; h^t) = a_i^s$ for $s \leq t-1$, and $\alpha_i^t(\sigma; h^t) = \sigma_i(o_i^t(h^t))$ for $s = t$. For $s \geq t+1$, $\alpha_i^s(\sigma; h^t)$ is determined iteratively as $\alpha_i^s(\sigma; h^t) = \sigma_i(o_i^s)$ where $o_i^t = o_i^t(h^t)$ and $o_i^s = (o_i^{s-1}, (\alpha_j^{s-1}(\sigma; h^t))_{j \in \overline{G}_i})$.

In the paper, we are interested in a sequential equilibrium. A *belief system* μ is a function which assigns each information set to a probability distribution on the histories in the information set. We denote $\mu(\cdot; o_i^t)$ as a distribution on o_i^t which μ assigns to o_i^t. Note that, since o_i^t has finite elements, $h^t \in \text{supp}(\mu(\cdot; o_i^t))$ if and only if $\mu(h^t; o_i^t) > 0$ and $\mu(h^t; o_i^t)$ with $h^t \in o_i^t$ is a probability of h^t when o_i^t is reached.[11]

A belief system μ is *consistent* with σ, if it is the limit of a sequence of belief systems which are generated by Bayesian updating of fully mixed behavioral strategies converging to σ.[12] A strategy σ is a *sequential equilibrium* if, for some belief system μ which is consistent with σ, it satisfies that: for each i and for each o_i^t,

$$\sum_{h^t \in o_i^t} \mu(h^t; o_i^t) U_i(\boldsymbol{\alpha}_N(\sigma; h^t)) \geq \sum_{h^t \in o_i^t} \mu(h^t; o_i^t) U_i(\boldsymbol{\alpha}_N(\sigma_i', \sigma_{-i}; h^t)) \text{ for all } \sigma_i' \in \Sigma_i. \quad (2)$$

If a strategy σ satisfies (2) for some μ, then it is said to be *sequentially rational* under μ.[13]

Given an information set o_i^t and a strategy σ, we define a *continuation payoff* CU_i of

[11] A pair of belief system μ and a strategy σ is called an *assessment*.

[12] A sequence $\{\mu_n\}_{n=1}^{\infty}$ of belief systems *converges* to μ if, for each o_i^t, $\mu_n(h^t; o_i^t) \to \mu(h^t; o_i^t)$ for all $h^t \in o_i^t$. A fully mixed behavioral strategy β_i of agent i is a function which assigns each information set o_i^t to a distribution $\beta_i(\cdot; o_i^t)$ on $\{C_0, C_1, D_0, D_1\}$ where $\beta_i(a; o_i^t) > 0$ for all $a \in \{C_0, C_1, D_0, D_1\}$. A sequence $\{\beta_{in}\}_{n=1}^{\infty}$ of fully mixed behavioral strategies of agent i converges to β_i, if for each o_i^t, $\beta_i(a; o_i^t) \to \beta_i(a; o_i^t)$ for all $a \in \{C_0, C_1, D_0, D_1\}$. If there is some $a_i^t \in \{C_0, C_1, D_0, D_1\}$ for each o_i^t such that $\beta_i(a_i^t; o_i^t) = 1$, then the behavioral strategy β_i is equivalent to the pure strategy σ_i such that $\sigma_i(o_i^t) = a_i^t$ for each o_i^t.

[13] Although Kreps and Wilson (1982) define a sequential equilibrium for finite extensive form games, we can extend their definition to infinite extensive games without any conceptual innovation. Many previous studies, such as Kandori (1992b), Sekiguchi (1997), and Xue (2004), adopt sequential equilibrium as a solution concept for infinite extensive form games.

agent i at o_i^t by

$$CU_i(\sigma; o_i^t) = \sum_{h^t \in o_i^t} \mu(h^t; o_i^t) \left[(1-\delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \sum_{j \in G_i} w(\alpha_i^\tau(\sigma; h^t), \alpha_j^\tau(\sigma; h^t)) \right].$$

Let $o_i^t = (a_i^s, a_{G_i}^s)_{s=1}^{t-1}$. Since, for all $h^t \in o_i^t$ and for all $\sigma \in \Sigma$,

$$\begin{aligned} &\sum_{h^t \in o_i^t} \mu(h^t; o_i^t) U_i(\boldsymbol{\alpha}_N(\sigma; h^t)) \\ = &\sum_{h^t \in o_i^t} \mu(h^t; o_i^t) \left[(1-\delta) \sum_{\tau=1}^{t-1} \delta^{\tau-1} \sum_{j \in G_i} w(a_i^\tau, a_j^\tau) \right] + \delta^{t-1} CU_i(\sigma; h^t), \end{aligned}$$

(2) holds if and only if $CU_i(\sigma; h^t) \geq CU_i(\sigma_i', \sigma_{-i}; h^t)$ for all $\sigma_i' \in \Sigma_i$.

3 Strategy σ^*

In this section, we define the phase for each information set, and then construct a strategy σ^* in which action at each information set o_i^t depends on the phase of o_i^t.

A *phase* of information set o_i^t is represented as

$$P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i},$$

where $\lambda_{ki}^t, \lambda_{ik}^t \in \{0, 1, \ldots, \Lambda\}$. We will determine Λ in Section 5 for which the strategy σ^* we construct in this section is a sequential equilibrium in the repeated prisoner's dilemma game with communication. Indeed, Λ is the length of periods when an agent plays defection to punish a deviator, and so it determines the strength of punishment for deviation.

To define a phase $P(o_i^t)$ for each o_i^t, we first define an *expectation function* $E : \{0, \ldots, \Lambda\} \to \{\{C_0, C_1\}, \{D_0\}, \{D_1\}\}$ by

$$E(\lambda) = \begin{cases} \{C_0, C_1\} & \text{if } \lambda = 0 \\ \{D_0\} & \text{if } \lambda = 1, \ldots, \Lambda - 1 \\ \{D_1\} & \text{if } \lambda = \Lambda \end{cases}$$

Given an information set o_i^t, let $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$. We can interpret $E(\lambda_{ki}^t)$ (*resp.* $E(\lambda_{ik}^t)$) as agent k's (*resp.* agent i's) expectation on agent i's action (*resp.* agent k's action) in period t. For example, if $E(\lambda_{ki}^t) = \{C_0, C_1\}$ for $k \in G_i$, then agent k expects that agent i plays C_0 or C_1 in period t. Furthermore, λ_{ki}^t (*resp.* λ_{ik}^t) can be interpreted as agent i's (*resp.* agent k's) expectation on how long agent k (*resp.* agent i) keeps playing D (D_1 or D_0) after period t (including period t). For each i and j with $ij \in G$, if $\lambda_{ij}^t \neq 0$, then agent j is said to be in *defection phase* under i's expectation, and if $\lambda_{ij}^t = 0$, then agent j is said to be in *cooperation phase* under i's expectation.

In period 1, $P(o_i^1)$ of agent i satisfies

$$(\lambda_{ki}^1, \lambda_{ik}^1) = (0, 0) \text{ for all } k \in G_i.$$

For each $t \geq 2$, let $o_i^t = (a_{\overline{G}_i}^s)_{s=1}^t$. The phase $P(o_i^t)$ of o_i^t is defined iteratively as follows. Let $P(o_i^{t-1}) = (\lambda_{ki}^{t-1}, \lambda_{ik}^{t-1})_{k \in G_i}$ be the phase for the information set $o_i^{t-1} = (a_{\overline{G}_i}^s)_{s=1}^{t-2}$ in which agent i observes the same actions as in o_i^t.

($P1$) In a case that $a_i^{t-1} \in E(\lambda_{ki}^{t-1})$ and $a_k^{t-1} \in E(\lambda_{ik}^{t-1})$,

if $\lambda_{ki}^{t-1} \in \{0, 1\}$, then $\lambda_{ki}^t = 0$
if $\lambda_{ki}^{t-1} \in \{2, \ldots, \Lambda\}$, then $\lambda_{ki}^t = \lambda_{ki}^{t-1} - 1$

and

if $\lambda_{ik}^{t-1} \in \{0, 1\}$, then $\lambda_{ik}^t = 0$
if $\lambda_{ik}^{t-1} \in \{2, \ldots, \Lambda\}$, then $\lambda_{ik}^t = \lambda_{ik}^{t-1} - 1$

($P2$) In a case that $a_i^{t-1} \notin E(\lambda_{ki}^{t-1})$ and $a_k^{t-1} \in E(\lambda_{ik}^{t-1})$,

if $a_i^{t-1} \neq D_1$, then $(\lambda_{ki}^t, \lambda_{ik}^t) = (\Lambda, \Lambda)$
if $a_i^{t-1} = D_1$, then $(\lambda_{ki}^t, \lambda_{ik}^t) = (\Lambda - 1, \Lambda)$

($P3$) In a case that $a_i^{t-1} \in E(\lambda_{ki}^{t-1})$ and $a_k^{t-1} \notin E(\lambda_{ik}^{t-1})$,

if $a_k^{t-1} \neq D_1$, then $(\lambda_{ki}^t, \lambda_{ik}^t) = (\Lambda, \Lambda)$
if $a_k^{t-1} = D_1$, then $(\lambda_{ki}^t, \lambda_{ik}^t) = (\Lambda, \Lambda - 1)$

($P4$) In a case that $a_i^{t-1} \notin E(\lambda_{ki}^{t-1})$ and $a_k^{t-1} \notin E(\lambda_{ik}^{t-1})$,

$(\lambda_{ki}^t, \lambda_{ik}^t) = (\Lambda, \Lambda)$

Given a history $h^t = (a_N^s)_{s=1}^{t-1}$, if $a_k^s \notin E(\lambda_{ik}^s)$ (*resp.* $a_i^s \notin E(\lambda_{ki}^s)$), we say that agent k (*resp.* agent i) *surprises* agent i (*resp.* agent k) in period s. Furthermore, if $a_k^s \notin E(\lambda_{ik}^s)$, then we call a_k^s a *surprise* to agent i by agent k.

In ($P1$), there is no surprise between i and k in period $t-1$. In this case, suppose that agents i and k do not surprise each other after period t. If $\lambda_{ik}^t \neq 0$, which means k is supposed to play D in period t, then agent i expects that agent k will keep playing

D for λ^t_{ik} periods and play C thereafter. If $\lambda^t_{ik} = 0$, which means agent k is supposed to play C, then agent i expects that k will play C forever.

In $(P2)$, agent i makes agent k surprised but agent k does not make agent i surprised in period $t-1$. In this case, if there is no other surprise between i and j in the future, then agent i expects that agent k will play D for Λ periods (D_1 in period t and D_0 for following $\Lambda - 1$ periods). Furthermore, if agent i played D_1 in period $t-1$ then agent k expects that agent i is in defection phase for $\Lambda - 1$ periods, and if agent i played D_0 in period $t-1$ then agent k expects that agent i starts a defection phase in period t. In $(P3)$, we just change the roles of agents i and k in $(P2)$.

In $(P4)$, agents i and k surprise each other in period $t-1$. In this case, if there is no other surprise between i and k in the future, each of them expects that the other agent plays D for Λ periods (D_1 in period t and D_0 for following $\Lambda - 1$ periods) and C thereafter.

Note that $(\lambda^t_{ki}, \lambda^t_{ki})$ depends only on i's and k's past actions. So, for each i and for each period t, $(\lambda^t_{ki}, \lambda^t_{ki})_{k \in G_i}$ depends only on $(a^s_{\overline{G}_i})^{t-1}_{s=1}$, so $P(o^t_i)$ is well defined for each o^t_i.

From the construction of P, it is not difficult to see that, for any o^t_i, $P(o^t_i) = (\lambda^t_{ki}, \lambda^t_{ik})_{k \in G_i}$ satisfies that, for each $k \in G_i$,

$$\lambda^t_{ik} - \lambda^t_{ki} \in \{-1, 0, 1\}. \tag{3}$$

Also, Lemma 1 provides another property of P which is used in constructing σ^*.

Lemma 1. *Let $t \geq 2$. For an information set o^t_i, let $P(o^t_i) = (\lambda^t_{ki}, \lambda^t_{ik})_{k \in G_i}$. If $\lambda^t_{ki} \neq \Lambda$ and $\lambda^t_{k'i} \neq \Lambda$ for $k \in G_i$ and $k' \in G_i$, then*

$$\lambda^t_{ki} = \lambda^t_{k'i}.$$

Proof. Let $o^t_i = (a^s_{\overline{G}_i})^t_{s=1}$ with $t \geq 2$. For $s < t$, let $P^s_i(o^s_i) = (\lambda^s_{ki}, \lambda^s_{ik})_{k \in G_i}$ where o^s_i is the information which is consistent with o^t_i. Let $\lambda^t_{ki} \neq \Lambda$ and $\lambda^t_{k'i} \neq \Lambda$ with $k \in G_i$ and $k' \in G_i$ and let $j \in \{k, k'\}$.

Suppose that $\{s : a^s_i = D_1,\ s < t\} = \varnothing$. Suppose in addition that $a^s_j \neq E(\lambda^s_{ij})$ for some $s < t$. By the construction of P, $\lambda^{s+1}_{ji} = \Lambda$. Since $a^{s+1}_i \notin \{D_1\} = E(\Lambda) = E(\lambda^{s+1}_{ji})$, we have $\lambda^{s+2}_{ji} = \Lambda$. Then, since $a^{s+2}_i \notin \{D_1\} = E(\Lambda) = E(\lambda^{s+2}_{ji})$, we have $\lambda^{s+3}_{ji} = \Lambda$. Continuing this procedure until $s + \tau = t$, we have $\lambda^t_{ji} = \Lambda$, which is a contradiction. Thus, $a^s_j \in E(\lambda^s_{ij})$ for all $s < t$. Suppose that $a^s_i \notin E(\lambda^s_{ji})$ for some $s < t$. Since $a^s_i \neq D_1$ and $a^s_i \notin E(\lambda^s_{ji})$, we have $\lambda^{s+1}_{ji} = \Lambda$. Then, since $a^{s+1}_i \notin \{D_1\} = E(\Lambda) = E(\lambda^{s+1}_{ji})$, we have $\lambda^{s+2}_{ji} = \Lambda$. Continuing this procedure until $s + \tau = t$ leads us to $\lambda^t_{ji} = \Lambda$ which is a contradiction. Thus, $a^s_i = E(\lambda^s_{ji})$ for all $s < t$. Since $a^s_i = E(\lambda^s_{ji})$ and $a^s_j = E(\lambda^s_{ji})$ for all $s < t$, by construction of P, we have $\lambda^t_{ji} = 0$.

Suppose that $\{s : a_i^s = D_1,\ s < t\} \neq \varnothing$. Let $\bar{s} = \max\{s : a_i^s = D_1,\ s < t\}$. If $a_j^s \notin E(\lambda_{ij}^s)$ for some s with $\bar{s} \le s < t$, then $\lambda_{ji}^{s+1} = \Lambda$. Since $a_i^{s+1} \notin \{D_1\} = E(\Lambda) = E(\lambda_{ji}^{s+1})$, we have $\lambda_{ji}^{s+2} = \Lambda$. Then, since $a_i^{s+2} \notin \{D_1\} = E(\Lambda) = E(\lambda_{ji}^{s+2})$, we have $\lambda_{ji}^{s+3} = \Lambda$. Continuing this procedure until $s+\tau = t$, we have $\lambda_{ji}^t = \Lambda$ which is a contradiction. Thus, $a_j^s \in E(\lambda_{ij}^s)$ for all s with $\bar{s} \le s < t$. If $a_i^s \notin E(\lambda_{ji}^s)$ for some s with $\bar{s} < s < t$, then $a_i^s \neq D_1$ and $a_i^s \notin E(\lambda_{ji}^s)$, which imply $\lambda_{ji}^{s+1} = \Lambda$. Then, since $a_i^{s+1} \notin \{D_1\} = E(\Lambda) = E(\lambda_{ji}^{s+1})$, we have $\lambda_{ji}^{s+2} = \Lambda$. Continuing this procedure until $s + \tau = t$, we have $\lambda_{ji}^t = \Lambda$ which is a contradiction. Thus, $a_i^s \in E(\lambda_{ji}^s)$ for all s with $\bar{s} < s < t$. Furthermore, $a_j^{\bar{s}} \in E(\lambda_{ij}^{\bar{s}})$ and $a_i^{\bar{s}} = D_1$ imply that $\lambda_{ji}^{\bar{s}+1} = \Lambda - 1$. Since $a_i^s \in E(\lambda_{ji}^s)$ and $a_j^s \in E(\lambda_{ij}^s)$ for all s with $\bar{s} < s < t$, by construction of P, we have $\lambda_{ji}^t = \max\{\Lambda - (t - \bar{s}), 0\}$ for $j \in \{k, k'\}$. ∎

Now, we are ready to define the strategy profile $\sigma^* = (\sigma_i^*)_{i \in N}$. Consider an agent i in an information set o_i^t with $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$. Lemma 1 implies that $P(o_i^t)$ satisfies one of the followings:

($S1$) for some $k \in G_i$, $\lambda_{ki}^t = \Lambda$.

($S2$) for all $k \in G_i$, $\lambda_{ki}^t = \lambda$ for some $\lambda \in \{0, 1, \ldots, \Lambda - 1\}$.

That means, agents i faces with only two situations: the situation ($S1$) where at least one neighbor expects that agent i plays D_1, or the situation ($S2$) where all his neighbors have the same expectation on his action.

The strategy σ_i^* of agent i is defined as follows: for each information set o_i^t,

- when $P(o_i^t)$ satisfies ($S1$), agent i plays D_1. That is, $\sigma^*(o_i^t) = D_1$.
- when $P(o_i^t)$ satisfies ($S2$), agent i plays D_0 if $E(\lambda) = \{D_0\}$, and C_0 if $E(\lambda) = \{C_0, C_1\}$. That is, $\sigma^*(o_i^t) = D_0$ if $E(\lambda) = \{D_0\}$, and $\sigma^*(o_i^t) = C_0$ if $E(\lambda) = \{C_0, C_1\}$.

In other words, agent i employing σ_i^* chooses D_1 if there is a neighbor who expects that agent i plays D_1, and follows his neighbors' expectation if his neighbors have the same expectation on agent i's action.

Let agents employ σ^* and consider an information set o_i^t with $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$. Suppose that $\lambda_{ki}^t = \Lambda$ for some $k \in G_i$, and $a_k^s \in E(\lambda_{ik}^s)$ and $a_i^s \in E(\lambda_{ki}^s)$ for all $k \in G_i$ and for all $s \ge t$. That is, there is an agent $k \in G_i$ who expects that agent i plays D_1 in period t and there is no surprise between i and k for all $k \in G_i$ after period t. Then, agent i plays D_1 in period t, D_0 in periods $t+1, \ldots, t+\Lambda-1$, and C_0 thereafter. If $\lambda_{ki}^t = \lambda \in \{1, \ldots, \Lambda-1\}$ for all $k \in G_i$ and there is no surprise between i and $k \in G_i$ after period t, then agent i plays D_0 in periods $t, \ldots, t + \lambda - 1$, and C_0 thereafter. Figure 2 provides examples of σ^*-path conditioning on history h^2 under a line-shaped network. In Figure 2 (a),

t	1	2	3	4	5	6
1	C_0	C_0	D_1	C_0	C_0	C_0
2	C_0	D_1	D_0	D_1	C_0	C_0
3	D_1	D_0	D_0	D_0	D_1	C_0
4	D_0	D_0	D_0	D_0	D_0	D_1
5	D_0	D_0	D_0	D_0	D_0	D_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Λ	D_0	D_0	D_0	D_0	D_0	D_0
$\Lambda+1$	D_0	D_0	C_0	D_0	D_0	D_0
$\Lambda+2$	D_0	C_0	C_0	C_0	D_0	D_0
$\Lambda+3$	C_0	C_0	C_0	C_0	C_0	D_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(a)

t	1	2	3	4	5	6	
1	C_0	C_0	D_0	C_0	C_0	C_0	h^2
2	C_0	D_1	D_1	D_1	C_0	C_0	
3	D_1	D_0	D_0	D_0	D_1	C_0	
4	D_0	D_0	D_0	D_0	D_0	D_1	
5	D_0	D_0	D_0	D_0	D_0	D_0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Λ	D_0	D_0	D_0	D_0	D_0	D_0	
$\Lambda+1$	D_0	D_0	D_0	D_0	D_0	D_0	
$\Lambda+2$	D_0	C_0	C_0	C_0	D_0	D_0	
$\Lambda+3$	C_0	C_0	C_0	C_0	C_0	D_0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

(b)

Figure 2: σ^*-path conditioning on history h^2

agent 3 surprises agents 2 and 4 by playing D_1 in period 1, so $(\lambda^2_{23}, \lambda^2_{32}) = (\Lambda - 1, \Lambda)$ and $(\lambda^2_{43}, \lambda^2_{34}) = (\Lambda - 1, \Lambda)$. Thus, agent 3 plays D_0 and agents 2 and 4 chooses D_1 in period 2. In Figure 2 (b), agent 3 surprises agents 2 and 4 by playing D_0 in period 1, so $(\lambda'^2_{23}, \lambda'^2_{32}) = (\Lambda, \Lambda)$ and $(\lambda'^2_{43}, \lambda'^2_{34}) = (\Lambda, \Lambda)$. Thus, agents 2, 3, and 4 play D_1 in period 2

Note that if agent i is an end agent with $G_i = \{k\}$, then $\sigma^*_i(o^t_i) = E(\lambda^t_{ki})$ for any o^t_i. In other words, an end agent i employing σ^* will not surprise his neighbor at any history. Also, for any history h^t, C_1 is never played in period $s \geq t$ along the σ^*-path conditioning on h^t.

Lemma 2 states that, under σ^*, C_0 is recovered in finite periods from any history. To prove Lemma 2, we need to define h^{*s} for each $s \geq 1$ which is the history in period s when σ^* is played given that h^t is realized. Given a history $h^t = (a^\tau_N)^{t-1}_{\tau=1}$ and the strategy σ^*, let

$$h^{*1} = \varnothing, \quad \text{and} \quad h^{*s} = (\alpha^\tau_N(\sigma^*; h^t))^{s-1}_{\tau=1} \text{ for } s \geq 2. \tag{4}$$

Note that, since $h^{*s} = (a^\tau_N)^{s-1}_{\tau=1}$ for each s with $2 \leq s \leq t$, h^{*s} with $s \leq t$ does not depend on σ^*.

Lemma 2. *Given the strategy σ^*, for any h^t, there exists $\bar{\tau} \geq 0$ such that for all $i \in N$, $\alpha^{t+\tau}_i(\sigma^*, h^t) = C_0$ for all $\tau \geq \bar{\tau}$.*

Proof. Fix a history $h^t = (a^s_N)^{t-1}_{s=1} \in o^t_i$. For each $s \geq 1$, let $P(o^s_i(h^{*s})) = (\lambda^s_{ki}, \lambda^s_{ik})_{k \in G_i}$, where h^{*s} is defined as in (4). Let $\kappa_i(0) = \{i\}$ and $\kappa_i(x) = \{k \in N \backslash \{i\} : d(k; i) = x\}$

for $x = 1, \ldots, \bar{M}$ where $\bar{M} = \max_{i \in N}\{\max\{d(k;i) : k \in N\backslash\{i\}\}\}$. Thus, $\kappa_i(x)$ is the set of agents who has distance x from i. Since G is minimally connected, $\bar{M} \geq 1$ and N is partitioned into $\kappa_i(0), \ldots, \kappa_i(\bar{M})$.

Consider agents i and k with $ik \in G$. We first show that there is no surprise between i and k after $t + \bar{M}$. Since $ik \in G$ is arbitrary, there is no surprise between i and his neighbors after $t + \bar{M}$. Then, by the construction of σ^*, agent i will play C_0 after period $t + \bar{M} + \Lambda$. Since $\bar{M}$ and Λ do not depend on i and k, letting $\bar{\tau} = \bar{M} + \Lambda$, we will complete the proof.

If $\bar{M} = 1$, then there are only two agents in N and every agent $i \in N$ is an end agent. Thus, for each $\tau \geq 0$, $\sigma_k^*(o_k^{t+\tau}(h^{*t+\tau})) \in E(\lambda_{ik}^{t+\tau})$ for all $k \in G_i$. Let $\bar{M} \geq 2$. Suppose that there is a surprise to i by k in period $t + \bar{M} + \tau$ for some $\tau \geq 0$. That is, $a_k^{t+\bar{M}+\tau} \notin E(\lambda_{ik}^{t+\bar{M}+\tau})$. Since $a_k^{t+\bar{M}+\tau} = \sigma_k^*(o_i^{t+\bar{M}+\tau}(h^{t+\bar{M}+\tau}))$, $a_k^{t+\bar{M}+\tau} = D_1$ and $\lambda_{ik}^{t+\bar{M}+\tau} \neq \Lambda$. Thus, there is an agent $k_2 \in G_k$ such that $k_2 \neq i$ and $\lambda_{k_2k}^{t+\bar{M}+\tau} = \Lambda$. Notice that $k_2 \in \kappa_i(2)$. Then, in period $t + \bar{M} + \tau - 1$, we have either (i) agent k surprises k_2 by playing $a_k^{t+\bar{M}+\tau-1} \neq D_1$, or (ii) agent k_2 surprises k. Since (i) implies the contradiction that $\sigma_k^*(o_i^{t+\bar{M}+\tau-1}(h^{t+\bar{M}+\tau-1})) \neq D_1$ and $\sigma_k^*(o_i^{t+\bar{M}+\tau-1}(h^{t+\bar{M}+\tau-1})) \notin E(\lambda_{k_2k}^{t+\bar{M}+\tau-1})$, (i) cannot be the case. Thus, agent k_2 surprises k in period $t + \bar{M} + \tau - 1$. Then, from the same argument as before, there is an agent $k_3 \in G_{k_2}$ with $k_3 \in \kappa_i(3)$ who surprises k_2 in period $t + \bar{M} + \tau - 2$. Continuing this procedure, we have that $k_m \in \kappa_i(m)$ is an end agent so there is no agent $k_{m+1} \in G_{k_m}$ with $k_{m+1} \neq k_{m-1}$ who surprises k_m in period $t + \bar{M} + \tau - m - 1$. Thus, k_m does not play $\sigma_{k_m}^*(o_{k_m}^{t+\bar{M}+\tau-m})$ in period $t + \bar{M} + \tau - m$, which is a contradiction. Therefore, there is no surprise to i by k in period $t + \bar{M} + \tau$ for some $\tau \geq 0$. Similarly, we can show that there is no surprise to k by i in period $t + \bar{M} + \tau$ for some $\tau \geq 0$. Then, since $ik \in G$ is an arbitrary link, there is no surprise between i and his neighbors after period $t + \bar{M}$. Therefore, by construction of σ^*, agent i will play C_0 after period $t + \bar{M} + \Lambda$. Letting $\bar{\tau} = \bar{M} + \Lambda$, we complete the proof. ∎

Kandori (1992b) introduces *global stability* as a desirable property for equilibrium. An equilibrium is globally stable if, for any finite history h^t, the continuation expected payoffs of agents eventually return to the payoffs the equilibrium sustains. In our notion, a equilibrium strategy σ is globally stable if, for any h^t,

$$\lim_{s \to \infty} CU_i(\sigma; o_i^s(h^{*s})) = CU_i(\sigma; o_i^1(h^1)) \text{ for all } i \in N,$$

where h^{*s} is constructed as in (4) for h^t. Since $CU_i(\sigma^*; o^1(h^1)) = \sum_{j \in G_i} w(C, C)$, Lemma 2 obviously implies that the strategy σ^* is a globally stable equilibrium if σ^* is an equilibrium.

4 Belief system μ

In this section, we construct a belief system μ which is consistent with σ^* and provide a property of μ.

In history h^t, an action $a_k^\tau \in h^t$ is a *mistake* if $a_k^\tau \neq \sigma_k^*(o_k^\tau(h^{*\tau}))$ where h^{*s} is defined as in (4). Given the strategy σ^*, the number of mistakes in history $h^t = (a_N^\tau)_{\tau=1}^{t-1}$ is denoted as $\rho(h^t)$. That is,

$$\rho(h^t) = |\{a_k^s \in h^t : a_k^s \neq \sigma_k^*(o_k^s(h^{*s})),\ k \in N\}|.$$

Let μ_ε be a belief system which is generated by Bayesian updating from behavioral strategy which assigns $(1-3\varepsilon)$ to $\sigma_i^*(o_i^t)$ and ε to each of other actions at each information set o_i^t. Let μ be the limit of μ_ε when $\varepsilon \to 0$. Trivially, μ is consistent with σ^*.

For each information set o_i^t and history $h^t \in o_i^t$, we have

$$\mu_\varepsilon(h^t; o_i^t) = \frac{\varepsilon^{\rho(h^t)}(1-3\varepsilon)^{|h^t|-\rho(h^t)}}{\sum_{\hat{h}^t \in o_i^t} \mu_\varepsilon(\hat{h}^t; o_i^t)}.$$

Given an information set o_i^t, let $h^t \in o_i^t$ and $\hat{h}^t \in o_i^t$ satisfy $\rho(\hat{h}^t) < \rho(h^t)$. Then, since

$$\frac{\mu_\varepsilon(h^t; o_i^t)}{\mu_\varepsilon(\hat{h}^t; o_i^t)} = \frac{\varepsilon^{\rho(h^t)}(1-3\varepsilon)^{|h^t|-\rho(h^t)}}{\varepsilon^{\rho(\hat{h}^t)}(1-3\varepsilon)^{|\hat{h}^t|-\rho(\hat{h}^t)}} \to 0 = \frac{\mu(h^t; o_i^t)}{\mu(\hat{h}; o_i^t)} \text{ as } \varepsilon \to 0,$$

we have $\mu(h^t; o_i^t) = 0$. Therefore, to conclude that a history $h^t \in o_i^t$ does not belong to the support of $\mu(\cdot; o_i^t)$, denoted $\text{supp}(\mu(\cdot; o_i^t))$, it is enough to find another history $\hat{h}^t \in o_i^t$ which has the smaller number of mistakes than h^t. This argument will prove Lemma 3.

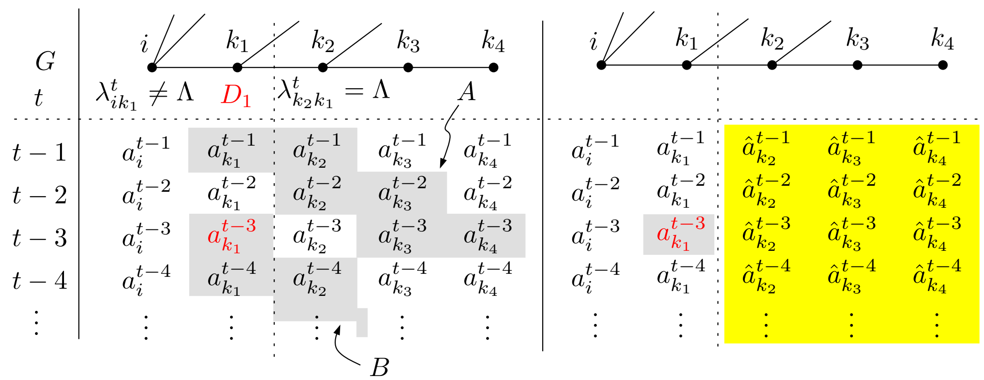
Lemma 3. *Consider an information set $o_i^t = (a_{\overline{G}_i}^s)_{s=1}^{t-1}$ and history $h^t = (a_N^s)_{s=1}^{t-1} \in o_i^t$. For each $s \geq 1$, let*

$$P(o_i^s(h^{*s})) = (\lambda_{ki}^s, \lambda_{ik}^s)_{k \in G_i},$$

where h^{*s} is defined as in (4). Suppose that $h^t \in \text{supp}(\mu(\cdot; o_i^t))$. Then, for each $\tau \geq 0$, $\sigma_k^*(o_k^{t+\tau}(h^{*t+\tau})) \in E(\lambda_{ik}^{t+\tau})$ for all $k \in G_i$.

The formal proof of Lemma 3 is found in the Appendix A. Here, we provide a sketch of proof.

Sketch of Proof. Suppose that, for some $\tau \geq 0$, $\sigma_{k_1}^*(o_{k_1}^{t+\tau}(h^{*t+\tau})) \notin E(\lambda_{ik_1}^{t+\tau})$ for some $k_1 \in G_i$. In Steps 1 and 2, we show that if there is a surprise $a_{k_1}^s$ to i by $k_1 \in G_i$ in period $s \geq 1$, then there is a mistake $a_{k_m}^{s'}$ where k_m is an agent who has distance m from i and $s' = s - m + 1$ or $s' = s - m$. For example, consider an agent i and a history h^t



(a) history h^t

(b) history $\hat{h}^t$

Figure 3: histories h^t and $\hat{h}^t$

described in Figure 3 (a). Suppose that $a^t_{k_1}$ is a surprise to agent i. Since it is not a mistake, $a^t_{k_1} = \sigma^*_k(o^t_i(h^t)) = D_1$ and $\lambda^t_{ik_1} \neq \Lambda$. Thus, there is an agent $k_2 \in G_{k_1}$ such that $k_2 \neq i$ and $\lambda^t_{k_2k_1} = \Lambda$. By the construction of P, there is a surprise between k_2 and k_1 in period $t-1$. Suppose that k_1 surprises k_2 in period $t-1$. Since $\lambda^t_{k_2k_1} = \Lambda$, we have $a^{t-1}_{k_1} \neq D_1$. Since $a^{t-1}_{k_1}$ is a surprise to agent i and $a^{t-1}_{k_1} \neq D_1$, by the construction of σ^*, $a^{t-1}_{k_1}$ is a mistake. Suppose that k_2 surprises k_1 in period $t-1$. Then, $a^{t-1}_{k_2}$ is a mistake, or there is an agent $k_3 \in G_{k_2}$ such that $k_3 \neq k_1$ and $\lambda_{k_3k_2} = \Lambda$. Continuing this procedure, it ends when $t-m = 1$ or k_m is an end agent. Therefore, for a surprise $a^t_{k_1}$ to agent i by k_1, there is a mistake which induces $a^t_{k_1}$ in shaded area A. In this case, we say that a surprise $a^t_{k_1}$ to agent i *is induced by the mistake* which we find. Similarly, if $a^{t-3}_{k_1}$ is a surprise to i by k_i, then there is a mistake which induces $a^{t-3}_{k_1}$ in shaded area B.

In Step 3, we show that a mistake can induce at most one surprise to agent i. Since there is no mistake in period $s \geq t$, mistakes in the history h^t are more than the actions which are agent i's mistakes, surprises to agent i by k, or C_1 played by $k \in G_i$.

In Step 4, we construct a history $\hat{h}^t = (\hat{a}^s_N)^{t-1}_{s=1}$ in which $(\hat{a}^s_k)^{t-1}_{s=1} = (a^s_k)^{t-1}_{s=1}$ for $k \in \overline{G}_i$ and $(\hat{a}^s_k)^{t-1}_{s=1} = (\sigma^*_k(o^s_k(\hat{h}^{*s})))^{t-1}_{s=1}$ for $k \notin \overline{G}_i$, which is described in Figure 3 (b). Trivially, $\hat{h}^t$ is in the same information set as h^t. Furthermore, surprises to agent i by $k \in G_i$ and $\hat{a}^s_k \in \hat{h}^t$ with $\hat{a}^s_k = C_1$ for $k \in G_i$ are mistakes. We also show that there is no other mistake in $\hat{h}^t$. Then, since an action of agent i in h^t is a mistake if and only if it is a mistake in $\hat{h}^t$, the number of mistakes in $\hat{h}^t$ is equal to the number of actions which are agent i's mistakes, surprises to i by $k \in G_i$, or C_1 played by $k \in G_i$. Therefore, the number of mistakes in $\hat{h}^t$ are smaller than that in h^t. The argument before Lemma 3 implies that $\mu(h^t; o^t_i) = 0$. ∎

Lemma 3 means that at any information set, agent i believes that none of his neighbors will surprise him in the future under σ^*. Thus, for each information set o_i^t, the future actions of his neighbors are uniquely determined from any history in $\text{supp}(\mu(\cdot; o_i^t))$, which makes it possible to calculate the continuation payoffs.

5 Sequential Equilibrium

In this section, we will show that σ^* is a sequential equilibrium with the belief system μ.

Given an information set o_i^t with $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$, define $K(\lambda, \lambda'; o_i^t)$ for each $\lambda, \lambda' \in \{0, 1, \ldots, \Lambda\}$ by

$$K(\lambda, \lambda'; o_i^t) = \{k \in G_i : \lambda_{ki}^t = \lambda, \lambda_{ik}^t = \lambda'\}.$$

If $k \in K(\lambda, \lambda'; o_i^t)$, then agent k expects that agent i will play $a_i^t \in E(\lambda)$ and agent i expects that agent k will play $a_k^t \in E(\lambda')$. For notational convenience, given an information set o_i^t, we denote $K_\lambda^{\lambda'}$ as $K(\lambda, \lambda'; o_i^t)$ if there is no confusion.

Given an information set o_i^t and the strategy σ_i^* of agent i, we denote $\sigma_i^*|_a^{o_i^t}$ as a strategy such that $\sigma_i^*|_a^{o_i^t}(o_i^t) = a$ and it agrees with σ_i^* at all other information sets. By the one deviation property of sequential equilibrium, to see that σ_i^* is a sequential equilibrium, it is enough to see that, for each o_i^t, $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ for all $a \in \{C_0, C_1, D_0, D_1\}$.

Because of (3) and Lemma 1, each information set o_i^t satisfies one of the following seven cases.

Case A. $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$ with $\lambda_{ki}^t = \Lambda$ for some $k \in G_i$.

- **Case 1.** G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_0^1, and K_0^0.
- **Case 2.** G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, $K_\lambda^{\lambda+1}$, K_λ^λ, and $K_\lambda^{\lambda-1}$, where $\lambda = 3, \ldots, \Lambda - 1$.
- **Case 3.** G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_2^3, K_2^2, and K_2^1.
- **Case 4.** G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_1^2, K_1^1, and K_1^0.

Case B. $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$ with $\lambda_{ki}^t = \lambda \neq \Lambda$ for all $k \in G_i$.

- **Case 5.** G_i is partitioned into K_0^1, and K_0^0.
- **Case 6.** G_i is partitioned into $K_\lambda^{\lambda+1}$, K_λ^λ, and $K_\lambda^{\lambda-1}$, where $\lambda \in \{2, \ldots, \Lambda - 1\}$.
- **Case 7.** G_i is partitioned into K_1^2, K_1^1, and K_1^0.

Note that, in Cases 1 $\sim$ 4, $K_\Lambda^\Lambda \cup K_\Lambda^{\Lambda-1} \neq \varnothing$. Also, it is possible that $K_0^1 \cup K_0^0 = \varnothing$ in Case 1, $K_\lambda^{\lambda+1} \cup K_\lambda^\lambda \cup K_\lambda^{\lambda-1} = \varnothing$ in Case 2, $K_2^3 \cup K_2^2 \cup K_2^1 = \varnothing$ in Case 3, and $K_1^2 \cup K_1^1 \cup K_1^0 = \varnothing$ in Case 4. Also, in Cases 5 $\sim$ 7, K_0^1, K_0^0, $K_\lambda^{\lambda+1}$, K_λ^λ, $K_\lambda^{\lambda-1}$, K_1^2, K_1^1, and K_1^0 can be empty

Under σ^* and μ, recall that, for any o_i^t, agent i believes that he will not be surprised by his neighbors in periods $s \geq t$. For example, consider Case 1. Since there is no surprise to agent i by his neighbors under σ^* and μ, agent i believes that agent $k \in K_\lambda^{\lambda'}$ plays D_1 if $\lambda' = \Lambda$, D_0 if $\lambda' \in \{1, \ldots, \Lambda - 1\}$, and C_0 if $\lambda' = 0$ in period t. If agent i follows σ_i^*, then he plays D_1 in period t which surprises agents in $K_0^1 \cup K_0^0$. Thus, $(\lambda_{ki}^{t+1}, \lambda_{ik}^{t+1}) = (\Lambda - 1, \Lambda)$ for $k \in K_0^1 \cup K_0^0$. Also, for agent $k \in K_\Lambda^\Lambda \cup K_\Lambda^{\Lambda-1}$, agents i and k do not surprise each other in period t. So, $(\lambda_{ki}^{t+1}, \lambda_{ik}^{t+1}) = (\Lambda - 1, \Lambda - 1)$ for $k \in K_\Lambda^\Lambda$ and $(\lambda_{ki}^{t+1}, \lambda_{ik}^{t+1}) = (\Lambda - 1, \Lambda - 2)$ for $k \in K_\Lambda^{\Lambda-1}$. After then, since there will be no more surprise between i and $k \in G_i$ in the future, the future actions of himself and his neighbors are uniquely determined along the σ^*-path conditioning on any history in $\text{supp}(\mu(\cdot; o_i^t))$. Therefore, we can calculate continuation payoffs of agent i for σ^* at o_i^t. Similarly, we also can derive the future actions of agent i and his neighbors and so calculate the continuation payoff of agent i at o_i^t when agents employ strategy $(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*)$ for each $a \in \{C_0, C_1, D_0, D_1\}$. For each case, the actions in periods $s \geq t$ under $(\sigma_i^*, \sigma_{-i}^*)$ and $(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*)$ are described in the Appendix B.

Claims 1 $\sim$ 7 state that, in each o_i^t, for sufficiently high δ, there is Λ for which agent i's continuation payoff of $(\sigma_i^*, \sigma_{-i}^*)$ is greater than that of $(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*)$ for all $a \in \{C_0, C_1, D_0, D_1\}$. All the proofs of Claims 1 $\sim$ 7 are found in the Appendix C. In the proofs, for each case, we calculate the continuation payoffs for $(\sigma_i^*, \sigma_{-i}^*)$ and $(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*)$ where $a \in \{C_0, C_1, D_0, D_1\}$ and compare them to find the condition on δ and Λ under which $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$. Then, by the one deviation property of sequential equilibrium, we can prove Proposition 1 which is the main result of this paper.

Claim 1. *In Case 1, if* $|K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l \leq 0$*, then there is* $\delta_{i1}' \in (0,1)$ *such that for all* $\delta \in (\delta_{i1}', 1)$*,* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*. In Case 1, if* $|K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l > 0$*, then for some* $\underline{\delta}_{i1} \in (0,1)$*, there is a function* $F_{i1} : (\underline{\delta}_{i1}, 1) \to \mathbb{R}$ *such that* $\Lambda \leq F_{i1}(\delta)$ *implies that* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*, and* $\lim_{\delta \to 1} F_{i1}(\delta) = \infty$*.*

Claim 2. *In Case 2, there is* $\underline{\delta}_{i2} \in (0,1)$ *such that for all* $\delta \in (\underline{\delta}_{i2}, 1)$*,* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*.*

Claim 3. *In Case 3, for some* $\underline{\delta}_{i3}$*, there is a function* $F_{i3} : (\underline{\delta}_{i3}, 1) \to \mathbb{R}$ *such that* $\Lambda \leq F_{i3}(\delta)$ *implies that* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*, and* $\lim_{\delta \to 1} F_{i3}(\delta) = \infty$*.*

Claim 4. *In Case 4, for some* $\underline{\delta}_{i4}$*, there is a function* $F_{i4} : (\underline{\delta}_{i4}, 1) \to \mathbb{R}$ *such that* $\Lambda \leq F_{i4}(\delta)$ *implies that* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*, and* $\lim_{\delta \to 1} F_{i4}(\delta) = \infty$.

Claim 5. *In Case 5, for some* $\underline{\delta}_{i5}$*, there is a function* $F_{i5} : (\underline{\delta}_{i5}, 1) \to \mathbb{R}$ *such that* $F_{i5}(\delta) \leq \Lambda$ *implies that* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*, and* $\lim_{\delta \to 1} F_{i5}(\delta) < \infty$.

Claim 6. *In Case 6,* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$.

Claim 7. *In Case 7, for some* $\underline{\delta}_{i7}$*, there is a function* $F_{i7} : (\underline{\delta}_{i7}, 1) \to \mathbb{R}$ *such that* $F_{i7}(\delta) \leq \Lambda$ *implies that* $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ *for all* $a \in \{C_0, C_1, D_0, D_1\}$*, and* $\lim_{\delta \to 1} F_{i7}(\delta) < \infty$.

In Proposition 1, we show that the strategy σ^* with the belief system μ can be a sequential equilibrium for sufficiently high δ. Then, because of Lemma 2, it is trivial that cooperation is recovered from any history in this equilibrium.

Proposition 1. *There is* $\delta^* \in (0, 1)$ *such that for any* $\delta \in (\delta^*, 1)$*, there is a sequential equilibrium which supports cooperation and in which cooperation is recovered in finite periods from any history.*

Proof. From Claims 1 $\sim$ 7, we know that, for all $i \in N$,

$$\lim_{\delta \to 1} F_{i1}(\delta) = \infty, \ \lim_{\delta \to 1} F_{i3}(\delta) = \infty, \ \lim_{\delta \to 1} F_{i4}(\delta) = \infty$$
$$\lim_{\delta \to 1} F_{i5}(\delta) < \infty, \text{ and } \lim_{\delta \to 1} F_{i7}(\delta) < \infty.$$

Thus, there exists $\delta^* \in (0, 1)$ such that, if $\delta \in (\delta^*, 1)$, then, for all i, $\delta > \delta'_{i1}$, $\delta > \underline{\delta}_{i2}$, and there is Λ such that

$$F_{i5}(\delta), F_{i7}(\delta) \leq \Lambda \leq F_{i1}(\delta), F_{i3}(\delta), F_{i4}(\delta). \tag{5}$$

The one deviation property of sequential equilibrium and Claims 1 $\sim$ 7 imply that for $\delta \in (\delta^*, 1)$, σ^* with Λ satisfying (5) is a sequential equilibrium. From Lemma 2, under σ^*, cooperation is recovered in finite periods from any history. ∎

6 Discussions

In the previous section, we show that σ^* is a sequential equilibrium in which cooperation is recovered in finite periods from any history. The role of local communication in σ^* is to enable agent to inform his neighbors that he starts a new defection phase. That is, an agent informs his neighbor that he will play D in Λ periods by sending a message 1 with playing D. Someone may be interested in a sequential equilibrium which supports cooperation under an environment without communication.

Consider a repeated prisoner's dilemma game without communication. That is, each agent has only two actions $\{C, D\}$ in each period. The payoffs of prisoner's dilemma game between two linked agents are given as in Table 1. Agent i's payoff in a stage game and his discounted average payoff in the repeated game are the same as those in the repeated prisoner's dilemma game with communication. In this environment, suppose that each agent employs a trigger strategy $\bar{\sigma}_i$.[14] That is, each agent plays D if and only if he observed D played by himself or his neighbor in the past history. In the environment without communication, one can show that the trigger strategy $\bar{\sigma}$ is a sequential equilibrium if $\delta \in [g/(1+g), g/(1+g) + l/((|G_i|-1)(1+g))]$ for all i. Trivially, $\bar{\sigma}$ supports cooperation. Also, one can show that if l is small enough and $g/(1+g)+l/((|G_i|-1)(1+g)) < \delta < 1$ for some i, then $\bar{\sigma}$ cannot be a sequential equilibrium. The intuitive reason is that an agent who observes a defection by his neighbor has an incentive to play C to block the spread of defection which spoils the future gain from cooperating with other neighbors.

Although the trigger strategy $\bar{\sigma}$ cannot be a sequential equilibrium for small l and high δ, Lemma 2 in Ellison (1994) provides an idea to construct a sequential equilibrium supporting cooperation for sufficiently high δ. That is, agents divide the game into T replica games where tth replica game is played in periods $t, T+t, 2T+t, \ldots$, and they play the trigger strategy $\bar{\sigma}$ in each replica game and ignore observations in other replica games. Here, T is chosen to satisfy $\delta^T \in [g/(1+g), g/(1+g) + l/((|G_i|-1)(1+g))]$ for all i. Although $\bar{\sigma}$ supports cooperation as an equilibrium, it is not stable to mistakes. That means, if an agent deviates from cooperation by a mistake in a replica game, then cooperation is never recovered in that replica game. This may not be a desirable property of equilibrium.

One may be tempted to find a sequential equilibrium which is stable to mistakes by considering a strategy with finite periods of punishment. For instance, consider a strategy $\hat{\sigma}_i$ for each agent i such that he plays C when he did not observe D in his past history. If he is surprised by his neighbor in period t, then he plays D in following Λ periods and C thereafter. If he makes his neighbor surprised by playing D, then he plays D in following $\Lambda - 1$ periods and C thereafter. If he makes his neighbor surprised by playing C, then he plays D in following Λ periods and C thereafter.[15] However, $\hat{\sigma}$ is not stable

[14] Xue (2004) discusses a trigger strategy under an environment where agents are located in a line-shaped network, while agents in our model are located in a minimally connected network. The argument in Xue (2004) can be applied to the environment with a generalized network.

[15] We can define $\hat{\sigma}$ formally in a similar way to define the strategy σ^*. That is, we first define the phase

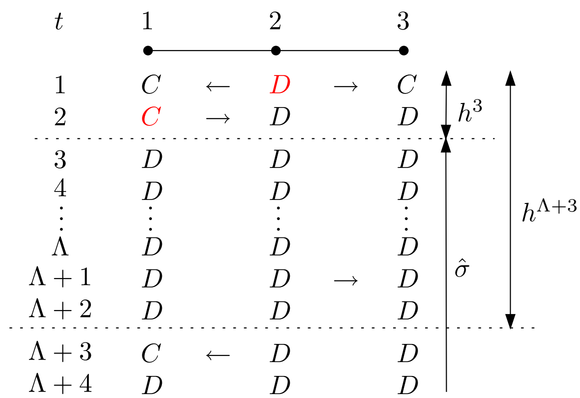
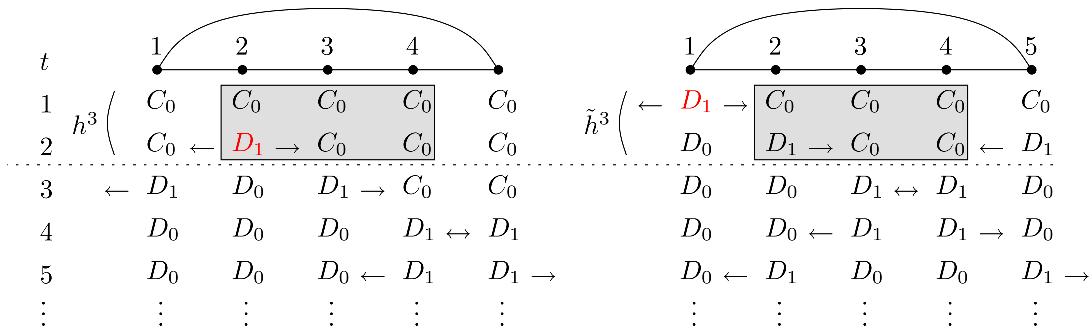


Figure 4: $\hat{\sigma}$-path conditioning on h^3

to the mistake, and moreover, it not a sequential equilibrium. For example, consider a network and $\hat{\sigma}$-path conditioning on h^3 which is described in Figure 4. Here, right-arrow (*resp.* left-arrow) represents that the agent in the left side (*resp.* right side) surprises the agent in the right side (*resp.* left side). In period 1, agent 2 makes agent 3 surprised by playing D and there is no surprise between 2 and 3 until period Λ. So, agent 3 expects that agent 2 plays D until period Λ and C in period $\Lambda + 1$. However, agent 1 makes agent 2 surprised by playing C in period 2. So, agent 2 will play D in period $\Lambda + 1$ to satisfy agent 1's expectation. This makes agent 3 surprised again and so agent 3 starts a defection phase in period $\Lambda + 2$ again. Since there is no period in which agent 1's and 3's expectations on the period when agent 2 ends the defection phase, C is never recovered after history h^3. To check the sequential rationality of $\hat{\sigma}$, notice that there is only two mistakes a_2^1 and a_1^2 in $h^{\Lambda+3}$. Also, we can see that $h^{\Lambda+3}$ is the only history which can survive in $\text{supp}(\hat{\mu}(\cdot; o_1^{\Lambda+3}(h^{\Lambda+3})))$ for any belief system $\hat{\mu}$ consistent with $\hat{\sigma}$. Thus, $h^{\Lambda+3}$ has the probability one under $\hat{\mu}(\cdot; o_1^{\Lambda+3}(h^{\Lambda+3}))$. Given the strategy $\hat{\sigma}_{-1}$ of the others, playing C in period $\Lambda + 3$ cannot be the best response for agent 1, because agent 2 never plays C is after period 3.

In the paper, we assume that G is minimally connected. Thus, given that agent i's information set o_i^t is reached, his neighbors' continuation actions along the σ^*-path are not random under $\mu(\cdot; o_i^t)$. This makes it possible to calculate the continuation payoffs for each strategies and to compare them. If we drop the assumption that G is minimally connected, the continuation actions may not be uniquely determined among the histories which are

$P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$ of each information set o_i^t in the same way as for σ^* with an expectation function $\hat{E} : \{0, 1, \ldots, \Lambda\} \to \{C, D\}$ given by $\hat{E}(0) = C$ and $\hat{E}(\lambda) = D$ for $\lambda \neq 0$. Notice that Lemma 1 does not hold in this case. The strategy $\hat{\sigma}_i$ is as follows: if $\hat{E}(\lambda_{ki}^t) = D$ for some $k \in G_i$, he plays D, and if $\hat{E}(\lambda_{ki}^t) = C$ for all $k \in G_i$, he plays C.



(a) $\alpha_N(\sigma^*; h^3)$ (b) $\alpha_N(\sigma^*; \tilde{h}^3)$

Figure 5: $\boldsymbol{\alpha}_N(\sigma^*; h^3)$ and $\boldsymbol{\alpha}_N(\sigma^*; \tilde{h}^3)$

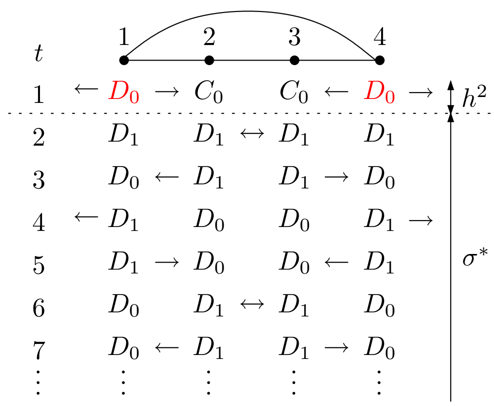


Figure 6: $\boldsymbol{\alpha}_N(\sigma^*; h^2)$

in the support of $\mu(\cdot; o_i^t)$. Too see this, consider a network $G = \{12, 23, 34, 45, 51\}$ and σ^*-paths conditioning on h^3 and $\tilde{h}^3$ which are given in Figure 5. Notice that histories h^3 and $\tilde{h}^3$ are in the same information set o_3^3 of agent 3 and each of them has only one mistake. Since o_3^3 cannot be reached without mistake, h^3 and $\tilde{h}^3$ should be in the support of $\mu(\cdot, o_3^3)$. However, as seen in Figure 5, actions after period 2 are different, which makes it difficult to calculate the continuation payoff for each strategy. Moreover, if the network is not minimally connected, then cooperation may not be recovered from some history under σ^*. Figure 6 provides an example of a network which is not minimally connected and history h^2 for which cooperation is not recovered under σ^*. In this example, the pattern of actions in periods $3 \sim 6$ are infinitely repeated along the σ^*-path conditioning on h^2.

7 Conclusion

In the paper, we consider the repeated prisoner's dilemma game with imperfect monitoring under a network. Since each agent cannot observe the action of other agents who are not directly connected to him, he cannot distinguish defections his neighbor plays between deviations and punishments. In this situation, it is already known that cooperation can be sustained as a sequential equilibrium. A trigger strategy such that observing a defection causes permanent punishment can be such an equilibrium. Although the efficient outcome can be obtained as an equilibrium in trigger strategy, it is not stable to mistakes. That means, if there is a small possibility for agents to choose defection by mistake, cooperation cannot be sustained any more.

The main contribution of this paper is to construct a sequential equilibrium which supports efficient outcome and is stable to mistakes by introducing local communication. Under the strategy we construct, cooperation is recovered in finite periods whatever the history is. The role of local communication is to enable agent to inform his neighbors that he starts a new defection phase, which makes it possible for cooperation to be recovered in contiguous periods. In the strategy we defined, agent's expectations on the actions of his neighbors plays an important role in the strategy, since a digression from expectation, called surprise, induces punishment in finite periods even if it is not a deviation.

As discussed in Section 6, the assumption of minimally connected network is crucial to show that the strategy is a sequential equilibrium. However, this assumption is somewhat restrictive, since we frequently observe that social networks in the real world are not minimally connected. The other assumption in this paper is that the benefit and the loss from defection are sufficiently small. If a prisoner's dilemma game does not satisfy this assumption, then the strategy we construct is not a sequential equilibrium. Thus, we may want to relax the assumption on the payoff in the prisoner's dilemma game. Furthermore, we can consider other games between two agents who are linked in the network instead of prisoner's dilemma game. It seems interesting to find a sequential equilibrium which results in an efficient outcome and is stable to mistakes under the model with general networks or with general two person games.

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A Proof of Lemma 3

Fix an information set $o_i^t = (a_{\overline{G}_i}^s)_{s=1}^{t-1}$ and a history $h^t = (a_N^s)_{s=1}^{t-1} \in o_i^t$. For convenience, let $h^{*\infty} = (\alpha_N^s(\sigma^*; h^t))_{s=1}^{\infty} = (a_N^s)_{s=1}^{\infty}$ and $P(o_i^s(h^{*s})) = (\lambda_{ki}^s, \lambda_{ik}^s)_{k \in G_i}$ for each $s \geq 1$.[16] Suppose that, for some $\tau \geq 0$, $\sigma_{k_1}^*(o_{k_1}^{t+\tau}(h^{*t+\tau})) \notin E(\lambda_{ik_1}^{t+\tau})$ for some $k_1 \in G_i$. If we find $\hat{h}^t \in o_i^t$ such that $\rho(\hat{h}^t) < \rho(h^t)$, then the argument before Lemma 3 implies that $h^t \notin \text{supp}(\mu(\cdot; o_i^t))$, which will complete the proof.

Let $\kappa_i(0) = \{i\}$ and $\kappa_i(x) = \{k \in N \backslash \{i\} : d(k; i) = x\}$ for $x = 1, \ldots, \bar{M}$ where $\bar{M} = \max_{i \in N}\{\max\{d(k; i) : k \in N \backslash \{i\}\}\}$. Thus, $\kappa_i(x)$ is the set of agents who has distance x from i. Since G is minimally connected, $\bar{M} \geq 1$ and N is partitioned into $\kappa_i(0), \ldots, \kappa_i(\bar{M})$. If $\bar{M} = 1$, then $G_k = \{i\}$ for each $k \neq i$, which means k is an end agent. Then, we have, for each $\tau \geq 0$, $\sigma_k^*(o_k^{t+\tau}(h^{*t+\tau})) \in E(\lambda_{ik}^{t+\tau})$ for all $k \in G_i$ which is a contradiction. Thus, $\bar{M} \geq 2$.

Step 1. *Let $k_{m-1} \in \kappa_i(m-1)$, $k_m \in \kappa_i(m)$, and $k_{m-1}k_m \in G$ where $2 \leq m+1 \leq \bar{M}$. Suppose that $a_{k_m}^s \notin E(\lambda_{k_{m-1}k_m}^s)$. If there is no agent $k_{m+1} \in G_{k_m}$ such that $k_{m+1} \in \kappa_i(m+1)$ and $a_{k_{m+1}}^{s-1} \notin E(\lambda_{k_m k_{m+1}}^{s-1})$, then either*

$$a_{k_m}^s \neq \sigma_{k_m}^*(o_{k_m}^s(h^{*s})), \text{ or} \tag{6}$$

$$a_{k_m}^{s-1} \neq \sigma_{k_m}^*(o_{k_m}^{s-1}(h^{*s-1})) \text{ and } a_{k_m}^{s-1} \in E(\lambda_{k_{m-1}k_m}^{s-1}). \tag{7}$$

Proof. Suppose that, for some m with $2 \leq m+1 \leq \bar{M}$, $a_{k_m}^s \notin E(\lambda_{k_{m-1}k_m}^s)$ and there is no agent $k_{m+1} \in G_{k_m}$ such that $k_{m+1} \in \kappa_i(m+1)$ and $a_{k_{m+1}}^{s-1} \notin E(\lambda_{k_m k_{m+1}}^{s-1})$. In addition, suppose that $a_{k_m}^s = \sigma_{k_m}^*(o_{k_m}^s(h^{*s}))$. Since $\sigma_{k_m}^*(o_{k_m}^s(h^{*s})) = a_{k_m}^s \notin E(\lambda_{k_{m-1}k_m}^s)$, by the construction of σ^*, $\sigma_{k_m}^*(o_{k_m}^s(h^{*s})) = D_1$ and $\lambda_{k_{m-1}k_m}^s \neq \Lambda$. Thus, there is an agent $k_{m+1} \in G_{k_m}$ such that $k_{m+1} \neq k_{m-1}$ and $\lambda_{k_{m+1}k_m}^s = \Lambda$. Then, only two cases are allowed in period $s-1$:

$$a_{k_{m+1}}^{s-1} \notin E(\lambda_{k_m k_{m+1}}^{s-1}) \tag{8}$$

$$a_{k_m}^{s-1} \notin E(\lambda_{k_{m+1}k_m}^{s-1}) \text{ and } a_{k_m}^{s-1} \neq D_1 \tag{9}$$

Since (8) contradicts our assumption, we have $a_{k_m}^{s-1} \notin E(\lambda_{k_m k_{m+1}}^{s-1})$ and $a_{k_m}^{s-1} \neq D_1$. By the construction of σ^*, $a_{k_m}^{s-1} \notin E(\lambda_{k_m k_{m+1}}^{s-1})$ and $a_{k_m}^{s-1} \neq D_1$ imply that $a_{k_m}^{s-1} \neq \sigma_{k_m}^*(o_{k_m}^{s-1}(h^{*s-1}))$. Furthermore, since $a_{k_m}^{s-1} \notin E(\lambda_{k_{m-1}k_m}^{s-1})$ and $a_{k_m}^{s-1} \neq D_1$ imply $\lambda_{k_{m-1}k_m}^s = \Lambda$ contradicting $\lambda_{k_{m-1}k_m}^s \neq \Lambda$, we have $a_{k_m}^{s-1} \neq \sigma_{k_m}^*(o_{k_m}^{s-1}(h^{*s-1}))$ and $a_{k_m}^{s-1} \in E(\lambda_{k_{m-1}k_m}^{s-1})$. ∎

Given an agent i, for each $k \in G_i$, we denote $\gamma_i(k)$ as the set of agents j such that the chain between i and j contains k. That is, $j \in \gamma_i(k)$ if and only if $k \in i \leftrightarrow j$. Since G is minimally connected, N can be partitioned into $\{i\}$ and $\gamma_i(k)$ for $k \in G_i$.

[16] By the definition of $\boldsymbol{\alpha}_N(\sigma^*; h^t)$, we can let $h^{*\infty} = (a_N^s)_{s=1}^{\infty}$ without conflicting with $h^t = (a_N^s)_{s=1}^{t-1}$. That is, $h^{*\infty}$ agrees with h^t for periods $s \leq t-1$.

Step 2. *Let $k_1 \in G_i$. If $a^s_{k_1} \notin E(\lambda^s_{ik_1})$ then, for some m with $1 \le m \le \bar{M}$, there is an agent $k_m \in \kappa_i(m) \cap \gamma_i(k_1)$ such that*

$$a^{s-m+1}_{k_m} \neq \sigma^*_{k_m}(o^{s-m+1}_{k_m}(h^{*s-m+1})) \text{ and } a^{s-m+1}_{k_m} \notin E(\lambda^{s-m+1}_{k_{m-1}k_m}), \text{ or} \tag{10}$$

$$a^{s-m}_{k_m} \neq \sigma^*_{k_m}(o^{s-m}_{k_m}(h^{*s-m})) \text{ and } a^{s-m}_{k_m} \in E(\lambda^{s-m}_{k_{m-1}k_m}). \tag{11}$$

Proof. Let $a^s_{k_1} \notin E(\lambda^s_{ik_1})$ for some $k_1 \in G_i$. If (10) and (11) do not hold for $m = 1$, then Step 1 implies that there is an agent $k_2 \in G_{k_1}$ such that $k_2 \in \kappa_i(2)$ and $a^{s-1}_{k_2} \notin E(\lambda^{s-1}_{k_1k_2})$. Then, if (10) and (11) do not hold for $m = 2$, there is an agent $k_3 \in G_{k_2}$ such that $k_3 \in \kappa_i(3)$ and $a^{s-2}_{k_3} \notin E(\lambda^{s-2}_{k_2k_3})$. Continuing this procedure, we eventually have a contradiction that there is no agent $k_m \in \kappa_i(m+1)$ such that $a^{s-m+1}_{k_{m+1}} \notin E(\lambda^{s-m+1}_{k_mk_{m+1}})$ because $s - m + 1 = 0$ or $m = \bar{M}$. This proves Step 2. ∎

Let $a^s_{k_1} \notin E(\lambda^s_{ik_1})$. From Step 2, we know that there is a mistake $a^{s-m+1}_{k_m}$ such that $a^{s-m+1}_{k_m} \neq \sigma^*_{k_m}(o^{s-m+1}_{k_m}(h^{*s-m+1}))$ and $a^{s-m+1}_{k_m} \notin E(\lambda^{s-m+1}_{k_{m-1}k_m})$, or $a^{s-m}_{k_m}$ such that $a^{s-m}_{k_m} \neq \sigma^*_{k_m}(o^{s-m}_{k_m}(h^{*s-m}))$ and $a^{s-m}_{k_m} \in E(\lambda^{s-m}_{k_{m-1}k_m})$, where $k_m \in \kappa_i(m) \cap \gamma_i(k_1)$. In this case, we say that a surprise $a^s_{k_1}$ is *induced* by mistake $a^{s-m+1}_{k_m}$ or $a^{s-m}_{k_m}$, respectively.

Step 3. *Let $k \in G_i$ and $k' \in G_i$, and let $a^s_k \in h^{*\infty}$ and $a^{s'}_{k'} \in h^{*\infty}$ satisfy that $a^s_k \notin E(\lambda^s_{ik})$ and $a^{s'}_{k'} \notin E(\lambda^{s'}_{ik'})$. Let a^s_k be induced by $a^{\hat{s}}_{\hat{k}}$ and $a^{s'}_{k'}$ be induced by $a^{\hat{s}'}_{\hat{k}'}$. If $s \neq s'$ or $k \neq k'$, then $\hat{s} \neq \hat{s}'$ or $\hat{k} \neq \hat{k}'$. That means, a mistake can induce at most one surprise to agent i.*

Proof. Suppose that $k \neq k'$, then $\hat{k} \in \gamma_i(k)$ and $\hat{k}' \in \gamma_i(k')$. Since $\gamma_i(k) \cap \gamma_i(k') = \varnothing$, we have $\hat{k} \neq \hat{k}'$. Suppose that $k = k'$ and $s \neq s'$. Without loss of generality, let $s > s'$. In addition, suppose that $\hat{k} = \hat{k}'$ and $\hat{s} = \hat{s}'$, so $a^{\hat{s}}_{\hat{k}} = a^{\hat{s}'}_{\hat{k}'}$. Since G is minimally connected, there is a unique chain $k \leftrightarrow \hat{k} = k' \leftrightarrow \hat{k}' = \{k_1, \ldots, k_m\}$ such that $k_1 = k = k'$, $k_m = \hat{k} = \hat{k}'$, and $k_lk_{l+1} \in G$ for all $l = 1, \ldots, m-1$. Then, $a^{s-m}_{k_m} = a^{\hat{s}}_{k_m} = a^{\hat{s}'}_{k_m} = a^{s'-m+1}_{k_m}$. From Step 2, we have $a^{\hat{s}}_{k_m} = a^{s-m}_{k_m} \in E(\lambda^{s-m}_{k_{m-1}k_m}) = E(\lambda^{\hat{s}}_{k_{m-1}k_m})$ and $a^{\hat{s}}_{k_m} = a^{s'-m+1}_{k_m} \notin E(\lambda^{s'-m+1}_{k_{m-1}k_m}) = E(\lambda^{\hat{s}}_{k_{m-1}k_m})$, which is a contradiction. ∎

From Steps 1 ∼ 3, we know that for each surprise $a^s_k \in h^{*\infty}$ to agent i by $k \in G_i$, there is a mistake which induces a^s_k and does not induce any other mistake. Furthermore, since C_1 is never played under σ^*, $a^s_k \in h^t$ for $k \in G_i$ satisfying $a^s_k = C_1$ is a mistake. Therefore, we have

$$\begin{aligned}
\rho(h^t) \quad &\ge \quad |\{a^s_i \in h^t : a^s_i \neq \sigma^*_i(o^s_i(h^{*s}))\}| + |\{a^s_k \in h^{*\infty} : a^s_k \notin E(\lambda^s_{ik}), k \in G_i\}| \\
&\qquad + |\{a^s_k \in h^t : a^s_k = C_1 \in E(\lambda^s_{ik}), k \in G_i\}| \\
&> \quad |\{a^s_i \in h^t : a^s_i \neq \sigma^*_i(o^s_i(h^{*s}))\}| + |\{a^s_k \in h^t : a^s_k \notin E(\lambda^s_{ik}), k \in G_i\}| \\
&\qquad + |\{a^s_k \in h^t : a^s_k = C_1 \in E(\lambda^s_{ik}), k \in G_i\}|.
\end{aligned}$$

Step 4. *There is a history $\hat{h}^t = (\hat{a}_N^s)_{s=1}^{t-1}$ such that $\hat{h}^t \in o_i^t$ and $\rho(\hat{h}^t) < \rho(h^t)$.*

Proof. We construct a history $\hat{h}^s = (\hat{a}_N^\tau)_{\tau=1}^{s-1}$ for each s with $1 \leq s \leq t$ iteratively as follows: Let $\hat{h}^1 = \varnothing$ and, for each s satisfying $2 \leq s \leq t$,

$$\hat{h}^s = (\hat{h}^{s-1}, \hat{a}_N^{s-1})$$

where

$$\begin{aligned} \hat{a}_k^{s-1} &= a_k^{s-1} && \text{if } k \in \overline{G}_i, \\ \hat{a}_k^{s-1} &= \sigma_k^*(o_k^{s-1}(\hat{h}^{s-1})) && \text{if } k \notin \overline{G}_i. \end{aligned}$$

By the construction of $\hat{h}^t$, we have $\hat{h}^t \in o_i^t(h^t)$. For each s, let $P(o_i^s(\hat{h}^s)) = (\hat{\lambda}_{ki}^s, \hat{\lambda}_{ik}^s)_{k \in G_i}$.

Notice that $\hat{a}_k^{s-1} \in \hat{h}^t$ is a surprise to agent i if and only if $a_k^{s-1} \in h^t$ is a surprise to i. Furthermore, since there is no mistake for agent $k \notin \overline{G}_i$, Step 2 implies that any surprise $\hat{a}_k^{s-1} \in \hat{h}^t$ to agent i is a mistake. That is, if $\hat{a}_k^{s-1} \notin E(\hat{\lambda}_{ik}^{s-1})$, then $\hat{a}_k^{s-1} \neq \sigma_k^*(o_k^{s-1}(\hat{h}^{s-1}))$. Also, since C_1 is never played under σ^*, an action $\hat{a}_k^s \in \hat{h}^t$ satisfying $\hat{a}_k^s \in E(\hat{\lambda}_{ik}^s)$ and $\hat{a}_k^s = C_1$ for $k \in G_i$ is a mistake.

Now, we want to show that, if $\hat{a}_k^s \in \hat{h}^t$ for $k \in G_i$ is a mistake, then $\hat{a}_k^s \notin E(\hat{\lambda}_{ik}^s)$ or $\hat{a}_k^s = C_1 \in E(\hat{\lambda}_{ik}^s)$. Suppose that $\hat{a}_k^s \in \hat{h}^t$ for $k \in G_i$ is a mistake where $\hat{\lambda}_{ik}^s = \hat{\lambda}_{k'k}^s = \lambda \in \{0, \ldots, \Lambda - 1\}$ for all $k' \in G_k$. Suppose in addition that $\hat{a}_k^s \in E(\hat{\lambda}_{ik}^s)$. Since $\hat{a}_k^s \in E(\hat{\lambda}_{k'k}^s)$ for all $k' \in G_k$ and $\hat{a}_k^s \neq \sigma_k^*(o_k^s(\hat{h}^s))$, we have $\hat{a}_k^s = C_1 \in E(\hat{\lambda}_{ki}^s)$. Suppose that there is a mistake $\hat{a}_k^s \in \hat{h}^t$ for $k \in G_i$ such that $\hat{a}_k^s \in E(\hat{\lambda}_{ik}^s)$ where $\hat{\lambda}_{k'k}^s = \Lambda$ for some $k' \in G_k$. Let $\bar{s}$ denote the earliest period when such a mistake exists. Let $k_1 \in G_i$ be an agent who makes the mistake $\hat{a}_k^{\bar{s}}$ in period $\bar{s}$ and $k_2 \in G_{k_1}$ be an agent with $\hat{\lambda}_{k_2 k_1}^{\bar{s}} = \Lambda$. Note that $\bar{s} > 1$ since $\hat{\lambda}_{k_2 k_1}^1 = 0$. Since $\hat{a}_{k_1}^{\bar{s}} \neq \sigma_{k_1}^*(o_{k_1}^{\bar{s}}(\hat{h}^{\bar{s}})) = D_1$ and $\hat{a}_{k_1}^{\bar{s}} \in E(\hat{\lambda}_{ik_1}^{\bar{s}})$, we have $\hat{\lambda}_{ik_1}^{\bar{s}} \neq \Lambda$, $i \neq k_2$. Since $\hat{\lambda}_{k_2 k_1}^{\bar{s}} = \Lambda$, by the construction of P, we have either (i) $\hat{a}_{k_2}^{\bar{s}-1} \notin E(\hat{\lambda}_{k_1 k_2}^{\bar{s}-1})$ or (ii) $\hat{a}_{k_1}^{\bar{s}-1} \notin E(\hat{\lambda}_{k_2 k_1}^{\bar{s}-1})$ and $a_{k_1}^{\bar{s}-1} \neq D_1$. If (i) is the case, then Step 2 implies that there is a mistake by some agent $k \notin G_i$ in $\hat{h}^t$, which contradicts the construction of $\hat{h}^t$. If (ii) is the case, then $\hat{a}_{k_1}^{\bar{s}-1}$ is the mistake and $\hat{a}_{k_1}^{\bar{s}-1} \in E(\hat{\lambda}_{ik_1}^{\bar{s}-1})$ since $\hat{\lambda}_{ik_1}^{\bar{s}} \neq \Lambda$. Then, by the definition of $\bar{s}$, we should have $\hat{\lambda}_{ik_1}^{\bar{s}-1} = \hat{\lambda}_{k'k_1}^{\bar{s}-1} = \lambda \in \{0, \ldots, \Lambda - 1\}$ for all $k' \in G_{k_1}$, which implies $\hat{a}_{k_1}^{\bar{s}-1} \in E(\hat{\lambda}_{k'k_1}^{\bar{s}-1})$ for all $k' \in G_{k_1}$. However, this contradicts that (ii) is the case. Therefore, if $\hat{a}_k^s \in \hat{h}^t$ for $k \in G_i$ is a mistake, then it is a surprise to agent i or it satisfies $\hat{a}_k^s = C_1 \in E(\hat{\lambda}_{ik}^s)$.

Furthermore, since $o_i^s(h^{*s}) = o_i^s(\hat{h}^s)$ for all $\tau \leq t$, $\hat{a}_i^s$ is a mistake in h^t if and only if a_i^s is a mistake in h^t, and for $k \in G_i$ and for $s \leq t-1$, $\hat{a}_k^s \notin E(\hat{\lambda}_{ik}^s)$ if and only if $a_k^s \notin E(\lambda_{ik}^s)$. Therefore,

$$\begin{aligned} \rho(\hat{h}^t) &= |\{\hat{a}_i^s \in \hat{h}^t : a_i^s \neq \sigma_i^*(o_i^s(\hat{h}^s))\}| + |\{\hat{a}_k^s \in \hat{h}^t : \hat{a}_k^s \neq \sigma_k^*(o_k^s(\hat{h}^s)), k \in G_i\}| \\ &= |\{\hat{a}_i^s \in \hat{h}^t : a_i^s \neq \sigma_i^*(o_i^s(\hat{h}^s))\}| + |\{\hat{a}_k^s \in \hat{h}^t : \hat{a}_k^s \notin E(\hat{\lambda}_{ik}^s), k \in G_i\}| \\ &\quad + |\{\hat{a}_k^s \in \hat{h}^t : \hat{a}_k^s = C_1 \in E(\hat{\lambda}_{ik}^s), k \in G_i\}| \\ &= |\{a_i^s \in h^t : a_i^s \neq \sigma_i^*(o_i^s(h^{*s}))\}| + |\{a_k^s \in h^t : a_k^s \notin E(\lambda_{ik}^s), k \in G_i\}| \\ &\quad + |\{a_k^s \in h^t : a_k^s = C_1 \in E(\hat{\lambda}_{ik}^s), k \in G_i\}| \\ &< \rho(h^t). \end{aligned}$$

This completes the proof. ∎

B Actions in period $s \geq t$ for each strategy

Case A. For an information set o_i^t such that $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$ and, for some $k \in G_i$, $\lambda_{ki}^t = \Lambda$,

Case 1. G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_0^1, and K_0^0.

		a_k^s,	$s \geq t$							
strategy	$k \in$	t	$t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_0	D_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_0^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$\{i\}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{D_0}^{o_i^t}, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{C}^{o_i^t}, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^1	D_0	C_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_0^0	C_0	C_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

Case 2. G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, $K_\lambda^{\lambda+1}$, K_λ^λ, and $K_\lambda^{\lambda-1}$, where $\lambda = 3, \ldots, \Lambda - 1$.

strategy	$k \in$	a_k^s, t	$s \geq t$ $t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_0	D_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda+1}$	D	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_λ^λ	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\vert_{D_0}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda+1}$	D	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_λ^λ	D_0	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
$(\sigma_i^*\vert_{C}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda+1}$	D	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_λ^λ	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

$D = D_1$ if $\lambda = \Lambda - 1$ and $D = D_0$ otherwise

Case 3. G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_2^3, K_2^2, and K_2^1.

strategy	$k \in$	a_k^s, t	$s \geq t$ $t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_0	D_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_2^3	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\vert_{D_0}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^3	D_0	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_2^2	D_0	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_2^1	D_0	C_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
$(\sigma_i^*\vert_{C}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^3	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_2^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

Case 4. G_i is partitioned into K_Λ^Λ, $K_\Lambda^{\Lambda-1}$, K_1^2, K_1^1, and K_1^0.

strategy	$k \in$	a_k^s, t	$s \geq t$ $t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_0	D_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$\{i\}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\vert_{D_0}^{o_i^t}, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_1^1	D_0	C_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_1^0	C_0	C_0	D_1	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\vert_{C}^{o_i^t}, \sigma_{-i}^*)$	K_Λ^Λ	D_1	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\Lambda^{\Lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

Case B. For an information set o_i^t such that $P(o_i^t) = (\lambda_{ki}^t, \lambda_{ik}^t)_{k \in G_i}$ and, for all $k \in G_i$, $\lambda_{ki}^t = \lambda \neq \Lambda$,

Case 5. G_i is partitioned into K_0^1, and K_0^0.

strategy	$k \in$	a_k^s, t	$s \geq t$ $t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$,	$\{i\}$	C	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{C_1}^{o_i^t}, \sigma_{-i}^*)$	K_0^1	D_0	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_0^0	C_0	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{D_1}^{o_i^t}, \sigma_{-i}^*)$	K_0^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$\{i\}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{D_0}^{o_i^t}, \sigma_{-i}^*)$	K_0^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_0^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C = C_0$ for $(\sigma_i^*, \sigma_{-i}^*)$ and $C = C_1$ for $(\sigma_i^*|_{C_1}^{o_i^t}, \sigma_{-i}^*)$

Case 6. G_i is partitioned into $K_\lambda^{\lambda+1}$, K_λ^λ, and $K_\lambda^{\lambda-1}$, where $\lambda \in \{2, \ldots, \Lambda - 1\}$.

strategy	$k \in$	a_k^s, t	$s \geq t$ $t+1$	$t+2$	$\cdots$	$t+\lambda-2$	$t+\lambda-1$	$t+\lambda$	$t+\lambda+1$	$\cdots$
	$\{i\}$	D_0	D_0	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	$K_\lambda^{\lambda+1}$	D	D_0	D_0	$\cdots$	D_0	D_0	D_0	C_0	$\cdots$
	K_λ^λ	D_0	D_0	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
strategy	$k \in$	t	$t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{D_1}^{o_i^t}, \sigma_{-i}^*)$	$K_\lambda^{\lambda+1}$	D	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_λ^λ	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{C}^{o_i^t}, \sigma_{-i}^*)$	$K_\lambda^{\lambda+1}$	D	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_λ^λ	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	$K_\lambda^{\lambda-1}$	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

$D = D_1$ if $\lambda = \Lambda - 1$ and $D = D_0$ otherwise

Case 7. G_i is partitioned into K_1^2, K_1^1, and K_1^0.

		a_k^s,	$s \geq t$							
strategy	$k \in$	t	$t+1$	$t+2$	$\cdots$	$t+\Lambda-1$	$t+\Lambda$	$t+\Lambda+1$	$t+\Lambda+2$	$\cdots$
$(\sigma_i^*, \sigma_{-i}^*)$	$\{i\}$	D_0	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_1^1	D_0	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
	K_1^0	C_0	C_0	C_0	$\cdots$	C_0	C_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{D_1}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	D_1	D_0	D_0	$\cdots$	D_0	C_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
$(\sigma_i^*\|_{C}^{o_i^t}, \sigma_{-i}^*)$	$\{i\}$	C	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^2	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^1	D_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$
	K_1^0	C_0	D_1	D_0	$\cdots$	D_0	D_0	C_0	C_0	$\cdots$

$C \in \{C_0, C_1\}$

C Proof of Claims in Section 5

For convenience, let $\pi_i(o_i^t)$ be a partition of G_i generated by o_i^t. For example, if o_i^t satisfies Case 1, then $\pi_i(o_i^t) = \{K_\Lambda^\Lambda, K_\Lambda^{\Lambda-1}, K_0^1, K_0^0\}$. Recall that $K_\lambda^{\lambda'}$ depends on o_i^t. We denote $\Pi_i(Z)$ as the set of all partitions of G_i in Case Z. For example, if a partition π_i is in $\Pi_i(4)$, that is $\pi_i \in \Pi_i(4)$, then π_i can be represented as $\pi_i = \{K_\Lambda^\Lambda, K_\Lambda^{\Lambda-1}, K_1^2, K_1^1, K_1^0\}$. Since G_i is finite, $\Pi_i(Z)$ is finite for each ω. Also, note that for each o_i^t, $\pi_i(o_i^t) \in \Pi_i(Z)$ if and only if o_i^t satisfies Case Z.

Proof of Claim 1. Note that

$$\begin{aligned}
CU_i(\sigma^*; o_i^t) &= (1-\delta)|K_0^0|(1+g) + (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
&\quad +(1-\delta)\delta^\Lambda(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_0^1| + |K_0^0|)l) + \delta^{\Lambda+1}|G_i|, \\
CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|K_0^0|(1+g) + \delta^{\Lambda+1}|G_i|, \text{ and} \\
CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)(|K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) \\
&\quad +(1-\delta)\delta(|K_0^1| + |K_0^0|)(1+g) \\
&\quad +(1-\delta)\delta^{\Lambda+1}((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^{\Lambda+2}|G_i|,
\end{aligned}$$

where $C \in \{C_0, C_1\}$.

Since $|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_0^1| + |K_0^0|)l \geq 1 - (|G_i| - 1)l \geq 0$, we have

$$|K_\Lambda^{\Lambda-1}|(1+g) + \delta(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_0^1| + |K_0^0|)l) \geq 0.$$

This implies that $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t)$.

To compare $CU_i(\sigma^*; o_i^t)$ and $CU_i(\sigma_i^*|_{C}^{o_i^t}; o_i^t)$, suppose that $|K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| +$

$|K_0^1|)l \le 0$. Then,

$$
\begin{aligned}
& CU_i(\sigma^*; o_i^t) \ge CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff \quad & (1-\delta)|K_0^0|(1+g) + (1-\delta)\delta^{\Lambda-1}|K_\Lambda^\Lambda|(1+g) \\
& \quad + (1-\delta)\delta^\Lambda((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^{\Lambda+1}|G_i| \\
& \ge (1-\delta)(|K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + (1-\delta)\delta(|K_0^1| + |K_0^0|)(1+g) \\
& \quad + (1-\delta)\delta^{\Lambda+1}((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^{\Lambda+2}|G_i| \\
\iff \quad & \delta^\Lambda(|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|) \\
& \ge \delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g).
\end{aligned}
$$

Since

$$
\begin{aligned}
& \delta^\Lambda(|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|) \\
& \to |K_\Lambda^\Lambda|(1+g) + |G_i| > 0, \text{and} \\
& \delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g) \\
& \to |K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l \le 0
\end{aligned}
$$

as $\delta \to 1$, there is $\delta'_{i1} \in (0,1)$ such that $CU_i(\sigma^*; o_i^t) \ge CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t)$ for all $\delta \in (\delta'_{i1}, 1)$.

Suppose that $|K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l > 0$. Then,

$$
\begin{aligned}
& CU_i(\sigma^*; o_i^t) \ge CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff \quad & \Lambda \ln\delta + \ln\left[|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|\right] \\
& \ge \ln\left[\delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g)\right] \\
\iff \quad & \Lambda \le \tfrac{1}{-\ln\delta} \ln\left[\tfrac{|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|}{\delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g)}\right]
\end{aligned}
$$

where δ is sufficiently large so that $\ln(\cdot)$ is well defined.

Since $|K_\Lambda^\Lambda|(1+g) + |G_i| \ge 1 + |K_0^0| + |K_0^1| > (|G_i|-1)g + |K_0^0| + |K_0^1| \ge |K_0^1|g + |K_0^0| + |K_0^1| = |K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l$, we have $-\ln\delta \to 0$ and

$$
\begin{aligned}
& \ln\left[\tfrac{|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|}{\delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g)}\right] \\
\to \quad & \ln\left[\tfrac{|K_\Lambda^\Lambda|(1+g) + |G_i|}{|K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l}\right] > 0
\end{aligned}
$$

as $\delta \to 1$. Thus, $\frac{1}{-\ln\delta} \ln\left[\frac{|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|}{\delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g)}\right] \to \infty$ as $\delta \to 1$. Let

$$
F_{i1}(\delta) = \min_{\pi_i \in \Pi_i(1)} \left\{ \tfrac{1}{-\ln\delta} \ln\left[\tfrac{|K_\Lambda^\Lambda|(1+g) + (1-\delta)\delta((|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}|) - (|K_0^1| + |K_0^0|)l) + \delta^2|G_i|}{\delta(-|K_0^0|g - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l) + \delta^2(|K_0^1| + |K_0^0|)(1+g)}\right] \right\}
$$

for sufficiently large δ. Here, the minimum is taken over the set $\{\pi_i \in \Pi_i(1) : \ |K_0^1|(1+g) + |K_0^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_0^1|)l \le 0\}$. Then, $F_{i1}(\delta) \to \infty$ and $\Lambda \le F_{i1}(\delta)$ implies that $CU_i(\sigma^*; o_i^t) \ge CU_i(\sigma_i^*|_a^{o_i^t}, \sigma_{-i}^*; o_i^t)$ for all $a \in \{C_0, C_1, D_0, D_1\}$. ∎

Proof of Claim 2. In Case 2, agent i's continuation payoff for each strategy is as follows:

$$\begin{aligned} CU_i(\sigma^*; o_i^t) &= (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\ &\quad +(1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + \delta^{\Lambda+1}|G_i|, \\ CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) \\ &\quad +\delta^{\Lambda+2}|G_i|, \text{ and} \\ CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= -(1-\delta)|G_i|l + \delta^{\Lambda+1}|G_i| \end{aligned}$$

where $C \in \{C_0, C_1\}$.

Since

$$\begin{aligned} & CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}; o_i^t) \\ \iff & \quad (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\ & \quad + (1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + \delta^{\Lambda+1}|G_i| \\ & \geq (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + \delta^{\Lambda+2}|G_i| \\ \iff & \quad \delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + \delta^{\Lambda+1}|G_i| \geq 0 \\ \iff & \quad \delta^{\Lambda}\left[|K_\Lambda^{\Lambda-1}|(1+g) + \delta^2|G_i| + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l)\right] \geq 0 \end{aligned}$$

and

$$\begin{aligned} & |K_\Lambda^{\Lambda-1}|(1+g) + \delta^2|G_i| + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) \\ \to \; & |K_\Lambda^{\Lambda-1}|(1+g) + |G_i| > 0 \end{aligned}$$

as $\delta \to 1$, we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t)$ for sufficiently large δ.

Since

$$\begin{aligned} & CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}; o_i^t) \\ \iff & \quad (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\ & \quad + (1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + \delta^{\Lambda+1}|G_i| \\ & \geq -(1-\delta)|G_i|l + \delta^{\Lambda+1}|G_i| \\ \iff & \quad \delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) + \delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) \geq -|G_i|l \\ \iff & \quad \delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) + \delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l) + |G_i|l \geq 0 \end{aligned}$$

and $|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_\lambda^{\lambda+1}| + |K_\lambda^{\lambda}| + |K_\lambda^{\lambda-1}|)l \geq 1 - (|G_i - 1)l > 0$, we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t)$. ∎

Proof of Claim 3. In Case 3, agent i's continuation payoff for each strategy is as follows:

$$\begin{aligned}
CU_i(\sigma^*; o_i^t) &= (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
&\quad +(1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+1}|G_i|, \\
CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)\delta|K_2^1|(1+g) \\
&\quad (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+2}|G_i|, \text{ and} \\
CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= -(1-\delta)|G_i|l + \delta^{\Lambda+1}|G_i|,
\end{aligned}$$

where $C \in \{C_0, C_1\}$.

Note that

$$\begin{aligned}
& CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff & \quad (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
& \quad + (1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+1}|G_i| \\
& \geq (1-\delta)\delta|K_2^1|(1+g) + (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+2}|G_i| \\
\iff & \quad \delta^{\Lambda}|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+2}|G_i| \\
& \geq \delta^2|K_2^1|(1+g) \\
\iff & \quad \Lambda \ln\delta + \ln\left[|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^2|G_i|\right] \\
& \geq \ln\left[\delta^2|K_2^1|(1+g)\right] \\
\iff & \quad \Lambda \leq \tfrac{1}{-\ln\delta}\ln\left[\tfrac{|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^2|G_i|}{\delta^2|K_2^1|(1+g)}\right]
\end{aligned}$$

where δ is sufficiently high so that $\ln(\cdot)$ is well defined.

Since $|K_\Lambda^{\Lambda-1}|(1+g) + |G_i| \geq |K_2^1| + 1 > |K_2^1| + (|G_i| - 1)\,g \geq |K_2^1| + |K_2^1|g = |K_2^1|(1+g)$, we have $-\ln\delta \to 0$ and

$$\ln\left[\tfrac{|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^2|G_i|}{\delta^2|K_2^1|(1+g)}\right] \to \ln\left[\tfrac{|K_\Lambda^{\Lambda-1}|(1+g) + |G_i|}{|K_2^1|(1+g)}\right] > 0$$

as $\delta \to 1$. Thus, $\tfrac{1}{-\ln\delta}\ln\left[\tfrac{|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^2|G_i|}{\delta^2|K_2^1|(1+g)}\right] \to \infty$ as $\delta \to 1$.

Furthermore, since

$$\begin{aligned}
& CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff & \quad (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
& \quad + (1-\delta)\delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + \delta^{\Lambda+1}|G_i| \\
& \geq -(1-\delta)|G_i|l + \delta^{\Lambda+1}|G_i| \\
\iff & \quad \delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) + \delta^{\Lambda}(|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) + |G_i|l \geq 0
\end{aligned}$$

and $|K_\Lambda^{\Lambda}| + |K_\Lambda^{\Lambda-1}| - (|K_2^3| + |K_2^2| + |K_2^1|)l) > 0$, we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}; o_i^t)$.

Letting

$$F_{i3}(\delta) = \min_{\pi_i \in \Pi_i(3)} \left\{ \frac{1}{-\ln\delta} \ln \left[\frac{|K_\Lambda^{\Lambda-1}|(1+g)+(1-\delta)\delta(|K_\Lambda^\Lambda|+|K_\Lambda^{\Lambda-1}|-(|K_2^3|+|K_2^2|+|K_2^1|)l)+\delta^2|G_i|}{\delta^2|K_2^1|(1+g)} \right] \right\}$$

for sufficiently large δ, we complete the proof. ∎

Proof of Claim 4. In Case 4, agent i's continuation payoff for each strategy is as follows:

$$\begin{aligned}
CU_i(\sigma^*; o_i^t) &= (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
&\quad +(1-\delta)\delta^{\Lambda}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+1}|G_i|, \\
CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta(|K_1^1| + |K_1^0|)(1+g) \\
&\quad +(1-\delta)\delta^{\Lambda+1}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+2}|G_i|, \text{ and} \\
CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)(|K_1^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_1^2| + |K_1^1|)l) + \delta^{\Lambda+1}|G_i|,
\end{aligned}$$

where $C \in \{C_0, C_1\}$.

Noth that

$$\begin{aligned}
& CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\Longleftrightarrow \quad & (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
& + (1-\delta)\delta^{\Lambda}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+1}|G_i| \\
& \geq (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta(|K_1^1| + |K_1^0|)(1+g) \\
& + (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+2}|G_i| \\
\Longleftrightarrow \quad & \delta^{\Lambda}|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta^{\Lambda+1}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+2}|G_i| \\
& \geq \delta^2(|K_1^1| + |K_1^0|)(1+g) \\
\Longleftrightarrow \quad & \Lambda \ln\delta + \ln\left[|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^2|G_i|\right] \\
& \geq \ln\left[\delta^2(|K_1^1| + |K_1^0|)(1+g)\right] \\
\Longleftrightarrow \quad & \Lambda \leq \frac{1}{-\ln\delta} \ln\left[\frac{|K_\Lambda^{\Lambda-1}|(1+g)+(1-\delta)\delta(|K_\Lambda^\Lambda|+|K_\Lambda^{\Lambda-1}|-(|K_1^2|+|K_1^1|+|K_1^0|)l)+\delta^2|G_i|}{\delta^2(|K_1^1|+|K_1^0|)(1+g)}\right]
\end{aligned}$$

where δ is sufficiently large so that $\ln(\cdot)$ is well defined.

Since $|K_\Lambda^{\Lambda-1}|(1+g) + |G_i| \geq |K_1^1| + |K_1^0| + 1 > |K_1^1| + |K_1^0| + (|G_i| - 1)\,g \geq (|K_1^1| + |K_1^0|)(1+g)$, we have $-\ln\delta \to 0$ and

$$\ln\left[\frac{|K_\Lambda^{\Lambda-1}|(1+g)+(1-\delta)\delta(|K_\Lambda^\Lambda|+|K_\Lambda^{\Lambda-1}|-(|K_1^2|+|K_1^1|+|K_1^0|)l)+\delta^2|G_i|}{\delta^2(|K_1^1|+|K_1^0|)(1+g)}\right] \to \ln\left[\frac{|K_\Lambda^{\Lambda-1}|(1+g)+|G_i|}{(|K_1^1|+|K_1^0|)(1+g)}\right] > 0$$

as $\delta \to 1$. Thus, $\frac{1}{-\ln\delta} \ln\left[\frac{|K_\Lambda^{\Lambda-1}|(1+g)+(1-\delta)\delta(|K_\Lambda^\Lambda|+|K_\Lambda^{\Lambda-1}|-(|K_1^2|+|K_1^1|+|K_1^0|)l)+\delta^2|G_i|}{\delta^2(|K_1^1|+|K_1^0|)(1+g)}\right] \to \infty$ as $\delta \to 1$.

To compare $CU_i(\sigma^*; o_i^t)$ and $CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t)$, note that

$$\begin{aligned}
& CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff \quad & (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta^{\Lambda-1}|K_\Lambda^{\Lambda-1}|(1+g) \\
& + (1-\delta)\delta^{\Lambda}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^{\Lambda+1}|G_i| \\
& \geq (1-\delta)(|K_1^0| - (|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_1^2| + |K_1^1|)l) + \delta^{\Lambda+1}|G_i| \\
\iff \quad & \delta^{\Lambda}|K_\Lambda^{\Lambda-1}|(1+g) + \delta^{\Lambda+1}(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) \\
& \geq -\delta(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| + |K_1^2| + |K_1^1|)l - \delta|K_1^0|g
\end{aligned}$$

Since $|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l > 0$, we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_C^{o_i^t}, \sigma_{-i}^*; o_i^t)$ for sufficiently large δ.

Letting

$$F_{i4}(\delta) = \min_{\pi_i \in \Pi_i(4)} \left\{ \frac{1}{-\ln\delta} \ln\left[\frac{|K_\Lambda^{\Lambda-1}|(1+g) + (1-\delta)\delta(|K_\Lambda^\Lambda| + |K_\Lambda^{\Lambda-1}| - (|K_1^2| + |K_1^1| + |K_1^0|)l) + \delta^2|G_i|}{\delta^2(|K_1^1| + |K_1^0|)(1+g)} \right] \right\}$$

for sufficiently high δ, we complete the proof. ∎

Proof of Claim 5. In Case 5, the payoff for each strategy is given as follows:

$$\begin{aligned}
CU_i(\sigma^*; o_i^t) &= CU_i(\sigma_i^*|_{o_i^t, C_1}, \sigma_{-i}^*; o_i^t) \\
&= (1-\delta)(|K_0^0| - |K_0^1|l) + \delta|G_i|, \\
CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|K_0^0|(1+g) - (1-\delta)\delta^{\Lambda}|G_i|l + \delta^{\Lambda+1}|G_i|, \text{ and} \\
CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|K_0^0|(1+g) + \delta^{\Lambda+1}|G_i|.
\end{aligned}$$

First, we want to compare $CU_i(\sigma^*; o_i^t)$ and $CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t)$. Note that

$$\begin{aligned}
& CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_0}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\
\iff \quad & (1-\delta)(|K_0^0| - |K_0^1|l) + \delta|G_i| \geq (1-\delta)|K_0^0|(1+g) + \delta^{\Lambda+1}|G_i| \\
\iff \quad & \delta|G_i| - (1-\delta)(|K_0^0|g + |K_0^1|l) \geq \delta^{\Lambda+1}|G_i| \\
\iff \quad & \Lambda \ln\delta \leq \ln\left[\delta|G_i| - (1-\delta)(|K_0^0|g + |K_0^1|l)\right] - \ln\delta|G_i| \\
\iff \quad & \Lambda \geq \frac{1}{-\ln\delta} \ln\left[\frac{\delta|G_i|}{\delta|G_i| - (1-\delta)(|K_0^0|g + |K_0^1|l)}\right],
\end{aligned}$$

for sufficiently high δ for which $\ln(\cdot)$ is well defined, and that

$$\frac{1}{-\ln\delta} \ln\left[\frac{\delta|G_i|}{\delta|G_i| - (1-\delta)(|K_0^0|g + |K_0^1|l)}\right] \to \frac{|K_0^0|g + |K_0^1|l}{|G_i|} < \infty$$

as $\delta \to 1$. Since $CU_i(\sigma_i^*|_{o_i^t,D_0}, \sigma_{-i}^*; o_i^t) \geq CU_i(\sigma_i^*|_{o_i^t,D_1}, \sigma_{-i}^*; o_i^t)$, letting

$$F_{i5}(\delta) = \max_{\pi_i \in \Pi_i(5)} \left\{ \frac{1}{-\ln \delta} \ln \left[\frac{\delta |G_i|}{\delta |G_i| - (1-\delta)(|K_0^0| g + |K_0^1| l)} \right] \right\}$$

for sufficiently high δ, we complete the proof. ∎

Proof of Claim 6. In Case 6, the payoff for each strategy is as follows:

$$\begin{aligned} CU_i(\sigma^*; o_i^t) &= (1-\delta)\delta^{\lambda-1}|K_\lambda^{\lambda-1}|(1+g) \\ &\quad +(1-\delta)\delta^\lambda(|K_\lambda^{\lambda-1}| + |K_\lambda^\lambda| - |K_\lambda^{\lambda+1}| l) + \delta^{\lambda+1}|G_i|, \\ CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)\delta^\Lambda |G_i|(-l) + \delta^{\Lambda+1}|G_i|, \text{and} \\ CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|G_i|(-l) + \delta^{\Lambda+1}|G_i|, \end{aligned}$$

where $C \in \{C_0, C_1\}$.

Since $\delta^{\lambda+1} - \delta^{\Lambda+1} > 0$, $|G_i| + |K_\lambda^{\lambda-1}| + |K_\lambda^\lambda| - \delta^\lambda |K_\lambda^{\lambda+1}| > 0$ and

$$\begin{aligned} & CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\ \iff \quad & (1-\delta)\delta^{\lambda-1}|K_\lambda^{\lambda-1}|(1+g) + (1-\delta)\delta^\lambda(|K_\lambda^{\lambda-1}| + |K_\lambda^\lambda| - |K_\lambda^{\lambda+1}| l) + \delta^{\lambda+1}|G_i| \\ & \geq (1-\delta)\delta^\Lambda |G_i|(-l) + \delta^{\Lambda+1}|G_i| \\ \iff \quad & (1-\delta)\delta^{\lambda-1}|K_\lambda^{\lambda-1}|(1+g) \\ & + (1-\delta)(|G_i| + |K_\lambda^{\lambda-1}| + |K_\lambda^\lambda| - \delta^\lambda |K_\lambda^{\lambda+1}|) l + (\delta^{\lambda+1} - \delta^{\Lambda+1})|G_i| \\ & \geq 0, \end{aligned}$$

we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t)$. Furthermore, since $CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t)$, we have $CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t)$. ∎

Proof of Claim 7. In Case 7, the payoff for each strategy is as follows:

$$\begin{aligned} CU_i(\sigma^*; o_i^t) &= (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta(|K_1^1| + |K_1^0| - |K_1^2| l) + \delta^2|G_i| \\ CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)|K_1^0|(1+g) - (1-\delta)\delta^\Lambda |G_i| l + \delta^{\Lambda+1}|G_i|, \text{ and} \\ CU_i(\sigma_i^*|_{C}^{o_i^t}, \sigma_{-i}^*; o_i^t) &= (1-\delta)(|K_1^0| - (|K_1^2| + |K_1^1|) l) + \delta^{\Lambda+1}|G_i|. \end{aligned}$$

Note that

$$\begin{aligned} & CU_i(\sigma^*; o_i^t) \geq CU_i(\sigma_i^*|_{D_1}^{o_i^t}, \sigma_{-i}^*; o_i^t) \\ \iff \quad & (1-\delta)|K_1^0|(1+g) + (1-\delta)\delta(|K_1^1| + |K_1^0| - |K_1^2| l) + \delta^2|G_i| \\ & \geq (1-\delta)|K_1^0|(1+g) - (1-\delta)\delta^\Lambda |G_i| l + \delta^{\Lambda+1}|G_i| \\ \iff \quad & (1-\delta)\delta(|K_1^1| + |K_1^0| - |K_1^2| l) + \delta^2 |G_i| \geq \delta^\Lambda[\delta|G_i| - (1-\delta)|G_i| l] \\ \iff \quad & \Lambda \geq \tfrac{1}{-\ln \delta} \ln \left[\tfrac{\delta|G_i| - (1-\delta)|G_i| l}{(1-\delta)\delta(|K_1^1| + |K_1^0| - |K_1^2| l) + \delta^2 |G_i|} \right], \end{aligned}$$

where δ is sufficiently high so that $\ln(\cdot)$ is well defined, and

$$\begin{aligned} & \tfrac{1}{-\ln\delta}\ln\left[\tfrac{\delta|G_i|-(1-\delta)|G_i|l}{(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|}\right] \\ \to\ & \tfrac{|G_i|-|K_1^1|-|K_1^0|+|K_1^2|l-|G_i|l}{|G_i|}<\infty. \end{aligned}$$

Also,

$$\begin{aligned} & CU_i(\sigma^*;o_i^t)\geq CU_i(\sigma_i^*|_C^{o_i^t},\sigma_{-i}^*;o_i^t) \\ \iff\ & \quad (1-\delta)|K_1^0|(1+g)+(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i| \\ & \geq(1-\delta)(|K_1^0|-(|K_1^2|+|K_1^1|)l)+\delta^{\Lambda+1}|G_i| \\ \iff\ & \quad \delta^{\Lambda+1}|G_i| \\ & \leq(1-\delta)|K_1^0|g+(1-\delta)(|K_1^2|+|K_1^1|)l \\ & \quad +(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i| \\ \iff\ & \quad \Lambda\ln\delta+\ln\delta|G_i| \\ & \leq\ln\left[(1-\delta)|K_1^0|g+(1-\delta)(|K_1^2|+|K_1^1|)l\right. \\ & \quad \left.+(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|\right] \\ \iff\ & \Lambda\geq\tfrac{1}{-\ln\delta}\ln\left[\tfrac{\delta|G_i|}{(1-\delta)|K_1^0|g+(1-\delta)(|K_1^2|+|K_1^1|)l+(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|}\right] \end{aligned}$$

where δ is sufficiently high so that $\ln(\cdot)$ is well defined, and

$$\begin{aligned} & \tfrac{1}{-\ln\delta}\ln\left[\tfrac{\delta|G_i|}{(1-\delta)|K_1^0|g+(1-\delta)(|K_1^2|+|K_1^1|)l+(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|}\right] \\ \to\ & \tfrac{|G_i|-|K_1^0|(1+g)-(|K_1^1|+|K_1^0|)-|K_1^1|l}{|G_i|}<\infty, \end{aligned}$$

as $\delta\to 1$.

Letting

$$F_{i7}(\delta)=\max_{\pi_i\in\Pi_i(7)}\left\{\max\left\{\begin{array}{c}\tfrac{1}{-\ln\delta}\ln\left[\tfrac{\delta|G_i|-(1-\delta)|G_i|l}{(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|}\right], \\ \tfrac{1}{-\ln\delta}\ln\left[\tfrac{\delta|G_i|}{(1-\delta)|K_1^0|g+(1-\delta)(|K_1^2|+|K_1^1|)l+(1-\delta)\delta(|K_1^1|+|K_1^0|-|K_1^2|l)+\delta^2|G_i|}\right]\end{array}\right\}\right\}$$

for sufficiently high δ, we complete the proof. ∎