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The Private Value of a Patent: A Cooperative Approach

Abstract We consider a game in characteristic form played by firms in a Cournot market and an outside patent holder of a cost-reducing innovation. The worth of a coalition of players is the total Cournot profit of the active firms within the coalition which results in the Nash equilibrium of the strategic game played by the coalition and its complement; each one of them chooses the number of firms to activate. Only firms in a coalition containing the patent holder are allowed to use the new technology. We prove that when the number of firms is large, the Shapley value of the patent holder approximates the non-cooperative result obtained previously in literature. When there is a positive fixed cost component, the Nash equilibrium must be mixed in some cases, but nevertheless the result still holds.

Introduction

The private value of a patent for cost reducing innovations was studied extensively in the literature.Most work on this subject models the interaction between the patent holder and the firms as a dynamic strategic game, using a non-cooperative approach and ignoring the fact that a patent holder can sign binding licensing agreements with one or more firms in the industry. See Kamien (1992) for comprehensive survey.

One exception to the non-cooperative approach is Tauman and Watanabe (2005) (hereafter TW) who analyzed the interaction of an outside patent holder of a cost reducing innovation with the firms in a Cournot market as a cooperative game in the first stage taking into account the non-cooperative Cournot competition of the second stage.

A key problem is how to define the worth of each coalition (a subset of firms which may or may not contain the patent holder). In TW the firms are allowed to merge, and the merged entity decides how many firms to actually operate in the market. If the patent holder is a member of a coalition every firm in this coalition can use the new technology. The traditional von Neuman - Morgenstern minmax approach defines the worth of a coalition as the total Cournot profit of the active firms in the coalition when the complement uses the most offensive approach (in terms of number of active firms) against it. TW studies the Shapley value of this game and showes that increasing the number of firms in the market, the Shapley value of the innovator approximates his payoff in the non-cooperative game where the innovator enjoys full bargaining power, traditionally studied in the literature. This asymptotically equivalent result shows the robustness of the private value of a patent for a sufficiently competetive Cournot market.

In this paper we define the worth of a coalition differently. We will examine the interaction of a coalition with its complement as a strategic game where the two coalitions choose simultaneously the number of firms in their coalition to activate and the number of firms to shut down. We then compute the Nash equilibrium of this game. The worth of a coalition is defined to be the total Cournot profit of the active firms in this coalition. We show that the asymptotic result of TW holds for this case, too.

In addition, we extend the model to the case where firms have a (small) fixed cost component, which is assumed to be zero in TW. This imposes a significant difficulty in the analysis since in some cases a pure strategy Nash equilibrium does not exist. In these cases we compute a mixed strategy equilibrium and again show that the asymptotic result holds.

The model

Consider the set $N = \{1,..,n\}$ of firms in a Cournot market. Each firm produces a homogeneous commodity. The production cost is c per unit. We consider a linear demand function $Q = \max(0, a-p)$, where p is a market price and $0 < c < a < \infty$.

An outside entity has a patent on new technology which reduces the unit cost of production from c to $c-\epsilon$, where $0 < \epsilon < c$. We consider a non-drastic innovation, namely $a-c-\epsilon > 0$.

Denote by q_i the output level of firm $i \in N$. The profit of an efficient firm is $q_i(p-c+\epsilon)$. The profit of an inefficient firm is $q_i(p-c)$. The innovator is denoted by 0. Let $N_0 = N \cup \{0\}$ be the set of players.

The firms are divided into two coalitions. One coalition includes the innovator-and every firm in this coalition has an access to the new technology. The other coalition contains all the inefficient firms (who use the old technology). Let $S \subseteq N$ be any coalition of firms. It can choose a number $m \leq |S|$ of firms to operate and to shut down the other $|S|-m$ firms. Similarly, the complement coalition will choose the number $l \leq |N \backslash S|$ of firms not in S to operate. The $m+l$ active firms will compete a la' Cornot (in quantities). Denote by $q_i^E(m,l)$ the Cournot output of an efficient firm i when there are m efficient firms and l inefficient firms. We denote by $q_i^{NE}(l,m)$ the output of an inefficient firm i when there are m efficient firms and l inefficient firms. Because of the symmetry, $q_i^E(m,l)$ and $q_i^{NE}(l,m)$ don't depend on i, and hence we can omit the index i.

Let $S \subseteq N$. The total profit of $S_0 = S \cup \{0\}$ is $\Pi_{S_0}(m,l) = mq^E(m,l)(p-c+\epsilon)$ and the total profit of S is $\Pi_S(l,m) = mq^{NE}(l,m)(p-c)$.

Let $K = (a-c)/\epsilon$. For non-drastic innovations $K > 1$ [the term of drastic innovation was introduced at first by Arrow(1962). He defined a drastic innovation to be one in which monopoly price $(a+c-\epsilon)/2$ with the new technology is lower than the competitive price,c, of the old technology. That is, the innovation is drastic iff $\frac{a-c}{\epsilon} > 1$].It's easy to verify, that if number of efficient firms is equal or greater than K, the inefficient firms are driven out of the market.

Consider the following game G between S_0 and $N\backslash S$. In the first stage both S_0 and $N\backslash S$ choose simultaneously the number of firms to activate. In the second stage the decisions of the first stage become commonly known and the active firms in S and $N\backslash S$ are all compete a la' Cournot. We will analyse the subgame perfect Nash equilibrium of G.

The worth of a coalition is defined to be the total profit of its active firms in the subgame perfect Nash equilibrium. This defines a game in a characteristic function. We will compute the Shapley value of this game .

Let $f_n \geq 0$ be the fixed cost component of every operating firm. We assume that this fixed cost is small enough, such that for every $m,l,m',l' \geq 0$, for every $S \subseteq N_0$, if $\Pi_S(m,l) > \Pi_S(m',l')$, then $\Pi_S(m,l) - f_n m > \Pi_S(m',l') - f_n m'$.

The magnitude of f_n depends on the total number n of firms. It is assumed that f_n decreases with n so that the last condition holds. Denote $\Pi_S^{f_n}(m,l) = \Pi_S(m,l) - f_n m$.

We analyze two cases :$f_n = 0$ and $f_n > 0$.

The characteristic functions

Case 1:$f_n = 0$

We first state three useful Lemmas.

Lemma 1 (TW)

Consider a market with $m+l$ firms where the first m operate with the new technology and the last l firms operate with the old technology. Let $f_n = 0$. Then, the Cournot outcome is

$$q^E(m,l) = \begin{cases} \frac{a-c+(l+1)\epsilon}{m+l+1}, m < K \\ \frac{a-c+\epsilon}{m+1}, m \geq K \end{cases}$$

$$q^{NE}(l,m) = \begin{cases} \frac{a-c-m\epsilon}{m+l+1}, m < K \\ 0, m \geq K \end{cases}$$

The profit of each firm is $(q_i)^2$,where $i = 1,..,m,..,m+l$.

Lemma 2 (TW)

Let $S \subseteq N$ with $|S| \geq 1$ be the set of licensee firms. Let $f_n = 0$. If the coalition $N \setminus S$ operates l non-licensee firms, where $1 \leq l \leq n - |S|$ then the optimal number $m(S,l)$ of licensee firms that S should operate is

$$m(S,l) = \begin{cases} \min(|S|, l+1) & , 1 \leq |S| < K \\ l+1 & , |S| \geq K > l+1 \\ K & , otherwise \end{cases}$$

The proof of Lemma 2 can be found in TW. Moreover, it is shown there that

$$m < K \rightarrow [\Pi_S(m,l) > \Pi_S(K,l) \Leftrightarrow mK > (l+1)^2] \qquad (1)$$

Lemma 3

Let $S \subseteq N$ be the set of non-licensee firms,let $|N \setminus S| \geq 1$and $f_n = 0$. If the coalition S operates m licensee firms, where $1 \leq m \leq |S|$, and $m < K$, then the optimal number $l(S,m)$ of licensee firms that $N \setminus S$ operates is $m(S,l) = \min(|N \setminus S|, m+1)$.

Proof By Lemma 1, the payoff of $N \setminus S$ is $\Pi_{N \setminus S}(m,l) = l(\frac{a-c+m\epsilon}{m+l+1})^2$, which is maximized for $l = m+1$, if $m+1 \leq |N \setminus S|$, and for $m = |N \setminus S|$, otherwise. ■

When $m \geq K$, the inefficient coalition can activate any number l of firms, and by Lemma 1 its Cournot profit will always be 0.

Let $V(S)$ be the worth of a coalition S . Namely, $V(S)$ is the total Cournot profit of the active firms in S in the subgame perfect Nash equilibrium of G.

Proposition 1 *Consider a non-drastic innovation,* $a - c - \epsilon > 0$*, and suppose that* $f_n = 0$*.* Let $S_0 = S \cup \{0\}$, $S \subseteq N$.Then,

(i) $K \leq \frac{n-1}{2}$.

$$V(S_0) = \begin{cases} (|S|)\left(\frac{a-c+(|S|+2)\epsilon}{2|S|+2}\right)^2 & , |S| < K \\ \frac{(a-c+\epsilon(|N \setminus S|+1))^2}{2(|N \setminus S|+1)} & , |S| \geq K, |N \setminus S|+1 < K \\ \epsilon(a-c) & , |S| > K, |N \setminus S|+1 \geq K \end{cases}$$

(ii) $K > \frac{n-1}{2}$.

$$V(S_0) = \begin{cases} (|S|)\left(\frac{a-c+(|S|+2)\epsilon}{2|S|+2}\right)^2 & ,|S| < \frac{n-1}{2} \\ |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2 & ,\frac{n-1}{2} < |S| < \frac{n+1}{2} \\ (n-|S|+1)\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2 & ,|S| \geq \frac{n+1}{2} \\ (n-|S|+1)\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2 & ,|S| \geq K, |N \setminus S|+1 < K \\ \epsilon(a-c) & ,K \leq |S|, |N \setminus S|+1 \geq K \end{cases}$$

(iii) $K \leq \frac{n-1}{2}$.

$$V(S) = \begin{cases} (n-|S|+1)\left(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2 & ,n-|S| < K \\ 0 & ,n-|S| \geq K, |S|+1 \geq K \\ |S|(\frac{a-c-|S|\epsilon}{2(|S|+1)})^2 & ,n-|S| \geq K \wedge |S|+1 < K \end{cases}$$

(iv) $K > \frac{n-1}{2}$

$$V(S) = \begin{cases} (|S|)\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+2}\right)^2 & ,n-|S| < K, |S| < \frac{n-1}{2} \text{ (this is possible only if } K > \frac{n+1}{2}) \\ |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2 & ,\frac{n-1}{2} < |S| < \frac{n+1}{2}, |S| > n-K \\ (n-|S|+1)\left(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2 & ,|S| \geq \frac{n+1}{2} \\ 0 & ,n-|S| \geq K, |S|+1 \geq K \\ |S|(\frac{a-c-|S|\epsilon}{2(|S|+1)})^2 & ,|S| < \frac{n-1}{2}, |S|+1 < K \end{cases}$$

Proof

See Appendix.

Case 2:$f_n > 0$

By our assumption that f_n is sufficiently small the analysis is the same as in the case $f_n = 0$, with one significant difference: suppose that $0 \in S$, $|S| \geq K$, $|N \backslash S| + 1 \geq K$. Then there is no equilibrium in pure strategies.First note that $l > 0$ is not possible in equilibrium. If $l > 0$ then $m = K$ and $q(m,l) = 0$ (this is derived by the same arguments used to prove Proposotion 1, together with Lemma 2). Hence $\Pi^{f_n}_{N \setminus S}(m,l) = 0 - lf_n$. But then $N \setminus S$ is better off operating no firms. The case where $l = 0$ is also not possible in equilibrium, because the optimal strategy for S_0 is $m = 1$ (a monopoly). But then, by Lemma 2, $N \setminus S$ can improve upon by operating at least one firm. Hence, there is no subgame perfect equilibrium point in pure strategies for this case. we need therefore to compute the mixed strategy equilibrium. In all other cases a pure strategies equilibrium exists and its outcome is unique. As we have noted above, the computation of the pure strategies equilibrium is similar to the case where the fixed cost is zero.

Let $f_n > 0$ and let $\alpha_1 = (p_0, \ldots, p_{|S|}, t_0, \ldots, t_{|N \setminus S|})$ be defined as follows:

$t_i = 0$, if $i \neq K-2, i \neq K-1$

$$p_K = 1 - \frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}$$

$p_i = 0, i \neq K-1, K$

$$t_{K-2} = \frac{K-(K-1)\left(\frac{2K}{2K-1}\right)^2 - \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2\right) - \left(\frac{2K}{2K-1}\right)^2)}$$

$$t_{K-1} = \frac{(K-1)\left(\frac{2K-1}{2K-2}\right)^2 - K + \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2 - \left(\frac{2K}{2K-1}\right)^2\right)}$$

$t_i = 0$, if $i \neq K-2, i \neq K-1$

Lemma 4

For $f_n > 0$ sufficiently small, the vector α_1 is a subgame perfect mixed strategy equilibrium.

Proof See Appendix

The worth of S_0 is the total profit of S_0 in the mixed strategy equilibrium. It is given in the following complicated formula :

$$V(S_0) = \left(\frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)\left(\frac{K-(K-1)\left(\frac{2K}{2K-1}\right)^2 - \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2 - \left(\frac{2K}{2K-1}\right)^2\right)}\right)\epsilon^2\left(\frac{2K-1}{2K-2}\right)^2(K-1) +$$

$$+\left(\frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)\left(\frac{(K-1)\left(\frac{2K-1}{2K-2}\right)^2 - K + \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2 - \left(\frac{2K}{2K-1}\right)^2\right)}\right)\epsilon^2\left(\frac{2K}{2K-1}\right)^2(K-1) -$$

$$-\left(\frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)(K-1)f_n +$$

$$+\left(1 - \frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)\left(\frac{K-(K-1)\left(\frac{2K}{2K-1}\right)^2 - \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2 - \left(\frac{2K}{2K-1}\right)^2\right)}\right)\epsilon^2 K +$$

$$+\left(1 - \frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)\left(\frac{(K-1)\left(\frac{2K-1}{2K-2}\right)^2 - K + \frac{f_n}{\epsilon^2}}{(K-1)\left(\left(\frac{2K-1}{2K-2}\right)^2 - \left(\frac{2K}{2K-1}\right)^2\right)}\right)\epsilon^2 K -$$

$$-\left(1 - \frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)Kf_n$$

Thus,

$$V(S_0) = \left(\frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)(\epsilon^2 K - f_n) - \left(\frac{f_n}{\epsilon^2\left(\frac{K-1}{(2K-1)^2} - \frac{K-2}{(2K-2)^2}\right)}\right)(K-1)f_n +$$

$+(1-\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})})\epsilon^2 K-(1-\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})})f_n K$

or

$$V(S_0)=\epsilon^2 K-Kf_n=\epsilon(a-c)-Kf_n \tag{2}$$

Similarly let $f_n>0$ and let

$\alpha_2=(p_0,\ldots,p_{|S|},t_0,\ldots,t_{|N\setminus S|})$ be defined as follows:

$p_{K-2}=\frac{K-(K-1)(\frac{2K}{2K-1})^2-\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)}$

$p_{K-1}=\frac{(K-1)(\frac{2K-1}{2K-1})^2-K+\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)}$

$p_i=0$, if $i\neq K-2, i\neq K-1$

$t_{K-1}=\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})}$

$t_K=1-\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})}$

$t_i=0$, if $i\neq K-1, i\neq K$

Then α_2 is an equilibrium point in the case where $0\in N\setminus S$, $n-|S|\geq K$ and $|S|+1\geq K$.

The total profit of S in this equilibrium is:

$V(S)=(\frac{K-(K-1)(\frac{2K}{2K-1})^2-\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)})(\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})})(\frac{1}{2K-2})^2\epsilon^2(K-2)+0-$

$-(\frac{K-(K-1)(\frac{2K}{2K-1})^2-\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)})f_n(K-2)+$

$+(\frac{(K-1)(\frac{2K-1}{2K-1})^2-K+\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)})(\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})})(\frac{1}{2K-1})^2\epsilon^2(K-1)+0-$

$-(\frac{(K-1)(\frac{2K-1}{2K-1})^2-K+\frac{f_n}{\epsilon^2}}{(K-1)((\frac{2K-1}{2K-2})^2-(\frac{2K}{2K-1})^2)})f_n(K-1).$

It can be verified that

$$V(S)=\frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})}\epsilon^2\frac{K-1}{(2K-1)^2}-f_n(K-1) \tag{3}$$

We summarize the above in the next proposition.

Proposition 2 *Consider a non-drastic innovation* $a-c-\epsilon>0$ *,and let* $f_n>0$. Then,

(i) $K\leq\frac{n-1}{2}$.

$$V(S_0)=\begin{cases}(|S|)\left(\frac{a-c+(|S|+2)\epsilon}{2|S|+2}\right)^2-|S|f_n & ,|S|<K\\ \frac{(a-c+\epsilon(|N\setminus S|+1))^2}{2(|N\setminus S|+1)}-|N\setminus S|f_n & ,|S|\geq K,|N\setminus S|+1<K\\ \epsilon(a-c)-Kf_n & ,K\leq|S|,|N\setminus S|+1\geq K\end{cases}$$

(ii) $K>\frac{n-1}{2}$.

$$V(S_0)=\begin{cases}(|S|)\left(\frac{a-c+(|S|+2)\epsilon}{2|S|+2}\right)^2-f_n|S| & ,|S|<\frac{n-1}{2}\\ |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2-f_n|S| & ,\frac{n-1}{2}<|S|<\frac{n+1}{2}\\ (n-|S|+1)(\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2-f_n) & ,|S|\geq\frac{n+1}{2}\\ (n-|S|+1)\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2-f_n(n-|S|+1) & ,|S|\geq K,|N\setminus S|+1<K\\ \epsilon(a-c)-f_nK & ,K\leq|S|,|N\setminus S|+1\geq K\end{cases}$$

(iii) $K\leq\frac{n-1}{2}$.

$$V(S)=\begin{cases}(n-|S|+1)\left(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2-(n-|S|+1)f_n & ,n-|S|<K\\ \frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})}\epsilon^2\frac{K-1}{(2K-1)^2}-f_n(K-1) & ,n-|S|\geq K,|S|+1\geq K\\ |S|(\frac{a-c-|S|\epsilon}{2(|S|+1)})^2-|S|f_n & ,n-|S|\geq K,|S|+1<K\end{cases}$$

(iv) $K>\frac{n-1}{2}$

$$V(S)=\begin{cases}(|S|)\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+2}\right)^2 & ,n-|S|<K,|S|<\frac{n-1}{2}\ \text{(is possible only if } K>\frac{n+1}{2}\text{)}\\ |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2 & ,\frac{n-1}{2}<|S|<\frac{n+1}{2},|S|>n-K\\ (n-|S|+1)(\left(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2-f_n) & ,|S|\geq\frac{n+1}{2}\\ |S|(\frac{a-c-|S|\epsilon}{2(|S|+1)})^2-f_n|S| & ,n-|S|\geq K,|S|+1<K\\ \frac{f_n}{\epsilon^2(\frac{K-1}{(2K-1)^2}-\frac{K-2}{(2K-2)^2})}\epsilon^2\frac{K-1}{(2K-1)^2}-f_n(K-1) & ,n-|S|\geq K,|S|+1\geq K\end{cases}$$

The Shapley value of V.

Let $\Re=(j_0,j_{1,..,}j_n)$ be an order of the players in N_0. Let $Þ_j^{\Re}=\{j'\in N_0|j'\Re j\}$ be the set of players in N_0 who precede player $j\in N_0$ in the order $\Re$. Let $(Sh_i(V))_{i\in N_0}$ be the Shapley value of V.

Fix an order $\Re(|S|+1)$ where the patent holder is located at $(|S|+1)$. Then $Þ_0^{\Re(|S|+1)}$ is the set of firms that precede the patent holder 0 in $\Re(|S|+1)$. Since $V(s)$ is a function of $|S|$:

$$Sh_0(V) = \frac{1}{n+1}\sum_{|S|=0}^{n}\Big[V\Big(Þ_0^{\Re(|S|+1)} \cup \{0\}\Big) - V\Big(Þ_0^{\Re(|S|+1)}\Big)\Big] \quad . \tag{5}$$

By efficiency and symmetry, for each firm i :

$$Sh_0(V) + nSh_i(V) = V(N_0) \tag{6}$$

We will compute the Shapley value for two cases.

Case 1: The zero fixed cost

Proposition 4 *Consider the game G with n firms. Let $f_n = 0$ be their fixed cost. Then,* $\lim_{n\to\infty} Sh_0(V) = \epsilon(a-c)$ *and* $\lim_{n\to\infty}\sum_{i\in N} Sh_i(V) = (\frac{a-c-\epsilon}{2})^2$.

Proof

Let n be big enough, $K < \frac{n-1}{2}$. Then $K < n-K$. By (1) and by parts (i) and (iii) of Proposition 1,

$$Sh_0(V) = \tfrac{1}{n+1}\sum_{|S|=1}^{K-2}(|S|)\Big(\Big(\tfrac{a-c+(|S|+2)\epsilon}{2|S|+2}\Big)^2 - \Big(\tfrac{a-c-|S|\epsilon}{2|S|+2}\Big)^2\Big) + \tfrac{1}{n+1}\Big((|S|)\Big(\Big(\tfrac{a-c+(|S|+2)\epsilon}{2|S|+2}\Big)^2\Big) +$$

$$+\tfrac{1}{n+1}\epsilon(a-c) + \tfrac{1}{n+1}\sum_{|S|=K+1}^{n-K}\epsilon(a-c) + \tfrac{1}{n+1}\Big(\epsilon(a-c) - (n-|S|+1)\Big(\tfrac{a-c-(n-|S|)\epsilon}{2(n-|S|+1)}\Big)^2\Big) +$$

$$+\tfrac{1}{n+1}\sum_{|S|=n-K+2}^{n-1}\Big\{(n-|S|+1)\Big(\tfrac{a-c+(n-|S|)\epsilon}{2(n-|S|)+2}\Big)^2 - (n-|S|+1)\Big(\tfrac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\Big)^2\Big\} +$$

$$+\tfrac{1}{n+1}\Big\{\Big(\tfrac{a-c+\epsilon}{2}\Big)^2 - \Big(\tfrac{a-c}{2}\Big)^2\Big\} \to \epsilon(a-c), \text{ as } n\to\infty.$$

Since $V(N_0) = (\frac{a-c+\epsilon}{2})^2$, by (6) we have $Sh_i(V) = \frac{1}{n}((\frac{a-c+\epsilon}{2})^2 - Sh_0(V))$.

$\sum_{i\in N} Sh_i(V) = (\frac{a-c+\epsilon}{2})^2 - Sh_0(V) \to (\frac{a-c+\epsilon}{2})^2 - \epsilon(a-c) = (\frac{a-c-\epsilon}{2})^2$. ■

Case 2: positive fixed cost

Proposition 5 *Consider the game game G with n firms. Let $f_n > 0$ be their fixed cost (which is small enough). Assume that $f_n \to 0$ as $n\to\infty$. Then* $\lim_{n\to\infty} Sh_0(v) = \epsilon(a-c)$ *and*

$\lim_{n\to\infty}\sum_{i\in N} Sh_i(v) = (\frac{a-c-\epsilon}{2})^2$.

Proof

Let n be sufficiently large, $K < \frac{n-1}{2}$. Then $K < n-K$. By (1) and by parts (i) and (iii) of Proposition 1,

$$Sh_0(v) = \tfrac{1}{n+1}\sum_{|S|=1}^{K-2}(|S|)\Big(\Big(\tfrac{a-c+(|S|+2)\epsilon}{2|S|+2}\Big)^2 - \Big(\tfrac{a-c-|S|\epsilon}{2|S|+2}\Big)^2\Big) + \tfrac{1}{n+1}\Big((|S|)\Big(\Big(\tfrac{a-c+(|S|+2)\epsilon}{2|S|+2}\Big)^2 -$$

$$-\Big(1 - \frac{f_n}{\epsilon^2\big(\frac{K-1}{(2k-1)^2} - \frac{K-2}{(2K-2)^2}\big)}\Big)\epsilon^2\tfrac{K-1}{(2K-1)^2} - f_n(K-1)\Big) + \tfrac{1}{n+1}\Big(\epsilon(a-c) - Kf_n -$$

$-(1-\frac{f_n}{\epsilon^2(\frac{K-1}{(2k-1)^2}-\frac{K-2}{(2K-2)^2})})\epsilon^2\frac{K-1}{(2K-1)^2}-$

$-f_n(K-1)) + \frac{1}{n+1}\sum_{|S|=K+1}^{n-K}(\epsilon(a-c)-Kf_n-$

$-(1-\frac{f_n}{\epsilon^2(\frac{K-1}{(2k-1)^2}-\frac{K-2}{(2K-2)^2})})\epsilon^2\frac{K-1}{(2K-1)^2}-f_n(K-1)) +$

$+\frac{1}{n+1}(\epsilon(a-c)-Kf_n-(n-|S|+1)(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|+1)})^2+(n-|S|+1)f_n)+$

$+\frac{1}{n+1}\sum_{|S|=n-K+2}^{n-1}(n-|S|+1)(\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2-\left(\frac{a-c+(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2) \to \epsilon(a-c)$ as $n\to\infty$ (and $f_n\to 0$).

Since $v(N_0)=(\frac{a-c+\epsilon}{2})^2-f_n$, by (6) we have $Sh_i=\frac{1}{n}((\frac{a-c+\epsilon}{2})^2-f_n-Sh_0(v))$.

$\sum_{i\in N}Sh_i(v)=(\frac{a-c+\epsilon}{2})^2-f_n-Sh_0(v)\to(\frac{a-c+\epsilon}{2})^2-\epsilon(a-c)=(\frac{a-c-\epsilon}{2})^2$. ■

Conclusions

We extended the result of Tauman and Watanabe (2005) to the case where the characteristic function is defined by the Nash equilibrium concept rather than by Minmax or the Maxmin concept. We prove that the equivalence result between the cooperative and the non-cooperative approach (with or without a small fixed cost) still holds for large markets.

Moreover, it can be shown, that when the coalitions determines a number of firms they operate non-simultaneously, and the coalition which includes the innovator takes its advantage and is the first to determine the number of its active firms, and innovation is significant enough, the equivalence result still holds. For less significant innovations, the equivalence doesn't hold, and value of a patent is smaller than in the simultaneous case.

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Appendix

Proof of proposition 1

Assume $f_n = 0$.

Let $S_0 = S \cup \{0\}, N_0 = N \cup \{0\}$.

Consider the case: $|S| < K$:

By Lemmas 2 and 3, the case $m < |S|, l < |N \setminus S|$ in the equilibrium is not possible (because every coalition can better off with operating one more firm). So, the only possible equilibriums are:

a. $|\boldsymbol{S}| > \frac{\boldsymbol{n}+\mathbf{1}}{\mathbf{2}} \Rightarrow |\boldsymbol{S}| > \boldsymbol{n} - |\boldsymbol{S}| + \mathbf{1} \Rightarrow |\boldsymbol{S}| > |\boldsymbol{N} \setminus \boldsymbol{S}| + \mathbf{1} \Rightarrow \boldsymbol{m} = \boldsymbol{l} + \mathbf{1}, \boldsymbol{l} = |\boldsymbol{N} \setminus \boldsymbol{S}| \Rightarrow \boldsymbol{l} = \boldsymbol{n} - |\boldsymbol{S}|$,

$m = n - |S| + 1$.

By Lemma 1, $\Pi_S(n - |S| + 1, n - |S|) = (n - |S| + 1)\left(\frac{a-c+(n-|S|+1)\epsilon}{2(n-|S|)+2}\right)^2$.

b. $|\boldsymbol{S}| \leq \frac{\boldsymbol{n}-1}{2} \Rightarrow \boldsymbol{n}-|\boldsymbol{S}| \geq |\boldsymbol{S}|+\mathbf{1} \Rightarrow |\boldsymbol{N} \setminus \boldsymbol{S}| \geq |\boldsymbol{S}|+\mathbf{1}$

This implies, by Lemmas 2 and 3, $m = |S|, l = |S|+1$.

$\Pi_S(|S|,|S|+1) = (|S|)\left(\frac{a-c+(|S|+2)\epsilon}{2|S|+2}\right)^2$.

c. $|\boldsymbol{S}| = \frac{\boldsymbol{n}}{2}, \boldsymbol{n}$ **is even** $\Rightarrow \boldsymbol{m} = |\boldsymbol{S}| = \boldsymbol{l} = |\boldsymbol{N} \setminus \boldsymbol{S}|$.

$\Pi_S(|S|,|S|) = |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2$.

Consider the case: $|S| \geq K$

By Lemma 2, for $l \geq 0,\ l+1 < K$, the optimal m for S_0 is $m = l+1$. For $l+1 \geq K$ the optimal strategy for S_0 is $m = K$ (note, that $q_j(m,l) = 0$ in this case). So, for $|N \setminus S|+1 < K$, we can apply also Lemma 3 for the area $l \leq |N \setminus S|$, and obtain the unique equilibrium in $m = |N \setminus S|+1, l = |N \setminus S|$.

$\Pi_S(|N \setminus S|+1, |N \setminus S|) = (|N \setminus S|+1)\left(\frac{a-c+\epsilon(|N \setminus S|+1)}{2(|N \setminus S|+1)}\right)^2 = \frac{(a-c+\epsilon(|N \setminus S|+1))^2}{2(|N \setminus S|+1)}$.

For $|N \setminus S|+1 \geq K$, there is no equilibrium for $m < K$ (by Lemmas 2 and 3). So, the equilibrium is $m = K, l = K-1$ or $l = K$. In this case $\Pi_S(K,l) = K\left(\frac{a-c+\epsilon}{K+1}\right)^2 = \epsilon(a-c)$.

Now, let $0 \in N_0 \setminus S$.

Consider the case: $|S| > n-K$

By arguments similar to those above, the case $m < |S|, l < |N \setminus S|$ in the equilibrium is not possible. So:

a. $|\boldsymbol{S}| \geq \frac{\boldsymbol{n}+1}{2} \Rightarrow |\boldsymbol{S}| > \boldsymbol{n}-|\boldsymbol{S}|+\mathbf{1} \Rightarrow |\boldsymbol{S}| > |\boldsymbol{N} \setminus \boldsymbol{S}|+\mathbf{1} \Rightarrow \boldsymbol{m} = \boldsymbol{l}+\mathbf{1},$

$l = |N \setminus S| \Rightarrow l = n-|S|, m = n-|S|+1$.

$\Pi_S(n-|S|+1, n-|S|) = (n-|S|+1)\left(\frac{a-c-(n-|S|)\epsilon}{2(n-|S|)+2}\right)^2$.

b. $|\boldsymbol{S}| \leq \frac{\boldsymbol{n}-1}{2} \Rightarrow \boldsymbol{n}-|\boldsymbol{S}| \geq |\boldsymbol{S}|+\mathbf{1} \Rightarrow |\boldsymbol{N} \setminus \boldsymbol{S}| \geq |\boldsymbol{S}|+\mathbf{1}$

By Lemmas 2 and 3, $m = |S|, l = |S|+1$.

$\Pi_S(|S|,|S|+1) = (|S|)\left(\frac{a-c-(|S|+1)\epsilon}{2|S|+2}\right)^2$.

c. $|\boldsymbol{S}| = \frac{\boldsymbol{n}}{2}, \boldsymbol{n}$ **is even** $\Rightarrow \boldsymbol{m} = |\boldsymbol{S}| = \boldsymbol{l} = |\boldsymbol{N} \setminus \boldsymbol{S}|$.

$\Pi_S(|S|,|S|) = |S|\left(\frac{a-c+(|S|+1)\epsilon}{2|S|+1}\right)^2$.

Consider the case: $|S| \leq n-K$.

By the arguments similar to those above we can show:

a. $|\boldsymbol{S}|+\mathbf{1} < \boldsymbol{K}$

The equilibrium is $m = |S|, l = |S|+1$, and

$\Pi_S(|S|,|S|+1) = |S|\left(\frac{a-c-|S|\epsilon}{2(|S|+1)}\right)^2$

b. $|\boldsymbol{S}|+\mathbf{1} \geq \boldsymbol{K}$

In equilibrium either $m = K - 1$ or $m = K$, and $l = K$. By Lemma 1, $\Pi_S(m,K) = 0$.

If we summarize all the above, we have Proposition 1.

Proof of Lemma 4

(1) It is easy to verify that:

$p_i \geq 0, q_i \geq 0$ for every i. $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} t_i = 1$. Thus, the vectors α_1 and α_2 are well defined.

(2) S is indifferent between $m = K - 1$ and $m = K$. For $m' \leq K - 2$, $m'K < (K-1)^2 = (l+1)^2$ and hence by (1) S obtains a payoff which is lower than its payoff when $m = K$ (or $m = K - 1$).

(3) $N \setminus S$ is indifferent between $l = K - 2$ and $l = K - 1$ (straightforward).

(4) Let us prove that $N \setminus S$ is worse off with $l < K - 2$ or with $l > K - 1$ relative to either $l = K - 2$ or $l = K - 1$.

Let $E\Pi_{N\setminus S}^{f_n}(l) = p_K \Pi_{N\setminus S}^{f_n}(K,l) + p_{K-1}\Pi_{N\setminus S}^{f_n}(K-1,l)$ be the expected payoff of $N \setminus S$ when S chooses the strategies $m = K$ and $m = K - 1$ with probabilities p_K and p_{K-1}, respectively.

The difference between the expected payoffs of $N \setminus S$ once with strategy l and once with strategy $l - 1$ is:

$$\Delta(l) = E\Pi_{N\setminus S}^{f_n}(l) - E\Pi_{N\setminus S}^{f_n}(l-1) = p_K\Pi_{N\setminus S}^{f_n}(K,l) + p_{K-1}\Pi_{N\setminus S}^{f_n}(K-1,l) - p_K\Pi_{N\setminus S}^{f_n}(K,l-1) -$$
$$-p_{K-1}\Pi_{N\setminus S}^{f_n}(K-1,l-1) = p_K\Pi_{N\setminus S}(K,l) + p_{K-1}\Pi_{N\setminus S}(K-1,l) -$$
$$-lf_n - p_K\Pi_{N\setminus S}(K,l-1) - p_{K-1}\Pi_{N\setminus S}(K-1,l-1) + (l-1)f_n =$$
$$= p_{K-1}\epsilon^2\left(\frac{l}{(K+l)^2} - \frac{l-1}{(K+l-1)^2}\right) - f_n.$$

The function $\Delta(l)$ decreases in l. This together with the fact that $\Delta(K-1) = 0$ (a straightforward calculation), we have for $l < K - 1$ that $\Delta(l)$ is positive. That means, that the expected payoff of $N \setminus S$ increases in l. So, the expected payoff of $N \setminus S$ with $l < K - 2$ is smaller than when $l = K - 2$. On the other hand, it requires that $\Delta(K) < 0$ hence the expected value with $l = K$ is smaller than when $l < K - 1$. ■