Announcement or Contribution?
The Relative Efficiency of Manipulated Lindahl Mechanisms

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Abstract

The private provision mechanism is individually incentive compatible but inefficient. The Lindahl mechanism is efficient but not incentive compatible. We contrast the outcome of the manipulated Lindahl mechanism to the private provision equilibrium. When the demand announcements of participants are unrestricted the Lindahl mechanism suffers from multiple equilibria. If the government removes the multiplicity by restricting the functional form of announcements the resulting Lindahl equilibrium can be made approximately efficient. Approximate efficiency is achieved by announcements that are one-dimensional regardless of the number of participants in the mechanism. This is in contrast to mechanisms that achieve exact efficiency but require announcements whose dimensionality increases at the same rate as the number of participants. The mechanism we describe benefits from simplicity at the cost of approximate efficiency.

JEL Classification: H3, H4

Keywords: Lindahl equilibrium; mechanism design, demand announcement; approximate efficiency

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1 Introduction

The Lindahl mechanism was introduced to economics by Lindahl (1919) and first formalized by Johansen (1963). The basis of the mechanism is that each participant announces a demand function for a public good with the cost share as the argument of the function. An equilibrium of the mechanism is a set of cost shares and a level of public good that simultaneously satisfy the demand functions and the need for the cost shares to sum to one. If all participants act honestly and announce demand functions that reflect preferences, the equilibrium is efficient: it selects a point from the "Samuelson set" of efficient allocations for the public good economy. Unfortunately, the Lindahl mechanism is not incentive compatible. By announcing a false demand function a participant in the mechanism can gain by increasing the share of the public good financed by other participants, even though this will reduce the quantity of public good in equilibrium. The private provision of a public good can be seen as an alternative allocation mechanism. In this mechanism the strategy of each participant is a level of contribution to the public good. The private provision mechanism is incentive compatible and, under standard assumptions, has a unique equilibrium (Bergstrom, Blume and Varian 1986, 1992). However, the private provision equilibrium is not efficient and a simultaneous increase in provision by all participants is Pareto-improving.

These properties of the Lindahl and private provision mechanisms are all well-known and several surveys are available (Cornes and Sandler 1996, Myles 1995). What is not clear in the literature are the properties of the equilibrium that arises if all participants in a Lindahl mechanism act strategically and, in particular, how the private provision mechanism performs relative to the manipulated Lindahl. Sertel and Sanver (1999) construct a set of conditions under which false announcement by all participants in the Lindahl mechanism leads to the private provision equilibrium. However, it is assumed that the government, who is the operator of the mechanism, knows the preferences of the participants and is uninformed only about endowments. What we wish to consider is the Lindahl mechanism when the government knows neither preferences nor endowments. We assume that all participants will attempt to manipulate the mechanism with freedom to choose from a general set of announcements. Our intention is to establish the existence of an equilibrium for this situation and to derive its properties, focussing in particular upon how it compares to the private provision equilibrium. We also wish to consider the role that government intervention can play in limiting manipulation when the government is uninformed.

Our motivation for pursuing this inquiry is that the Lindahl mechanism is an eminently practical method of determining public good provision. Since it relies only on the announcement of a demand function by each participant it is a simple and practical way of eliciting valuations and determining an equilibrium allocation. Its usefulness is amplified by that fact that it does not require any information on preferences or endowments. At the heart of our analysis is the question of whether we can retain any of these appealing properties and at the

same time reduce the manipulation of the mechanism. Looking ahead, our main results show that by placing restrictions upon the demand functions that can be announced we can construct a simple mechanism that achieves an equilibrium allocation that is approximately efficient. It should be noted at this point that the Lindahl mechanism (with honest revelation) decentralizes a single point of the Samuelson set of efficient allocations. This feature of the Lindahl mechanism is often overlooked in the literature. The mechanism we construct also approximately decentralizes a single point but this need not be the same point as for the Lindahl mechanism with honest revelation: both points are Pareto efficient but may differ in the cost share of the public good assigned to each participant. Although the equilibria are in the Samuelson set there may be nothing especially attractive about the distributional aspects of either allocation. What we do show in this respect is that, under certain conditions, the approximately efficient point Pareto-dominates the private provision equilibrium.

The paper contributes to the extensive literature on mechanisms for the decentralization of efficient equilibria with public goods. Bergstrom (1970) described the *distributive Lindahl mechanism* in which each participant announces a set of cost shares (for each good, for each participant) and prices are adjusted to achieve an equilibrium which is necessarily efficient. This mechanism has recently been generalized by Tian (2003) to ensure that it is safe from the formation of coalitions. Varian (1994) provides a *compensation mechanism* that decentralizes efficient equilibria. This mechanism involves each participant announcing a vector of Pigouvian taxes (equal in dimensionality to the number of participants). Each participant is then subjected to the taxes announced by other participants plus a side-payment based on the deviation of his Pigouvian taxes from those of the other players. Anderlini and Siconolfi (2004) also construct a mechanism that decentralizes the set of efficient equilibria. In their mechanism the government announces a set of tax shares, then each consumer announces the contribution they wish to make to the public good as an addition to that already announced by other consumers plus a set of (non-negative) transfers of endowment they wish to make. It is shown that this achieves an efficient outcome provided that there are at least three consumers (the same requirement appears in the analysis by Cornes and Sandler (2000) of Pareto-improvements in the private contribution model). However, the mechanism does not determine the tax shares nor generate the information necessary to derive these shares from welfare maximization. It should be noted that the strategy announced by participants in each of these mechanisms has dimensionality at least equal to the number of participants, and sometimes equal to the square of this number. As a consequence the strategies becomes increasingly complex (as measured by dimensionality) as the number of participants increases. The complexity of messages is avoided in the mechanism described in Groves and Ledyard (1977, 1980). This is achieved by changing the message space from the communication of prices (or taxes) to the communication of quantities. Their "optimal government" utilizes a quadratic rule for computing the charge levied on each consumer given the messages received (increments to the level of public good), and decentralizes an efficient allocation. Further developments of this

class of mechanisms are surveyed in Groves and Ledyard (1987).

Where we depart from this literature is that we remain with the central Lindahlian concept of consumers announcing public good demand functions which, in turn, determine tax shares and the level of public good. It is known that the equilibrium of the Lindahl mechanism is not incentive compatible so announced demands will not be true demands. What has not been clarified in the literature is the equilibrium that emerges when all participants in the Lindahl mechanism attempt to manipulate it by announcing false demands, or even whether an equilibrium exists in such a case. The closest existing work is by Otani and Sicilian (1982) who study the equilibrium of demand announcements in private good economies. We prove that an equilibrium does exist, and use this fact as a starting point for analyzing how the Lindahl mechanism can be improved if the manipulation is taken into account. We show that by restricting the permissible functional form of the demand announcement it is possible to construct a mechanism in which each participant announces a single parameter and the resulting equilibrium is approximately efficient. We do not use approximately efficient here to mean that it approaches efficiency as population size becomes large, but instead to refer to the limit as a parameter in the announced demand tends to a specific value. The important feature of the mechanism is that each participant makes a one-dimensional announcement regardless of the size of the population. The practical value of the mechanism is that the announcement is no more that the statement of a demand for the public good as a function of the tax share. Hence, our mechanism can elicit the necessary information with a single question that can be easily implemented. Compared to other mechanisms in the literature we obtain simplicity at the cost of approximate efficiency.

The second section of the paper introduces the notation and briefly describes the private provision model and the Lindahl mechanism. Section 3 presents the general results on the existence and multiplicity of equilibria with general demand announcements in the Lindahl mechanism. The consequences of using parametric representation of possible announcements are considered in Section 4. Sections 5 and 6 demonstrates how approximate efficiency can be achieved for mechanisms with linear and hyperbolic announcements respectively. Conclusions are given in Section 7.

2   Equilibrium provision

The economy we analyze has a single private good and a single public good. Production of both goods is subject to constant returns to scale. The units of measurement are chosen so that the price of both goods is constant at 1. We call the agents that are involved in the allocation mechanisms *participants*. This neutral terminology is chosen to capture the fact that the participants can be consumers, firms, or countries. There are H participants indexed $h = 1, ..., H$. The income of participant h is fixed at $M_h \geq 0$. Participant h has preferences represented by the utility function

$$U^h = U^h\left(x_h, G\right), \tag{1}$$

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where x_h is the consumption of the private good and G the total quantity of public good. In their analysis of private provision Bergstrom, Blume, and Varian (1986) place restrictions upon the choices arising from (1). The conditions of Assumption 1 can be shown to be sufficient to imply the restrictions of Bergstrom, Blume and Varian.

Assumption 1 *The utility function is twice continuously differentiable and satisfies:*
(i) $U^h (x_h, G)$ is strictly concave;
(ii) $U^h_{xG} \geq 0$.

2.1 Private provision mechanism

In the private provision mechanism participant h makes a contribution $g_h \geq 0$ to the public good and $G = \sum_{h=1}^{H} g_h$. If $g_h > 0$ participant h is termed a *contributor* and is a *non-contributor* if $g_h = 0$. The contribution towards the public good by all participants other than h, $\overline{G}_h$, is defined by $\overline{G}_h = G - g_h$. Using the budget constraint $x_h + g_h = M_h$, utility can be written in terms of $\overline{G}_h$ and g_h as

$$U^h (x_h, G) = U^h \left(M_h - g_h, g_h + \overline{G}_h \right). \qquad (2)$$

Participant h chooses g_h to maximize (2) given $\overline{G}_h$ and subject to $g_h \in [0, M_h]$. The Nash reaction function can be written as $g_h = \rho_h \left(\overline{G}_h \right), \rho_h \left(\overline{G}_h \right) \in [0, M_h]$. The equilibrium of the private provision mechanism occurs at a set of choices for the participants such that all the reaction functions are simultaneously satisfied.

Definition 1 *A private provision equilibrium is an array of contributions $\{\widehat{g}_h\}$, $\widehat{g}_h \in [0, M_h]$, such that $\widehat{g}_h = \rho_h \left(\overline{G}_h \right)$ for all $h = 1, ..., H$, with $\overline{G}_h = \sum_{j=1, j \neq h}^{H} \widehat{g}_j$.*

Bergstrom, Blume and Varian (1986, 1992) prove that a private provision equilibrium exists under Assumption 1. Furthermore, they demonstrate that a sufficient condition for uniqueness of equilibrium is that both private and public goods are normal. This condition is implied by Assumption 1. The private provision mechanism is also an *aggregative game* (Cornes and Hartley, 2004), meaning that the equilibrium is dependent on the sum of contributors' incomes and is invariant to income redistributions that do not change the set of contributors.

2.2 Lindahl mechanism

The Lindahl mechanism requires that each participant announce a demand function for the public good. Denote by τ_h, $0 \leq \tau_h \leq 1$, the share of cost of the public good paid by h. If a quantity G of the public good is provided the budget constraint of h is

$$x_h + \tau_h G = M_h. \qquad (3)$$

The Lindahl demand function of h is denoted $\varphi_h (\tau_h)$, where $\varphi_h : [0, 1] \rightarrow \Re_+$. A Lindahl equilibrium given a set of announced demand functions can now be

defined. The Lindahl equilibrium can be shown to exist under weak assumptions upon the Lindahl demand functions. We give such a proof in the next section.

Definition 2 *A Lindahl equilibrium given a set of announced demand functions* $\{\varphi_1(\tau_1), ..., \varphi_H(\tau_H)\}$ *is an array of shares and a level of public good,* $\{\hat{\tau}_1, ..., \hat{\tau}_H, G\}$, *such that for all h*

$$\begin{cases} G \geq \varphi_h(\hat{\tau}_h), \\ \hat{\tau}_h \geq 0, \end{cases}$$

with complementary slackness and

$$\sum_{h=1}^{H} \hat{\tau}_h = 1.$$

The announced Lindahl demand function is the true Lindahl demand function (participants act honestly) if for all $\tau_h \in [0,1]$

$$\varphi_h(\tau_h) = \arg \max_{\{G\}} U^h(M_h - \tau_h G, G). \tag{4}$$

When all participants announce true Lindahl demand functions the equilibrium of the mechanism is efficient. Ignoring corner solutions for simplicity a demonstration follows by observing that the necessary condition for the maximization in (4) is

$$\frac{U_G^h}{U_x^h} = \tau_h. \tag{5}$$

Summing over participants gives

$$\sum_{h=1}^{H} \frac{U_G^h}{U_x^h} = \sum_{h=1}^{H} MRS_{Gx}^h = \sum_{h=1}^{H} \tau_h = 1. \tag{6}$$

This is the Samuelson rule for the economy and completes the demonstration that the Lindahl equilibrium with honest announcement is Pareto efficient.

The efficiency of the Lindahl mechanism with honest announcements is shown in figure 1. The true Lindahl demand functions are the loci of the vertical points on the indifference curves and the equilibrium is found at their intersection. At this point the indifference curves for the two participants are tangential and the equilibrium is Pareto efficient. Note for later reference that the Samuelson set is given by the dashed locus of tangency points.

It is well known that the Lindahl mechanism is not incentive compatible. By announcing a demand function that differs from the true demand function a participant in the mechanism can beneficially modify the outcome. This is shown in figure 2 where it is assumed that participant 1 acts honestly and participant 2 knows the demand announcement of 1. Honesty on the part of participant 2 would lead to the equilibrium e_L. However, by announcing a false demand function the equilibrium can be driven to point e_M which represents the maximization of 2's utility given the Lindahl demand function of 1.
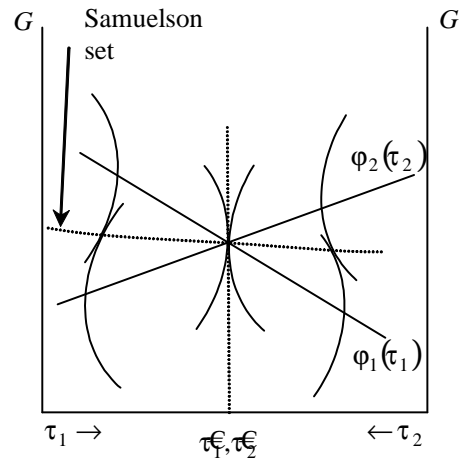
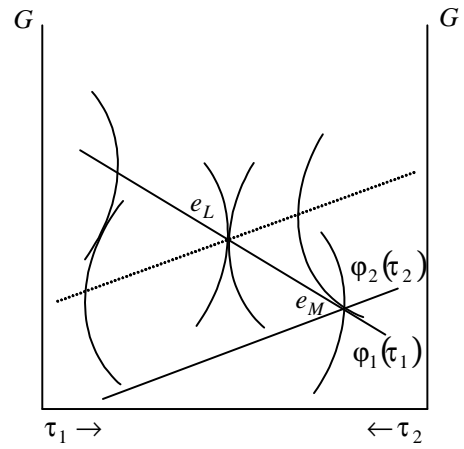Figure 1: Lindahl equilibrium with true demands

Figure 2: Incentive incompatibility

3   Announcement equilibrium

The observation that the Lindahl mechanism is not incentive compatible raises questions about the equilibrium that emerges if all participants strategically choose false announcements. The example of the previous section has shown that if one participant is strategic the equilibrium level of public good is reduced relative to the level with honest announcement. It might be thought that strategic behavior by all participants drives the level of public good even lower. Surprisingly, this issue does not seem to have been addressed in the existing literature.

Each participant h, $h = 1, ..., H$, in the mechanism announces a Lindahl demand function $\varphi_h(\tau_h)$ where τ_h, $0 \leq \tau_h \leq 1$, is the Lindahl share of h in the cost of the public good. We impose the following assumption upon the announced demand function.

Assumption 2 *For all $h = 1, ..., H$, a Lindahl demand function $\varphi_h(\tau_h) : [0, 1] \rightarrow \Re_+$ satisfies:*
(i) $\varphi_h(\tau_h)$ is continuous;
(ii) $\varphi_h(\tau_h)$ is strictly monotonically decreasing whenever $\varphi_h(\tau_h) > 0$.

Relaxing restriction (ii) to allow weakly decreasing demand would not be too difficult but would require some of the results to be modified. We denote the set of demand functions satisfying Assumption 2 by Φ, and a set of announcements by $\varphi = (\varphi_1(\tau_1), ..., \varphi_H(\tau_H)) \in \Phi^H \equiv \times_H \Phi$.

We have introduced the concept of a Lindahl share for each participant. The requirement that these shares are consistent with the allocation of the full cost of the public good between participants is captured in Definition 3.

Definition 3 *A vector $\tau = (\tau_1, ... \tau_H)$ of Lindahl shares satisfies:*
(i) $\tau_h \geq 0$ all h;
(ii) $\sum_h \tau_h = 1$.

Given demand announcements φ the resulting set of Lindahl shares is a solution (if one exists) to the system

$$\begin{cases} \varphi_h(\tau_h) \leq \max\{\varphi_1(\tau_1), ..., \varphi_H(\tau_H)\}, \text{ all } h, \\ \tau_h \geq 0, \end{cases} \tag{7}$$

where the inequalities hold with complementary slackness. At a solution to (7) the level of public good is determined as $G = \varphi_k(\tau_k)$ for some k with $\tau_k > 0$, and we write

$$G = \Gamma(\varphi). \tag{8}$$

The first result proves the existence of a vector of Lindahl shares for any set of announcements.

Lemma 1 *Under Assumption 2 there exists a unique vector of Lindahl shares satisfying (7).*

Denote the set of announcements $\{\varphi_1, ..., \varphi_{i-1}, \varphi_{i+1}, ..., \varphi_H\}$ by φ_{-i}. An announcement equilibrium is defined as follows.

Definition 4 *An announcement equilibrium is a set of announcements $\hat{\varphi} \in \Phi^H$ and a vector of Lindahl shares $\hat{\tau}$ satisfying (7) given $\hat{\varphi}$ such that:*

$$\hat{\varphi}_h \in \arg\max\left\{U^h(M_h - \hat{\tau}_h\Gamma\left(\varphi_h, \hat{\varphi}_{-h}\right), \Gamma\left(\varphi_h, \hat{\varphi}_{-h}\right))\right\}.$$

With these definitions it is now possible to prove the existence of an announcement equilibrium.

Theorem 1 *Under Assumptions 1 and 2 there exists an announcement equilibrium.*

The argument of Theorem 1 shows that a set of equilibrium announcements exists which support the private provision equilibrium as an announcement equilibrium. Reviewing the argument it can be seen that the properties of the private provision equilibrium were only used to simplify the equation system that was to be solved. The assumed continuity shows that the equation system must also have a strictly positive solution in some neighborhood of the private provision equilibrium. Therefore, there must be other allocations which can be supported as announcement equilibria. When there are only two consumers this fact is easily demonstrated. Consider figure 3. Select a point such as a_1 where two indifference curves cross with positive gradient. Then add a linear announcement for participant 1 which is tangential to the indifference curve of participant 2, and a linear announcement for participant 2 which is tangential to the indifference curve of participant 1. The same construction can also be applied to derive announcements that support a_2 as an equilibrium. These announcements constitute an equilibrium since neither participant has an incentive to deviate. These observations lead to the following corollary.

Corollary 1 *The Lindahl mechanism has multiple announcement equilibria.*

This non-uniqueness of announcements when participants are strategic was observed in a related context by Varian (1994). In fact, it is possible to support any allocation where the indifference curves have positive gradient, which for some forms of preferences will be the entire set of allocations below the Samuelson locus in the Lindahl allocation space. Once we allow the participants freedom to make strategic announcements the model is unable to provide a prediction upon the equilibrium allocation.

4 Parametric announcements

The proof of existence for an announcement equilibrium involved supporting a chosen allocation by linear announcements. The next result demonstrates that the use of linear announcements is a general route to a simplified analysis of the equilibrium since any equilibrium in announcements that satisfy Assumption 2
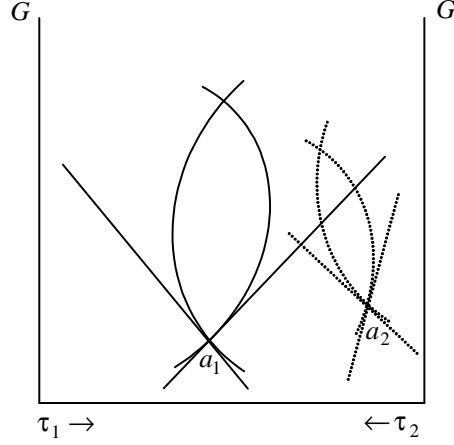
Figure 3: Multiple equilibria

can be achieved by announcements which are linear. For simplicity it is assumed that all shares are positive in equilibrium but the proof can be extended in the same manner as that of Theorem 1 to accommodate participants with zero shares.

Lemma 2 *(i) If $\hat{\varphi}$ is a set of equilibrium announcements for the Lindahl mechanism, then $\left\{ \hat{a}_h - \hat{b}_h \tau_h \right\}$ is also a set of equilibrium announcements if $\hat{\varphi}_h \left(\hat{\tau}_h \right) = \hat{a}_h - \hat{b}_h \hat{\tau}_h$ and $\hat{b}_h = \hat{\varphi}'_h \left(\hat{\tau}_h \right)$.*
    (ii) If $\hat{a}_h - \hat{b}_h \tau_h$, with $\hat{b}_h > 0$, satisfies

$$\hat{a}_h - \hat{b}_h \tau_h \in \arg\max \left\{ U^h (M_h - \hat{\tau}_h \Gamma \left(a_h - b_h \tau_h, \hat{\varphi}_{-h} \right), \Gamma \left(a_h - b_h \tau_h, \hat{\varphi}_{-h} \right)) \right\},$$

then there exists $\hat{\varphi}_h \in \Phi^H$ such that $\hat{\varphi}$ is a set of equilibrium announcements.

The implication of this result is that the search for equilibrium in terms of announcements satisfying Assumption 2 can be replaced by a search among the subset of linear announcements without altering the set of equilibria. All that is important is the point of intersection of the announcements, and the gradient of the announcements at the point of intersection.

However, there still remains the problem of multiplicity. We need to be careful to check that the replacement of general announcements by linear announcements has not reduced the set of allocations that can be supported. For instance, if the optimization of utility over the choice of linear announcements resulted in a unique choice for each participant then the multiplicity of equilibria would have been removed and linear announcement would be more restrictive then general announcements. This will not occur if the optimization over linear announcements does not uniquely determine the two parameters of the linear announcement for each participant. The next result addresses this question.

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To see what can be identified assume that the announcement of participant h is a function defined by a vector of n parameters (for instance a non-degenerate polynomial of degree n). Let this function be denoted $f\left(\tau_h; c_{h1}, ..., c_{hn}\right)$ so that participant h announces $c_h = \left(c_{h1}, ..., c_{hn}\right)$. Assume that there is no redundancy so $f_{c_{hk}} \neq 0$. For any set of announcements from the participants the shares solve

$$f\left(\tau_h; c_h\right) = C, \tag{9}$$

for some C and $\sum_{h=1}^H \tau_h = 1$. Since the announcements are strictly decreasing in τ_i the implicit function theorem applies and

$$\tau_h = \nu^h\left(c_1, ..., c_H\right). \tag{10}$$

The level of public good is

$$G = f\left(\nu^h\left(c_1, ..., c_H\right); c_h\right) = g\left(c_1, ..., c_H\right). \tag{11}$$

The choice problem for participant i is then

$$\max_{\{c_h\}} U\left(M_h - \nu^h\left(c_1, ..., c_H\right) g\left(c_1, ..., c_H\right), g\left(c_1, ..., c_H\right)\right). \tag{12}$$

Lemma 3 *The optimization problem (12) can determine at most one parameter in each vector c_h.*

The conclusion of this section is that the general announcements can be replaced by linear announcements. However, the problem of non-uniqueness still arises. The necessary conditions for the choice of parametrized announcement do not uniquely determine the parameters of the linear announcement. In fact, at most one parameter of any parametrized announcement can be found. This reflects the multiplicity of equilibria but causes problems for the mechanism without government intervention. One positive aspect is that the path is opened for the government to modify the mechanism by restricting one or more parameters of the announcement in order to enhance the efficiency of the equilibrium.

5 Intervention with linear announcements

The consequence of Lemma 3 is that the announcement mechanism will only have a unique equilibrium if announcements are restricted to be monoparametric functions. This fact can be exploited by the operator of the mechanism in the following way. The operator can insist that participants make announcements with a restricted functional form. If the set of permissible announcements has one free parameter that is chosen by the participants then a unique equilibrium can be ensured. What then becomes interesting is the extent to which the operator can control the equilibrium that results.

In this section we consider the consequence of restricting permissible announcements to be linear. Two forms of linear announcement are studied.

Definition 5 *(i) In an announcement mechanism with fixed gradient the announcement of each participant h must be of the form $\varphi_h(\tau_h) = a_h - b\tau_h$.*

(ii) In an announcement mechanism with fixed intercept the announcement of each participant h must be of the form $\varphi_h(\tau_h) = a - b_h\tau_h$.

For the announcement mechanism with fixed gradient the intercept, a_i, of the announcement becomes the single choice variable of participant i. Conversely, with fixed intercept it is the gradient, b_i, that is the choice variable. The interesting question is how the equilibrium of the mechanism is modified through the selection by the government of the fixed parameter. To state the result we need to define the concept of approximate efficiency.

Definition 6 *Let ζ be the parameter fixed by the government. An announcement equilibrium is approximately efficient if $lim_{\zeta \to \Lambda} \sum_{h=1}^{H} \frac{U_G^h(\zeta)}{U_x^h(\zeta)} = 1$ for some constant Λ.*

It should be noted that the constant Λ may be finite or infinite, and that the equilibrium may not be well defined at the limiting value. A general result on approximate efficiency with linear announcements is established when the following additional assumption is imposed upon preferences.

Assumption 3 *The utility function satisfies $U_G^h(x_h, G) \to \infty$ as $G \to 0$ for any x_h.*

Theorem 2 *Under Assumptions 1 and 3*

(i) The equilibrium of the announcement mechanism with fixed gradient b is approximately efficient with $\zeta = b$ and $\Lambda = \infty$.

(ii) The equilibrium of the announcement mechanism with fixed intercept a is approximately efficient with $\zeta = a$ and $\Lambda = \infty$.

Theorem 2 shows that the government can control the announcement equilibrium by restricting the form of announcement that is permissible. By insisting the announcements are linear, and by appropriately setting either the gradient or the intercept of the announcement, it is possible to achieve approximate efficiency. Linear announcements are not the only functional form that can achieve approximate efficiency, and the next section introduces an alternative form that has some appealing features.

6   Hyperbolic announcements

This section shows that the restriction to hyperbolic demand functions can also be used to obtain approximate efficiency. These announcements retain the feature of the linear case that the message is one-dimensional regardless of the number of participants in the mechanism.

6.1 Approximate efficiency

The choice of an announcement can be expressed in a different form to motivate the construction. Assume there are two participants in the mechanism. Given the announcement of participant i, participant j chooses the point on i's announcement that maximizes j's utility. Hence, taking $\varphi_i(\tau_i)$ as given, participant j solves

$$\max_{\{\tau_j\}} U^j\left(M_j - \tau_j\varphi_i(1-\tau_j), \varphi_i(1-\tau_j)\right). \tag{13}$$

This optimization generates the first-order condition

$$\frac{U_G^j}{U_x^j} = \tau_j - \frac{\varphi_i(1-\tau_j)}{\varphi_i'(1-\tau_j)}. \tag{14}$$

The simultaneous solution to (14) and the analogous first-order condition for participant i determines an announcement equilibrium. Alternatively, for fixed values of $\frac{U_G^h}{U_x^h}$, $h = 1, 2$, the first-order conditions are differential equations that solve for announcements which lead to the chosen values in equilibrium. Proceeding in this way permits the outcome to be engineered.

The basis of the approximate efficiency result is built on the next theorem which relates the announcement equilibrium and the private provision equilibrium.

Lemma 4 *The announcement equilibrium coincides with the private provision equilibrium if the only permissible announcements are of the hyperbolic form* $\varphi_h(\tau_h) = \dfrac{C_h}{\tau_h}$.

Lemma 4 implies that if the participants are required to announce the constant C_i in a hyperbolic demand function then the private provision equilibrium is achieved. The attainment of the private provision equilibrium through announcement of hyperbolic demands is interesting, but does not provide a case in favor of using such announcements to allocate the public good. Instead, it suggests that private provision should be employed as a simpler mechanism that requires no government intervention. However, a modification of hyperbolic announcement can improve upon the private provision equilibrium.

Define an announcement mechanism with modified hyperbolic demand as follows.

Definition 7 *An announcement mechanism with modified hyperbolic demand requires each participant to announce the parameter C_i in a demand function of the form* $G = \dfrac{C_i}{\tau_i - \xi}$, *for given $\xi \geq 0$.*

We can then find circumstances in which the modified hyperbolic demand can be used to generate a Pareto improvement over the private provision equilibrium.

Lemma 5 *Let all participants have identical preferences. For any $\xi \in \left[0, \frac{1}{H}\right)$ an increase in ξ generates a Pareto improvement.*

The reasoning behind this lemma is based on the fact that the private provision mechanism is an aggregative game. Hence, the equilibrium outcome is determined by the sum of incomes. When participants have identical preferences the attainment of a Pareto improvement starting from $\xi = 0$ does not depend upon the distribution of income. Furthermore, this argument can be applied at every value of ξ until a point in the Samuelson set is achieved.

A Pareto improvement may not occur with an increase in ξ if the participants do not have identical preferences. The reason for this is that an increase in ξ moves the economy closer to the efficient allocation with equal shares. If the shares are very dissimilar at the private provision equilibrium then moving to equal shares may cause a loss for participants whose initial shares were low. However, as Lemma 6 shows, this cannot happen for all participants.

Lemma 6 *If preferences are not identical then for $\xi \in \left[0, \frac{1}{H}\right)$ there is at least one h such that $\dfrac{\partial U^h}{\partial \xi} > 0$.*

Using the necessary condition for choice of C_i from the proof of Lemma 5 the sufficient condition for $\dfrac{\partial U^h}{\partial \xi} > 0$ can be written in the intuitive form

$$\frac{\dfrac{U_G^h}{\alpha_h}}{\dfrac{1}{H}\sum_{j=1}^{H}\dfrac{U_G^j}{\alpha_j}} < \frac{(H-1)(1-H\xi) + 1 - H\dfrac{\beta_h}{\alpha_h}}{1 - \sum_{j=1}^{H}\dfrac{\beta_j}{\alpha_j}}. \tag{15}$$

where α_j and β_j are defined in (20)-(21). This condition can be interpreted as stating that participants whose preferences are not very different from "the average" are likely to benefit from an increase in ξ.

The increase in ξ may not always generate a Pareto improvement since the implied redistribution can offset the efficiency gain. What can be shown is that it is always possible to obtain approximate efficiency by appropriate choice of ξ. The central theorem is the following.

Theorem 3 *The equilibrium of the announcement mechanism with modified hyperbolic demands is approximately efficient with $\zeta = \xi$ and $\Lambda = \frac{1}{H}$.*

The limit equilibrium is efficient and has equal shares. More importantly, the mechanism requires each participant to announce a single parameter regardless of the number of participants. Furthermore, the mechanism achieves approximate efficiency without the government requiring any information on incomes or preferences. Note, in particular, that this result is not the same as simply telling participants that there are equal shares and asking for an announcement

$$G \qquad \text{Lindahl equilibrium} \qquad \text{Announcement equilibrium} \qquad G$$

$$\tau_1 \rightarrow \qquad \tau_1^e = 1/2 \qquad \leftarrow \tau_2$$
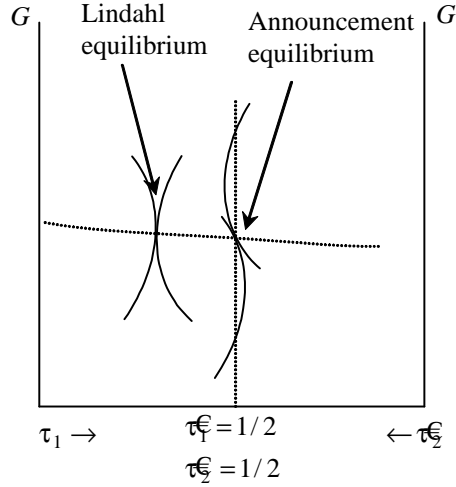$$\tau_2^e = 1/2$$

Figure 4: Announcement equilibrium

of public good demand. If the government were to do this, there would be free-riding and inefficiency would result.

Two points are worth noting. First, the limit as $\xi \to \dfrac{1}{H}$ is efficient but will not coincide with the Lindahl equilibrium with honest announcement unless that has $\hat{\tau} = \frac{1}{H}$. This is shown in figure 4. The locus of efficient allocations is shown in the tax share diagram. These all occur at tangencies of the indifference curves. The Lindahl equilibrium is identified as the efficient allocation where the indifference curves are vertical. The announcement equilibrium with modified hyperbolic announcements tends to the point with shares equal $1/2$. Second, it might be thought that $\dfrac{C_h}{\tau_h - \xi}$ will tend to $\dfrac{C_h}{\tau_h}$ if $\xi = \frac{1}{H}$ and $H \to \infty$, thus giving the private provision equilibrium for large economies. This is not the case since C_h and τ_h are dependent on ξ through the equilibrium of the mechanism.

6.2 Coalition proofness

The analysis so far has looked at the strategic incentives of individual participants. It is also possible for coalitions to deviate. The mechanism must be tested to see whether it is safe from manipulation by coalitions. The importance of coalition proofness has been stressed by Bernheim *et al.* (1987). However, their recursive concept of coalition-proofness is not straightforward to apply. Instead we investigate whether the announcement equilibrium meets the strong Nash equilibrium criterion of Aumann (1959). An equilibrium is strong Nash if no coalition can profitably deviate taking as given the choices of the participants who are not in the coalition.

Suppose that the first $m < H$ participants form a coalition. Assume that the coalition's welfare function is the weighted sum of utilities of its members with the sum of weights normalized to unity. In this case the coalition solves

$$\max_{\{C_1,\ldots,C_m\}} \sum_{i=1}^{m} \mu_i U^i (x_i, G), \quad \sum_{i=1}^{m} \mu_i = 1, \tag{16}$$

while non-members solve

$$\max_{\{Cj\}} U^j (x_j, G), \quad j = m+1,\ldots,H. \tag{17}$$

The set of first-order conditions for non-members remains the same. For the members $i = 1,\ldots,m$ of the coalition the necessary conditions can be written as

$$-U_x^i \left[1 - (H-1)\xi\right] + \sum_{j \neq i}^{m} \mu_j \left(U_x^i \left[1 - (H-1)\xi\right] - U_x^j \xi\right) + \sum_{j=1}^{m} \mu_j U_G^j = 0. \tag{18}$$

If the coalition members have identical preferences and the weights in the coalition welfare function are equal then $U_x^i = U_x^j$, $U_G^i = U_G^j$ and the set of the first-order conditions becomes

$$-U_x^i \left[1 - (H-1)\xi\right] + (1 - H\xi)(m-1)U_x^i + U_G^i = 0. \tag{19}$$

In the limit, as $\xi \to \dfrac{1}{H}$, the second term disappears, and the solution converges to the non-cooperative outcome. Hence, possible gains from forming a coalition converge to zero and the mechanism has a strong Nash equilibrium.

7   Conclusions

The Lindahl mechanism promises much as a means of determining what quantity of a public good should be supplied and how the cost should be distributed. This promise is undermined by the strategic behavior of participants in the mechanism since making a false announcement of preferences is individually rational. This manipulation can damage the functioning of the mechanism to such an extent that it may be dominated by the inefficient mechanism of private provision.

We model the manipulation of the mechanism through the announcement of demand functions for the public good. We have shown that an equilibrium in announcements exists and that the equilibrium is not unique. In fact, there is an uncountable infinity of equilibria. The non-uniqueness is reflected in the fact that in a parametrized version of the announcement the first-order conditions for choice can determine at most one parameter. This leads into the idea of a mechanism where the permissible structures of the demand announcements are restricted by the operator of the mechanism. When the demand announcements

must be linear fixing either the slope or the intercept of the announcements allows approximate efficiency to be achieved. It was also shown that approximate efficiency could be achieved with modified hyperbolic demand announcement and the participants could be brought arbitrarily close to the efficient equilibrium with equal shares.

The value of these results is to show how it is possible to move close to efficiency with no information on preferences or incomes and, in the limit, to obtain an equilibrium that is in the Samuelson set. It is also important to stress that the announcements required to obtain approximate efficiency are one-dimensional regardless of the number of participants. This needs to be contrasted to mechanisms that can achieve exact efficiency but use messages with dimensionality at least as great as the number of participants (and sometimes greater). The limitation of our mechanism is the we can decentralize only a single point in the Samuelson set.

Our analysis has remained true to the spirit of the Lindahl mechanism as the announcement of public good demand functions with equilibrium determined by cost shares. What we have shown is that judicious restrictions on the forms of demand announcements can overcome the consequences of manipulation. The mechanism trades simplicity for approximate efficiency, with the simplicity suggesting that it is possible to envisage this mechanism being employable in practice.

Appendix

Proof of Lemma 1

Define the function $V = \sum_{i=1}^{H} \tau_i \left(\max_{\{j \in H\}} \left\{ \varphi_j \left(\tau_j \right) \right\} - \varphi_i \left(\tau_i \right) \right)$. V is a continuous function of the τ_i and has a minimum on the simplex which is a compact set. Since all terms in the sum are non-negative the minimum value V can achieve is zero. The proof establishes that V will always achieve the minimum of 0 on the simplex given Assumption 2.

Assume that for some set of announcements, φ, V is minimized by $\{\tau^*\}$ and that the minimized value $V^* > 0$. For $V^* > 0$ there must be some i for which $\max_{\{j \in H\}} \left\{ \varphi_j \left(\tau_j^* \right) \right\} - \varphi_i \left(\tau_i^* \right) > 0$ and $\tau_i^* > 0$. Select the i for which $\max_{\{j \in H\}} \left\{ \varphi_j \left(\tau_j^* \right) \right\} - \varphi_i \left(\tau_i^* \right)$ is greatest (if i is not unique the argument extends by selecting all) and denote this $\tilde{h}$. Let $h \in K \subset H$ if $\max_{\{j \in H\}} \left\{ \varphi_j \left(\tau_j^* \right) \right\} = \varphi_h \left(\tau_h^* \right)$. Index the members of this set by $k = 1, ..., \hat{H}$.

Now define a new set of shares $\{\tilde{\tau}\}$. Let $\tilde{\tau}_{\tilde{h}} = \tau_{\tilde{h}}^* - \varepsilon$, and for $k \in K$ let $\tilde{\tau}_k = \tau_k^* + \varepsilon_k$, $\sum_K \varepsilon_k = \varepsilon$. The continuity of φ_j allows the values of ε_k to be selected so that $\max_{\{j \in H\}} \left\{ \varphi_j \left(\tilde{\tau}_j \right) \right\} - \varphi_{\tilde{h}} \left(\tilde{\tau}_{\tilde{h}} \right) > 0$, and $\max_{\{j \in H\}} \left\{ \varphi_j \left(\tilde{\tau}_j \right) \right\} = \varphi_k \left(\tilde{\tau}_k \right)$, $k \in K$. Define $\Delta \varphi_{\tilde{h}} = \varphi_{\tilde{h}} \left(\tau_{\tilde{h}}^* \right) - \varphi_{\tilde{h}} \left(\tilde{\tau}_{\tilde{h}} \right)$ and $\Delta \varphi_k = \varphi_k \left(\tau_k^* \right) - \varphi_k \left(\tilde{\tau}_k \right)$. By (ii) of Assumption 2 we have $\Delta \varphi_{\tilde{h}} < 0$ and $\Delta \varphi_k > 0$.

Let $\tilde{V}$ denote the value of V at $\{\tilde{\tau}\}$. Then observe

$$
\begin{aligned}
V^* - \tilde{V} &= \sum_{i=1}^{H} \left[\tau_i^* \left(\max\left\{ \varphi_j \left(\tau_j^* \right) \right\} - \varphi_i \left(\tau_i^* \right) \right) - \tilde{\tau}_i \left(\max\left\{ \varphi_j \left(\tilde{\tau}_j \right) \right\} - \varphi_i \left(\tilde{\tau}_i \right) \right) \right] \\
&= \tau_{\tilde{h}}^* \left(\varphi_k \left(\tau_k^* \right) - \varphi_{\tilde{h}} \left(\tau_{\tilde{h}}^* \right) \right) - \tilde{\tau}_{\tilde{h}} \left(\varphi_k \left(\tilde{\tau}_k \right) - \varphi_{\tilde{h}} \left(\tilde{\tau}_{\tilde{h}} \right) \right) \\
&= \tau_{\tilde{h}}^* \left(\varphi_k \left(\tau_k^* \right) - \varphi_{\tilde{h}} \left(\tau_{\tilde{h}}^* \right) \right) - \left(\tau_{\tilde{h}}^* - \varepsilon \right) \left(\varphi_k \left(\tau_k^* \right) - \Delta \varphi_k - \varphi_{\tilde{h}} \left(\tau_{\tilde{h}}^* \right) + \Delta \varphi_{\tilde{h}} \right) \\
&= \tau_{\tilde{h}}^* \left(\Delta \varphi_k - \Delta \varphi_{\tilde{h}} \right) + \varepsilon \left(\varphi_k \left(\tau_k^* \right) - \Delta \varphi_k - \varphi_{\tilde{h}} \left(\tau_{\tilde{h}}^* \right) + \Delta \varphi_{\tilde{h}} \right) > 0.
\end{aligned}
$$

Since $V^* - \tilde{V} > 0$ choice $\{\tau^*\}$ could not minimize V. Hence, the minimized value of V is 0 and the equation system has a solution. The strict monotonicity implies that this solution is unique.

Proof of Theorem 1

The proof shows that the private provision equilibrium can be supported as an announcement equilibrium. This establishes that the announcement equilibrium exists.

Under Assumption 1 a unique equilibrium exists for the private provision mechanism. Assume initially that $g_h > 0$ $\forall h$ at this equilibrium. The last part of the argument will relax this assumption. The private provision equilibrium is then equivalent to a Lindahl equilibrium with cost shares $\tau_h = \frac{g_h}{\sum_{i=1}^{H} g_h} > 0$, $h = 1, ..., H$, and public good level $G = \sum_{i=1}^{H} g_h$.

Assume all participants other than h have made linear announcements, $\varphi_i = a_i - b_i \tau_i$, with $b_i > 0$. Such announcements satisfy Assumption 2 in the region in which $a_i - b_i \tau_i > 0$. Now consider participant h. The locus of values of $\{\tau_{-h}\}$ consistent with equilibrium (the intersection of the loci $\{\varphi_{-h}\}$) is defined by

$$
a_i - b_i \tau_i = a_j - b_j \tau_j, \quad i = 1 \text{ if } h \neq 1, \ i = 2 \text{ if } h = 1, \ j = 1, ..., H.
$$

Solving these equations successively

$$
\tau_k = \frac{a_k - a_{k+1}}{b_k} + \frac{b_{k+1}}{b_k} \tau_{k+1}, \quad k = 1, ..., H - 1, \ k \neq h.
$$

Recursively substituting for $h = 1, ..., H - 1$, using the fact that the shares must sum to 1, solving for τ_H, then substituting into the announcement of H shows that given the announcements $\{\varphi_{-h}\}$, h can select the equilibrium from the choice locus

$$
G = \frac{\sum_{i=1, i \neq h}^{H} \frac{a_i}{b_i}}{\sum_{i=1, i \neq h}^{H} \frac{1}{b_i}} + \frac{1}{\sum_{i=1, i \neq h}^{H} \frac{1}{b_i}} \tau_h.
$$

The same construction holds for all $h = 1, ..., H$.

A necessary condition for the private provision equilibrium to be supported by the announcements is that the gradient of the choice locus is equal to the gradient of the indifference curve of participant h at the point $\{\tau_h, G\}$. The gradient of an indifference curve of h is

$$
\frac{dG}{d\tau_h} \Big|_{U^h = const.} \equiv \nabla_h = \frac{U_x^h G}{U_G^h - U_x^h \tau_h} > 0.
$$

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Since $g_h > 0$ at the private provision equilibrium it follows that $U_x^h = U_G^h$ and $\tau_h < 1$. Therefore $\nabla_h = \frac{G}{1-\tau_h} > 0$. Since this applies for all h it is necessary to prove that there is a solution with $b_i > 0$ to the equation system

$$\frac{1}{\sum_{i=1, i \neq h}^{H} \frac{1}{b_i}} = \nabla_h, \quad h = 1, ..., H.$$

Solving the system gives

$$b_h = \frac{1}{\frac{1}{H-1} \sum_{i=1}^{H} \frac{1}{\nabla_i} - \frac{1}{\nabla_h}} = \frac{1}{\frac{G}{H-1} \sum_{i=1, i \neq h}^{H} \frac{1}{1-\tau_i}} > 0.$$

Hence there is a strictly positive solution.

The Lindahl equilibrium corresponding to the private provision equilibrium has

$$\tau_i = \frac{g_i}{G},$$

so

$$b_h = \frac{1}{G^2 \sum_{i=1, i \neq h}^{H} \frac{1}{G-g_i}}.$$

Given these values for b_h the values of a_h, $h = 1, ..., H$, can be chosen to ensure that the announcements satisfy

$$a_h - b_h \tau_h = a_{h'} - b_{h'} \tau_{h'}, \quad \text{all } h, h'.$$

Hence

$$a_h - \frac{1}{G^2 \sum_{i=1, i \neq h}^{H} \frac{1}{G-g_i}} \frac{g_h}{G} = a_{h'} - \frac{1}{G^2 \sum_{i=1, i \neq h'}^{H} \frac{1}{G-g_i}} \frac{g_{h'}}{G}, \quad \text{all } h, h',$$

must be satisfied when evaluated at the private provision equilibrium $\{g_h\}$.

Consider now a private provision equilibrium where the contribution is 0 for some participants. Partition the set of participants into the set H_1 for whom $g_h > 0$ and the set H_2 for whom $g_h = 0$. Apply the argument given above to support the choices of the participants in set H_1. The intersection of their announcements determines the locus of potential allocations facing each participant in the set H_2 given that all other in the set choose $g_h = 0$. Since non-contribution was privately optimal for set H_2 in the private provision equilibrium, it will remain so in the announcement equilibrium. Therefore assign an announcement to each member of the set H_2 that intersects the locus of potential allocations at $G < 0$. This results in the cost shares being 0 for all $h \in H_2$ which is an equilibrium outcome.

Proof of Lemma 2

(i) Assumption 2 restricts φ_i to be strictly decreasing in τ_i. Hence, the equilibrium must be unique and occur at the point $\hat{\varphi}_i(\hat{\tau}_i) = \hat{\varphi}_j(\hat{\tau}_j)$. The linear announcements $\hat{a}_i - \hat{b}_i \tau_i$, $i = 1, ..., H$, are constructed to also have a unique

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intersection at $\{\hat{\tau}_i\}$. By definition no participant will deviate from the equilibrium with announcements $\{\hat{\varphi}_i(\tau_i)\}$ and the equality $\hat{b}_i = \hat{\varphi}'_i(\hat{\tau}_i)$ guarantees that this is also true for the linear announcements. (ii) This is obvious since the given linear function satisfies Assumption 2 and is a member of the set of possible announcements.

Proof of Lemma 3

Observe that $g_{c_{ik}} = f_{\tau_i}\nu^i_{c_{ik}} + f_{c_{ik}}$, and $\nu^i_{c_{ik}} = -\frac{f_{c_{ik}}}{f_{\tau_i}}\left[1 - \frac{\frac{1}{f_{\tau_i}}}{\sum_{h=1}^{H}\frac{1}{f_{\tau_h}}}\right]$. Hence, the necessary condition for c_{ik} in the optimization (12) is $0 = -U^i_x\left[\nu^i_{c_{ik}}g + \nu^i g_{c_{ik}}\right] + U^i_G g_{c_{ik}}$, which can be written as $0 = gU^i_x\left[\sum_{h=1}^{H}\frac{1}{f_{\tau_h}} - \frac{1}{f_{\tau_i}}\right] + \left[U^i_G - \nu^i U^i_x\right]$. This condition does not depend on k so there is a single independent necessary condition and the optimization can determine at most one parameter of the announcement for all $k = 1, ..., n$.

Proof of Theorem 2

(i) The choice of a_h is defined by $a_h \equiv \arg\max U^h\left(M_h - \tau_h G, G\right)$, where in equilibrium $G = \frac{b}{H}\left(\frac{\sum_{i=1}^{H}a_i}{b} - 1\right)$, and $\tau_h = \frac{a_h}{b} - \frac{\sum_{i=1}^{H}a_i}{Hb} + \frac{1}{H}$. The first-order condition for choice of a_h is

$$-U^h_x\left(\tau_h\frac{\partial G}{\partial a_h} + G\frac{\partial \tau_h}{\partial a_h}\right) + U^h_G\frac{\partial G}{\partial a_h} = 0.$$

But $\frac{\partial G}{\partial a_h} = \frac{1}{H}$, $\frac{\partial \tau_h}{\partial a_h} = \frac{H-1}{Hb}$, so $\frac{U^h_G}{U^h_x} = \tau_h + \frac{H-1}{b}G$. Taking the sum of the first-order conditions for all participants, and noting that the shares must sum to one, we obtain

$$\sum_{i=1}^{H}\frac{U^h_G}{U^h_x} = 1 + (H-1)\left(\frac{\sum_{i=1}^{H}a_i}{b} - 1\right).$$

Under Assumption 3 it is never possible to have a zero public good in equilibrium. Also, because the number of participants is finite and their incomes are bounded, the quantity of public good is finite. Hence, for every $b > 0$, in equilibrium $0 < G < \infty$. Now let $b \to \infty$. For G to remain finite it must be the case that

$$\frac{\sum_{i=1}^{H}a_i}{b} - 1 \to 0 \quad \text{as} \quad b \to \infty.$$

Hence, it follows that

$$\sum_{i=1}^{H}\frac{U^h_G}{U^h_x} \to 1 \quad \text{as} \quad b \to \infty.$$

Therefore, as $b \to \infty$ the equilibrium is approximately efficient.

(ii) Given a value of a the choice of b_h for participant h is defined by $b_h \equiv \arg\max U^h\left(M_h - \tau_h G, G\right)$. The first-order condition with respect to $\dfrac{1}{b_h}$ is

$$\frac{\partial U^h}{\partial \left(1/b_h\right)} \equiv -U_x^h\left(\tau_h \frac{\partial G}{\partial \left(1/b_h\right)} + G\frac{\partial \tau_h}{\partial \left(1/b_h\right)}\right) + U_G^h \frac{\partial G}{\partial \left(1/b_h\right)} = 0.$$

In equilibrium $G = a\left(1 - \dfrac{1}{a\sum_{i=1}^{H}\frac{1}{b_i}}\right)$, $\quad \tau_h = \dfrac{\frac{1}{b_h}}{\sum_{i=1}^{H}\frac{1}{b_i}}$, so

$$\frac{\partial G}{\partial \left(1/b_h\right)} = \frac{1}{\left(\sum_{i=1}^{H}\frac{1}{b_i}\right)^2}, \quad \frac{\partial \tau_h}{\partial \left(1/b_h\right)} = \frac{1}{\sum_{i=1}^{H}\frac{1}{b_i}} - \frac{\frac{1}{b_h}}{\left(\sum_{i=1}^{H}\frac{1}{b_i}\right)^2}.$$

Using these derivatives the necessary condition can be rewritten as $\dfrac{U_G^h}{U_x^h} = \tau_h + G\left(\sum_{i=1}^{H}\frac{1}{b_i} - \frac{1}{b_h}\right)$. Taking the sum over all participants, $\sum_{i=1}^{H}\dfrac{U_G^h}{U_x^h} = 1 + (H-1)\left(a\sum_{i=1}^{H}\frac{1}{b_i} - 1\right)$. Since in equilibrium $0 < G < \infty$ for every a, it must be the case that

$$1 - \frac{1}{a\sum_{i=1}^{H}\frac{1}{b_i}} \to 0 \quad \text{as} \quad a \to \infty.$$

This, in its turn, implies

$$a\sum_{i=1}^{H}\frac{1}{b_i} \to 1 \quad \text{and} \quad \sum_{i=1}^{H}\frac{U_G^h}{U_x^h} \to 1 \quad \text{as} \quad a \to \infty.$$

Therefore as $a \to \infty$ the equilibrium is approximately efficient.

Proof of Lemma 4

The private provision equilibrium is described by $\dfrac{U_G^i}{U_x^i} = 1$, $i = 1, ..., H$. Using the necessary condition (14) with the change of variable $\tau = 1 - z$ this equilibrium will be achieved if $(1-z) - \dfrac{\varphi_i(z)}{\varphi_i'(z)} = 1$. Solving the implied differential equation, $\varphi_i(z) = \dfrac{C_i}{z}$.

The constants of integration, C_i, become the choice variables for the participants. To see this note that at equilibrium we have $\dfrac{C_i}{\tau_i} = \dfrac{C_j}{\tau_j}$, all i, j, so $\tau_i = \dfrac{C_i}{\sum_{h=1}^{H} C_h}$ and $G = \sum_{h=1}^{H} C_h$. The optimization facing participant i is then $\max_{\{C_i\}} U^i\left(M_i - C_i, \sum_{h=1}^{H} C_h\right)$, which is precisely the objective function when C_i is the level of contribution at the private provision equilibrium.

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Proof of Lemma 5

The demand announcement of h is given by $\varphi(\tau_h) = \frac{C_h}{\tau_h - \xi}$, for some $\xi \in [0, \frac{1}{H})$. In equilibrium $\varphi(\tau_h) = G$ for all h, and $\sum_{h=1}^{H} \tau_h = 1$ which imply

$$G = \frac{\sum_{h=1}^{H} C_h}{1 - H\xi}, \quad \tau_h = \xi + (1 - H\xi) \frac{C_h}{\sum_{j=1}^{H} C_j}, \quad x_h = M_h - C_h - \frac{\xi}{1 - H\xi} \sum_{j=1}^{H} C_j.$$

The choice of participant h is defined by $C_h \equiv \arg\max U^h(M_h - \tau_h G, G)$, so it satisfies the necessary condition $-U_x^h[1 - (H-1)\xi] + U_G^h = 0$.

We need to show that for any ξ, $0 \leq \xi < \frac{1}{H}$, a Pareto improvement can be achieved by increasing ξ. Computing $\dfrac{\partial U^h}{\partial \xi}$ we have

$$\frac{\partial U^h}{\partial \xi} = U_x^h \left[(H-1)G + \sum_{j=1}^{H} C_j' - C_h' \right],$$

where C_h' denotes $\dfrac{\partial C_h}{\partial \xi}$. The expression for C_h' is obtained by total differentiation of the necessary condition. This gives

$$C_h' = -\frac{1}{1 - H\xi} \left(\xi - \frac{\beta_h}{\alpha_h} \right) \sum_{j=1}^{H} C_j' + (H-1) \frac{U_x^h}{\alpha_h} - \frac{G}{1 - H\xi} \left(1 - H\frac{\beta_h}{\alpha_h} \right),$$

where for convenience we use the notation

$$\begin{align}
\alpha_h &= -[1 - (H-1)\xi] U_{xx}^h + U_{Gx}^h, & (20)\\
\beta_h &= -[1 - (H-1)\xi] U_{Gx}^h + U_{GG}^h. & (21)
\end{align}$$

Hence

$$\frac{\partial U^h}{\partial \xi} = \frac{U_x^h (H-1)}{1 - \sum_{j=1}^{H} \frac{\beta_j}{\alpha_j}} \left[-\frac{U_x^h}{\alpha_h} \left(1 - \sum_{j=1}^{H} \frac{\beta_j}{\alpha_j} \right) + \left(1 - (H-1)\xi - \frac{\beta_h}{\alpha_h} \right) \sum_{j=1}^{H} \frac{U_x^j}{\alpha_j} \right].$$

Assumption 1 and the fact that $\xi \in [0, \frac{1}{H})$ imply $\alpha_h > 0$ and $\beta_h < 0$ for all h. Hence, it is sufficient for $\dfrac{\partial U^h}{\partial \xi} > 0$ that

$$\frac{\frac{U_x^h}{\alpha_h}}{1 - (H-1)\xi - \frac{\beta_h}{\alpha_h}} < \frac{\sum_{j=1}^{H} \frac{U_x^j}{\alpha_j}}{1 - \sum_{j=1}^{H} \frac{\beta_j}{\alpha_j}}.$$

For identical consumers this condition becomes

$$\frac{\frac{U_x^h}{\alpha_h}}{1 - (H-1)\xi - \frac{\beta_h}{\alpha_h}} < \frac{H\frac{U_x^h}{\alpha_h}}{1 - H\frac{\beta_h}{\alpha_h}},$$

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which holds as long as $\xi < \frac{1}{H}$.

Proof of Lemma 6

Observe that the expression for $\dfrac{\partial U^h}{\partial \xi}$ in the proof of Lemma 5 can be used to write

$$\frac{1}{H-1}\sum_{h=1}^{H}\frac{1}{U_x^h}\frac{\partial U^h}{\partial \xi} = \frac{\sum_{j=1}^{H}\dfrac{U_x^j}{\alpha_j}}{1-\sum_{j=1}^{H}\dfrac{\beta_j}{\alpha_j}}\left(H-1\right)\left(1-H\xi\right).$$

The right-hand side is positive as long $\xi < \frac{1}{H}$. Therefore, for any $\xi \in \left[0, \frac{1}{H}\right)$ the derivative $\dfrac{\partial U^h}{\partial \xi}$ must be positive for at least one participant.

Proof of Theorem 3

From the proof of Lemma 5 we know that the choice of announcement must satisfy the necessary condition $-U_x^h\left[1-\left(H-1\right)\xi\right] + U_G^h = 0$, for all $h = 1,\ldots,H$. In the limit, as $\xi \rightarrow \dfrac{1}{H}$ the necessary conditions becomes $\frac{U_x^h}{U_G^h} = \dfrac{1}{H}$, which characterizes the point in the Samuelson set with equal shares.

References

Anderlini, L. and Siconolfi, P. (2004) "Efficient provision of public goods with endogenous redistribution", *Review of Economic Design*, **8**, 413 - 447.

Aumann, R. (1959) "Acceptable points in general cooperative n-person games" in A.W. Tucker and R.D.Luce (eds.), *Contributions to the Theory of Games* IV (Princeton: Princeton University Press).

Bergstrom, T.C. (1970) "A "Scandinavian consensus" solution for efficient income distribution among nonmalevolent consumers", *Journal of Economic Theory*, **2**, 383 -398.

Bergstrom, T.C., Blume, L. and Varian, H.R. (1986) "On the private provision of public goods", *Journal of Public Economics*, **29**, 25 - 49.

Bergstrom, T.C., Blume, L. and Varian, H.R. (1992) "Uniqueness of Nash equilibrium in private provision of public goods", *Journal of Public Economics*, **49**, 391 - 392.

Bernheim, B.D., Peleg, B, and Whinston, M.D. (1987) "Coalition-proof Nash equilibria I: Concepts", *Journal of Economic Theory*, **42**, 1 - 12.

Cornes, R. and Hartley, R. (2004) "Well-behaved aggregative games" (unpublished).

Cornes, R. and Sandler, T. (1996) *The theory of externalities, public goods and club goods*, Cambridge: Cambridge University Press.

Cornes, R. and Sandler, T. (2000) "Pareto improving redistribution in the pure public good model", *German Economic Review*, **1**, 169 - 186.

Groves, T. and Ledyard, J. (1977) "Optimal allocation of public goods: a solution to the "free rider" problem", *Econometrica,* **45**, 783 - 809.

Groves, T. and Ledyard, J. (1980) "The existence of efficient and incentive compatible equilibria with public goods", *Econometrica,* **48**, 1487 - 1506.

Groves, T. and Ledyard, J. (1987) "Incentive compatibility since 1972", in: T. Groves and R. Radner, eds., *Essays in Honor of Leonid Hurwicz* (University of Minnesota Press, Minneapolis).

Johansen, L. (1963) "Some notes on the Lindahl theory of determination of public expenditures", *International Economic Review*, **4**, 346 358.

Lindahl, E. (1919) *Positive Losung, die Gerechtigkeit der Besteuerung*, Lund, reprinted as "Just Taxation - A Positive Solution" in R.A. Musgrave and A.T. Peacock (eds.), *Classics in the Theory of Public Finance*, London: Macmillan.

Myles, G.D. (1995) *Public Economics*, Cambridge: Cambridge University Press.

Otani, Y. and Sicilian, J. (1982) "Equilibrium allocations of Walrasian preference games", *Journal of Economic Theory*, **27**, 47 - 68.

Sertel, M.R. and Sanver, M.R. (1999) "Equilibrium outcomes of Lindahl-endowment pretension games", *European Journal of Political Economy*, **15**, 149 - 162.

Tian, G. (2003) "A solution to the problem of consumption externalities", *Journal of Mathematical Economics*, **39**, 831 - 847.

Varian, H.R. (1994) "A solution to the problem of externalities when agents are well-informed", *American Economic Review*, **84**, 1278 - 1293.