First-Best Allocations & the Signup Game: A New Look at Incomplete Information

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March 21, 2006
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Abstract

Mechanism design problems are sensitive to the solution concept used. It is well known that by assuming a dominant strategy solution concept in principal agent problems or for monopolies with incomplete information, the principal can extract rent arbitrarily close to the second-best outcomes (second degree price discrimination).

This paper introduces the signup game where the principal has the ability to define contracts according to the number of agents 'signing-up' for it. In a finite agent population, if the agents' underlying distribution, F(v), is the true realized distribution, by implementing a signup game, the principal can obtain rent arbitrarily close to the first-best or the perfect price discriminatory solution. This need only assume a solution concept of iterative elimination of strictly dominated strategies.

The main contribution of this paper is that in a finite agent population, if the agents' realized distribution is, one from a set of possible distributions which do not (first order) stochastically dominate each other, the principal is able to construct a sign-up game whose Pareto superior equilibrium has the principal extracting rent arbitrarily close to the first-best case.

Introduction

Much theoretical work has been done in the area of the principal agent problem where parties are constrained by asymmetric information. Many issues, price discrimination with quantity discounts (Goldman, Leland and Sibley, 1984; Roberts, 1979; Spence, 1977; Maskin and Riley 1984), monopoly pricing of goods of differing quality (Mussa and Rosen, 1978), lender borrower financial contracts (Freixas and Laffont; 1990) and labor contracts (Hart 1983) are all analogous to the basic principal agent problem with similar solutions.

The problem boils down to the principal constructing a sorting mechanism to extract as much rent as possible, given his information about the agents. In the principal-agent problem and models of monopolies with incomplete information, the principal or the monopoly is assumed only to know the distribution, F(v), of skill or tastes, v, of the population. The two common interpretations of the information of the principal or the monopoly are, that either each agent or buyer is drawn randomly from the underlying distribution or that F(v) is the true realized distribution of the agent or buyer population in question. The latter is the more traditional interpretation. This paper will show that if the principal or the monopoly has the ability to define contracts according to the number of agents or buyers 'signing-up' for it, the traditional interpretation would yield the first-best or perfect price discriminatory solutions.

A solution as above has been implemented for tax schedule in Piketty (1993). The paper's main contribution is that in a finite agent population, if the agents' realized distribution is, one from a set of possible distributions which do not (first order) stochastically dominate each other, the principal is able to construct a sign-up game whose Pareto superior equilibrium has the principal extracting rent arbitrarily close of the first-best case when any of the possible distributions are realized.

The Model

Following Maskin and Riley (1984), this paper will characterize the problem as a monopoly with incomplete information[1].

A monopolist produces a good at a constant marginal cost, c. A buyer of type i has preference represented by the utility function

$$U_i(q, -T) = \int_0^q p(x; v_i)dx - T, \tag{1}$$

Figure 1. copied from Maskin and Riley (1984)

where q is the number of units purchased from the monopolist and T is the payment made for those units. Thus, we will take the standard approach where the monopolist will be trying to extract the consumer surplus. The parameter v_i encapsulates all the information about the taste of the buyer for the good. The monopolist does not observe v, but knows F(v), the distribution of the buyers' preferences. Following Maskin and Riley (1984), we shall also assume that higher levels of v are associated with higher demand and that the demand price p(q; v) is decreasing in q and that there is some $q^e(v)$ for which demand price exceeds marginal cost. Therefore, for each v, $q^e(v)$ is the efficient consumption level.
That is:

Assumption 1. **(i) For all feasible v the demand price function $p(q; v)$ is nonincreasing in q and nonnegative, and there exists $q^e(v) \geqslant 0$ such that $p(q; v)$ is decreasing in q for $q \leqslant q^e(v)$, and $p(q; v) \geqslant c$ if and only if $q \leqslant q^e(v)$. (ii) $p(q; v)$ is twice continuously differentiable for $q \leqslant q^e(v)$. (iii) $p(q; v)$ is strictly increasing in v whenever $p(q; v)$ is positive.**

Figure 2. copied from Maskin and Riley (1984)

In the standard analysis, this results in a selling procedure that is a schedule of pairs $\langle \hat{q}_s, \hat{T}_s \rangle_{s \in S}$, which the seller offers to the buyers. If a buyer chooses s, from the available S pairs, he receives q_s and pays a total of T_s. Therefore, the return to the seller from the buyer is

[1] Maskin and Riley (1984) show that solution to the problem also holds for the principal agent problem.

$$\hat{R}_s = \hat{T}_s - c\hat{q}_s. \qquad (2)$$

Figure 3. copied from Maskin and Riley (1984)

We shall assume throughout that the buyer always has the option <0,0> available to him. This would be outside option of not buying.

Combining (1) and (2), we can rewrite the utility of a buyer of type i as

$$U(q, R; v_i) = \int_0^q p(x; v_i)dx - cq - R \equiv N(q; v_i) - R, \qquad (3)$$

where $N(q; v_i)$ is the social surplus generated by the sale. Thus, we can think of the trades between the seller (the "principal") and buyers ("agents") as giving each buyer the entire surplus less a fee R. The selling procedure is then a schedule of pairs $\langle q_s, R_s \rangle_{s\in S}$ offered to each of the buyers. This latter formulation proves more convenient.[4]

Figure 4. copied from Maskin and Riley (1984)

The Sign-Up Game

Now suppose the monopolist can construct the following game of contracts.

The monopolist offers contracts (or bundles) that consumers are to signup for. The monopolist can make the outcome of the contracts dependent on the number of consumers who signup for them and this rule is known ex ante signing up. Given the set of contracts and the rules determining the outcomes, the consumers would be playing a signup game among themselves. If they wish not to play they get a payoff of the outside option, 0 (same as getting a <0, 0> contract).

CLAIM:

If the monopolist can implement a signup game, and he knows the fractions of each type in the finite population, (1) he can implement a signup game whose Nash Equilibria include one where he gets the rent that he would have gotten from perfect price discrimination. (2) he can implement a signup game with a unique Nash Equilibrium which gets him arbitrarily close to perfect price discriminatory rent (when consumers play strategies that survive iterative deletion of strictly dominated strategies).

First let's start with an example.

Example.

Consumers utility function = $q(\theta\text{-}q) - T$

Monopolist's marginal cost = 0

There are two consumers in the economy,

one with $\theta = 1$ (low type) and the other with $\theta = 2$ (high type).

Therefore, the utility maximizing quantities (first best) for the low type is $q = 0.5$ and for the high type is $q = 1$. If the monopolist knows the consumers type, he can offer them the corresponding quantities and extract all the rent, ¼ from the low type and 1 from the high type.
But the monopoly only knows that there is one low type and one high type out there.

Solution:

The monopolist offers two contracts, Option 1 and Option 2, for the two consumers to signup for, whose implemented bundle ($< q, T >$) is as shown in the table.

#(consumers signed up for Option 2)	Option 1	Option 2
1 or 2	$< \frac{1}{2}, \frac{1}{4} - \varepsilon >$	$< 1, 1-\varepsilon >$
0	$<0, 0>$	$< 1, 1-\varepsilon >$

where $\varepsilon \geq 0$ (and close to 0). Basically, Option 2 is always the same but Option 1 is not implemented if no consumers have signed up for Option 2.
For the above contract rules, the consumers will be playing the following game:

Player 1 (low type)	Player 2 (high type):	Stay out	Option 1	Option 2
	Stay out	0 , 0	0, 0	0, ε
	Option 1	0 , 0	0, 0	**ε, ε**
	Option 2	$-1 + \varepsilon$, 0	$-1 + \varepsilon$, $\frac{1}{2} + \varepsilon$	$-1 + \varepsilon$, ε

For $\varepsilon > 0$ and close to 0 ($\varepsilon < 1$), the above game has only one Nash Equilibrium, (Option 1, Option 2) which could be obtained by the iterative elimination of **strictly** dominated strategies. In this case, the monopolist obtains perfect price discriminatory rent - 2ε. Therefore, choosing ε arbitrarily close to 0, the monopolist can extract rent arbitrarily close to the perfect price discriminatory rent.

If $\varepsilon = 0$, all pairs (s1, s2) where s1 is an element of {stay out, option 1} and s2 is an element of {stay out, option 1, option 2} are Nash Equilibria. This also includes the (Option 1, Option 2) equilibrium where the monopolist gets the same rent as in perfect price discrimination.

Let us now generalize the above signup game rule devised by the monopolist for the case with n type of consumers.

Let there be n types $\theta_1, \ldots, \theta_n$

wlog let us assume that $\theta_1 < \ldots < \theta_n$

Let the fraction of the population of type $\theta_i = f(\theta_i)$

Let the fraction of the population of type $\theta \leq \theta_i = F(\theta_i)$

Let the perfect price discriminatory bundle for θ_i type consumers be $< q_i, T_i >$

The Signup (Implementation) Rule the monopolist should implement

Create n Options (Option 1, …, Option n) for the consumers to sign up for (Option i would be intended for θ_i type consumers).

Let us define some conditions for that the monopolist will find useful.

Condition n: fraction of the consumers that have signed up for Option n = $p(n) \geq f(\theta_n)$

Condition n-1: fraction of the consumers that have signed up for Option n-1 + Option n
$$= p(n-1) + p(n) \geq f(\theta_{n-1}) + f(\theta_n) = 1 - F(\theta_{n-2})$$
…

Condition i: fraction of the consumers that have signed up for Option i + …+ Option n
$$= p(i) + .. + p(n) \geq f(\theta_i) + .. + f(\theta_n) = 1 - F(\theta_{i-1})$$
…

Therefore, condition 1: fraction of the consumers that have signed up for all the options
$$= p(1) + .. + p(n) \geq f(\theta_1) + .. + f(\theta_n) = 1$$

Option implementation rule

Option I, $< q , T >$, is said to be in its implemented state if either q or T >0.

Option n = $< q_n, T_n+\varepsilon>$ always

Options n-1 $< q_{n-1}, T_{n-1}+\varepsilon>$ if condition n is met & Option n is in the implemented state
$< 0, 0>$ otherwise

..

Options i $< q_i, T_i +\varepsilon>$ if condition i+1 is met & option i+1 is in the implemented state
$< 0, 0>$ otherwise

Options 1 $< q_1, T_1 +\varepsilon>$ if condition 2 is met and option 2 is in the implemented state
$< 0, 0>$ otherwise

Notice that Option i will be implemented only if Option i+1 is implemented.

CLAIM:

Given the above signup game, for $\varepsilon > 0$ (and close to zero) iterative elimination of strictly dominated strategies makes consumers of θ_i type sign up for option i, and all options will be implemented and the monopoly would extract perfect price discrimination rent $-n\varepsilon$. Consumers of θ_i type signing up for option i is still a Nash equilibrium when $\varepsilon = 0$ and in that case the monopoly extracts the same rent as in perfect price discrimination.

Outline for $\varepsilon > 0$ (iterative elimination of strictly dominated strategies)

Signing up for option n is strictly dominated for all types θ_i where $i<n$ by staying out. Therefore eliminate option n for all types θ_i, $i<n$. Notice that, now only for θ_n type consumers have the strategy option n survived. Also notice that, therefore, none of the other options (especially option n-1) would be implemented unless all θ_n type consumers

sign up for option n. Thus, for θ_n type consumers, in the current set up option n yields them a payoff of ε while all other strategies yield a payoff of 0. Therefore, option n is a strictly dominant strategy for θ_n type consumers. Thus, eliminate all other strategies for θ_n type consumers.

Now, given that all θ_n type consumers sign up for option n, and all other θ_i, $i < n$ types have option n eliminated. The case has been reduced to a game with n-1 types with the associated fractions.

That is, now for all consumers θ_i, $i<n-1$ type consumers option n-1 is strictly dominated by staying out. Therefore, eliminating option n-1 for all types θ_i, $i<n-1$ makes option n-1 a strictly dominant strategy for θ_{n-1} type consumers … and so on.

This results in θ_i type consumers signing up for option i and the monopoly ends up extracting rent = perfect price discriminatory rent - $n\varepsilon$. The way the signup game is defined, for $\varepsilon = 0$, θ_i type consumers signing up for option i is a Nash Equilibrium in which case all consumers get a payoff of 0 and the monopoly ends up extracting rent = perfect price discriminatory rent. In this Nash Equilibrium, given the other consumers strategies, each consumer of type θ_i would be indifferent between choosing any of the options j , $j \leq i$.

Comparison to perfect price discrimination under complete information

Under complete information, the monopoly extracts the reservation price from all the consumers but the consumers are indifferent to buying and staying out. But if a consumer chooses not to participate, the monopoly will only lose that consumer's rent.

Under the incomplete information case above, with the signup game in place, the monopoly extracts all rent when the consumers signup for their intended options. Given others signup, all consumers are indifferent between signing up and staying out. But if a consumer (of type θ_i) chooses not to participate, the monopoly will lose more than that consumer's rent since other options (option j such that $j < i$) will not be implemented.

Implementation

Monopoly has to be able to commit to the implementation rule. That is, after the monopoly defines the rules of the signup game including the implementation rule, he should not be able to renegotiate with the consumers. This is the same restriction required in traditional mechanism design problems as when the principal cannot renegotiate after an agent (low type) has chosen a contract even though it could be beneficial to both parties. Thus, here also we will assume that a court of law will see to it that the monopoly will implement the announced signup game with its implementation rules.

In traditional principal agent problems the principal only has to define the contracts and the agents are free to choose any of the contracts at any time afterwards. In the above procedure however, contracts are implemented only after all the agents have signed up for some contract (including having decided to stay out). It may be the case that the agent does not know whether the contract he is signing up for is going to get implemented. Where could you find a situation to implement as above?

Since the principal needs to keep track of the agents signing up, an online set up might be ideal. The principal announces the sign up game with the implementation rule for n agents who have to sign up before midnight and so they get to verify whether the contract was implemented in the morning.

In mechanism design problems we are only interested to see whether the agents are revealing their types in equilibrium in the desired solution concept.

In view of that, if we are only interested in Nash Equilibrium, another simpler way to implement the above problem would be to have the implementation rule as follows:

If the fraction of the consumers that has signed up for option i = p(i) = $f(\theta_i)$ for all i

Implement all contracts
Otherwise implement no contracts.

Even with the above implementation rule, truth revealing would be an equilibrium. The above rule would prove useful in designing an implementation rule for more complicated incomplete informational settings.

A more complicated informational setting

Let us assume that the principal has the following information about distribution of v among the finite agent population.
The realized distribution of v = F(v) is an element of the set $\{F_1(v), \ldots., F_k(v)\}$. It could be the case that the principal might even have a probability associated with each of the possible distributions.

CLAIM:
Let the principal allow the agents to sign up for options. Let option i be implemented in its first-best state for type θ_i, if the fractions signed up for each option i, matches the fractions of the population of type θ_i, for one of the possible distributions $F_j(v)$. Then, if each of the possible distributions does not (first order) stochastically dominate any of the possible distributions, truth revealing is a Nash Equilibrium in which case the principal is able to extract all the rent. Furthermore, it is not a Nash equilibrium for the population, $F_a(v)$, to sign up according to some other possible distribution, $F_j(v)$. Thus, in all possible Nash Equilibria, the agents' payoff will be the same as the outside option.

Proof

Take the scenario when all the agents, from a realized distribution $F_a(v)$, are truth revealing. Then, the fractions signed up for each option will match the fractions of the population of $F_a(v)$, and the contracts they have signed up for are implemented. For this not to be Nash Equilibrium, an agent of type θ_i, would find it beneficial to, sign up for some other option j. We know that for any agent of type θ_i, signing up for an option j such that j >i, would always result in the agent getting a non-positive payoff (zero if the option is not implemented and a negative payoff if the option is implemented) thus it is never beneficial for a low type to reveal himself as a high type. Thus, now we only have to show that for all agents of type θ_i it is not better to reveal themselves as $j < i$ if everyone else is truthfully revealing themselves.

If an agent of type θ_i is better off, signing up for an option j, $j < i$, then it must be the case that, this option is implemented (since not being implemented would yield the same payoff) and therefore, this new signed up distribution must be represented by some one of the other possible distributions, $F_b(v)$.

Now let us look at signed up distributions that correspond to $F_a(v)$ and $F_b(v)$

The only difference is that one agent signing for option i in distribution $F_a(v)$ has signed up for option j to yield distribution $F_b(v)$.

Therefore,

$F_a(x)= F_b(x)$ for $x < \theta_j$ since the signed up distributions are identical for options x, $x < j$.
$F_a(x)< F_b(x)$ for $\theta_j \leq x < \theta_i$ since an extra agent has signed up for option j.
$F_a(x)= F_b(x)$ for $\theta_i \leq x$ since all the options k, $k > i$ have the same number of agents signed up

This implies that F_b stochastically dominated F_a. A contradiction. Therefore, truth revealing is a Nash Equilibrium.

Now let us show that it is not an equilibrium for a population from F_a to sign up to correspond to some other distribution F_b.

All we have to show here is that for a population of a distribution $F_a(v)$, revealing itself to be of a distribution $F_b(v)$ is not a Nash Equilibrium.

If the population reveals itself to be of a distribution of $F_b(v)$, all options will be in their implemented state. We know then that all agents of type θ_i have to sign up for options j such that $j \leq i$.

Therefore,
Fraction of options j, $j \leq i$ available to be signed up = $F_b(\theta_i) \geq F_a(\theta_i)$ for all θ_i.

This implies that F_b (first order) stochastically dominate F_a. A contradiction.

Therefore, it can never be the case that a population from a realized distribution F_a, would want to imitate to be of a distribution of F_b, if F_b does not first order stochastically dominate F_a.

Also note that if a population is in equilibrium with the options in their implemented state, then it must be the case that the agents have revealed their types truthfully. This is because options will only be implemented if it corresponds to one of the possible distributions and above we showed that it is not an equilibrium for any population to reveal itself to be of another distribution. So if they are in equilibrium with the options implemented, it must be the case that they have revealed their types truthfully.

If the principal decides to offer $\varepsilon>0$, for each agent if the options are implemented, then the truth revealing equilibrium Pareto dominates (Pareto superior to) all other possible equilibria, since in all other possible equilibria options are not implemented and agents get a payoff of 0.

Example:

Table 1. The First-Best Options for Agents of Different Types with $U(q,T,\theta) = q(\theta-q) - T$

θ	First best q^e	T^e
1 (low)	½	¼
2 (mid)	1	1
3 (high)	1½	2¼

Table 2. Payoff for different of agents with the first-best options

Option < q, T>	U(θ=1)	U(θ=2)	U(θ=3)
< ½, ¼>	0	½	1
< 1, 1>	-1	0	1
< 1½, 2¼>	-3	-1½	0

Let us assume there are two agents and they will either both be of type $\theta = 2$ or one each from $\theta=1$ and $\theta=3$. We can denote this by two distributions D1 and D2

D1 = (f(θ=1), f(θ=2), f(θ=3)) = (0, 1, 0) D2= (½, 0, ½)

Note that neither of the distributions first order stochastically dominates the other. The principal only has this information and thus will not know whether the realized state is D1 or D2.

Let us look at the signup game the agents will be playing if the principal has the implementation rule as in the above claim.

That is, options will be in their implemented state if the fractions signed up for all the options correspond to either D1 or D2.

Case 1: The realized state is D1.

Player 2 (mid type) θ=2

		Stay out	Option 1	Option 2	Option 3
Player 1 (mid type) θ=2	Stay out	**0 , 0**	**0, 0**	0, 0	0, 0
	Option 1	**0 , 0**	**0, 0**	0, 0	½ + ε, -1½ + ε
	Option 2	0, 0	0, 0	**ε, ε**	0, 0
	Option 3	0, 0	-1½ + ε, ½ + ε	0, 0	0, 0

Case 2: The realized state is D2. (w.l.o.g. let player 1 be the low type player)

		Player 2 (high type) θ=3			
		Stay out	Option 1	Option 2	Option 3
Player 1 (low type) θ=1	Stay out	**0 , 0**	**0, 0**	**0, 0**	0, 0
	Option 1	0 , 0	0, 0	0, 0	**ε, ε**
	Option 2	0, 0	0, 0	-1 + ε, 1+ ε	0, 0
	Option 3	0, 0	-3+ ε, 1+ ε	0, 0	0, 0

In this example above, it seems that agents could come up at the truth revealing equilibrium through iterative elimination of weakly dominated strategies, but this is the case for this special example with 2 agents and not the case for all scenarios. All possible Nash equilibria are bolded. Note that the truth revealing equilibria is Pareto superior to any of the other possible equilibria.

Notice that in the above design, the probability of realizing D1 over D2 did not matter. This would not be the case with traditional mechanism design. Traditionally, the principal would need to know the underlying probability of realizing each of the possible types and then would go about creating the second-best options. However, as long as the realizable distributions do not first order stochastically dominate another, a signup game can be designed such that, if the agents choose strategies that correspond to the only positive payoff yielding (Pareto superior) equilibrium possible, the principal can obtain rent arbitrarily close to the first-best rent.

What does this say about a population of agent drawn for a distribution?

Take a simple example of a population of n agents being randomly drawn from a distribution, G = (f(θ=1), f(θ=2)) = (p, 1-p). As one can see the possible realizable distributions first order stochastically dominate each other. Therefore, the signup game with the above implementation rule would not be of use here.

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