On a non-existence example of a wdom-vNM set in the Shapley-Scarf housing economy

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Abstract

We consider the exchange economy E of Shapley and Scarf (1974). Economy E has a finite number of agents. Each agent is endowed with one differentiated object like a house. They exchange objects to obtain preferable ones. Each agent needs exactly one object, and his preference ordering over the objects may contain indifferences. No monetary transfers are allowed.

For economy E, a von-Neumann-Morgenstern set defined by strong domination may not exist even if the core is nonempty. On the other hand, the strict core becomes the unique vNM set by weak domination (wdom-vNM set) if it is nonempty. Roth and Postlewaite (1977) proved this property by assuming strict preferences. Wako (1991) proved it by allowing indifferent preferences. However, if we allow indifferences, the strict core may be empty. For such cases, it was unknown whether a wdom-vNM set always exists in economy E.

We give an example in which each feasible allocation is individually rational (IR), and a wdom-vNM set does not exist. A nice property of vNM sets enabled us to find the example in a much shorter time than doing a full check. Let X be the set of Pareto efficient allocations of a given example, and X' the set that we have after removing from X each allocation which does not weakly dominate any allocations in X. We can show that a wdom-vNM set exists in X if and only if a wdom-vNM set exists in X', and that each wdom-vNM set in X can be recovered from wdom-vNM sets in X'. Applying this property to X iteratively, we could examine the existence of a wdom-vNM set by checking a reduced set of X. Konishi-Quint-Wako (2001) considered extended models of economy E, and showed examples with empty cores. Their examples have no wdom-vNM sets. The non/existence of a wdom-vNM set of the original economy E was thus a remaining question, which was answered by our example. Our investigation also found an example with a unique wdom-vNM set that consists of only allocations which are not IR.

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Purpose

- We consider the **House exchange economy** of Shapley-Scarf (1974) to investigate whether a **wdom-vNM set** (a vNM set defined by weak domination) always exists.

➢ In this economy,

if the strict core (the core defined by weak domination) is non-empty, then it is the unique wdom-vNM set.

[Roth-Postlewaite 1977, Wako 1991]

But it was unknown whether a wdom-vNM set always exists when the strict core is empty.

Results

- ☐ Checking every possible case by computers, we found

• Any economy with *less than 5* agents has a wdom-vNM set

• There exist 5-agent economies with *no* wdom-vNM set

• There exist 4-agent economies with a unique wdom-vNM set consisting of only *not-individually-rational* allocations.

- ☐ A sufficient condition for a wdom vNM set to exist

"Each agent has a (1,0,-1)-preference relation"

↑

The preference relation studied by Roth-Sönmez-Ünver (2005)

House Exchange Economy (Shapley-Scarf 1974)

- n agents $\qquad N = \{1,2,...,n\}$
- Each agent i has one object initially: "*house i*" ("*object i*")
  The n objects can be differentiated.
- Each agent i needs exactly one object, and has a preference ordering R_i over the n objects: $\quad R_i \subseteq N \times N$

  * Indifferences are allowed.
- There is neither money nor other medium of exchange.
- The agents swap objects among themselves in a mutually beneficial way. They cannot throw away objects.
  - An allocation is a permutation mapping $x : N \to N$.
    $x = (x(1),...,x(n))$: a vector representation.

Weak domination between allocations

X = the whole set of allocations

$x, y \in X$ allocations

$A \subseteq X$ a subset of allocation

$S \subseteq N$ a coalition (nonempty subset of N)

- x **weakly dominates** y **via** S [x *wdom(S)* y] if
  1) $\{x(i) \mid i \in S\} = S$
  2) $x(i)\, R_i\, y(i)$ for each $i \in S$, and
     $x(i)\, P_i\, y(i)$ for at least one $i \in S$

- *WDOM(A)*=$\{ y \in X \mid x$ *wdom(S)* y for some $x \in A$
  and some $S \subseteq N \}$

Strict Core

The **strict core** SC is the set of allocations that are **not weakly dominated** by any allocation via any coalition.

$$SC = X \setminus WDOM(X)$$

- x is **Pareto efficient** if there is no y in X with $y\ wdom(N)\ x$.
  PO = the set of Pareto efficient allocations
- x is **individually rational** if $x(i) R_i\ i$ for each $i \in N$.

  IR = the set of individually rational allocations

➢ The strict core SC_A *in* A: $SC_A = A - WDOM(A)$.

$$SC = SC_X = SC_{IR} \subseteq SC_{PO} \subseteq PO$$

Wdom-vNM set (vNM set defined by weak domination)

A **wdom-vNM set** V is a nonempty subset of X with

1) internal stability: $WDOM(V) \cap V = \phi$, and
2) external stability: $(X - V) \subseteq WDOM(V)$

$$V = X \setminus WDOM(V)$$

➢ Define a wdom-vNM set V_A <u>*in*</u> A as $V_A = A \setminus WDOM(V_A)$.

• A wdom-vNM set V_{IR} in IR

$$V_{IR} = IR \setminus WDOM(V_{IR}).$$

V_{IR} is different from $V(=V_X)$

Core and a sdom-vNM set (a vNM defined by strong domination)

- x **strongly dominates** y **via** S [$x\ sdom(S)\ y$]   if
  1) $\{x(i) \mid i \in S\} = S$
  2) $x(i)\, P_i\, y(i)$ for each $i \in S$
- $SDOM(A) = \{ y \in X \mid x\ sdom(S)\ y$ for some $x \in A$ and some $S \subseteq N \}$

- The **core** $C = X \setminus SDOM(X)$

✧ The core is not empty for any house exchange economy.
(Shapley-Scarf 1974)

- A **sdom-vNM set** is a nonempty subset of X with

$$V = X \setminus SDOM(V)$$

Example 1. Non-existence of a sdom-vNM set

$N = \{1,2,3\}.$ 6 allocations

1) $2 \ P_1 \ 3 \ P_1 \ 1$ x=(2,3,1)

2) $3 \ P_2 \ 1 \ P_2 \ 2$ y=(2,1,3), z=(1,3,2), u =(3,2,1)

3) $1 \ P_3 \ 2 \ P_3 \ 3$ v =(3,1,2), w=(1,2,3)

- *y sdom(12) u, u sdom(13)z, z sdom(23) y,* and *SDOM(x)*={ v ,w} $\Rightarrow$ C={x}
- If a sdom-vNM set V exists, $x \in V$.
- {x} is lack of external stability.
- *y, z* and *u* generate an odd number wdom cycle.

➢ No sdom-vNM set exists.

- *SC*={x}, and *SC* is the unique wdom-vNM set.

Wdom-matrix

b \ a	x	y	z	u	v	w
x	0	1	1	1	1	1
y	0	0	0	1	1	1
z	0	1	0	0	1	1
u	0	0	0	0	1	1
v	0	0	0	0	0	1
w	0	0	0	0	0	0

a *wdom*(S) b for some S $\rightarrow$ 1

Otherwise $\rightarrow$ 0

- Column x is a zero vector $\Rightarrow$ $SC=\{x\}$
- Cell with dark shadow: Internal stability of $\{x\}$
- Cells with pale shadow: External stability of $\{x\}$
  - ➢ $V=\{x\}=SC$ is a wdom-vNM set

Example 2. $SC=\emptyset$ and a wdom-vNM set exists

$N=\{1,2,3\}$

1) $2 \ P_1 \ 3 \ P_1 \ 1$
2) $3 \ I_2 \ 1 \ P_2 \ 2$
3) $2 \ P_3 \ 1 \ P_3 \ 3$

allocations

$x=(1,2,3)$, $y=(1,3,2)$
$z=(2,1,3)$, $u=(2,3,1)$
$v=(3,1,2)$, $w=(3,2,1)$

a \ b	x	y	z	u	v	w
x	0	0	0	0	0	0
y	1	0	1	0	1	1
z	1	1	0	1	0	1
u	1	1	0	0	0	1
v	1	0	1	0	0	1
w	1	0	0	0	0	0

- No zero column vector ➢$SC=\emptyset$
- $V=\{u,v\}$ is a wdom-vNM set
- $y=(1,3,2)$ and $z=(2,1,3)$ are *top trading cycle* allocations, but not contained in V.

Numbers of possible cases

Size of Economy (agents)	Preference patterns	Cases to check (Combination of pref. pat'n)	Size of a wdom matrix (# of allocations)	Number of subsets of allocations
2	3	$3^2 = 9$	2!=2	$2^2 - 1 = 3$
3	13	$13^3 = 2197$	3!=6	$2^6 - 1 = 63$
4	75	$75^3 =$ $3.16E + 07$	4!=24	$2^{24} - 1 =$ $1.68E + 07$
5	541	$541^5 =$ $4.63E + 13$	5!=120	$2^{120} - 1 =$ $1.33E + 36$

Properties for reducing the number of checks and a size of a wdom matrix

[1] If a wdom matrix is a **zero** or **symmetric** matrix, the allocations listed in the wdom matrix form a wdom-vNM set in the set of those allocations.

[2] $V_X = V_{PO}$

- We can make a wdom matrix only out of Pareto efficient allocations.

• Let A be any allocation set with $|A| \geq 2$.

[3] Suppose there is $x \in A$ such that

$$\forall y \in A \setminus \{x\}, \quad y \; wdom(S) \; x \quad \text{for some } S.$$

Then,

V is a wdom-vNM set in A $\Leftrightarrow$ V is a wdom-vNM set in $A \setminus \{x\}$

[4] Suppose $\exists x \in A$ s.t.

x does not weakly dominate any $y \in A$

1) If V is a wdom-vNM set in A, then

$V \setminus \{x\}$ is a wdom-vNM set in $A \setminus \{x\}$

2) If V is a wdom-vNM set in $A \setminus \{x\}$, then

one of V or $V \bigcup \{x\}$ is a wdom-vNM set in A

✧ Applying [3] and [4] recursively, we can reduce the size of a wdom matrix without losing information on vNM sets.

Wdom matrix of Ex. 2

a \ b	x	y	z	u	v	w
x	0	0	0	0	0	0
y	1	0	1	0	1	1
z	1	1	0	1	0	1
u	1	1	0	0	0	1
v	1	0	1	0	0	1
w	1	0	0	0	0	0

✧ Applying [3] and [4] recursively, we can reduce the size of a wdom matrix without losing information on vNM sets.

Wdom matrix of Ex. 2

a \ b	y	z	u	v	w
y	0	1	0	1	1
z	1	0	1	0	1
u	1	0	0	0	1
v	0	1	0	0	1
w	0	0	0	0	0

✧ Applying [3] and [4] recursively, we can reduce the size of a wdom matrix without losing information on vNM sets.

Wdom matrix of Ex. 2

b / a	x	y	z	u	v	w
x	0	0	0	0	0	0
y	1	0	1	0	1	1
z	1	1	0	1	0	1
u	1	1	0	0	0	1
v	1	0	1	0	0	1
w	1	0	0	0	0	0

$\Rightarrow$

b / a	y	z	u	v
y	0	1	0	1
z	1	0	1	0
u	1	0	0	0
v	0	1	0	0

Furthermore,

[5] *In our experience*, arranging the allocations in a wdom matrix in order of the numbers of 1's in rows works quite well to reduce a "timeout" from happening before completing checks.

Example 3. Five-agent economy with no wdom-vNM set

$N = \{1,2,3,4,5\}$

1) $4 \; I_1 \; 5 \; P_1 \; 2 \; P_1 \; 3 \; I_1 \; 1$

2) $4 \; I_2 \; 5 \; P_2 \; 3 \; P_2 \; 1 \; I_2 \; 2$

3) $4 \; I_3 \; 5 \; P_3 \; 1 \; P_3 \; 2 \; I_3 \; 3$

4) $1 \; I_4 \; 2 \; I_4 \; 3 \; P_4 \; 5 \; I_4 \; 4$

5) $1 \; I_5 \; 2 \; I_5 \; 3 \; P_5 \; 4 \; I_5 \; 5$

- Agents 1, 2 and 3 compete for objects 4 and 5.
- The agent who could not get the best object tries to get the second best object utilizing indifferences of other agents.

➢ In this example, such efficient trading generates a one-way wdom cycle of 3 outcomes.

Example 3. Five-agent economy with no wdom-vNM set

$N = \{1,2,3,4,5\}$

1) $4 \;\; I_1 \;\; 5 \;\; P_1 \;\; 2 \;\; P_1 \;\; 3 \;\; I_1 \;\; 1$
2) $4 \;\; I_2 \;\; 5 \;\; P_2 \;\; 3 \;\; P_2 \;\; 1 \;\; I_2 \;\; 2$
3) $4 \;\; I_3 \;\; 5 \;\; P_3 \;\; 1 \;\; P_3 \;\; 2 \;\; I_3 \;\; 3$
4) $1 \;\; I_4 \;\; 2 \;\; I_4 \;\; 3 \;\; P_4 \;\; 5 \;\; I_4 \;\; 4$
5) $1 \;\; I_5 \;\; 2 \;\; I_5 \;\; 3 \;\; P_5 \;\; 4 \;\; I_5 \;\; 5$

- There are wdom cycles of 3 allocations.

➢ No wdom-vNM exists.

<Reduced wdom matrix>

x=(5,3,4,2,1), y=(4,3,5,1,2)
z=(5,4,1,2,3), v=(4,5,1,3,2)
u=(2,5,4,3,1), w=(2,4,5,1,3)

a \ b	x	y	z	v	u	w
x	0	0	0	0	1	1
y	0	0	0	0	1	1
z	1	1	0	0	0	0
v	1	1	0	0	0	0
u	0	0	1	1	0	0
w	0	0	1	1	0	0

Example 3. Five-agent economy with no wdom-vNM set

$N = \{1,2,3,4,5\}$

1) $4 \; I_1 \; 5 \; P_1 \; 2 \; P_1 \; 3 \; I_1 \; 1$

2) $4 \; I_2 \; 5 \; P_2 \; 3 \; P_2 \; 1 \; I_2 \; 2$

3) $4 \; I_3 \; 5 \; P_3 \; 1 \; P_3 \; 2 \; I_3 \; 3$

4) $1 \; I_4 \; 2 \; I_4 \; 3 \; P_4 \; 5 \; I_4 \; 4$

5) $1 \; I_5 \; 2 \; I_5 \; 3 \; P_5 \; 4 \; I_5 \; 5$

- every allocation of Ex. 3 is individually rational.

➢ Example 3 is also an example with no V_{IR} (a wdom-vNM set in IR).

✧ Every example with *less than 5* agents has a wdom-vNM set. [checked by our computer program]

Example 4. Four-agent economy with nonIR wdom-vNM set

$N = \{1,2,3,4\}$

1) $4 \; I_1 \; 3 \; I_1 \; 2 \; I_1 \quad 1$
2) $1 \; I_2 \; 3 \; P_2 \; 4 \; P_2 \quad 2$
3) $1 \; P_3 \; 3 \; P_3 \; 2 \; I_3 \quad 4$
4) $1 \; I_4 \; 3 \; P_4 \; 2 \; P_4 \quad 4$

Pareto efficient allocations

x=(2,1,4,3), w=(4,3,2,1)
v=(4,1,2,3), y =(2,3,4,1)
u=(4,3,1,2), z=(2,4,1,3)

- V={x,w,v,y} is the unique wdom-vNM set of this example.
- However, its components are allocations that are not individually rational.

a \ b	x	w	v	y	u	z
x	0	0	0	0	0	1
w	0	0	0	0	1	0
v	0	0	0	0	0	0
y	0	0	0	0	0	0
u	0	0	0	0	0	0
z	0	0	0	0	0	0

Example 4. Four-agent economy with nonIR wdom-vNM set

$N = \{1,2,3,4\}$

1) $4 \;\; I_1 \;\; 3 \;\; I_1 \;\; 2 \;\; I_1 \;\; 1$
2) $1 \;\; I_2 \;\; 3 \;\; P_2 \;\; 4 \;\; P_2 \;\; 2$
3) $1 \;\; P_3 \;\; 3 \;\; P_3 \;\; 2 \;\; I_3 \;\; 4$
4) $1 \;\; I_4 \;\; 3 \;\; P_4 \;\; 2 \;\; P_4 \;\; 4$

Pareto efficient allocations

$x=(2,1,4,3)$, $w=(4,3,2,1)$
$v=(4,1,2,3)$, $y=(2,3,4,1)$
$u=(4,3,1,2)$, $z=(2,4,1,3)$

- In each of $\{v, y, u, z\}$, *4-agent trading cycle* is formed. These allocations do not weakly dominate other Pareto efficient allocation. We can remove them from a wdom matrix.
- We then know any wdom-vNM set must contain $\{x,y\}$, but not $\{u,z\}$. Adding v and y one by one, we know $V=\{x,w,v,u\}$ is the unique wdom-vNM set

<u>A sufficient condition for a wdom vNM set to exist</u>

We consider a sufficient condition for a wdom vNM set to exist in any economy with n agents.

- An agent has a **(1,0,-1)-preference relation** if his preference relation can be represented by a utility function that evaluates objects as follows:

  his own initial object $\longrightarrow$ 0

  any other agent's object $\longrightarrow$ 1 <u>or</u> -1

* Even if each agent has a (1,0,-1)-preference relation, the strict core can be empty.

- ☐ Assume that each of n agents has a (1,0,-1)-preference relation.

Proposition 1.

$$PO \bigcap TTC = IR \cap PO,$$

where TTC is the set of top trading cycle allocations.

Proposition 2. For any $x \in IR \cap PO$, the following set V is a wdom-vNM set of the economy:

$$V = \{x\} \cup \{y \in IR \cap PO \mid x(i)\, I_i\, y(i) \text{ for each } i \in N\}.$$

Example 5. (1,0,-1)-preference relation

$N = \{1,2,3\}$ allocations

1) $2 \;\; P_1 \;\; 1 \;\; P_1 \;\; 3$ $\quad x=(2,3,1), \quad y=(3,1,2)$

2) $3 \;\; I_2 \;\; 1 \;\; P_2 \;\; 2$ $\quad z=(2,1,3), \quad v=(1,3,2)$

3) $2 \;\; P_3 \;\; 3 \;\; P_3 \;\; 1$ $\quad u=(3,2,1), \quad w=(1,2,3)$

- $IR \cap PO = \{z, v\}$
- $SC = \varnothing$
- $V_1 = \{z\}, \;\; V_2 = \{v\}$ are wdom-vNM sets.
- x and y are Pareto efficient, but not individually rational.

Statistics

Size	Cases	$SC \neq \varnothing$	$\exists$ vNM	$\nexists$vNM
3	2,197 100%	2,143 97.54%	54 2.46%	0 0%
4	3.16E+07 100%	2.97 E+07 93.75%	1.98E+06 6.25%	0 0%
5 (*)	1.65E+08 100%	1.20E+08 72.59%	4.52E+07 27.41%	1,344 8.15E-04%
5 full	4.63E+13 100%	$\nexists V_{IR}$ and $SC_{PO} = \varnothing$ 9.18E+08 (1.98E-03%)		

(*) The cases in which each agent's preference is of the form:

P4I5, **P4I5P**, or 4I5P.