# Strong constrained egalitarian allocations: How to find them *

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**Abstract**

This paper provides a geometrical decomposition theorem for the *strong Lorenz core* (Dutta and Ray, 1991). As a consequence, we characterize the existence of the set of *strong constrained egalitarian allocations* and we give an algorithm to find, for any TU game, the strong constrained egalitarian allocations. Moreover, we characterize the connectivity of the strong Lorenz core.

## 1 Introduction

The *egalitarian* solution for cooperative TU-games (ES for short), introduced by Dutta and Ray (1989), unifies the two conflicting concepts of egalitarianism and individual interest. They show that the ES is a singleton or the empty set and describe, for the class of convex games, an algorithm to locate the unique egalitarian allocation, showing that in this framework it belongs to the core. Nevertheless, the ES may not exist even for balanced games and, in general, is difficult to find it.

In order to obtain egalitarian solutions existing for larger classes of games than just convex games, Dutta and Ray (1991) introduced the *strong constrained egalitarian solution* (SCES for short). The definitions of ES and SCES are identical, except in the concept of

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blocking that in the SCES requires every member of the blocking coalition be strictly better off. However, as shown by Dutta and Ray (1991), this slight modification on the blocking concept imply meaningful qualitative differences between the ES and the SCES. Precisely, the SCES exists under very mild conditions, but not posses the uniqueness property. Of course, both solutions are related, since there is always an strong constrained egalitarian allocation which Lorenz dominates the egalitarian allocation. For convex games and all four-player games, every SCEA Lorenz-dominates the ES. The idea of combining stability and egalitarianism has been also the aim of Arin and Iñarra (2001), Hougaard et all (2001), and Arin et all (2004) who suggest to select Lorenz maximal imputations in the core arguing that their approach takes into account the coalitional stability as a natural restriction on the outcome.

This paper will focus on the SCES, that is, on Lorenz maximal allocations with respect to the *strong Lorenz core*. Dutta and Ray (1991) show the coincidence between the strong Lorenz core and the *equal division core* introduced by Selten in 1972 to explain outcomes of experimental cooperative games. From his experimental observations, Selten (1987) argued that "the evidence suggests that equity considerations have a strong influence on observed payoff divisions". From this coincidence, the strong Lorenz core is not only justified from the theoretical idea of satisfying participation constraints for the coalitions if the norm of egalitarianism is treated consistently across coalitions, but also from an experimental point of view. An axiomatic justification of the equal division core can be found in Bhattacharya (2004).

Therefore, on one hand it seems natural to consider Lorenz-maximal solutions with respect to the strong Lorenz core arguing that in this way egalitarianism is treated in a 'consistent' manner across coalitions, existence is guaranteed for a large class of games and there is experimental evidence. On the other hand, Dutta and Ray's construction of the SCES as a recursive structure being related to the strong Lorenz core suggests that it should be not easy to implement. So, the aim of this paper is to make the analysis of the SCES and the strong Lorenz core even clearer by extending their work in two directions.

First, we show that the strong Lorenz core can be decomposed as a union of a finite number of simple polyhedrons. As a consequence we characterize the non-emptiness and the connectivity of this set.

Second, we provide an algorithm, for any game and for any number of players, to locate all the set of SCEA in a finite number of elementary computations and therefore easy to implement.

The plan of the paper is as follows. In Section 2 we provide some notation. In Section 3 we present the decomposition theorem for the strong Lorenz core and in Section 4 we give the direct consequences of the decomposition theorem, the characterization of the non-emptiness and connectivity of the strong Lorenz core, and the non-emptiness of the SCES. In Section 5 we give an algorithm that computes the set of strong constrained egalitarian allocations in a finite number of elementary computations. In Section 6 we give some final remarks for the extension of this work to the asymmetric case and, finally in Section 7 the concluding comments.

## 2 Preliminaries

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. By $N \subseteq \mathbb{N}$ we denote a finite set of players, in general $N = \{1, \ldots, n\}$. A *transferable utility coalitional game (a game)* is a pair $(N, v)$ where $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$ and $2^N$ denotes the set of all subsets (coalitions) of $N$. A game $(N, v)$ is *nonnegative* if $v(S) \geq 0$, for any coalition $S \subseteq N$. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by $\Gamma$. Given a coalition $S \subset N, S \neq \emptyset$ and $(N, v) \in \Gamma$, we define the subgame $(S, v_S)$ by $v_S(Q) := v(Q)$, for all $Q \subseteq S$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by $N$, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_T$ denotes the restriction of $x$ to $T$: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. In addition, we define $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x > 0\}$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ denotes that $x_i \geq y_i$, for all $i \in N$. We say that $x > y$, if and only if $x \geq y$ and for some $j \in N$, $x_j > y_j$. By $z = \max\{x, y\}$, we denote the vector $z \in \mathbb{R}^N$ such that $z_i = \max\{x_i, y_i\}$, for all $i \in N$.

The *pre-imputation set* of a game $(N, v)$ is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$. A *solution* on a set $\Gamma$ of games is a mapping $\sigma$ which associates with any game $(N, v)$ a subset $\sigma(N, v)$ of the set $X(N, v)$. Notice that the solution set $\sigma(N, v)$ is allowed to be empty. For a game $(N, v)$, the set of *imputations* is given by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq v(i), \ \forall\, i \in N\}$. The *core* of a game $(N, v)$ is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S)$ for all $S \subseteq N\}$. The *equal division core (EDC)* (Selten, 1972) is an extension of the core containing those imputations which can not be improved upon by the equal division allocation of any subcoalition, formally $EDC(N, v) := \left\{x \in I(N, v) \mid \forall\, \varnothing \neq S \subset N, \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|}\right\}$. For any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained by rearranging the coordinates in a non-decreasing order, that is, $\hat{x}_1 \leq \hat{x}_2 \leq \ldots \leq \hat{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$, we say that $y$ *Lorenz-dominates* $x$, $(y \succ_L x)$, if $\sum_{j=1}^{k} \hat{y}_j \geq \sum_{j=1}^{k} \hat{x}_j$, for every $k = 1, \ldots, n$, with at least one strict inequality. Given a game $(N, v)$, the *strong Lorenz Core ($L^*$)* (Dutta and Ray, 1991) is defined in a recursive way. The strong Lorenz core of a singleton coalition is $L^*(\{i\}) =$

$\{v(\{i\})\}$. Now suppose that the strong Lorenz core for all coalitions of cardinality $k$ or less have been defined, where $1 < k < |N|$. The strong Lorenz core of a coalition of size $(k+1)$ is defined by $L^*(S) = \{x \in \mathbb{R}^S \mid x(S) = v(S),$ and $\nexists\, T \subset S$ and $y \in EL^*(T)$ s.t. $y \geq x_T\}$, where $EL^*(T) = \{x \in L^*(T) \mid \nexists\, y \in L^*(T)$ s.t. $y \succ_L x\}$. Theorem 1 in Dutta and Ray (1991) states that $L^*(N) = EDC(N,v)$, for any game $(N,v)$. Hence, given a game $(N,v)$, the *strong constrained egalitarian allocations* is the set $EL^*(N) = \{x \in EDC(N,v) \,|\, \nexists\, y \in EDC(N,v)$ such that $y \succ_L x\}$.

A game with a non-empty set of imputations is called *essential*. A game with a non-empty core is called *balanced* and, if all its subgames have non-empty cores, the game is said to be *totally balanced*. A game $(N,v)$ is *convex* (Shapley, 1971) if, for every $S,T \subseteq N$, $v(S)+v(T) \leq v(S\cup T)+v(S\cap T)$. A game $(N,v)$ is *superadditive* if, for every $S,T \subseteq N$, $S\cap T = \emptyset$, $v(S)+v(T) \leq v(S\cup T)$. A game is $N$-*superadditive* if for all partition $\{S_1,\ldots,S_m\}$ of $N$, it holds $v(S_1)+\ldots+v(S_m) \leq v(N)$. A game $(N,v)$ is said to be *modular* if there exists a vector $x = (x_1,\ldots,x_n) \in \mathbb{R}^N$ such that for every $S \subseteq N$, $v(S) = \sum_{i\in S} x_i$. To indicate the modular game generated by $x \in \mathbb{R}^N$ we will use $(N, v_x)$.

An ordering $\theta = (i_1,\ldots,i_n)$ of $N$, where $|N| = n$, is a bijection from $\{1,\ldots,n\}$ to $N$. We denote by $\mathcal{S}_N$ the set of all orderings of $N$.

## 3 A decomposition theorem for the strong Lorenz core

The main result of this section is that the strong Lorenz core (or the equal division core) can be decomposed as the union of a finite number of simple polyhedrons. This will be the basic tool for our following results. In order to find those polyhedrons we introduce the concept of *equal share worth vectors*.

**Definition.** Let $(N,v)$ be a game and $\theta = (i_1,\ldots,i_n) \in \mathcal{S}_N$. We define the equal share worth vector associated to $\theta$, denoted by $\bar{x}^\theta(v) \in \mathbb{R}^N$, as follows:

$$\bar{x}^\theta_{i_k}(v) := \max_{S\in P_{i_k}} \left\{\frac{v(S)}{|S|}\right\}, \text{ for } k = 1,\ldots,n,$$

where $P_{i_1} := \{S \subseteq N \mid i_1 \in S\}$ and $P_{i_k} := \{S \subseteq N \mid i_1,\ldots,i_{k-1} \notin S,\ i_k \in S\}$, for $k = 2,\ldots,n$.

**Remark 3.1.** Notice that for all $\theta = (i_1,\ldots,i_n) \in \mathcal{S}_N$, the set $\{P_{i_1},\ldots,P_{i_n}\}$ forms a partition of the set $2^N \setminus \emptyset$. In addition, for all $i \in N$, $\bar{x}^\theta_i(v) \geq v(\{i\})$, and for any non-empty coalition $S \subseteq N$, there is a player $i \in S$ such that $\bar{x}^\theta_i(v) \geq \frac{v(S)}{|S|}$. However, in general $\bar{x}^\theta(v)$ is not an efficient vector and so it does not belong to the strong Lorenz core.

Next we define the polyhedrons that will give rise to the decomposition of the strong Lorenz core.

**Definition.** Let $(N,v)$ be a game and $\theta=(i_1,\ldots,i_n)\in\mathcal{S}_N$. We define the polyhedron generated by the equal share worth vector $\bar{x}^\theta(v)$, denoted by $\Delta^{\bar{x}^\theta}(v)$, as the convex hull of all $\bar{x}^\theta(v)+\delta_{\bar{x}^\theta}\,e_i$, where $\delta_{\bar{x}^\theta}=v(N)-\bar{x}^\theta(v)(N)$ and, for any $i\in N$, $e_i$ is the $i$-th canonical vector of $\mathbb{R}^N$. That is,

$$\Delta^{\bar{x}^\theta}(v):=\text{convex}\,\{\bar{x}^\theta(v)+\delta_{\bar{x}^\theta}\,e_i,\ \text{for all}\ i\in N\}.$$

Given a game $(N,v)$ and $\theta\in\mathcal{S}_N$ such that $\delta_{\bar{x}^\theta}\geq 0$, it is easy to see that

$$\Delta^{\bar{x}^\theta}(v)=\{x\in X(N,v)\mid x\geq\bar{x}^\theta(v)\}. \tag{1}$$

To decompose the strong Lorenz core we only need to work with a special kind of the polyhedrons defined above, those generated by the equal share worth vectors associated to $\theta\in\mathcal{S}_N$ with $\delta_{\bar{x}^\theta}\geq 0$, and minimal with respect to the usual order in $\mathbb{R}^N$.

**Lemma 3.2.** *Let $(N,v)$ be a game and $\bar{x}^\theta(v)$, $\bar{x}^{\theta'}(v)$ such that $\delta_{\bar{x}^\theta}\geq 0$, $\delta_{\bar{x}^{\theta'}}\geq 0$. Then, the following statements are equivalent:*

1. $\bar{x}^\theta(v)\leq\bar{x}^{\theta'}(v)$.
2. $\Delta^{\bar{x}^{\theta'}}(v)\subseteq\Delta^{\bar{x}^\theta}(v)$.

Proof: From expression (1) it follows straightforward that $\bar{x}^\theta(v)\leq\bar{x}^{\theta'}(v)$ implies $\Delta^{\bar{x}^{\theta'}}(v)\subseteq\Delta^{\bar{x}^\theta}(v)$. Next we prove the converse. Assuming $\Delta^{\bar{x}^{\theta'}}(v)\subseteq\Delta^{\bar{x}^\theta}(v)$ we deduce that $\bar{x}^{\theta'}(v)+\delta_{\bar{x}^{\theta'}}\,e_k\in\Delta^{\bar{x}^\theta}(v)$, for all $k\in N$. Hence, again from expression (1), $\bar{x}^{\theta'}(v)+\delta_{\bar{x}^{\theta'}}\,e_k\geq\bar{x}^\theta(v)$. Finally, let $j\in N$ and take $k\neq j$, then $\bar{x}^{\theta'}_j(v)\geq\bar{x}^\theta_j(v)$, getting the result.□

**Definition.** Let $(N,v)$ be a game. We define the set of minimal equal share worth vectors as follows:

$$\begin{aligned}\mathcal{M}(v):=\{x\in\mathbb{R}^N\mid x=\bar{x}^\theta(v)\ &\text{for some}\ \theta\in\mathcal{S}_N,\ \delta_{\bar{x}^\theta}\geq 0\,\text{and}\ \nexists\ \theta'\in\mathcal{S}_N,\\ &\text{s. t. }\bar{x}^{\theta'}(v)<\bar{x}^\theta(v)\}.\end{aligned} \tag{2}$$

Now we have all the tools to state a decomposition theorem for the strong Lorenz core (or the equal division core) in terms of the above polyhedrons.

**Theorem 3.3.** *The strong Lorenz core is the union of a finite set of polyhedrons. In particular, given a game $(N,v)$*

$$L^*(N)=\bigcup_{x\in\mathcal{M}(v)}\Delta^x(v).$$

PROOF: Let $x \in L^*(N)$. We construct a specific order $\theta \in \mathcal{S}_N$ and an equal share worth vector $\bar{x}^\theta(v)$ such that $x \geq \bar{x}^\theta(v)$. This order $\theta$ is generated by the following algorithm. First, choose a coalition $S_1 \in 2^N$, $S_1 \neq \emptyset$, such that $\frac{v(S_1)}{|S_1|} = \max_{\emptyset \neq C \in 2^N} \left\{ \frac{v(C)}{|C|} \right\}$. Having chosen $S_1$, since we suppose that $x \in L^*(N)$, there exists a player $i_1 \in S_1$ such that $x_{i_1} \geq \frac{v(S_1)}{|S_1|}$. Second, choose $S_2 \in 2^{N \setminus \{i_1\}}$, $S_2 \neq \emptyset$, such that $\frac{v(S_2)}{|S_2|} = \max_{\emptyset \neq C \in 2^{N \setminus \{i_1\}}} \left\{ \frac{v(C)}{|C|} \right\}$. As before, since $x \in L^*(N)$, there exists a player $i_2 \in S_2$ such that $x_{i_2} \geq \frac{v(S_2)}{|S_2|}$. Following this process we obtain an ordering $\theta = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_N$ such that

$$x \geq \bar{x}^\theta(v), \tag{3}$$

where $\bar{x}^\theta_{i_j}(v) = \frac{v(S_j)}{|S_j|}$, $j = 1, 2, \ldots, n$.
Since $x \in X(N, v)$, from (3) it follows that $\delta_{\bar{x}^\theta} \geq 0$. Hence, from expression (1) we have $x \in \Delta^{\bar{x}^\theta}(v)$. If $\bar{x}^\theta(v) \in \mathcal{M}(v)$, we are finished. If not, we can find an order $\theta'$ such that $\bar{x}^{\theta'}(v) < \bar{x}^\theta(v)$ with $\bar{x}^{\theta'}(v) \in \mathcal{M}(v)$. But then, from Lemma 3.2, $\Delta^{\bar{x}^\theta}(v) \subseteq \Delta^{\bar{x}^{\theta'}}(v)$, and so $x \in \Delta^{\bar{x}^{\theta'}}(v)$.

To show the reverse inclusion, let $x \in \Delta^{\bar{x}^\theta}(v)$, where $\Delta^{\bar{x}^\theta}(v)$ is generated by $\bar{x}^\theta(v) \in \mathcal{M}(v)$. Then, from , $x \in X(N, v)$ and $x \geq \bar{x}^\theta(v)$. Recall that for all $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$, the set $\{P_{i_1}, \ldots, P_{i_k}\}$ as described in Definition 3 forms a partition of the set $2^N \setminus \{\emptyset\}$ (see Remark 3.1). Now let $S \in 2^N \setminus \{\emptyset\}$, and $i_r \in S$ be the first player in $S$ with respect to the ordering $\theta$. Then, $S \in P_{i_r}$, and so $x_{i_r} \geq \bar{x}^\theta_{i_r}(v) = \max_{C \in P_{i_r}} \left\{ \frac{v(C)}{|C|} \right\} \geq \frac{v(S)}{|S|}$. Hence, we can conclude that $x \in L^*(N)$. $\Box$

# 4 Consequences of the decomposition theorem

In this part we are going to use the decomposition theorem stated in the section above in order to characterize non-emptiness and connectivity of the strong Lorenz core, and non-emptiness of the strong constrained egalitarian solution.

## 4.1 Non-emptiness

First of all, notice that since the strong Lorenz core is a compact set, the non-emptiness of the strong Lorenz core and the strong constrained egalitarian solution are equivalent.

Since the strong Lorenz core is an extension of the core, balancedness (Bondareva, 1963 and Shapley, 1967) gives a first condition to guarantee the non-emptiness. However, it is well-known that strong Lorenz core can be non-empty even if the core is empty. On the other hand, Dutta and Ray (1991) state that for weakly superadditive games both the strong Lorenz core and the strong constrained egalitarian solution are non-empty. Nevertheless, as we will see in the following example, weak superadditivity does not characterize non-emptiness.

**Example 1.** *Let* $(N, v)$ *be a three-person game, and* $v(\{i\}) = 0$*, for all* $i = 1, 2, 3$*,* $v(\{1, 2\}) = 2$*,* $v(\{1, 3\}) = v(\{2, 3\}) = 0$*,* $v(N) = 1$*.*
*Notice that this game is not weakly superadditive since* $v(\{1, 2\}) + v(\{3\}) > v(\{1, 2, 3\})$ *and the strong Lorenz core* $L^*(N) = \{x \in I(N, v) \mid x_1 \geq 1 \text{ or } x_2 \geq 1\}$ *is a non-empty set.*

So, our objective is now to characterize non-emptiness using Theorem 3.3. Since the strong Lorenz core is a compact set, the non-emptiness of $L^*(N)$ implies the non-emptiness of the set of strong constrained egalitarian allocations, $EL^*(N)$. Hence, it follows immediately the following result.

**Theorem 4.1.** *Let* $(N, v)$ *be a game. Then, the following statements are equivalent:*

1. $L^*(N) \neq \emptyset$.
2. $EL^*(N) \neq \emptyset$
3. *There exists* $x \in \mathcal{M}(v)$ *such that* $x(N) \leq v(N)$*.*

## 4.2 Connectedness

In general, the strong Lorenz core is not a connected set. To show this consider the following example.

**Example 2.** *Let* $(N, v)$ *be a three-person game, and* $v(\{i\}) = 0$*, for all* $i \in N$*,* $v(\{1, 2\}) = 0$*,* $v(\{1, 3\}) = v(\{2, 3\}) = v(N) = 1$*. The set of minimal equal share worth vectors is* $\mathcal{M}(v) = \{x = (0.5, 0.5, 0), y = (0, 0, 0.5)\}$*, and from Theorem 3.3 we can express* $L^*(N) = \Delta^x(v) \cup \Delta^y(v)$*, where* $\Delta^x(v) = \{(0.5, 0.5, 0)\}$ *and* $\Delta^y(v) = \text{convex}\ \{(0.5, 0, 0.5), (0, 0.5, 0.5), (0, 0, 1)\}$*. Now it is easy to see that* $L^*(N)$ *is not connected.*

Let us consider a game with a connected strong Lorenz core.

**Example 3.** *Let* $(N, v)$ *be a four-player glove market game. Assume player* 1 *and* 2 *own one right-hand glove each, and player* 3 *and* 4 *own one left-hand glove each. Single gloves (or several gloves from the same hand) are worthless; any pair of gloves can, however, be sold for* 1$ *(it is assumed that the gloves are identical except for the left-right distinction). Thus, the characteristic function of the game is* $v(\{i\}) = 0$ *for all* $i = 1, \ldots, 4$*,* $v(\{1, 2\}) = v(\{3, 4\}) = 0$*,* $v(N) = 2$ *and* $v(S) = 1$ *otherwise.*

In this example $\mathcal{M}(v) = \{x = (0.5, 0.5, 0, 0),\ y = (0, 0, 0.5, 0.5)\}$. Again from Theorem 3.3, we know that $L^*(N) = \Delta^x(v) \cup \Delta^y(v)$, where $\Delta^x(v) = \text{convex}\ \{(1.5, 0.5, 0, 0), (0.5, 1.5, 0, 0), (0.5, 0.5, 1, 0), (0.5, 0.5, 0, 1)\}$ and $\Delta^y(v) = \text{convex}\{(1, 0, 0.5, 0.5), (0, 1, 0.5, 0.5), (0, 0, 1.5, 0.5), (0, 0, 0.5, 1.5)\}$.

Figure 1 represents the core and the strong Lorenz core of the game in the efficiency hyperplane (of dimension 3). The strong Lorenz core corresponds to the two shadowed pyramides and the core is the discontinuous black segment.

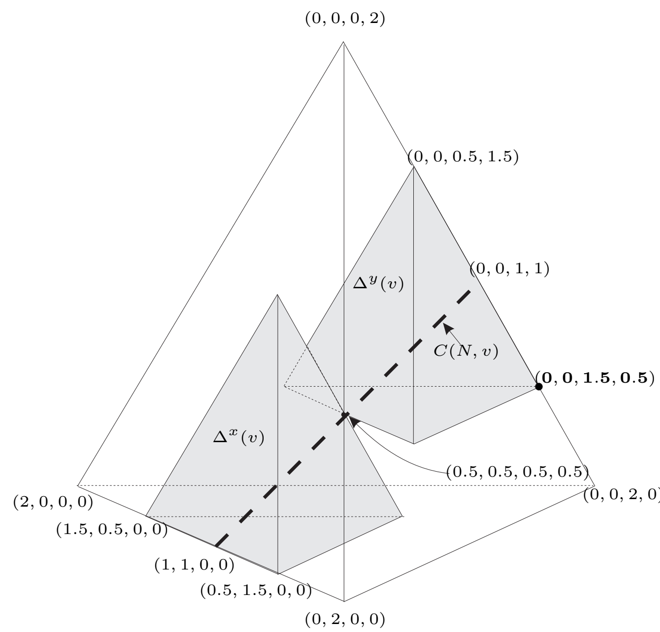


Figure 1: The strong Lorenz core and the core of Example 3.

What is most important for our purpose is to see what kind of relationship has the minimal share worth vectors in the examples above. Notice that in example 2, the vector $z = \max\{0.5, 0.5, 0), (0, 0, 0.5)\} = (0.5, 0.5, 0.5)$ is not Pareto optimal, that is, $z(N) = 1.5 > v(N)$. While, in example 3, $z = \max\{(0.5, 0.5, 0, 0), (0, 0, 0.5, 0.5)\} = (0.5, 0.5, 0.5, 0.5)$ satisfies Pareto optimality. As we will see, this "max-relation" between minimal equal share worth vectors will allow us to characterize connectedness. First we need to introduce some additional notation.

Given a game $(N, v)$ with non-empty strong Lorenz core, we define the finite *graph* $(V, E)$, where $V := \{x \in \mathbb{R}^N \mid x \in \mathcal{M}(v)\}$ is the set of vertices and $E := \{\{x, y\} \mid x, y \in \mathcal{M}(v)$ and $\sum_{i \in N} \max\{x_i, y_i\} \leq v(N)\}$ is the set of edges. Two minimal equal share worth vectors $x, y \in V$, $x \neq y$, are *connected* if there exists a sequence of vertices $\{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$, with $x_{i_k} \in V$ for all $k \in \{1, 2, \ldots, t\}$, such that $x_{i_1} = x$, $x_{i_t} = y$, and $\{x_{i_k}, x_{i_{k+1}}\} \in E$, for all $k \in \{1, 2, \ldots, t-1\}$. Finally, a graph is connected if for any two vertices $x, y$ there is a sequence of connected vertices from $x$ to $y$.

**Theorem 4.2.** *Let $(N, v)$ be a game such that $L^*(N) \neq \emptyset$. Then, the following statements are equivalent:*

1. *$L^*(N)$ is connected.*

2. *The graph $(V,E)$ is connected.*

PROOF:

$1 \Rightarrow 2$) Let $x, y \in V$. Since, by hypothesis, $L^*(N)$ is connected, from Theorem 3.3 it follows the existence of a sequence of vectors $\{x_1, \ldots, x_k\} \in V$ such that $x = x_1$, $y = x_k$, and $\emptyset \neq \Delta^{x_l}(v) \cap \Delta^{x_{l+1}}(v)$, for all $l = 1, \ldots, k-1$. Let $z \in \Delta^{x_l}(v) \cap \Delta^{x_{l+1}}(v)$. Then, from expression (1), $z(N) = v(N)$ and $z \geq \max\{x_l, x_{l+1}\}$, which implies $\{x_l, x_{l+1}\} \in E$, for all $l = 1, \ldots, k-1$, and so the graph $(V,E)$ is connected.

$2 \Rightarrow 1$) Let $x_1, x_2 \in L^*(N)$. From Theorem 3.3, we know that there exist $y_1, y_2 \in \mathcal{M}(v)$ such that $x_1 \in \Delta^{y_1}(v)$ and $x_2 \in \Delta^{y_2}(v)$. Since we suppose that $(V,E)$ is connected, then there exist a sequence of vertices $\{z_1, \ldots, z_k\} \in V$ such that $y_1 = z_1$, $y_2 = z_k$, and $\{z_l, z_{l+1}\} \in E$, for all $l = 1, \ldots, k-1$. But then, $v(N) \geq \sum_{i \in N} \max\{z_{l,i}, z_{l+1,i}\}$, for all $l = 1, \ldots, k-1$, which implies $\Delta^{z_l}(v) \cap \Delta^{z_{l+1}}(v) \neq \emptyset$. Let $w_l \in \Delta^{z_l}(v) \cap \Delta^{z_{l+1}}(v)$, $l = 1, \ldots, k-1$. Since for all $j = 1, \ldots, k$, $\Delta^{z_j}(v)$ is a convex set, $[x_1, w_1] \subset \Delta^{z_1}(v)$, $[w_l, w_{l+1}] \subset \Delta^{z_{l+1}}(v)$, for $l = 1, \ldots, k-2$, and $[w_{k-1}, x_2] \subset \Delta^{z_k}(v)$. But then, $[x_1, w_1] \cup [w_1, w_2] \cup \ldots \cup [w_{k-1}, x_2] \subset L^*(N)$, which prove the connectivity of the strong Lorenz core $L^*(N)$.$\square$

Now we examine conditions under which the strong Lorenz core is non-empty and connected.

**Definition.** Let $(N,v)$ be a game and $\emptyset \neq S \subseteq N$. Then, $S$ is an equity coalition if $\frac{v(S)}{|S|} \geq \frac{v(T)}{|T|}$, for all $T \subseteq S$.

**Proposition 4.3.** *Let $(N,v)$ be a game. Then, the following statements are equivalent:*

1. *$N$ is an equity coalition.*

2. *$L^*(N)$ is connected and $e_N \in L^*(N)$, where $e_N = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$.*

PROOF:

$1 \Rightarrow 2$) Let $x \in \mathcal{M}(v)$. Since $N$ is an equity coalition, $e_N \geq x$. Hence, from expression 1 and Theorem 3.3, we have that $e_N \in \bigcap_{x \in \mathcal{M}(v)} \Delta^x(v) \subseteq L^*(N)$, and thus $L^*(N)$ is a non-empty connected set.

$2 \Rightarrow 1$) Let $\emptyset \neq S \subset N$. Since $e_N \in L^*(N)$, there exists a player $i \in S$ such that $e_{N,i} \geq \frac{v(S)}{|S|}$, and so $N$ is an equity coalition.$\square$

# 5 An algorithm for locating the strong constrained egalitarian allocations

In this section we will describe an algorithm to compute de strong constrained egalitarian allocations in any game with non-empty strong Lorenz core. We start by introducing some definitions. Let $(N, v)$ be a game with $L^*(N) \neq \emptyset$. A payoff vector $x$ is *Lorenz maximal* in $L^*(N)$ if $x \in L^*(N)$ and there exists no $z \in L^*(N)$ such that $z$ Lorenz dominates x.

**Definition.** Let $(N, v)$ be a game, $x \in \mathcal{M}(v)$, and $\theta = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_N$ such that $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_n}$. Given $k \in \{1, \ldots, n-1\}$, $n \geq 2$, we define the vectors $y^k$ and $y_x$ as follows:

1. For $j = 1, \ldots, n$

$$y^k_{i_j} := \begin{cases} x_{i_j} & \text{if } j \leq k \\ \frac{v(N) - x_{i_1} - \ldots - x_{i_k}}{n-k} & \text{otherwise.} \end{cases} \tag{4}$$

2. $y_x := y^{k^*}$, where $k^* = \min\{k \in \{1, \ldots, n-1\} \mid y^k \in \Delta^x(v)\}$.

**Remark 5.1.** Notice that $y^{n-1}$ is an extreme point of the polyhedron $\Delta^x(v)$. Since $\{1, \ldots, n-1\}$ is a finite set, there exists a minimal $k^* \in \{1, \ldots, n-1\}$ such that $y^{k^*} \in \Delta^x(v)$. Hence, $y^k$ is well defined. Moreover, if $x$ is an efficient vector, then $\Delta^x(v) = \{x\}$ and $x = y^{n-1}$.

**Lemma 5.2.** *Let $(N, v)$ be a game with non-empty $L^*(N)$, and $x \in \mathcal{M}(v)$. Then, $y_x$ Lorenz dominates any other element $z \in \Delta^x(v)$, $z \neq y_x$.*

PROOF: See appendix.

Now combining Theorem 3.3 and Lemma 5.2 we get the following result.

**Theorem 5.3.** *Let $(N, v)$ be a game. Then, the strong constrained egalitarian allocations are the Lorenz maximal elements in $\mathcal{L}_{max}(\Delta)$, where $\mathcal{L}_{max}(\Delta) := \{y_x \mid x \in \mathcal{M}(v)\}$.*

**Remark 5.4.** From the result above, we have a way to compute the strong constrained egalitarian allocations working as follows.
1. Let $(N, v)$ be a game. Then, find the set of minimal equal share worth vectors, $\mathcal{M}(v)$.
2. For any $x \in \mathcal{M}(v)$, construct the allocation $y_x$ (see expression (4)).
3. Finally, check the Lorenz maximal allocations in $\mathcal{L}_{max}(\Delta)$.
Notice that the above procedure is easy to implement since it only requires a finite number of elementary operations.

## 6 Final remarks

The notion of the equity core of a cooperative TU-game was introduced by Selten (1978) as a generalization of the equal division core or the strong Lorenz core to the asymmetric case by taking into account exogenous and positive weights of the players. And, of course, when all players have the same weight both notions coincide.

The equity core, with respect to a positive vector $w \in \mathbb{R}^N_{++}$ of weights of the players, is the set of payoff vectors efficient for the grand coalition and such that there is no any coalition being able to block by the proportional allocation to $w$. In other words, a payoff vector is in the equity core if no coalition can divide its value proportionally to $w$ among its members and, in this way, give more to all its members than the amount they receive in the payoff vector.

It is worth emphasizing that, by using equivalent reasonings, we can extend the decomposition theorem obtained for the strong Lorenz core to the asymmetric case and we obtain a decomposition theorem for the equity core. Therefore, all the results we have obtained about existence and connectivity of the strong Lorenz core based on the decomposition theorem can also be extended for the equity core.

## 7 Concluding comments

The notion of SCES has an intrinsic difficulty. We provide a geometric approach of the strong Lorenz core and an intuitive method to find the SCES for any game. It seems natural to think that, by using similar arguments it could be possible to find a decomposition theorem for the Lorenz core and so an algorithm to locate the egalitarian solution, if it exists, for any game. Dutta and Ray already gave a simple algorithm to locate the egalitarian solution for the particular class of convex games which has been widely used in the literature.

## 8 Appendix

The aim of this appendix is to provide a proof of Lemma 5.2. The Lemma is proved in three stages. First, we show the rearranged vector $\hat{y}_x$ in a non-decreasing order. Second, we check the inequalities to prove the Lorenz dominance and third we check that at least one of those inequalities is strict.

If $|N| = 1$, then $\Delta^x(v) = \{x\}$ with $x = v(N)$. If $|N| \geq 2$, let $x \in \mathcal{M}(v)$ and suppose, without loss of generality,

$$x_1 \geq x_2 \geq \cdots \geq x_n. \tag{5}$$

Take $k^* = \min\{k \in \{1, \ldots, n-1\} \mid y^k \in \Delta^x(v)\}$ and $y_x = y^{k^*}$ as defined in (4). Step 1: We prove that $\hat{y}^{k^*} = \left(\frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}, \ldots, \frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}, x_{k^*}, \ldots, x_1\right)$. We must distinguish two cases:

Case 1: If $k^* = 1$. In this situation $y_x = y^1 = \left(x_1, \frac{v(N)-x_1}{n-1}, \ldots, \frac{v(N)-x_1}{n-1}\right)$. Suppose, on the contrary, that $\frac{v(N)-x_1}{n-1} > x_1$. Then, $\frac{v(N)}{n} > x_1$ which is a contradiction since, by definition, $x_1 \geq \frac{v(N)}{n}$.

Case 2: If $2 \leq k^* \leq n-1$. In this case, $y^{k^*} \in \Delta^x(v)$ and, for $h = 1, \ldots, k^*-1$, $y^h \notin \Delta^x(v)$. Hence, from expression (4) we know that for some $i \in \{k^*, \ldots, n\}$ it holds

$$\frac{v(N) - x_1 - \cdots - x_{k^*-1}}{n-(k^*-1)} < x_i \leq x_{k^*}$$

$$v(N) - x_1 - \cdots - x_{k^*-1} < (n-(k^*-1))x_{k^*}$$

$$\frac{v(N) - x_1 - \cdots - x_{k^*-1} - x_{k^*}}{n-k^*} < x_{k^*} \tag{6}$$

Then, taking into account that $x_{k^*} \leq x_{k^*-1} \leq \cdots \leq x_2 \leq x_1$ (see expression (5)), from (6) it follows $\hat{y}^{k^*} = \left(\frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}, \ldots, \frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}, x_{k^*}, \ldots, x_1\right)$.

Step 2: We prove that for any $z \in \Delta^x(v)$, $z \neq y^{k^*}$, it holds: $\hat{y}_1^{k^*} + \cdots + \hat{y}_j^{k^*} \geq z_{i_1} + \cdots + z_{i_j}$ for $j = 1, \ldots, n$ where $z_{i_1} \leq z_{i_2} \leq \cdots \leq z_{i_n}$. Once again, we distinguish two cases:

Case 1: If $j \in \{n-k^*, \ldots, n\}$. By efficiency, we have $\hat{y}_1^{k^*} + \cdots + \hat{y}_n^{k^*} = z_{i_1} + \cdots + z_{i_n}$. We want to see that for any $j \in \{n-k^*, \ldots, n-1\}$, $\hat{y}_1^{k^*} + \cdots + \hat{y}_j^{k^*} \geq z_{i_1} + \cdots + z_{i_j}$, which is equivalent to

$$v(N) - \hat{y}_{j+1}^{k^*} - \cdots - \hat{y}_n^{k^*} \geq v(N) - z_{i_{j+1}} - \cdots - z_{i_n},$$

equivalently

$$\hat{y}_{j+1}^{k^*} + \cdots + \hat{y}_n^{k^*} \leq z_{i_{j+1}} + \cdots + z_{i_n},$$

which is true because

$$\hat{y}_{j+1}^{k^*} + \cdots + \hat{y}_n^{k^*} \leq x_{n-j} + \cdots + x_1 \leq z_{n-j} + \cdots + z_1 \leq z_{i_{j+1}} + \cdots + z_{i_n},$$

where the last before inequality follows from the fact that $z \in \Delta^x(v)$.

Case 2: If $j \in \{1, \ldots, n-k^*-1\}$. Here we use an induction argument.

- For $j = n-k^*-1$: Suppose, on the contrary, that

$$\hat{y}_1^{k^*} + \cdots + \hat{y}_{n-k^*-1}^{k^*} < z_{i_1} + \cdots + z_{i_{n-k^*-1}}. \tag{7}$$

From Case 1, we know that $\hat{y}_1^{k^*} + \cdots + \hat{y}_{n-k^*}^{k^*} \geq z_{i_1} + \cdots + z_{i_{n-k^*}}$. This inequality together with expression (7) implies $\hat{y}_{n-k^*}^{k^*} > z_{i_{n-k^*}}$. But then, since $z_{i_{n-k^*}} \geq \cdots \geq z_{i_2} \geq z_{i_1}$, we have $(n-k^*-1)\hat{y}_{n-k^*}^{k^*} > z_{i_1} + \cdots + z_{i_{n-k^*-1}}$. From Step 1, we know that $\hat{y}_1^{k^*} = \cdots = \hat{y}_{n-k^*-1}^{k^*} = \hat{y}_{n-k^*}^{k^*}$, and so $\hat{y}_1^{k^*} + \cdots + \hat{y}_{n-k^*-1}^{k^*} > z_{i_1} + \cdots + z_{i_{n-k^*-1}}$, which contradicts (7). Hence, we can conclude that

$$\hat{y}_1^{k^*} + \cdots + \hat{y}_{n-k^*-1}^{k^*} \geq z_{i_1} + \cdots + z_{i_{n-k^*-1}}.$$

- Induction hypothesis: $\hat{y}_1^{k^*} + \cdots + \hat{y}_{l+1}^{k^*} \geq z_{i_1} + \cdots + z_{i_{l+1}}$.
- Now we have to see that $\hat{y}_1^{k^*} + \cdots + \hat{y}_l^{k^*} \geq z_{i_1} + \cdots + z_{i_l}$ for $l = n-k^*-2, \ldots, 1$. To show this it is enough to follow the reasoning used for $j = n-k^*-1$, taking into account the induction hypothesis.

Step 3: Finally, we have to prove that, if $y^{k^*} \neq z$ then at least one of the inequalities stated in Step 2 is strict. Equivalently, we show that if

$$\hat{y}_1^{k^*} + \cdots + \hat{y}_j^{k^*} = z_{i_1} + \cdots + z_{i_j} \text{ for } j = 1, \ldots, n \tag{8}$$

then $y^{k^*} = z$. Indeed, if expression (8) holds, then $\hat{y}_j^{k^*} = z_{i_j}$, for $j = 1, \ldots, n$ and so

$$\begin{aligned} z_{i_1} = \cdots = z_{i_{n-k^*}} &= \frac{v(N) - x_1 - \cdots - x_{k^*}}{n-k^*}, \\ z_{i_{n-k^*+1}} &= x_{k^*} \\ z_{i_{n-k^*+2}} &= x_{k^*-1} \\ &\vdots \\ z_{i_{n-1}} &= x_2 \\ z_{i_n} &= x_1 \end{aligned} \tag{9}$$

Suppose that $z_1 \neq y_1^{k^*} = x_1$, then from (9): $z_1 = x_j$ for some $j \in \{2, \ldots, k^*\}$, or $z_1 = \frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}$. In any case $z_1 \leq x_1$. But $z \in \Delta^x(v)$, which implies $z_1 \geq x_1$. Hence $z_1 = x_1 = y_1^{k^*}$. Now suppose that $z_2 \neq y_2^{k^*} = x_2$. Then, $z_2 = x_j$ for some $j \in \{3, \ldots, k^*\}$ or $z_2 = \frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*}$. In any case $z_2 \leq x_2$. But $z \in \Delta^x(v)$, and so $z_2 \geq x_2$. Hence, $z_2 = x_2 = y_2^{k^*}$. Following a symmetric reasoning we get $z_3 = x_3 = y_3^{k^*}, \ldots, z_{k^*} = x_{k^*} = y_{k^*}^{k^*}$. But then, from (9), it follows straightforward $z_{k^*+1} = \cdots = z_n = \frac{v(N)-x_1-\cdots-x_{k^*}}{n-k^*} = y_{k^*+1}^{k^*} = \cdots = y_n^{k^*}$. Hence, we have proved that $z = y^{k^*}$.

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