

Farsightedly Stable Networks*

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Abstract

We propose a new concept, pairwise farsighted stable set, in order to predict which networks may be formed among farsighted players. A set of networks G is pairwise farsighted stable (i) if all possible pairwise deviations from any network $g \in G$ are deterred by the threat of ending worse off or equal off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) if there is no proper subset of G satisfying conditions (i) and (ii). We show that a pairwise farsighted stable set always exists and we provide the necessary and sufficient condition such that a unique pairwise farsighted stable set consisting of a single network exists. We find that the pairwise farsighted stable sets and the set of strongly efficient networks, those which are socially optimal, may be disjoint if the allocation rules have nice properties. Finally, we study the relationship between pairwise farsighted stability and other concepts such as the largest consistent set.

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1 Introduction

The organization of individual agents into networks and groups or coalitions has an important role in the determination of the outcome of many social and economic interactions. For instance, networks of personal contacts are important in obtaining information about job opportunities. Goods can be traded and exchanged through networks, rather than markets, of buyers and sellers. The partitioning of societies into groups is also important in many contexts, such as the provision of public goods and formation of alliances, cartels and federations.¹

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that individuals not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). There are alternative approaches to modeling network stability. One is to explicitly model a game by which links form and then to solve that game using the Nash equilibrium or one of its refinements. Aumann and Myerson (1988) take such an approach in the context of communication games, where individuals sequentially propose links. However, such an approach has the disadvantage that the game is necessarily ad hoc and is quite sensitive to the exact network formation process. Dutta and Mutuswami (1997) analyze a link formation game where individuals simultaneously choose all the links they wish to be involved in. But this approach is static and myopic. Individuals cannot be forward-looking in the sense that they do not forecast how others might react to their actions. For instance, individuals might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original individuals. A dynamic (but still myopic) network formation process has been recently studied by Jackson and Watts (2002) who have proposed a dynamic process in which individuals form and sever links based on the improvement that the resulting network offers them relative to the current network. This deterministic dynamic process may end at stable networks or in some cases may cycle.²

We propose a new concept, pairwise farsighted stable set, in order to predict which networks may be formed among farsighted players. A set of networks G is pairwise farsighted stable (i) if all possible pairwise deviations from any network $g \in G$ are deterred by the threat of ending worse off or equal off, (ii) if there exists a *farsighted improving path* from any network outside the set leading to some network in the set, and (iii) if there

¹Jackson (2003, 2005) provides a survey of models of network formation.

²Watts (2001) has extended the Jackson and Wolinsky model to a dynamic process but she has limited attention to the specific contest of the connections model and a particular deterministic dynamic.

is no proper subset of G satisfying conditions (i) and (ii). A farsighted improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network. We show that a pairwise farsighted stable set always exists and we provide the necessary and sufficient condition such that a unique pairwise farsighted stable set consisting of a single network exists. We find that the pairwise farsighted stable sets and the set of strongly efficient networks, those which are socially optimal, may be disjoint if the allocation rules have nice properties. Finally, we study the relationship between pairwise farsighted stability and other concepts such as the largest consistent set, a notion due to Chwe (1994). By means of examples we show that there is no relationship between (i) the pairwise farsighted stable set and the coalitional largest consistent set, (ii) the pairwise farsighted stable set and the pairwise largest consistent set, (iii) the pairwise largest consistent set and the coalitional largest consistent set.

Although the literature on stability in networks is well established and growing (see Jackson, 2005), the literature on farsighted stability is still in its infancy. Page, Wooders and Kamat (2005) have addressed the issue of farsighted stability in network formation by extending Chwe's (1994) result on the nonemptiness of farsightedly consistent sets. In order to demonstrate the existence of farsightedly consistent directed networks, they have provided a new framework that extends the standard notion of a directed network and also introduces the notion of a supernetwork. A supernetwork specifies how the different directed networks are connected via coalitional moves and coalitional preferences, and thus provides a network representation of agent preferences and the rules governing network formation (that is, a supernetwork is equivalent to the social environment studied by Chwe (1994) where the set of outcomes is replaced by the set of directed networks). Given the rules governing network formation and agents preferences as represented via the supernetwork, a directed network (i.e., a particular node in the supernetwork) is said to be farsightedly consistent if no agent or coalition of agents is willing to alter the network (via the addition, subtraction, or replacement of arcs) for fear that such an alteration might induce further network alterations by other agents or coalitions that in the end leave the initially deviating agent or coalition no better off, and possibly worse off. They have shown that for any supernetwork corresponding to a given collection of directed

networks, the set of farsightedly consistent networks is nonempty.³ Dutta, Ghosal and Ray (2005) have studied a model of dynamic network formation where individuals are farsighted and evaluate the desirability of a move in terms of its consequences on the entire discounted stream of payoffs. Only special coalitions are active at any date. They have shown that a Markovian equilibrium process of network formation exists. They have demonstrated that there are valuation structures in which no equilibrium strategy profile yields paths that are absorbed solely into a set of efficient networks. This can be viewed as the dynamic counterpart of the conflict between (static) stability and efficiency demonstrated by Jackson and Wolinsky (1996). They have finally provided two conditions on the valuation structure that guarantee that there is some equilibrium profile at which the complete graph is reached in the limit from all initial networks.⁴

The paper is organized as follows. In Section 2 we introduce some notations and basic properties and definitions for networks. In Section 3 we define the notion of pairwise farsighted stable network and we study its properties. In Section 4 we propose a set-valued extension, the pairwise farsighted stable set of networks. In Section 5 we analyze the relationship between pairwise farsighted stable networks and the (pairwise or coalitional) largest consistent set. In Section 6 we conclude.

2 Networks

Let $N = \{1, \dots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. If we are considering a pair of individuals i and j , then $\{i, j\} \in g$ indicates that i and j are linked under the network g . For simplicity, we write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under

³Page and Wooders (2005) have introduced the farsighted network formation game induced by a supernetwork. They have shown that in any farsighted network formation game the feasible set of networks contains a unique, finite, disjoint collection of nonempty subsets having the property that each subset forms a strategic basin of attraction. These basins of attraction contain all the networks that are likely to emerge and persist if individuals behave farsightedly in playing the network formation game. A von Neumann Morgenstern stable set of the farsighted network formation game (or farsighted basis) is constructed by selecting one network from each basin of attraction. The core of the farsighted network formation game (the farsighted core) is constructed by selecting one network from each basin of attraction containing a single network. The farsighted core is shown to be nonempty if and only if there exists at least one farsighted basin of attraction containing a single network.

⁴Other approaches to farsightedness in network formation are suggested by the work of Xue (1998), Herings, Mauleon and Vannetelbosch (2004), and Mauleon and Vannetelbosch (2004).

the network g . Let g^N be the set of all subsets of N of size 2. \mathbb{G} denotes the set of all possible networks or graphs on N , with g^N being the complete network. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$. For any network g , let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g .

A path in a network $g \in \mathbb{G}$ between i and j is a sequence of players i_1, \dots, i_K such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. A nonempty network $g' \subseteq g$ is a component of g , if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j , and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$. The set of components of g is denoted as $C(g)$. We can partition the players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by g . That is, $S \in \Pi(g)$ if and only if either there exists $h \in C(g)$ such that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.

Different network configurations lead to different values of overall production or overall utility to players. These various possible valuations are represented via a value function. A *value function* is a function $v : \mathbb{G} \rightarrow \mathbb{R}$. The set of all possible value functions is denoted \mathcal{V} . A value function only keeps track of how the total societal value varies across different networks. We also wish to keep track of how that value is allocated or distributed among the players forming a network. An allocation rule is a function $Y : \mathbb{G} \times \mathcal{V} \rightarrow \mathbb{R}^N$ such that $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g . It is important to note that an allocation rule depends on both g and v . This allows an allocation rule to take full account of a player i 's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure.

Jackson and Wolinsky (1996) have proposed basic properties of value and allocation functions. A value function is *component additive* if $v(g) = \sum_{g' \in C(g)} v(g')$ for all $g \in \mathbb{G}$. Component additive value functions are ones for which the value of a network is the sum of the value of its components. An allocation rule Y is *component balanced* if for any component additive v , $g \in \mathbb{G}$, and $g' \in C(g)$, we have $\sum_{i \in N(g')} Y_i(g', v) = v(g')$. Component balance only makes requirements on Y for v 's that are component additive, and Y can be arbitrary otherwise. Given a permutation of players π and any $g \in \mathbb{G}$, let $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\}$. Thus, g^π is a network that shares the same architecture as g but with the specific players permuted. A value function is *anonymous* if for any permutation π and any $g \in \mathbb{G}$, $v(g^\pi) = v(g)$. Given a permutation π , let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in \mathbb{G}$. An allocation rule Y is *anonymous* if for any v , $g \in \mathbb{G}$,

and permutation π , we have $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.⁵

In evaluating societal welfare, we may take various perspectives.⁶ A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \subseteq \mathbb{G}$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i . This definition of efficiency of a network takes Y as fixed, and hence can be thought of as applying to situations where no intervention is possible. A network $g \subseteq \mathbb{G}$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \subseteq \mathbb{G}$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

The network-theoretic literature uses two different notions of a coalition deviation. *Pairwise deviations* (Jackson and Wolinsky, 1996) are deviations on a single link at a time and deviations by at most a pair of players at a time. That is, link addition is bilateral (two players that would be involved in the link must agree to adding the link), link deletion is unilateral (at least one player involved in the link must agree to delete the link), and network changes take place one link at a time. *Coalitionwise deviations* (Jackson and van den Nouweland, 2005) are deviations on several links at a time and deviations by some group of players at a time. Link addition is bilateral, link deletion is unilateral, but multiple link changes can take place at a time. Whether a pairwise deviation or a coalitionwise deviation makes more sense will depend on the setting network formation takes place. We will mainly restrict our analysis to pairwise deviations. A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 1 *A network g is pairwise stable with respect to value function v and allocation rule Y if*

- (i) *for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and*
- (ii) *for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.*

⁵Anonymous value functions are those such that the architecture of a network matters, but not the labels of individuals. Anonymity of an allocation rule requires that if all that has changed is the labels of the agents and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling.

⁶Throughout the paper we use the notation \subseteq for weak inclusion and \subsetneq for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

Let us say that g' is adjacent to g if $g' = g + ij$ or $g' = g - ij$ for some ij . A network g' defeats g if either $g' = g - ij$ and $Y_i(g', v) > Y_i(g, v)$, or if $g' = g + ij$ with $Y_i(g', v) \geq Y_i(g, v)$ and $Y_j(g', v) \geq Y_j(g, v)$ with at least one inequality holding strictly. Pairwise stability is equivalent to saying that a network is pairwise stable if it is not defeated by another (necessarily adjacent) network.⁷

3 Pairwise farsighted stable sets of networks

The following example shows that a network that is both pareto-dominant and pairwise stable can be "less farsightedly stable" than another network.

Example 1. *Criminal networks.*⁸ Each player is a criminal. If two players are connected, then they are part of the same criminal network. Each group of connected criminals has a positive probability of winning the loot. The loot is divided among the connected criminals based on the network architecture. Let n_i be the number of links criminal i has. Let B be the loot. Criminal i 's payoff is given by

$$Y_i(v, g) = p_i(g) \cdot (y_i(v, g) - \phi) + (1 - p_i(g)) \cdot y_i(v, g) = y_i(v, g) - p_i(g) \cdot \phi \quad (1)$$

where $y_i(v, g)$ is i 's expected share of the loot, $p_i(g)$ is i 's probability of being caught, and ϕ the corresponding fine. Beside being competitors in the crime market, criminals may also benefit from having criminal mates. It is assumed that (i) the bigger the group of connected criminals, the higher its probability of getting the loot, and (ii) the higher the criminal connections to a criminal, the lower his individual probability of being caught. Thus, it is assumed that $p_i(g)$, the individual probability of being caught, is decreasing with the number of links criminal i has, n_i . Criminal i 's expected share of the loot is given by

$$y_i(v, g) = \frac{|S|}{\sum_{S' \in \Pi(g)} |S'|} \cdot \frac{n_i}{\sum_{j \in S} n_j} \cdot B$$

where $|S| \cdot [\sum_{S' \in \Pi(g)} |S'|]^{-1}$ is the probability that the group S will win the loot, and

⁷Jackson and van den Nouweland (2005) have proposed a refinement of pairwise stability where coalitionwise deviations are allowed: the strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component.

⁸It is a simplified version of Calvó-Armengol and Zenou's (2004) model where, in addition to forming links with criminal mates, criminals choose their level of criminal activities and whether or not to be involved in criminal activities.

$n_i \cdot [\sum_{j \in S} n_j]^{-1}$ is the share of the loot criminal $i \in S$ would obtain.⁹ In Figure 1 we have depicted the 3-player case with $B = 6$ and $p_i(g) = (n - 1 - n_i)/n$. For $\phi < \frac{3}{2}$, both the partial networks (g_1, g_2, g_3) and the complete network (g_7) are pairwise stable networks. For $\phi \geq \frac{3}{2}$, the complete network is the unique pairwise stable network. \square

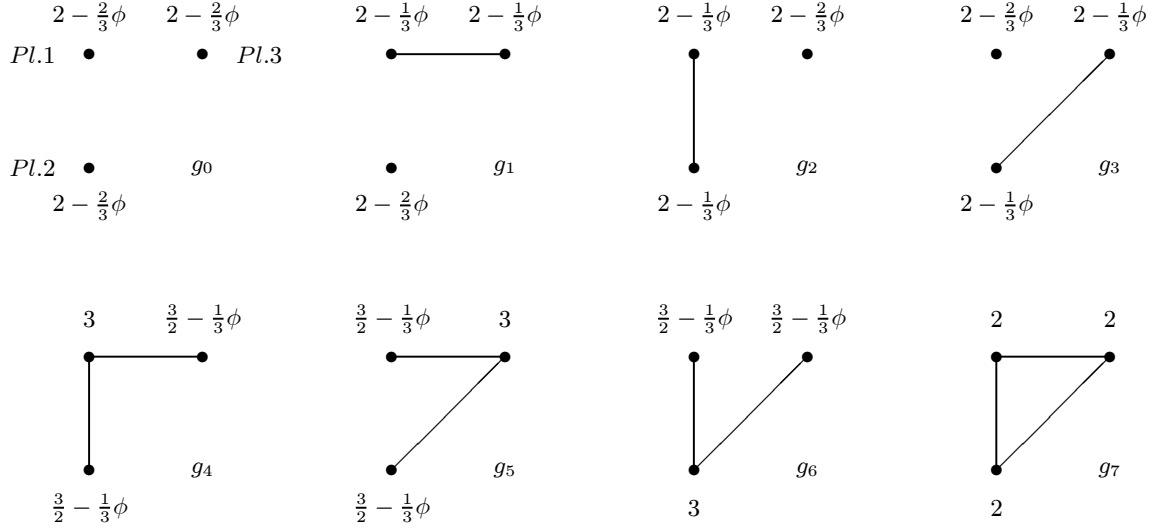


Figure 1: Farsightedness in criminal networks.

Take ϕ being smaller than $\frac{3}{2}$ in Example 1. Suppose that two links are added from any partial network to form the complete network g_7 . Then, from g_7 no farsighted improving path will go back to the partial network. A *farsighted improving path*¹⁰ is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network, with at least one of the two strictly preferring the end network. If a link is deleted, then it must be that at least one of the two players involved in the link strictly prefers the end network. Suppose now that two or three links are deleted to the complete network to form g' . Then, from g' no myopic improving path go back to the complete network but there are farsighted improving paths that go back to the complete network. Moreover, from any $g \neq g_7$ there is a farsighted improving path going to g_7 . Thus, we say that the partial networks are "less farsightedly

⁹This assumption captures the idea that delinquents learn from other criminals belonging to the same network how to commit crime in a more efficient way by sharing the know-how about the technology of crime (see Calvó-Armengol and Zenou, 2004).

¹⁰Jackson and Watts (2002) have provided a myopic definition of an improving path. A "myopic" improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network.

stable" than the complete network, while they are pairwise stable.

Let us introduce the formal definition of a farsighted improving path.

Definition 2 *A farsighted improving path from a network g to a network g' is a finite sequence of graphs g_1, \dots, g_K with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \dots, K-1\}$ either:*

- (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_K) > Y_i(g_k)$, or
- (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K) > Y_i(g_k)$ and $Y_j(g_K) \geq Y_j(g_k)$.

If there exists a farsighted improving path from g to g' , then we use the symbol $g \rightarrow g'$. For a given network g , let $F(g) = \{g' \subseteq g^N \mid g \rightarrow g'\}$. This is the set of networks for which there is a farsighted improving path leading to from g . Thus, $g \rightarrow g'$ means that g' should be the endpoint of a farsighted improving path if g is its initial point, and that at least one farsighted improving path from g should go to g' .

We now introduce our new concept, pairwise farsighted stable sets. A set of networks G is pairwise farsighted stable (i) if all possible *pairwise deviations* from any network $g \in G$ are deterred by the threat of ending worse off or equal off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) if there is no proper subset of G satisfying conditions (i) and (ii) (minimality condition). Formally, pairwise farsighted stable sets are defined as follows.

Definition 3 *A set of networks $G \subseteq \mathbb{G}$ is pairwise farsighted stable with respect to allocation rule Y and value function v if*

- (i) $\forall g \in G$,
- (ia) $\forall ij \notin g$ such that $g+ij \notin G$, $\exists g' \in G \cap F(g+ij)$ such that $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ or $Y_i(g', v) < Y_i(g, v)$ or $Y_j(g', v) < Y_j(g, v)$,
- (ib) $\forall ij \in g$ such that $g-ij \notin G$, $\exists g', g'' \in G \cap F(g-ij)$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,
- (ii) $\forall g' \notin G$ we have $g \in F(g')$ for some $g \in G$,
- (iii) $\nexists G' \subsetneq G$ such that G' satisfies (ia), (ib) and (ii).

Part (i) in Definition 3 requires that all possible pairwise deviations from any network $g \in G$ are deterred by the threat of ending worse off or equal off. Consider a "pairwise" deviation from $g \in G$ to an adjacent network that does not belong to G . There might be

further pairwise deviations which end up at $g' \in G$, where $g + ij \rightarrow g'$ or $g - ij \rightarrow g'$. If either i or j is worse off or both are equal off at g' compared to the original network g then the "pairwise" deviation is deterred. Notice that the set \mathbb{G} (trivially) satisfies **(ia)**, **(ib)** and **(ii)** in Definition 3. This motivates the requirement of a minimality condition, namely condition **(iii)**.

Theorem 1 *There always exists at least one pairwise farsighted stable set of networks.*

Proof. Let us proceed by contradiction. Let us assume that there does not exist any set of networks $G \subseteq \mathbb{G}$ that is pairwise farsighted stable. This means that for any $G^0 \subseteq \mathbb{G}$ that satisfies **(i)** and **(ii)** in Definition 3 (there always exist such a G^0), we can find a proper subset G^1 that satisfies **(i)** and **(ii)**. But again for G^1 , we can find a proper subset G^2 that satisfies **(i)** and **(ii)**. Iterating the reasoning we can build an infinite (decreasing) sequence $\{G^k\}_{k \geq 0}$ of **distinct** elements of \mathbb{G} satisfying **(i)** and **(ii)**. But since $\#\mathbb{G} < \infty$, this is not possible; so the proof is completed. ■

In Example 1 with $n = 3$, $B = 6$ and $p_i(g) = (n - 1 - n_i)/n$, the set consisting of the complete network is the unique pairwise farsighted stable set whatever the fine ϕ . For $\phi < \frac{3}{2}$, $F(g_0) = \{g_1, g_2, g_3, g_7\}$, $F(g_1) = \{g_2, g_3, g_7\}$, $F(g_2) = \{g_1, g_3, g_7\}$, $F(g_3) = \{g_1, g_2, g_7\}$, $F(g_4) = \{g_1, g_2, g_3, g_7\}$, $F(g_5) = \{g_1, g_2, g_3, g_7\}$, $F(g_6) = \{g_1, g_2, g_3, g_7\}$, and $F(g_7) = \emptyset$; for $\phi \geq \frac{3}{2}$, $F(g_0) = \mathbb{G} \setminus \{g_0\}$, $F(g_1) = \{g_4, g_5, g_7\}$, $F(g_2) = \{g_4, g_6, g_7\}$, $F(g_3) = \{g_5, g_6, g_7\}$, $F(g_4) = \{g_7\}$, $F(g_5) = \{g_7\}$, $F(g_6) = \{g_7\}$, and $F(g_7) = \emptyset$. So, $g_7 \in \bigcap_{g \in \mathbb{G} \setminus \{g_7\}} F(g)$ and $\nexists g' \neq g_7$ such that $g' \in \bigcap_{g \in \mathbb{G} \setminus \{g'\}} F(g)$. Thus, $\{g_7\}$ is the unique pairwise farsighted stable set.

Now we give the necessary and sufficient condition for the existence of a singleton set that is pairwise farsightedly stable.

Theorem 2 *$\{g\}$ is a pairwise farsighted stable set if and only if $\forall g' \in \mathbb{G} \setminus \{g\}$ we have $g \in F(g')$.*

Proof. If $\{g\}$ is a pairwise farsighted stable set then by **(ii)** in Definition 3 we have that $\forall g' \in \mathbb{G} \setminus \{g\}$ we have $g \in F(g')$. Now suppose that $\forall g' \in \mathbb{G} \setminus \{g\}$ we have $g \in F(g')$. Since $g \in F(g + ij)$ and $g \in F(g - ij)$, **(ia)** and **(ib)** hold for $g' = g$ and $g'' = g$. Finally, **(iii)** is satisfied because $\{g\}$ is a singleton; so the proof is completed. ■

Theorem 2 tells us that $\{g\}$ is a pairwise farsighted stable set if and only if there exists a farsighted improving path from any network leading to g .

Example 2. *Symmetric Connections Model* (Jackson and Wolinsky, 1996). Players form links with each other in order to exchange information. If player i is connected to player j , by a path of t links, then player i receives a payoff of δ^t from his indirect connection with player j . It is assumed that $0 < \delta < 1$, and so the payoff δ^t decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost c to both i and j . This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Formally, player i 's payoff from network g is given by

$$Y_i(v, g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j:ij \in g} c \quad (2)$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). In Figure 2 we have depicted the 3-player case where (i) for $c < \delta(1 - \delta)$, the complete network (g_7 in Figure 2) is the unique pairwise stable network, (ii) for $\delta(1 - \delta) < c < \delta$, the star networks (g_4, g_5, g_6 in Figure 2) are pairwise stable, (iii) for $c > \delta$, the empty network is the unique pairwise stable network. Applying our newly defined concept to the symmetric connections model with three players, we obtain that a network g is pairwise stable if and only if $\{g\}$ is pairwise farsighted stable. Indeed, we have (i) for $c < \delta(1 - \delta)$, $F(g_7) = \emptyset$, $F(g_6) = \{g_4, g_5, g_7\}$, $F(g_5) = \{g_4, g_6, g_7\}$, $F(g_4) = \{g_5, g_6, g_7\}$, $F(g_3) = \{g_4, g_5, g_6, g_7\}$, $F(g_2) = \{g_4, g_5, g_6, g_7\}$, $F(g_1) = \{g_4, g_5, g_6, g_7\}$, and $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$; (ii) for $\delta(1 - \delta) < c < \delta$, $F(g_7) = \{g_4, g_5, g_6\}$, $F(g_6) = \{g_4, g_5\}$, $F(g_5) = \{g_4, g_6\}$, $F(g_4) = \{g_5, g_6\}$, $F(g_3) = \{g_4, g_5, g_6\}$, $F(g_2) = \{g_4, g_5, g_6\}$, $F(g_1) = \{g_4, g_5, g_6\}$, and $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$; (iii) for $c > \delta$, $F(g_7) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6\}$, $F(g_6) = \{g_0, g_2, g_3\}$, $F(g_5) = \{g_0, g_1, g_3\}$, $F(g_4) = \{g_0, g_1, g_2\}$, $F(g_3) = \{g_0\}$, $F(g_2) = \{g_0\}$, $F(g_1) = \{g_0\}$, and $F(g_0) = \emptyset$. \square

Thus, Example 1 and Example 2 suggest that pairwise farsighted stability may be a refinement of Jackson and Wolinsky (1996) pairwise stability notion. The next example is a counter-example where a pairwise farsighted stable set does not include the unique pairwise stable network.

Example 3. Consider a situation where three players can form links. The payoffs they obtained from the different network configurations are given in Figure 3. The network g_6 is the unique pairwise stable network but $\{g_7\}$ and $\{g_4, g_6\}$ are the pairwise farsighted stable sets of networks. Indeed, $F(g_7) = \{g_4, g_6\}$, $F(g_6) = \{g_7\}$, $F(g_5) = \{g_4, g_6, g_7\}$, $F(g_4) = \{g_7\}$, $F(g_3) = \{g_4, g_5, g_6, g_7\}$, $F(g_2) = \{g_4, g_5, g_6, g_7\}$, $F(g_1) = \{g_4, g_5, g_6, g_7\}$, and $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. \square

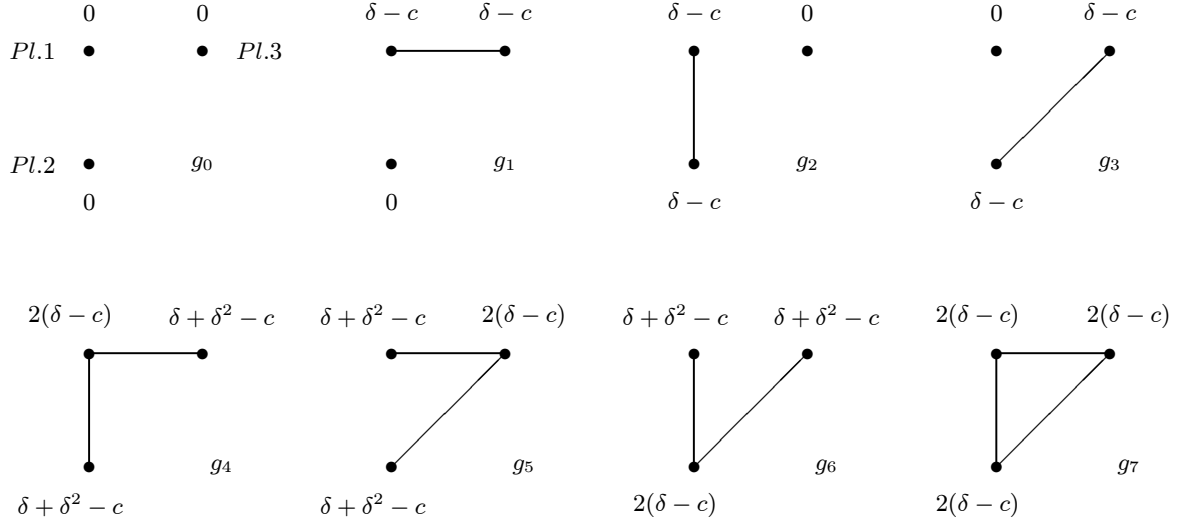


Figure 2: The symmetric connections model with three players.

We observe that, in Example 3, if $\{g\}$ is a pairwise farsighted stable set then g does not belong to any other pairwise farsighted stable sets. This result holds in general as is shown in Lemma 1.

Lemma 1 *If $\{g\}$ is a pairwise farsighted stable set then there does not exist $G \subseteq \mathbb{G}$ such that $g \in G$ and G is a pairwise farsighted stable set.*

Proof. Obviously if $\{g\}$ is a pairwise farsighted stable set and $g \in G$ then G cannot be a pairwise farsighted stable set because the minimality condition **(iii)** in Definition 3 would be violated. ■

Theorem 3 *Suppose $\{g\}$ is a pairwise farsighted stable set. Then, $\{g\}$ is the unique pairwise farsighted stable set if and only if $F(g) = \emptyset$.*

Proof. (\Rightarrow) First we show that if $\{g\}$ is a pairwise farsighted stable set and $F(g) = \emptyset$, there does not exist another pairwise farsighted stable set G , $G \neq \{g\}$. By Lemma 1 we know that $g \notin G$. In order G to be a pairwise farsighted stable set we need that $\forall g' \notin G$, $F(g') \in G$. Since $F(g) = \emptyset$, condition **(ii)** in Definition 3 is violated.

(\Leftarrow) We have to show now that if $\{g\}$ is the unique pairwise farsighted stable set then $F(g) = \emptyset$. Suppose however that $\{g\}$ is the only pairwise farsighted stable set but $F(g) \neq \emptyset$, for example, $F(g) = G$ with G containing at least one network g^* . Then, we need to show that there exists G^* such that G^* is a pairwise farsighted stable set and $g \notin G^*$ (by Lemma 1). Since $g \notin G^*$, G^* should contain at least one network $g^* \in G$ with $g^* \in F(g)$. Let $\overline{G} = \{\overline{g} \in \mathbb{G} \text{ such that } g^* \notin F(\overline{g})\}$. Different cases should be considered.

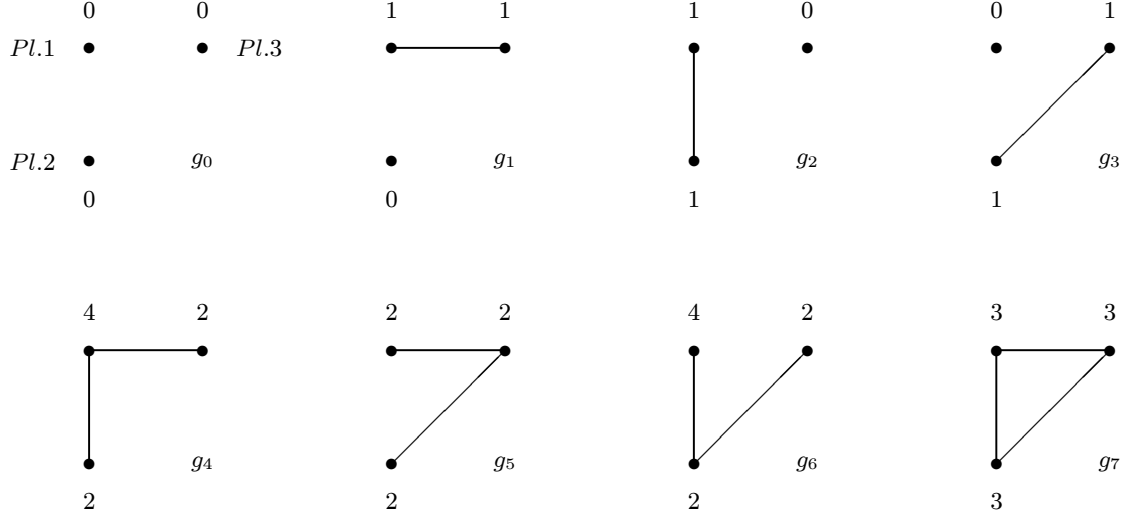


Figure 3: Pairwise stable networks and pairwise farsighted stable sets.

(a) $\overline{G} = \emptyset$. Then, $G^* = \{g^*\}$ satisfies **(ii)** in Definition 3 and is a pairwise farsighted stable set. Indeed, consider any pairwise deviation from g^* to g' , $g' \notin G^*$. Then, $g^* \in F(g')$ and the deviation is deterred. Also **(iii)** in Definition 3 is satisfied.

(b) $\overline{G} = \{\overline{g}\}$. Then, $G^* = \{\overline{g}, g^*\}$ satisfies **(ii)** in Definition 3. Now we have to show that conditions **(ia)** and **(ib)** in Definition 3 are satisfied for \overline{g} and g^* . Consider any pairwise deviation from \overline{g} to g' , $g' \notin G^*$. We know that $g^* \in F(g')$. Then, conditions **(ia)** and **(ib)** in Definition 3 are satisfied because otherwise we will have a sequence of pairwise deviations starting at \overline{g} and finishing at g^* and such that the initial deviating players prefer g^* to \overline{g} ; i.e., $g^* \in F(\overline{g})$. Consider any pairwise deviation from g^* to g' , $g' \notin G^*$. Then, $g^* \in F(g')$ and the deviation is deterred. Finally, we have to show that **(iii)** in Definition 3 is satisfied: $\nexists G' \subsetneq G^*$ such that G' satisfies **(ia)**, **(ib)** and **(ii)**. Indeed, $G' = \{\overline{g}\}$ violates condition **(ii)** since $\overline{g} \notin F(g)$, for $\{g\}$ being a pairwise farsighted stable set; and $G' = \{g^*\}$ violates condition **(ii)** since $g^* \notin F(\overline{g})$.

(c) $\overline{G} = \{\overline{g}, \widehat{g}\}$. Two cases should be considered.

(c.1.) $\overline{g} \notin F(\widehat{g})$ and $\widehat{g} \notin F(\overline{g})$. Then, $G^* = \{\overline{g}, \widehat{g}, g^*\}$ satisfies **(ii)** in Definition 3. Consider any pairwise deviation from \overline{g} (or from \widehat{g}) to g' , $g' \notin G^*$. We know that $g^* \in F(g')$. Then, conditions **(ia)** and **(ib)** in Definition 3 are satisfied because otherwise $g^* \in F(\overline{g})$ (or $g^* \in F(\widehat{g})$). Consider any pairwise deviation from g^* to g' , $g' \notin G^*$. Then, $g^* \in F(g')$ and the deviation is deterred. Finally, we have to show that $\nexists G' \subsetneq G^*$ such that G' satisfies **(ia)**, **(ib)** and **(ii)**. Indeed, $G' = \{\overline{g}, \widehat{g}\}$ violates condition **(ii)** since $\overline{g} \notin F(g)$ and $\widehat{g} \notin F(g)$, for $\{g\}$ being a pairwise farsighted stable set; $G' = \{g^*\}$ violates condition **(ii)** since $g^* \notin F(\overline{g})$ and $g^* \notin F(\widehat{g})$; and $G' = \{\overline{g}, g^*\}$ (or $G' = \{\widehat{g}, g^*\}$) violates

condition **(ii)** since $\bar{g} \notin F(\hat{g})$ (or $\hat{g} \notin F(\bar{g})$) and $g^* \notin F(\hat{g})$.

(c.2.) $\bar{g} \in F(\hat{g})$ (or $\hat{g} \in F(\bar{g})$ or both, $\bar{g} \in F(\hat{g})$ and $\hat{g} \in F(\bar{g})$). Then, $G^* = \{\bar{g}, g^*\}$ (or $G^* = \{\hat{g}, g^*\}$) satisfies **(ii)** in Definition 3. Consider any pairwise deviation from \bar{g} to g' , $g' \neq \hat{g}$, $g' \notin G^*$. We know that $g^* \in F(g')$. Then, conditions **(ia)** and **(ib)** in Definition 3 are satisfied because otherwise $g^* \in F(\bar{g})$. Consider the pairwise deviation from \bar{g} to \hat{g} (if such a deviation is possible). Since $\bar{g} \in F(\hat{g})$ such deviation is deterred. Consider any pairwise deviation from g^* to g' , $g' \neq \hat{g}$, $g' \notin G^*$. Then, $g^* \in F(g')$ and the deviation is deterred. Consider the pairwise deviation from g^* to \hat{g} (if such a deviation is possible). But $\bar{g} \in F(\hat{g})$. Thus, either $\bar{g} \notin F(g^*)$ and the deviation is deterred, or $\bar{g} \in F(g^*)$ and the deviation is not deterred. In the first case, condition **(iii)** in Definition 3 is satisfied (as in case (b)) and then $G^* = \{\bar{g}, g^*\}$ (or $G^* = \{\hat{g}, g^*\}$) is a pairwise farsighted stable set. In the second case, $G^* = \{\bar{g}, \hat{g}, g^*\}$ will be a pairwise farsighted stable set satisfying **(ii)** and **(ia)** and **(ib)** in Definition 3 (as in case (c.1)) and also satisfying condition **(iii)**. Indeed, $G' = \{\bar{g}, \hat{g}\}$ violates condition **(ii)** since $\bar{g} \notin F(g)$ and $\hat{g} \notin F(g)$, for $\{g\}$ being a pairwise farsighted stable set; $G' = \{g^*\}$ violates condition **(ii)** since $g^* \notin F(\bar{g})$ and $g^* \notin F(\hat{g})$; and $G' = \{\bar{g}, g^*\}$ (or $G' = \{\hat{g}, g^*\}$) violates **(ia)** or **(ib)** since we have shown that a pairwise deviation from g^* to \hat{g} is not deterred.

(d) $\bar{G} = \{\bar{g}, \hat{g}, \tilde{g}\}$. Two cases should be considered.

(d.1.) $\bar{g} \notin F(\hat{g})$, $\bar{g} \notin F(\tilde{g})$, $\hat{g} \notin F(\bar{g})$, $\hat{g} \notin F(\tilde{g})$, $\tilde{g} \notin F(\bar{g})$ and $\tilde{g} \notin F(\hat{g})$. Then, $G^* = \{\bar{g}, \hat{g}, \tilde{g}, g^*\}$ satisfies **(ii)** in Definition 3. Consider any pairwise deviation from \bar{g} (from \tilde{g} or \hat{g}) to g' , $g' \notin G^*$. We know that $g^* \in F(g')$. Then, conditions **(ia)** and **(ib)** in Definition 3 are satisfied because otherwise $g^* \in F(\bar{g})$ ($g^* \in F(\hat{g})$ or $g^* \in F(\tilde{g})$). Consider any pairwise deviation from g^* to g' , $g' \notin G^*$. Then, $g^* \in F(g')$ and the deviation is deterred. Finally, we have to show that $\nexists G' \subsetneq G^*$ such that G' satisfies **(ia)**, **(ib)** and **(ii)**. Indeed, $G' = \{\bar{g}, \hat{g}, \tilde{g}\}$ violates condition **(ii)** since $\bar{g} \notin F(g)$, $\hat{g} \notin F(g)$ and $\tilde{g} \notin F(g)$, for $\{g\}$ being a pairwise farsighted stable set; $G' = \{g^*\}$ violates condition **(ii)** since $g^* \notin F(\bar{g})$, $g^* \notin F(\hat{g})$ and $g^* \notin F(\tilde{g})$; $G' = \{\bar{g}, \hat{g}, g^*\}$ (or $G' = \{\bar{g}, \tilde{g}, g^*\}$ or $G' = \{\hat{g}, \tilde{g}, g^*\}$) violates condition **(ii)** since $\bar{g} \notin F(\hat{g})$ and $g^* \notin F(\tilde{g})$; and $G' = \{\bar{g}, g^*\}$ (or $G' = \{\hat{g}, g^*\}$ or $G' = \{\tilde{g}, g^*\}$) violates condition **(ii)** since $\bar{g} \notin F(\hat{g})$ (or $\bar{g} \notin F(\tilde{g})$) and $g^* \notin F(\hat{g})$ (or $g^* \notin F(\tilde{g})$).

(d.2.) Some network of \bar{G} farsightedly dominates some other network in \bar{G} , for example, $\bar{g} \in F(\hat{g})$. Then, as in case (c.2.), either $G^* = \{\bar{g}, \tilde{g}, g^*\}$ or $G^* = \{\bar{g}, \hat{g}, \tilde{g}, g^*\}$ is a pairwise farsighted stable set. Indeed, both sets satisfy **(ii)** in Definition 3. Moreover, if $\bar{g} \notin F(g^*)$ and $\tilde{g} \notin F(g^*)$, then $G^* = \{\bar{g}, \tilde{g}, g^*\}$ satisfies **(ia)**, **(ib)** and **(iii)** in Definition 3; while in case that $\bar{g} \in F(g^*)$ then $G^* = \{\bar{g}, \hat{g}, \tilde{g}, g^*\}$ satisfies all conditions in Definition 3 (see the argument in (c.2.)). ■

Next we show that the set of pairwise farsighted stable networks and the set of strongly efficient networks, those which are socially optimal, may be disjoint if the allocation rules are component balanced and anonymous. Bhattacharya (2005) has obtained a similar result with respect to the notion of the coalitional largest consistent set.

Proposition 1 *There exists a value function such that for every component balanced and anonymous rule, strongly efficient networks are not included in the pairwise farsighted stable sets with respect to the value function and the allocation rule.*

Proof. Take the following value function defined for any $g \in \mathbb{G}$: $v(\{12, 23, 13\}) = 9$, $v(\{12, 23\}) = 0$, $v(\{12, 13\}) = 0$, $v(\{23, 13\}) = 0$, $v(\{12\}) = 8$, $v(\{23\}) = 8$, $v(\{13\}) = 8$, $v(\emptyset) = 0$, and $v(g) = 0$ to any g which has a link involving a player other than players 1, 2 and 3. Fix any component balanced and anonymous allocation rule Y . Then, by component balance and anonymity,

- (i) $Y_1(\{12, 23, 13\}, v) = Y_2(\{12, 23, 13\}, v) = Y_3(\{12, 23, 13\}, v) = 3$,
- (ii) $Y_1(\{12, 23\}, v) = c$, $Y_3(\{12, 23\}, v) = c$, $Y_2(\{12, 23\}, v) = -2c$, $Y_2(\{12, 13\}, v) = c$, $Y_3(\{12, 13\}, v) = c$, $Y_1(\{12, 13\}, v) = -2c$, $Y_1(\{13, 23\}, v) = c$, $Y_2(\{13, 23\}, v) = c$, $Y_3(\{13, 23\}, v) = -2c$,
- (iii) $Y_1(\{12\}, v) = Y_2(\{12\}, v) = 4$, $Y_3(\{12\}, v) = 0$, $Y_1(\{13\}, v) = Y_3(\{13\}, v) = 4$, $Y_2(\{13\}, v) = 0$, $Y_2(\{23\}, v) = Y_3(\{23\}, v) = 4$, $Y_1(\{23\}, v) = 0$,
- (iv) $Y_1(\emptyset, v) = Y_2(\emptyset, v) = Y_3(\emptyset, v) = 0$,
- (v) $Y_1(g, v) = Y_2(g, v) = Y_3(g, v) = 0$ for any g which has a link involving a player other than players 1, 2 and 3, and
- (vi) for $i \in N \setminus \{1, 2, 3\}$, $Y_i(g, v)$ for all $g \in \mathbb{G}$.¹¹

The unique strongly efficient network is $\{12, 23, 13\}$. We have (i) $F(\{12, 23, 13\}) = \{\{12, 23\}, \{23, 13\}, \{12, 13\}, \{12\}, \{23\}, \{13\}\}$ for $c > 3$ and $F(\{12, 23, 13\}) = \{\{12\}, \{23\}, \{13\}\}$ otherwise; (ii) $F(\{12, 23\}) = \{\{12, 23, 13\}, \{12\}, \{23\}, \{13\}\}$ for $c < 3$, $F(\{12, 23\}) = \{\{12\}, \{23\}, \{13\}\}$ for $3 \leq c < 4$, and $F(\{12, 23\}) = \{\{12\}, \{23\}\}$ for $c \geq 4$. $F(\{12, 13\}) = \{\{12, 23, 13\}, \{12\}, \{23\}, \{13\}\}$ for $c < 3$, $F(\{12, 13\}) = \{\{12\}, \{23\}, \{13\}\}$ for $3 \leq c < 4$, and $F(\{12, 13\}) = \{\{12\}, \{13\}\}$ for $c \geq 4$. $F(\{13, 23\}) = \{\{12, 23, 13\}, \{12\}, \{23\}, \{13\}\}$ for $c < 3$, $F(\{13, 23\}) = \{\{12\}, \{23\}, \{13\}\}$ for $3 \leq c < 4$, and $F(\{13, 23\}) = \{\{13\}, \{23\}\}$ for $c \geq 4$; (iii) $F(\{12\}) = \{\{13\}, \{23\}\}$, $F(\{13\}) = \{\{12\}, \{23\}\}$, $F(\{23\}) = \{\{12\}, \{13\}\}$; (iv) $F(\emptyset) = \{\{12, 23, 13\}, \{12\}, \{23\}, \{13\}\}$; (v) for any g which has a link involving a

¹¹This network formation game is a slight modification of the 3-player game given in the proof of Theorem 2 in Dutta, Ghosal and Ray (2005) and in the proof of Proposition 1 in Bhattacharya (2005) that we have extended to $|N| > 3$ by assigning $v(g) = 0$ to any g which has a link involving a player other than players 1, 2 and 3.

player other than players 1, 2 and 3, $\{\{12\}, \{23\}, \{13\}\} \subseteq F(g)$. Thus, $\{\{12\}\}$, $\{\{23\}\}$ and $\{\{13\}\}$ are the only pairwise farsighted stable sets. ■

Remark 1 An allocation rule is said to be egalitarian if for every $v \in \mathcal{V}$ and $g \in \mathbb{G}$, $Y_i(g, v) = v(g)/n$. Suppose that Y is the egalitarian rule and there is a unique strongly efficient network g^e . Then, $\{g^e\}$ is the unique pairwise farsighted stable set.

Before studying the relationship between pairwise farsighted stable sets and other farsighted solution concepts we analyze some classical examples.

Example 4. Co-author Model (Jackson and Wolinsky, 1996). Each player is a researcher who spends time writing papers. If two players are connected, then they are working on a paper together. The amount of time researcher i spends on a given project is inversely related to the number of projects, n_i , that he is involved in. Formally, player i 's payoff is given by

$$Y_i(v, g) = \sum_{j:ij \in g} \frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \quad (3)$$

for $n_i > 0$. For $n_i = 0$ we assume that $Y_i(g) = 0$. In Figure 4 we have depicted the 3-player case where the complete network g_7 is the unique pairwise stable network. □

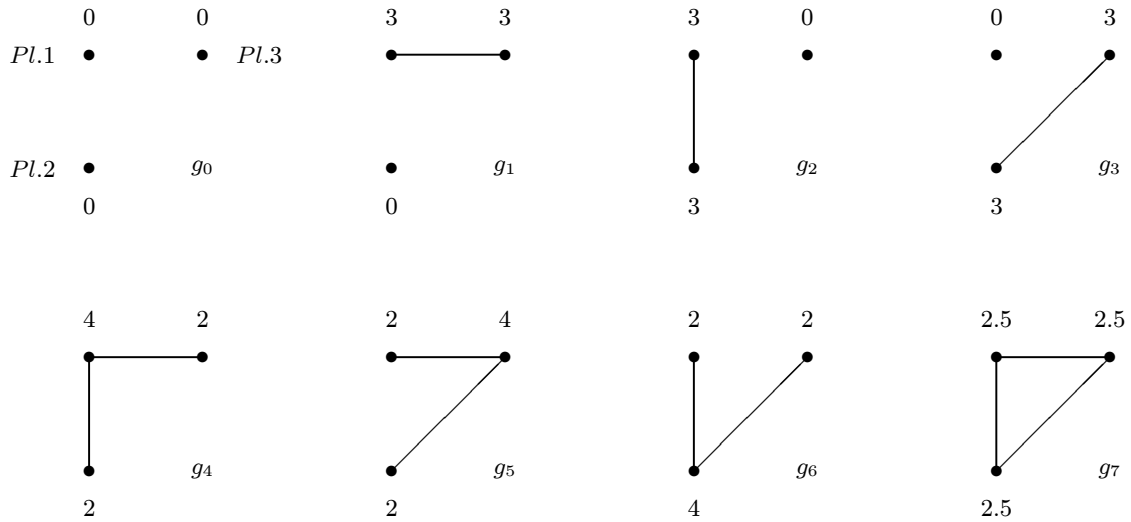


Figure 4: The co-author model with three players.

Unfortunately, no singleton set is pairwise farsighted stable in Example 4. Indeed, there is no network such that there is a farsighted improving path from any other network leading to it. Precisely, $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $F(g_1) = \{g_4, g_6\}$, $F(g_2) = \{g_5, g_6\}$, $F(g_3) = \{g_4, g_5\}$, $F(g_4) = \{g_7\}$, $F(g_5) = \{g_7\}$, $F(g_6) = \{g_7\}$, and $F(g_7) = \emptyset$. However,

a set formed by the complete and two star networks is a pairwise farsighted stable set of networks. Precisely, the pairwise farsighted stable sets are $\{g_4, g_5, g_7\}$, $\{g_4, g_6, g_7\}$ and $\{g_5, g_6, g_7\}$ in the co-author model with three players.

Consider again the symmetric connections model of Example 2 but now with four players (see Figure 5). For $c < \delta(1-\delta)$, the complete network (g_7 in Figure 5) is the unique pairwise stable and $\{g_7\}$ is the unique pairwise farsighted stable set, (ii) for $\delta(1-\delta) < c < \delta(1-\delta^2)$, the star and circle networks (g_2, g_3 in Figure 5) are pairwise stable and $\{g_2\}$ and $\{g_3\}$ are pairwise farsighted stable sets, (iii) for $\delta(1-\delta^2) < c < \delta$, the star and chain networks (g_2, g_1 in Figure 5) are pairwise stable and $\{g_1\}$ and $\{g_2\}$ are pairwise farsighted stable sets, (iv) for $\delta + \frac{1}{2}\delta^2 < c$, the empty network is the unique pairwise stable and $\{g_0\}$ is the unique pairwise farsighted stable set. However, for $\delta < c < \delta + \frac{1}{2}\delta^2$ a pairwise farsighted stable set that is singleton fails to exist while the empty network is the unique pairwise stable network. Indeed, there is no network such that there is a farsighted improving path from any other network leading to it. The empty network $\{g_0\}$ is not a pairwise farsighted stable set because $g_0 \notin F(g_1)$ and $g_0 \notin F(g_3)$. But, there is a unique pairwise farsighted stable set that includes only the empty network (g_0), the circle network (g_3) and the chain networks (similar to g_1). Thus, similarly to Watts (2002), we have that if players are farsighted then the circle network is stable.¹²

4 The largest consistent set

In this section we study the relationship between pairwise farsighted stability and two definitions of the largest consistent set: (i) the pairwise largest consistent set where only pairwise deviations or moves are allowed, and (ii) the coalitional largest consistent set where coalitionwise deviations or moves are allowed.

4.1 The pairwise largest consistent set

First we define the largest consistent set when only pairwise deviations and moves are possible.

Definition 4 *Let $Z^0 \equiv \mathbb{G}$. Then, Z^k ($k = 1, 2, \dots$) is inductively defined as follows: $g \in Z^{k-1}$ belongs to Z^k with respect to Y and v if and only if*

¹²Watts (2002) has analyzed the dynamic formation of networks by self-interested players who can form and sever links in the symmetric connections model with n players. Assuming that players are initially unconnected and that $\delta < c$, she has shown that if players are non-myopic then it is possible for a network shaped like a circle to form.

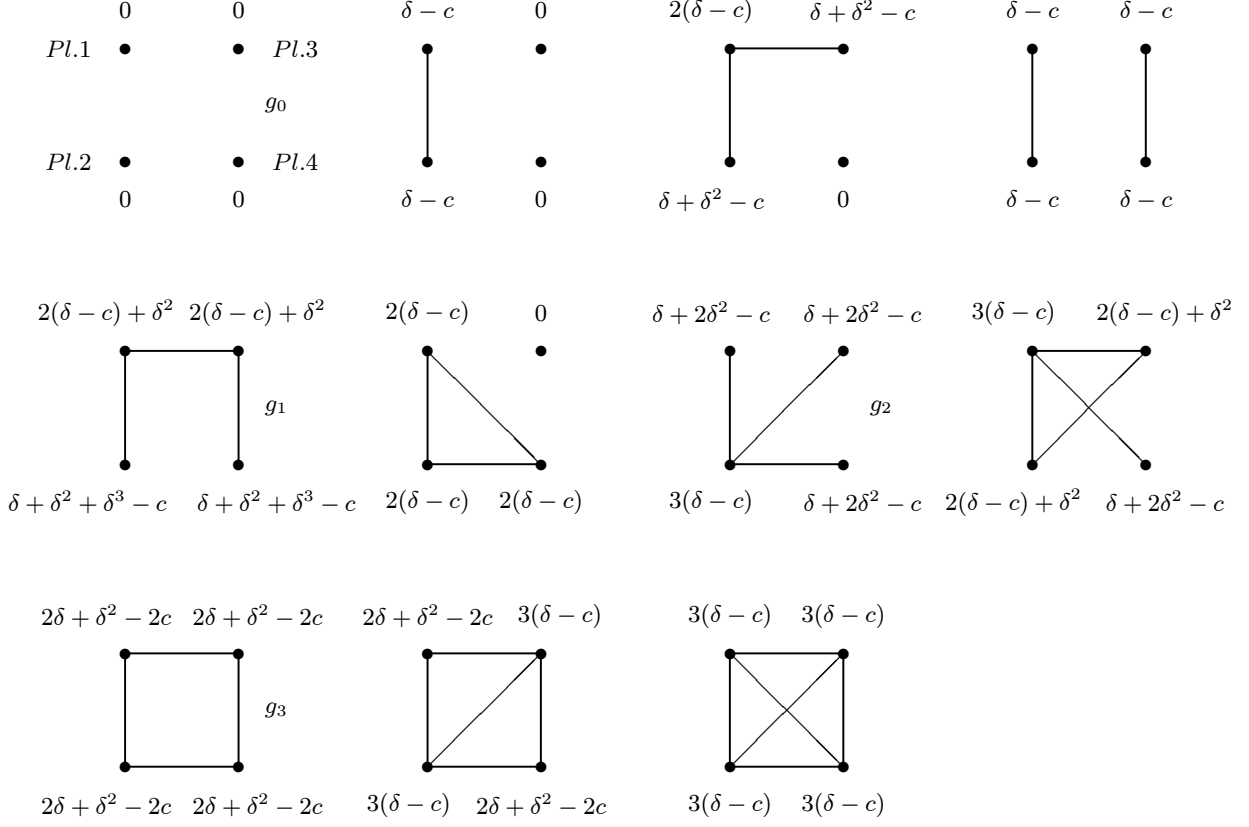


Figure 5: The symmetric connections model with four players.

(ia) $\forall ij \notin g, \exists g' \in Z^{k-1}$ where $g' = g + ij$ or $g' \in F(g + ij)$ such that $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ or $Y_i(g', v) < Y_i(g, v)$ or $Y_j(g', v) < Y_j(g, v)$,

(ib) $\forall ij \in g, \exists g', g'' \in Z^{k-1}$ where $g' = g - ij$ or $g' \in F(g - ij)$, $g'' = g - ij$ or $g'' \in F(g - ij)$, such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$.

The pairwise largest consistent set $PLCS(\mathbb{G})$ is $\bigcap_{k \geq 1} Z^k$.

That is, a network $g \in Z^{k-1}$ is stable (at step k) and belongs to Z^k , if all possible pairwise deviations are deterred. Consider a pairwise deviation from g to an adjacent network by coalition S . There might be further pairwise deviations which end up at g' , where $g + ij \rightarrow g'$ or $g - ij \rightarrow g'$. If either i or j is worse off or both are equal off at g' compared to the original network g then the pairwise deviation is deterred. Since \mathbb{G} is finite, there exists $m \in \mathbb{N}$ such that $Z^k = Z^{k+1}$ for all $k \geq m$, and Z^m is the pairwise largest consistent set $PLCS(\mathbb{G})$. If a network is not in the pairwise largest consistent set, it cannot be stable.

Theorem 4 *If $\{g\}$ is a pairwise farsighted stable set then g belongs to the pairwise largest consistent set $PLCS(\mathbb{G})$.*

Proof. Since $\{g\}$ is a pairwise farsighted stable set we have that for all $ij \notin g : g \in F(g + ij)$ and for all $ij \in g : g \in F(g - ij)$. So $g \in Z^1$. By induction, $g \in Z^k$ for $k \geq 1$. So, $g \in PLCS(\mathbb{G})$. ■

Remember that two networks g and g' are adjacent if they differ by one link.

No indifference Y and v exhibit no indifference if for any g and g' that are adjacent either g defeats g' or g' defeats g .

Proposition 2 *Suppose that Y and v exhibit no indifference. If g is pairwise stable then it belongs to the pairwise largest consistent set.*

Proof. Since Y and v exhibit no indifference, we have that a pairwise stable network g defeats (i) $g + ij$ for all $ij \notin g$ and (ii) $g - ij$ for all $ij \in g$. Thus, $g \in F(g + ij)$ and $g \in F(g - ij)$. So $g \in Z^1$. By induction $g \in Z^k$ for $k \geq 1$. So, $g \in PLCS(\mathbb{G})$. ■

4.2 The coalitional largest consistent set

Second we define the largest consistent set when coalitional deviations and moves are possible. A network $g' \in \mathbb{G}$ is obtainable from $g \in \mathbb{G}$ via deviations by coalition $S \subseteq N$, denoted $g \mapsto_S g'$, if

- (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subset S$, and
- (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \emptyset$.

Definition 5 *A network $g \in \mathbb{G}$ is indirectly weakly dominated by $g' \in \mathbb{G}$, or $g \ll g'$, if there exists a sequence g^0, g^1, \dots, g^m (where $g^0 = g$ and $g^m = g'$) and a sequence S_0, S_1, \dots, S_{m-1} such that $g^j \mapsto_{S_j} g^{j+1}$, $Y_i(g', v) \geq Y_i(g^j, v)$ for all $i \in S_j$ and $Y_i(g', v) > Y_i(g^j, v)$ for some $i \in S_j$, for $j = 0, 1, \dots, m - 1$.*

Based on the indirect weak dominance relation, the coalitional largest consistent set $CLCS(\mathbb{G})$ is defined in an iterative way (see Mauleon and Vannetelbosch, 2004). Chwe (1994) has shown that there uniquely exists a coalitional largest consistent set.

Definition 6 *Let $Z^0 \equiv \mathbb{G}$. Then, Z^k ($k = 1, 2, \dots$) is inductively defined as follows: $g \in Z^{k-1}$ belongs to Z^k with respect to allocation rule Y and value function v if and only if $\forall g', S$ such that $g \mapsto_S g'$, $\exists g'' \in Z^{k-1}$, where $g' = g''$ or $g' \ll g''$, such that we do*

not have $Y_i(g, v) \leq Y_i(g'', v)$ for all $i \in S$ and $Y_i(g, v) < Y_i(g'', v)$ for some $i \in S$. The coalitional largest consistent set $CLCS(\mathbb{G})$ is $\bigcap_{k \geq 1} Z^k$.

That is, a network $g \in Z^{k-1}$ is stable (at step k) and belongs to Z^k , if all possible deviations are deterred. Consider a deviation from g to g' by coalition S . There might be further deviations which end up at g'' , where $g' \ll g''$. There might not be any further deviations, in which case the end network $g'' = g'$. In any case, the end network g'' should itself be stable (at step $k-1$), and so, should belong to Z^{k-1} . If some member of coalition S is worse off or all of them are equal off at g'' compared to the original network g , then the deviation is deterred. Since \mathbb{G} is finite, there exists $m \in \mathbb{N}$ such that $Z^k = Z^{k+1}$ for all $k \geq m$, and Z^m is the coalitional largest consistent set $CLCS(\mathbb{G})$. If a network is not in the coalitional largest consistent set, it cannot be stable. The coalitional largest consistent set is the set of all networks which can possibly be stable.¹³

Theorem 5 *If $\{g\}$ is a pairwise farsighted stable set then g belongs to the coalitional largest consistent set $CLCS(\mathbb{G})$.*

Proof. Since $\{g\}$ is pairwise farsighted stable we have that for all $g' \neq g$ it holds that $g \in F(g')$. So $g \in Z^1$. By induction $g \in Z^k$ for $k \geq 1$. So, $g \in CLCS(\mathbb{G})$. ■

Example 5. Consider a slightly modified version of the co-author model with three players. The payoffs they obtained from the different network configurations are given by (3) except in the complete network g_7 where $Y_i(g_7) = 2$ for all $i \in N$ (see Figure 6). We have $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $F(g_1) = \{g_4, g_6\}$, $F(g_2) = \{g_5, g_6\}$, $F(g_3) = \{g_4, g_5\}$, $F(g_4) = \emptyset$, $F(g_5) = \emptyset$, $F(g_6) = \emptyset$, and $F(g_7) = \emptyset$. There is a unique pairwise farsighted stable set: $\{g_4, g_5, g_6, g_7\}$, but none of these networks belong to the coalitional largest consistent set $CLCS$. It consists of g_1, g_2 , and g_3 . □

¹³Bhattacharya (2005) has studied the conflict between stability and efficiency in network formation when players are farsighted. He has used the social environment introduced by Chwe (1994) to define an environment of social networks in which the feasible coalitional moves are assumed to be as in Jackson and van den Nouweland (2005). Identifying the stable networks as the ones in the coalitional largest consistent set, it is shown that there exists a value function such that for every component balanced and anonymous allocation rule, the corresponding coalitional largest consistent set does not contain any strongly efficient network. Relaxing the requirement of component balanced or anonymity leads to some possibility results. It is also shown that there exists an environment of social networks (with a component balanced and anonymous allocation rule) such that the corresponding largest consistent set does not contain any Pareto efficient network. Finally, it is shown that the largest consistent set with respect to the component-wise egalitarian allocation rule contains at least a Pareto efficient network, and it contains every strongly efficient network whenever the value function is top-convex; i.e., if some strongly efficient network also maximizes the per-capita value among individuals.

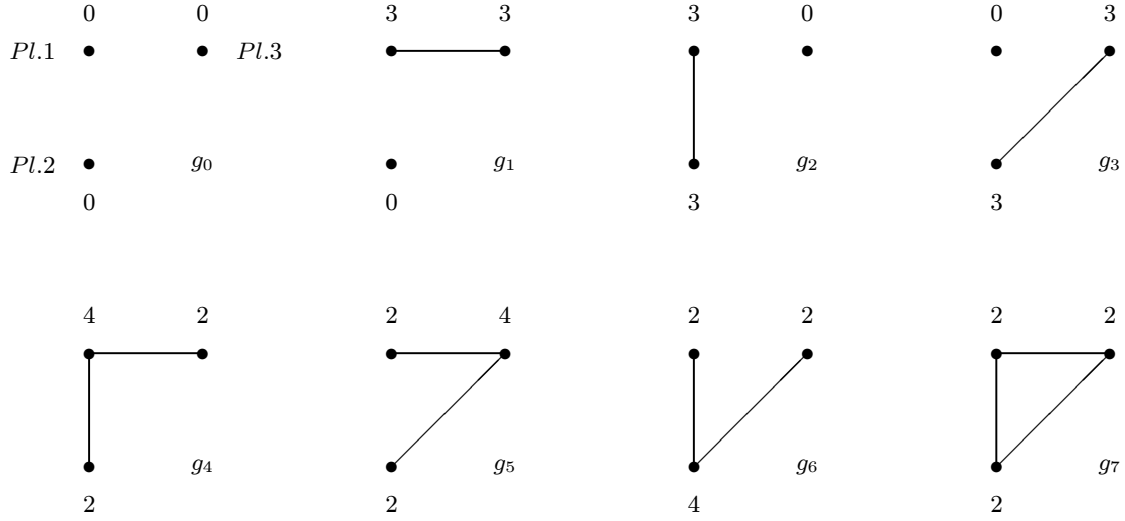


Figure 6: An example where the pairwise farsighted stable set and the coalitional largest consistent set are disjoint.

Concept	Example 4, $n = 3$	Example 5
Pairwise stability	$\{g_7\}$	$\{g_4, g_5, g_6, g_7\}$
Pairwise farsighted stable sets of networks	$\{g_4, g_5, g_7\}, \{g_4, g_6, g_7\}, \{g_5, g_6, g_7\}$	$\{g_4, g_5, g_6, g_7\}$
<i>PLCS</i>	$\{g_1, g_2, g_3, g_7\}$	$\{g_4, g_5, g_6, g_7\}$
<i>CLCS</i>	$\{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$	$\{g_1, g_2, g_3\}$

Table 1: The (no)-relationships among solution concepts for network stability.

Remark 2 *There is no relationship between (i) the pairwise farsighted stable set and the coalitional largest consistent set, (ii) the pairwise farsighted stable set and the pairwise largest consistent set, (iii) the pairwise largest consistent set and the coalitional largest consistent set.*

5 Conclusion

We have proposed a new concept, pairwise farsighted stable set, in order to predict which networks may be formed among farsighted players. A set of networks G is pairwise farsighted stable (i) if all possible pairwise deviations from any network $g \in G$ are deterred by the threat of ending worse off or equal off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) if there is no proper subset of G satisfying conditions (i) and (ii). We have shown that a pairwise

farsighted stable set always exists and we have provided the necessary and sufficient condition such that a unique pairwise farsighted stable set consisting of a single network exists. We have found that the pairwise farsighted stable sets and the set of strongly efficient networks, those which are socially optimal, may be disjoint if the allocation rules have nice properties. Finally, we have studied the relationship between pairwise farsighted stability and other concepts such as the largest consistent set, a notion due to Chwe (1994). By means of examples we have shown that there is no relationship between (i) the pairwise farsighted stable set and the coalitional largest consistent set, (ii) the pairwise farsighted stable set and the pairwise largest consistent set, (iii) the pairwise largest consistent set and the coalitional largest consistent set.

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