Many-to-one Matching When Colleagues Matter.

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Abstract :

This paper studies many-to-one matching markets in which each agent's preferences not only depend on the institution that hires her, but also on the group of her colleagues, which are matched to the same institution. With an unrestricted domain of preferences the non-emptiness of the core is not guaranteed. We show that under certain conditions on agents' preferences two possible situations in which al least one stable allocation exists, emerge. In both situations, at least one stable allocation exists. The first one reflects real-life situations in which agents are more concerned about an acceptable set of colleagues than about the firm hiring them. The second one refers to markets in which a workers' ranking is accepted by workers and firms present in such market.

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Running Title: Matching when colleagues matter.

1 Introduction

Matching models has been successfully used to describe many real-life situations like college admissions problems, centralized job markets such National Resident Matching Program (NRMP). In these situations, students (or workers) are often concerned not only about the characteristics of the institutions they are assigned to, but also about colleagues at each institution. (See Roth and Sotomayor [9] for a detailed discussion). This paper deals with matching markets in which agents on one side of the market care about their colleagues. In particular, our purpose is to construct a model combining these two kind of problems: the many-to-one matching problems, introduced by Gale and Shapley [8] and the hedonic coalition formation problems, introduced by Drèze and Greenberg [5]. Coalition formation models have been used to describe the formation of academic departments, research groups, medical teams and many other real-life examples for groups of workers. Therefore, if a worker receives a job offer, in order to decide to accept it or not, she should consider both features: the firm and the set (or coalition) of colleagues. We call this kind of problem Coalition-Matching problems. In these models, the non-emptyness of the core is not guaranteed.

Regarding to the matching models, the literature provides two sufficient conditions, each one guarantees the existence of stable matchings. No restriction is imposed on individuals' preferences over outcomes, and simple and very intuitive restrictions on institutions' preferences over allocations are enough. Both conditions, called responsiveness and substitutability, introduce a kind of separability on the institutions' preferences over groups of individuals to be matched with.

Concerning the second class of problems, the hedonic coalition formation problems, the literature also provides some conditions to guarantee the existence of stable allocations. Banerjee et al. [3] and Bogomolnaia and Jackson [4] provide conditions over individuals' preference profiles under which stable allocations do exist. Alcalde and Romero-Medina [2] and Alcalde and Revilla [1] study conditions on individuals' preferences that guarantee the existence of stable outcomes for coalition formation problems.

To obtain some positive stability results in this combined setting is much harder that we can think at first. One might be tempted to think about the following possibility to solve our general problem. Let us assume that institutions' preferences satisfy substitutability, and individuals' preferences fullfil the tops responsiveness condition (in the sense defined in the paper by Alcalde and Revilla [1]). The reader can think that the combination of both properties yields the existence of stable allocations. Nevertheless, this straight conclusion is not true! The same argument for *wrong straight conclusions* could be provided for others combinations of conditions on institutions' preferences yielding stable allocations in the classical many-to-one matching problem and, the conditions provided for the coalition formation problem.

A first approach to this problem was introduced by Dutta and Massó [6]. Dutta and Massó obtain two positive results. The first positive result is obtained in a particular case of the model in which only couples are allowed, when some conditions on preferences are combined: togetherness and group substitutability. They also obtain some negative results when coalitions of any size are allowed, except with $\mathcal{F}$-lexicographic preferences. When workers' preferences are $\mathcal{F}$-lexicographic (the workers only cares about the colleagues when the firm is fixed) the model is very close to the classical many-to-one matching in which the co-workers do not appear in the workers' preferences.

The main criticism to the Dutta-Massó approach is that they consider that couples are exogenously given. In this paper we extend their analysis in two ways. First, couples or colleagues groups, are not exogenously given. Second, we do not focus on couples but on groups of individuals. Therefore, we do not restrict ourselves to the case in which groups of colleagues are composed of only two individuals.

In this paper we explore a condition which is sufficient for the existence of stable allocations. This condition comes from a generalization of Dutta and Massó's *Togetherness*, we call it *Group Togetherness*. Finally, we also present a way to avoid some of the negative results.Dutta and Massó**??** present a condition called Unanimous Ranking According to Desirability which is applied to the individuals' preferences. In this paper a different version of this condition is used over the preferences of individuals and institutions and a positive result can be obtained.

In a recent paper, F. Echenique and M. Yenmez [7] present a method to obtain the core allocations, if any exists, in a similar framework but without any assumption over preferences. In particular they propose some new solutions in case that the core is empty.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 shows that stronger conditions than those used in the two classical problems, are needed in a coalition matching problem. Sections 4 and 5 present positive results in two different frameworks. Section 6 briefly concludes. The appendix includes two examples without stable allocations that satisfy other conditions.

2 The Model

There is a set $F = \left\{f^1, \dots, f^\ell\right\}$ of firms and a set $W = \{w_1, \dots, w_n\}$ of workers. Each firm f^j's preferences are a linear ordering $P(f^j)$ defined over $2^W \cup \left\{f^j\right\}$. Thus firms only care about the set of its employees. Workers' preferences are defined over pairs consisting of one firm and a set of workers. Each worker w_i's preferences are a linear ordering $P(w_i)$ defined over $(F \times W^i) \cup \{w_i\}$, where $W^i = \{S \mid S \subseteq W, w_i \in S\}$. Here, $\{w_i\}$ represents an unemployed worker. In this case, we consider that the unemployed worker has no colleagues. A preference profile is a list $P = (P(x))_{x \in W \cup F}$. A coalition configuration is a partition of W.

A **coalition matching problem** is a many-to-one matching problem in which each agent cares about the agents matched to the same firm as she is. Then a coalition matching problem is fully described by a set of firms, a set of workers, and a preference profile, i.e. a list $\{F, W, P\}$.

A **matching** specifies who works where. Formally, a matching μ is a correspondence from $F \cup W$ to itself such that

1) for all $f^j \in F$, if $\mu\left(f^j\right) \notin W$, then $\mu\left(f^j\right) = f^j$,

2) for all $w_i \in W$, if $\mu(w_i) \notin F$, then $\mu(w_i) = \{w_i\}$, and

3) for all $\left(f^j, w_i\right) \in F \times W$, $\mu(w_i) = f^j$ if, and only if $w_i \in \mu\left(f^j\right)$.

Let $M(F, W, P)$ be the set of all matchings for $\{F, W, P\}$. Given $\mu \in M(F, W, P)$

and $w_i \in W$, we denote by $\mu^2(w_i)$, the set of worker i's colleagues. Here, $f^j = \mu(w_i)$, $\mu^2(w_i) = \mu(f^j)$.[1]

A matching is individually rational if no agent prefers to be unmatched to his assignment at the matching.

A matching $\mu \in M(F, W, P)$ is **Individually Rational (IR) for $\{\mathbf{F}, \mathbf{W}, \mathbf{P}\}$**, if for all $f^j \in F$ and all $w_i \in W$:

$$1)\ (\mu(w_i),\ \mu^2(w_i))\ P(w_i)\ \{w_i\}.$$

$$2)\ \mu(f^j)\ \ P(f^j)\ \ \{f^j\}.$$

Let $\mathcal{I}(F, W, P)$ be the set of IR matchings for $\{F, W, P\}$.

A wide class of concepts of stability exists in the literature on matching and coalition formation. For our problem, we propose a concept of stability that is very similar to standard concepts of core stability.[2] Given $\{F, W, P\}$, a matching $\mu \in M(F, W, P)$ is **stable for $\{\mathbf{F}, \mathbf{W}, \mathbf{P}\}$** if there is no $\widetilde{\mu} \in M(F, W, P)$ and a set $V \subset F \cup W$ such that:

$$\begin{aligned}
&1)\ (\widetilde{\mu}(w_i), \widetilde{\mu}^2(w_i))P(w_i)((\mu(w_i), \mu^2(w_i)), && \text{for all } w_i \in V.\\
&2)\ \widetilde{\mu}(f^j)P(f^j)\mu(f^j), && \text{for all } f^j \in V.\\
&3)\ \widetilde{\mu}(w_i) \in V, && \text{for all } w_i \in V.\\
&4)\ \widetilde{\mu}(f^j) \subset V, && \text{for all } f^j \in V.
\end{aligned}$$

We say that such a **V blocks $\boldsymbol{\mu}$**.[3] Let $\mathcal{C}(F, W, P)$ be the set of stable matchings for $\{F, W, P\}$. Obviously, $\mathcal{C}(F, W, P) \subset \mathcal{I}(F, W, P)$.

3 Well-known solutions.

In this section we show that even when two conditions, coalitional substitutability and F-essentiality, are required, a stable matching may not exist. Substitutability

[1] To be precise, $\mu^2(w_i)$ is the set of w_i's colleagues with w_i inclosed. Note that the w_i's preferences are defined over elements from W^i.

[2] A deviation by a worker might produce a reaction from her old colleagues (and her new colleagues) who could prefer another firm and group of co-workers.Thus, the standard concept of pairwise stability used in the many-to-one matching problems literature can not be applied to coalition matching problems.

[3] Notice that 3) and 4) imply that $\widetilde{\mu}^2(w_i) \subset V$; for all $w_i \in V$.

is a sufficient condition on firms preferences for the existence of stable matchings in the classical many-to-one matching model (Roth and Sotomayor [9]). Coalitional substitutability, the counterpart for our model of the property called group substitutability by Dutta and Massó [6], is stronger than substitutability. F-essentiality is the counterpart for our model of a sufficient condition for the existence of stable coalition formation structures proposed by Alcalde and Romero-Medina [2]. In the coalition formation literature, tops responsiveness (Alcalde and Revilla [1]), a weaker condition than F-essentiality, is sufficient for stability. Example 6 of the appendix satisfies tops responsiveness and coalitional substitutability and no stable matching exists.

Given a set $S \subset W$, and a firm f^j with preferences $P\left(f^j\right)$, let $Ch_{f^j}\left(S\right)$ denote the most preferred subset of W for f^j.

Firm f^j's preference, $P(f^j)$, satisfies **substitutability** if for all $S \subseteq W$, and all $w_i, w_h \in S$, $(i \neq h)$, $w_i \in Ch_{f^j}\left(S\right)$ implies $w_i \in Ch_{f^j}\left(S \backslash \{w_h\}\right)$, with $Ch_{f^j}\left(S\right)$ being maximal on S for $P\left(f^j\right)$. Our definition below differs from the original notion of substitutability because it states conditions on worker sets rather than on workers. Coalitional substitutability is stronger than substitutability.

Firm f^j's preference, $P\left(f^j\right)$ satisfies **Coalitional Substitutability** if for all $S \subseteq W$, all partition of S, $\widehat{S} = \{S_1, ..., S_k\}$, and all S_l, $S_h \in \widehat{S}$, $(l \neq h)$, if $S_l \subseteq Ch_{f^j}\left(S\right)$, then $S_l \subseteq Ch_{f^j}\left(S \backslash S_h\right)$.[4]

We need to introduce additional restrictions on preferences to reach our objective. We extent the essentiality condition presented by Alcalde and Romero-Medina [2] to our current framework.

Let $f^j \in F$ and $w_i \in W$. The coalition $T^j \in W^i$, containing worker w_i, is

[4] Dutta and Massó [6] propose a condition on the institutions' preferences called Group Substitutability, but they only apply that condition for groups of at most two agents, i.e. couples.

essential relative to f^j **for** w_i if and only if, for all $T, T' \in W^i$:

1) If $T^j = \{w_i\}$, and $T \neq \{w_i\}$, then $\{w_i\}\ P(w_i)\ (f^j, T)$.

2) If $T^j \neq \{w_i\}$, then

(a) if and only if $T^j \nsubseteq T$ then $(f^j, \{w_i\})\ P(w_i)\ (f^j, T)$, and

(b) if $T^j \subseteq T \subset T'$, then $(f^j, T)\ P(w_i)\ (f^j, T')$.

A preference profile satisfies **F-essentiality** for $\{F, W, P\}$, if and only if for all $w_i \in W$ and all $f^j \in F$, there exists a coalition that is essential for w_i relative to f^j.

For all $w_i \in W$, the coalition whose existence is stated in the condition may differ according to which firm the worker is matched to. In other words, the set of colleagues that is essential for a worker in general, depends on the firm to which she is assigned.

Coalitional substitutability and F-essentiality are not sufficient to guarantee the existence of a stable matching. This is shown by the following example.

Example 1: Let $F = \{f^1, f^2, f^3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be the preference profile given by the following table, where elements are ranked in descending order of preference and only acceptable partners are listed:

f^1	f^2	f^3
$\{w_2, w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$
$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$
$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2\}$
$\{w_3\}$	$\{w_2\}$	$\{w_3\}$
$\{w_2\}$	$\{w_1\}$	$\{w_1\}$
$\{w_1\}$	$\{w_3\}$	$\{w_2\}$
$\{f^1\}$	$\{f^2\}$	$\{f^3\}$

w_1	w_2	w_3
$\{f^1, w_1, w_3\}$	$\{f^2, w_1, w_2\}$	$\{f^2, w_2, w_3\}$
$\{f^2, w_1, w_2\}$	$\{f^2, w_1, w_2, w_3\}$	$\{f^3, w_2, w_3\}$
$\{f^2, w_1, w_2, w_3\}$	$\{f^1, w_2, w_3\}$	$\{f^1, w_1, w_3\}$
$\{f^1, w_1, w_2, w_3\}$	$\{f^1, w_1, w_2, w_3\}$	$\{f^1, w_1, w_2, w_3\}$
$\{f^3, w_1, w_3\}$	$\{f^3, w_2, w_3\}$	$\{f^2, w_1, w_2, w_3\}$
$\{f^3, w_1, w_2, w_3\}$	$\{f^3, w_1, w_2, w_3\}$	$\{f^3, w_1, w_2, w_3\}$
$\{w_1\}$	$\{w_2\}$	$\{w_3\}$

Firms' preferences satisfy coalitional substitutability. Workers' preferences satisfy F-essentiality but not separability.

Claim: there is no stable matching.

i) No matching μ such that $\mu\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\in F$ is stable, since $\left\{w_1,w_3,f^1\right\}$ blocks it.

ii) No matching μ' such that $\mu'\left(f^j\right)=W$ for some f^j is stable, since for all $f^j\in F$, $\left\{f^j\right\}\ P\left(f^j\right)\ \{w_1,w_2,w_3\}$.

iii) Let $\mu''\in\mathcal{I}(F,W,P)$. Then for all $f^j\in F$, $\left|\mu''\left(f^j\right)\cap W\right|\in\{0,2\}$, i.e. each firm is assigned two individuals or none. Hence, for all $\mu''\in\mathcal{C}(F,W,P)$;

$$\mu''\left(f^j\right)=\{w_i,w_h\}\ \text{ for some } f^j\in F \text{ and}$$

$$\mu''\left(f^k\right)=\left\{f^k\right\}\ \text{ for all } f^k\in F\diagdown\left\{f^j\right\}.$$

To show that there is no stable matching, we find that no matching having the above structure is stable. So, let us consider the remaining cases:

1) $\mu''\left(f^1\right)=\{w_1,w_2\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^1$.

2) $\mu''\left(f^2\right)=\{w_1,w_3\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^2$.

3) $\mu''\left(f^3\right)=\{w_1,w_2\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^3$.

These matchings are blocked by $\{w_1\}$.

4) $\mu''\left(f^2\right)=\{w_2,w_3\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^2$.

This matching is blocked by $\{w_2\}$.

5) $\mu''\left(f^1\right)=\{w_2,w_3\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^1$.

6) $\mu''\left(f^3\right)=\{w_1,w_3\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^3$.

These matchings are blocked by $\{w_3\}$.

7) $\mu''\left(f^1\right)=\{w_1,w_3\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $f^j\neq f^1$.

This matching is blocked by $\left\{f^2,w_2,w_3\right\}$.

8) $\mu''\left(f^2\right)=\{w_1,w_2\}$ and $\mu''\left(f^j\right)=\left\{f^j\right\}$ for all $j\neq 2$.

This matching is blocked by $\left\{f^1,w_1,w_3\right\}$.

9) $\mu''(f^3) = \{w_2, w_3\}$ and $\mu''(f^j) = \{f^j\}$ for all $j \neq 3$.

This matching is blocked by $\{f^2, w_1, w_2\}$.

♦

4 Positive result: Group Togetherness.

In this section, we present conditions that ensure the existence of stable matchings when workers only care about an acceptable group of colleagues. In many situations individuals prefer matchings in which they are together to matchings in which they are not (Dutta and Massó [6]). We present a generalized version of togetherness that applies to settings in which groups may be of any size. Our condition, which is stronger than F-essentiality, is called *Essentiality.* The difference between Essentiality and F-essentiality may also be presented as the result of adding a new requirement to F-essentiality: *separability.* In our framework, separability implies that workers' preferences over firms are independent of their preferences over sets of colleagues.

Let $w_i \in W$. Here, w_i's preferences are **separable** if for all $S, S' \in W^i$ and all f^j, $f^k \in F$, we have

$$(f^j, S)\ P(w_i)\ (f^j, S') \Longleftrightarrow (f^k, S)\ P(w_i)\ (f^k, S') \text{ and}$$

$$(f^j, S)\ P(w_i)\ (f^k, S) \Longleftrightarrow (f^j, S')\ P(w_i)\ (f^k, S').$$

Note that, under separability, the preferences of each worker w_i, $P(w_i)$, induce two binary relations, her preferences over firms (let us denote this relation as P_i^F), and her preferences over colleagues (let us denote this relation as P_i^C). These relations are defined as follows:

1) $f^j\ P_i^F\ f^k$ if $(f^j, S)\ P(w_i)\ (f^k, S)$ for all f^j, $f^k \in F$ and all $S \in W^i$, and

2) $S\ P_i^C\ S'$ if $(f^j, S)\ P(w_i)\ (f^j, S')$ for all S, $S' \in W^i$ and all $f^j \in F$.

From now on, we assume that workers' preferences are separable.

It is easy to see that coalitional substitutability and separability do not guarantee the existence of a stable matching. This is shown by Example 6 in the appendix. If separability and F-essentiality are imposed, the coalition that is essential for a worker has to be the same whatever firm hires her. This is what motives the following definition.

Let $P(w_i)$ be a separable and linear ordering for worker w_i. Coalition T_i^e, containing worker w_i, is **essential** for her if and only if her restricted preferences over coalitions are such that, for all T, $T' \subset W^i$:

1) If $T_i^e = \{w_i\}$, then $\{w_i\}\ P_i^C\ T_i$ for all $T \neq \{w_i\}$ and

2) If $T_i^e \neq \{w_i\}$, then

(a) $\{w_i\}\ P_i^C\ T$ if and only if $T_i^e \nsubseteq T$, and

(b) if $T_i^e \subseteq T \subset T'$ then $T_i\ P_i^C\ T_i'$.

A worker's preferences satisfy *essentiality* whenever there exists a coalition that is essential for her. Note that, under *essentiality*, whatever firm the worker works for, the essential coalition is the same.

To obtain positive results concerning the existence of stable matchings, we need to introduce a further property: *Group Togetherness.* This additional requirement refers to the intensity of workers' preferences concerning a particular item on her preferences. A worker prefers the matchings in which she is matched to an acceptable set of colleagues more than the others. But if she compares two matchings with acceptable sets of colleagues, it does not matter which coalition is better for her. In the last case, the worker only cares about the firm.

Let (f^j, T), (f^h, T') be such that $T, T' \subset W^i$. Then $P(w_i)$ satisfies **Group Togetherness (GT)** if

1) If $\{w_i\}\ P_i^C\ T$ and $\{w_i\}\ P_i^C\ T'$ then $(f^j, T)\ P_i\ (f^h, T')$ iff $f^j\ P_i^F\ f^h$.

2) If $T\ P_i^C\ \{w_i\}$ and $T'\ P_i^C\ \{w_i\}$ then $(f^j, T)\ P_i\ (f^h, T')$ iff $f^j\ P_i^F\ f^h$.

3) If $T\ P_i^C\ \{w_i\}$ and $\{w_i\}\ P_i^C\ T'$ then $(f^j, T)\ P_i\ (f^h, T')$.

To introduce the result in this section we need an algorithm that yields stable matchings under some of the above mentioned conditions: essentiality and group togetherness. In that algorithm, a coalition configuration of workers is obtained in a first part and a matching between those coalitions and firms is obtained in the second part. If there are some workers' coalition that could not find a firm that hires them, this coalition is broken up and a new second part of the añgorithm has to be applied. The algorithm and their properties are shown in the following.

We present an algorithm, which can be understood as the conjunction of two well-known algorithms:

The first one is the *ess*-algorithm defined by Alcalde and Romero-Medina [2] which, when applied to a coalition formation problem in which essentiality is satisfied, produces a stable coalition configuration of workers.

The other is the multistage-deferred-acceptance algorithm defined by Gale and Shapley [8] for matching problems. It is applied to a matching problem in which the agents are the firms and the coalitions of workers that have been obtained in the previous *ess*-algorithm. These coalitions make offers to the firms as in Dutta and Massó [6], i.e. if there is any coalition that is not assigned to any firm then a new deferred-acceptance algorithm is applied with this coalition broken up.

Group deferred-acceptance algorithm.

Part 1: Let $\sigma : 2^W \to 2^W$ be the function that associates with each set of workers, $T \subseteq W$, the coalition

$$\sigma(T) = \cup_{w_i \in T} \{S \subseteq W \mid S \text{ is essential for } w_i\}.$$

For each $w_i \in W$, let $S_i^0 = \{w_i\}$.

Stage 1: Let $S_i^1 = \sigma(\{w_i\})$. If $S_i^1 = S_i^0$, the algorithm stops. The outcome is $S_i = S_i^1$. Otherwise, go to stage 2.

Stage k : Let $S_i^k = \sigma\left(S_i^{k-1}\right)$. If $S_i^k = S_i^{k-1}$, the algorithm stops. The outcome is $S_i = S_i^k$. Otherwise, go to stage $k+1$.

The coalition $T_{w_i}^\sigma$ is defined for each worker:

$$T_{w_i}^\sigma = \begin{cases} S_i & \text{if } S_i \subseteq S_m \text{ for all } m \in N \text{ such that } m \in S_i. \\ \{w_i\} & \text{otherwise.} \end{cases}$$

Let $T^\sigma = \left(T_{w_i}^\sigma\right)_{w_i \in W}$. It is a partition of the set of workers if the coalition formation problem satisfies essentiality (Alcalde and Romero-Medina [2]).

Part 2: Let the matching problem in which the set of agents on one side is F and on the other one is T^σ, the partition obtained in the first part of the algorithm. The algorithm takes as preferences of each element of T^σ, the preferences of one worker in each coalition. Let T_q^σ be a coalition from T^σ, and $\widetilde{w}_q$ be the agent with the lowest subindex in T_q^σ. The preferences of T_q^σ in the matching problem are the preferences of $\widetilde{w}_q$ but restricted to that the set of colleagues of $\widetilde{w}_q$ would be supersets of T_q^σ.[5] Formally: $P\left(T_q^\sigma\right) = P\left(\widetilde{w}_q\right)$ defined over $F \times W^{T_q^\sigma}$.[6] Denote the matching problem defined in this way as $\mathcal{M}^1$.

Stage 1: Let $\mathcal{M}^1$ be the many-to-one matching problem. Each coalition of workers, T_q^σ, makes offers to their most preferred firms according to $\widetilde{w}_q$'s preferences. Firms accept the offers if they are acceptable, otherwise reject. Let $\widehat{\mu}^1$ be the resulting matching. Let $\widehat{T}^1 = \left\{T_q^\sigma \mid \widehat{\mu}^1\left(T_q^\sigma\right) \in F\right\}$. If for all $S \in T^\sigma$ such that $|S| > 1$, $S \in \widehat{T}^1$, the algorithm stops. The matching is $\widehat{\mu}^1$.

Otherwise, there is an unmatched coalition of workers from T^σ, say T_q^σ. Go to stage 2.

Stage 2: Let $\mathcal{M}^2$ be the many-to-one matching problem that is obtained when such T_q^σ breaks up into single workers. Let $E^1 = \left\{T_q^\sigma \mid \left|T_q^\sigma\right| > 1 \text{ and } \widehat{\mu}^1\left(T_q^\sigma\right) \notin F\right\}$ be the set coalitions that have to be broken up. The set of individuals in the matching problem $\mathcal{M}^2$ is $\left(T^\sigma \diagdown E^1\right) \cup \left\{w_i \mid w_i \in T_q^\sigma,\ \forall T_q^\sigma \in E^1\right\}$. The coalitions that are not accepted by any firm in the first stage are replaced by the workers who are in that coalitions. Apply the deferred-acceptance algorithm with workers or coalitions making offers to $\mathcal{M}^2$.

Stage k: Let $\mathcal{M}^k$ be the many-to-one matching problem with the set of firms, F, and the set of individuals: $\widehat{T}^{k-1} \cup \left\{w_i \mid w_i \in T_q^\sigma,\ \forall T_q^\sigma \in E^{k-1}\right\}$ where $\widehat{T}^{k-1} = \left\{T_q^\sigma \mid \widehat{\mu}^{k-1}\left(T_q^\sigma\right) \in F\right\}$ is the set of coalitions matched with a firm in the previous stage, and $E^{k-1} = \left\{T_q^\sigma \mid \left|T_q^\sigma\right| > 1 \text{ and } \widehat{\mu}^{k-1}\left(T_q^\sigma\right) \notin F\right\}$ is the

[5] A family of algorithms can be defined depending on how the worker, whose preferences will be used, is selected. For instance, the worker with the highest subindex can be selected, or someone randomly.

[6] Note that, as separability is required, the algorithm only needs to take the restricted preferences of $\widetilde{w}_q$ over institutions. In other words, the preferences of T_q^ε are P_q^F.

set of unmatched coalitions with more that one worker in the previous stage. Consider as preferences for the members of $\widehat{T}^{k-1}$ the preferences of the worker with the lowest subindex in each coalition, and for remaining agents (firms and single workers) their true preferences. Individuals (workers and coalitions) make offers to their most preferred firms, and firms accept (or reject) the offers if they are acceptable (or unacceptable). Let $\widehat{\mu}^k$ be the resulting matching. Let $\widehat{T}^k = \left\{T_q^\sigma \mid \widehat{\mu}^k\left(T_q^\sigma\right) \in F\right\}$. If for all $S \in T^\sigma$ such that $|S| > 1$, $S \in \widehat{T}^k$, the algorithm stops. The matching is $\widehat{\mu}^k$. Otherwise, go to the stage $k+1$.

We denote the matching resulting from this algorithm as μ^*, and the the coalition of colleagues assigned to w_i as $T_i^* = \mu^*\left(\mu^*(w_i)\right)$.

To illustrate this algorithm, consider the following example.

Example 2: Let $F = \left\{f^1,\ f^2,\ f^3\right\}$ and $W = \{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6\}$, with preferences satisfying GT and coalitional substitutability. Firms' preferences are given in the following table:

f^1	f^2	f^3
$\{w_2, w_6\}$	$\{w_1, w_2, w_3\}$	$\{w_3, w_5, w_6\}$
$\{w_2, w_5\}$	$\{w_1, w_2\}$	$\{w_3, w_5\}$
$\{w_5, w_6\}$	$\{w_1, w_3\}$	$\{w_5, w_6\}$
$\{w_2\}$	$\{w_2, w_3\}$	$\{w_3, w_6\}$
$\{w_6\}$	$\{w_1\}$	$\{w_3\}$
$\{w_5\}$	$\{w_2\}$	$\{w_5\}$
	$\{w_3\}$	$\{w_6\}$
		$\{w_4\}$

Workers' preferences satisfy separability and are such that

$$\begin{array}{ll} w_1 : P_1^C & Ch_1(W) = \{w_1, w_2\}. \\ w_2 : P_2^C & Ch_2(W) = \{w_1, w_2, w_3\}. \\ w_3 : P_3^C & Ch_3(W) = \{w_1, w_3\}. \\ w_4 : P_4^C & Ch_4(W) = \{w_4, w_5\}. \\ w_5 : P_5^C & Ch_5(W) = \{w_4, w_5\}. \\ w_6 : P_6^C & Ch_6(W) = \{w_3, w_5, w_6\}. \end{array}$$

$$\text{for all } w_i \in W; \quad f^1 P_i^F f^2 P_i^F f^3.$$

The algorithm selects the following sets for each worker,

$$T_1^\sigma = \{w_1, w_2, w_3\} = T_2^\sigma = T_3^\sigma.$$

$$T_4^\sigma = \{w_4, w_5\} = T_5^\sigma.$$

$$T_6^\sigma = \{w_6\}.$$

Then the modified deferred-acceptance algorithm is applied.

- T_1^σ, T_4^σ and T_6^σ proposes to f^1. Firm f^1 only accepts $\{w_6\}$.

- T_1^σ and T_4^σ propose to f^2. Firm f^2 only accepts T_1^σ.

- T_4^σ proposes to f^3. Firm f^3 rejects T_4^σ.

Then T_4^σ is broken up and a new deferred-acceptance algorithm's stage is applied with the sets of workers: T_1^σ, T_6^σ, $\{w_4\}$, $\{w_5\}$. The resulting matching is given by:

$$\mu^* \left(f^1\right) = \{w_5, w_6\}.$$

$$\mu^* \left(f^2\right) = \{w_1, w_2, w_3\}.$$

$$\mu^* \left(f^3\right) = \{w_4\}.$$

The matching μ^* is stable.♦

Next we show that if workers' preferences satisfy essentiality and GT, the algorithm yields a stable matching. We need two lemmas.

Lemma 1: *The Group deferred-acceptance algorithm always terminates.*

The *ess*-algorithm always terminates in finitely many (Alcalde and Romero-Medina [2]) and when firms' preferences satisfy substitutability, the deferred-acceptance algorithm always terminates in a finite number of stages (Gale and Shapley[8].) Hence, the *ess*-algorithm produces a partition of the set of workers that will be assigned to a firm later on.

Lemma 2: *The Group deferred-acceptance algorithm yields a matching.*

In the *ess*-algorithm, the final stage is a simple deferred-acceptance algorithm in which the workers are replaced by coalitions of colleagues that have been previously formed. It is well known that a matching is obtained (Gale and Shapley [8]). Here, as no worker can be in two coalitions in the outcome of the *ess*-algorithm, a matching is obtained.

The main result of this section is the following.

Theorem 1 *If firms' preferences satisfy Coalitional Substitutability, workers' preferences satisfy Essentiality and GT, the Group deferred-acceptance algorithm produces a stable matching.*

Proof. By lemmas 1 and 2 we only need to prove that the matching is stable. In order to prove stability we show that the matching is IR and that there is no blocking group.

Step 1. *The matching belongs to* $\mathcal{I}(F, W, P)$: This is so for every firm, by coalitional substitutability. If a firm has hired any group of workers then it does not want to fire everyone. For every worker who is working with her essential set (or a superset of it), by GT the matching is preferred to be unemployed. For every worker who is not working with her essential set (or a superset) also by GT, the matching is preferred to be unemployed.

Step 2. *No group blocks the matching*: suppose that there exists (f^j, S) that blocks the matching that results in the algorithm, μ^*. We check that in all possible cases, a contradiction appears.

- Let $w_i \in S$ and $T_i^e \subseteq S \subset \mu^{*2}(w_i)$. If $f^j = \mu^*(w_i)$, then $\mu^*\left(f^j\right) \ P\left(f^j\right) \ S$ (otherwise f^j does not hire $\mu^*\left(f^j\right) \backslash S$). This contradicts the assumption that (f^j, S) blocks μ^*. Thus $\left(f^j, S\right)$ does not block μ^*. If $f^j \neq \mu^*(w_i)$, there is $w_j \in T_i^*$ such that $\mu^*(w_j) \ P(w_j) \ f^j$, and either $w_j \in T_i^e \subseteq S$ ($\left(f^j, S\right)$ does not block μ^*) or $w_j \notin T_i^e$. If $w_j \notin T_i^\sigma$ then as $w_j \in T_i^*$ there are two possibilities:

1) If $T_i^* \subset S$ then $T_i^*\ P^C(w_i)\ S$. Since $w_i \in S$, this contradicts the assumption that (f^j, S) blocks μ^*.

2) If $T_i^* \nsubseteq S$ then there is w_h such that $w_h \in T_i^e \subset S$ and $T_i^*\ P(w_h)\ \emptyset\ P(w_h)S$.

Then w_h does not block μ^*. As $w_h \in S$, this contradicts that (f^j, S) blocks μ^*.

- Let $w_i \in S$, and $T_i^e \nsubseteq S$ and $T_i^e \subseteq \mu^{*2}(w_i)$. Then by GT, a contradiction exists.
- Let $w_i \in S$, and $T_i^e \subset \mu^{*2}(w_i) \subset S$. Then there is $S' \subset \left\{S \backslash \mu^{*2}(w_i)\right\}$ and f^j does not hire S' in the algorithm. Then either $\mu\left(f^j\right)\ P\left(f^j\right)\ S$ or $f^j \neq \mu^*(w_i)$. In the last case there is $w_j \in T_i^e \subset S$ such that $\mu^*(w_i)\ P(w_j)\ f^j$. It contradicts that (f^j, S) blocks μ^*.
- Let $w_i \in S$, and $T_i^e \nsubseteq \mu^{*2}(w_i)$ then $f^j\ P(w_i)\ \mu^*(w_i)$ (otherwise w_i does not block μ^*) but by coalitional substitutability and the ess-algorithm f^j does not prefer to hire w_i. Then either $S = \mu^{*2}(w_i)$ or we reach a contradiction. But if $f^j \neq \mu^*(w_i)$ then there is $w_j \in S$, such that $\mu^*(w_i)\ P(w_j)\ f^j$. This contradicts that $(f^j,\ S)$ blocks μ^*.

Then no $\left(f^j, S\right)$ can block, and thus $\mu^* \in \mathcal{C}(F, W, P)$.■

In the paper by Alcalde and Revilla [1], it is shown that a property, called Tops Responsiveness condition (TRC), weaker than essentiality, is sufficient for the non-emptiness of the core in the Coalition Formation Problem. In this framework, it is easy to show that a stable matching may not exist if we require TRC instead of essentiality. Note that the example 6 in the appendix also satisfies separability and coalitional substitutability.

5 Positive result: Common Best Colleague.

Sometimes people are concerned about their own colleagues rather than the firms which hire them in a more absolute way than GT indicates. Think of young researchers who have to choose among some research centers or institutes in which the labor conditions are very similar. The first question for most of them is: If I choose that institute, who are the researchers I can work with? That means that the researchers evaluate a matching in a *lexicographic* way. First, they consider the group of colleagues. And if they are indifferent, then they consider other features of the institute or center. In this section we consider the case in which the workers have $\mathcal{W}$-lexicographic preferences, as defined here.

The worker w_i's preferences are $\mathcal{W}$**-lexicographic** if, for all S, $T \in W^i$, $S \neq T$ and for all f^j, $f^h \in F$, the following condition is satisfied:

$$(f^j, S)P(w_i)(f^h, T) \Leftrightarrow SP_i^C T.$$

We need other conditions over preferences to ensure stability. Dutta and Massó [6] show that if the workers' preferences satisfy a condition called Unanimous Ranking According to Desirability, a stable matching may not exist in their framework. We define a new condition that if the workers' preferences restricted to colleagues satisfy, guarantee the existence of stable matchings. We call it Common best Colleague condition (CBC). A new result is obtained: If the firms' preferences consider the same ranking over workers then a stable matching exists. From now on, we assume that a common ranking over workers exists, and that the set of workers is ordered according to such a ranking. So, the subindex of each workers reflects her position in that ranking. In order to define CBC, we need additional notation.

Let $S, T \subseteq W$ be such that $S \neq T$. Suppose that a ranking over the workers denoted by the subindexes exists. We define Ψ^1 as follows:

$$\Psi^1(S,T) = \{w_i\}\text{; such that } w_i \in S \diagdown T\text{; and } i = \inf\{h : w_h \in S \diagdown T\}.$$

Similarly, we denote $\Psi^2(S,T) = \Psi^1(T,S)$. In words: Ψ^1 selects the worker from the first set with the lowest subindex that is not in the other set. Note that if $S \subset T$, $S \diagdown T = \emptyset$. Then $\Psi^1(S,T) = \emptyset$. And similarly for $\Psi^2(S,T)$ when $T \subset S$.

Then, we can compare two sets only by looking at the two workers with the lowest subindex from each set that are not in the other set. This allows us to present a property that explains how the agents can compare two sets of workers if a complete ranking over workers is commonly assumed.

In words: if w_i's preferences satisfy Common Best Colleague condition, w_i chooses between two groups of colleagues, S and T, that group containing the agent with the lower subindex who is not in the other set. And if S is a subset of T, w_i chooses T if there is any agent in $T \diagdown S$ who is below w_i in the ranking.[7]

A worker's (say w_i) preferences satisfy the **Common Best Colleague Condition (CBC)** if for all $S, T \in W^i$ and $S, T \neq \emptyset$:

1) If $S \diagdown T \neq \emptyset$, $T \diagdown S \neq \emptyset$, and $\Psi^1(S,T)$ has a lower subindex than $\Psi^2(S,T)$,

then $S \; P_i^C \; T$.

2) If $S \subset T$ and there is $w_k \in T \backslash S$ such that $k < i$, then $T \; P_i^C \; S$.

The following examples show the preferences of a worker when CBC is satisfied.

Example 3: Let $S = \{w_1, w_2, w_3, w_5\}$ and $T = \{w_1, w_3, w_4, w_6\}$. Let be a ranking denoted by the subindexes. Here, $S \diagdown T = \{w_2, w_5\}$ and $T \diagdown S = \{w_4, w_6\}$. In this case: $\Psi^1(S,T) = \{w_2\}$ and $\Psi^2(S,T) = \Psi^1(T,S) = \{w_4\}$, respectively. So, if every worker's preferences satisfies CBC, $S \; P_i^C \; T$, for each worker in $S \cup T$.♦

Example 4: Let $S = \{w_1, w_2, w_3, w_5\}$ and $T = \{w_1, w_3\}$. Here, $\Psi^1(S,T) = \{w_2\}$ and $\Psi^2(S,T) = \Psi^1(T,S) = \emptyset$. Assume that CBC is satisfied by the preferences of all the agents. So, $S \; P_3^C \; T$ because w_2 has a higher subindex than w_3. But it is possible that $T \; P_1^C \; S$, because w_2 has a lower position in the ranking than w_1.♦

We say that CBC is fulfilled in a coalition matching problem if every worker satisfies CBC with the same ranking. It can be shown that if the workers' preferences satisfy CBC and we only require Coalitional Substitutability for the firms' preferences a stable matching may not exist (Example 7 in the appendix).

However, a positive result can be obtained if the CBC requirements are extended to the firms' preferences. As CBC has been defined for workers we define CBC for

[7] Note that w_i only compares S and T if w_i belongs to both sets. Then w_i is not a member of $\Psi^1(S,T)$ or $\Psi^2(S,T)$.

firms.

A firm's (say f^j) preferences satisfy the **Common Best Colleague Condition (CBC)** if for all S, $T \subseteq W$:

If $S \diagdown T \neq \emptyset,\ T \diagdown S \neq \emptyset$, and $\Psi^1(S,T)$ has a lower subindex than $\Psi^2(S,T)$,

then $S\ P\left(f^j\right)\ T$.

If $S \subset T$ nothing is required for the firms' preferences.

In order to prove the existence of stable matchings we need an algorithm that selects a stable matching when it is applied. We call this algorithm, CBC algorithm. We can think of a real life situation in which a leader, a worker who obtains the first position in the common ranking, exists. The leader may be a researcher who proposes to other researchers form a research group, and that looks for a University or an Institute (we call it firm) that hires the whole group. If everyone (University and researchers) agree the contract is signed and these agents are retired from algorithm. Otherwise, the leader tries to form a new group. Each stage in the algorithm, reflecting the process in which each leader tries to form a group and find a center that hires them, has 3 steps.

To present the CBC algorithm we need additional notation. Let $\Phi_i(F,W)$ be the set of possible pairs of firm and set of coworkers for agent w_i when the set of firms is F and the set of workers is W. Formally, $\Phi_i(F,W) = \left\{\left(f^j, S\right) \in F \times W^i\right\} \cup \{w_i\}$. Then each element from $\Phi_i(F,W)$ is a pair of one firm and a set of workers that includes w_i or w_i remaining alone. For notational convenience we consider the last element as a pair $\left(f^0, w_i\right)$ where $f^0 = \emptyset$.

The **CBC algorithm**, is defined as follows:

Stage 1: Let $F_1 = F$ and $T_1 = W$.

Step 1: Let w_1, the worker with the lowest subindex in T_1, be the **leader**. Let $D_1^1 = \Phi_1(F_1, W_1)$.

Step 2: Let $\left(\widetilde{f}_1, \widetilde{S}_1\right) \in \Phi_1(F_1, W_1)$ be such that $\left(\widetilde{f}_1, \widetilde{S}_1\right) P(w_1) \left(f^j, T\right)$, for all $\left(f^j, T\right) \in F \times W^1$. So $\left(\widetilde{f}_1, \widetilde{S}_1\right)$ is the preferred pair from the set of possible pairs of firm and set of coworkers for the leader. From now on, $\left(\widetilde{f}_1, \widetilde{S}_1\right)$ is called the **proposal.**

Case a) If for all worker and firm included in the proposal, the proposal is preferred to remaining unmatched, then the set of workers $\widetilde{S}_1$ is matched with the firm $\widetilde{f}_1$. The remaining individuals and firms, $F_2 = F \diagdown \left\{\widetilde{f}_1\right\}$ and $T_2 = W \diagdown \widetilde{S}_1$, go to stage 2.[8]

Case b) If there is one agent included in the proposal for them this proposal is worse than remaining unmatched, then the proposal is rejected. The set of possible pairs for the leader is reduced in that element take $\Phi_1(F, W) \diagdown \left(\widetilde{f}_1, \widetilde{S}_1\right)$ as the new set of possible pairs for w_1 and a new round in step 2 begins.

Stage t: Let $\left(\widetilde{f}_{t-1}, \widetilde{S}_{t-1}\right)$ be the proposal accepted at stage $t-1$. Let $F_t = F_{t-1} \diagdown \left\{\widetilde{f}_{t-1}\right\}$ and $T_t = T_{t-1} \diagdown \widetilde{S}_{t-1}$.

Step 1: Let w_i, the worker with the lowest subindex in T_t, be the leader. Let $D_t^1 = \Phi_i(F_t, T_t)$.

Step 2: Let $\left(\widetilde{f}_t, \widetilde{S}_t\right)$ be the leader's most preferred pair from D_t^k.[9]

Case a) $\left(\widetilde{f}_t, \widetilde{S}_t\right) P(w_h) \{w_h\}$, for all $w_h \in \widetilde{S}_t$, and $\left(\widetilde{f}_t, \widetilde{S}_t\right) P(\widetilde{f}_t) \left\{\widetilde{f}_t\right\}$. Then $\mu\left(\widetilde{f}_t\right) = \widetilde{S}_t$. Let $F_{t+1} = F_t \diagdown \left\{\widetilde{f}_t\right\}$ and $T_{t+1} = T_t \diagdown \widetilde{S}_t$, if $F_{t+1} \neq \emptyset$ and $T_{t+1} \neq \emptyset$ go to the stage $t+1$.

Case b) $\left\{\widetilde{f}_t\right\} P(\widetilde{f}_t) \left(\widetilde{f}_t, \widetilde{S}_t\right)$ or there is a worker w_h such that $\{w_h\} P(w_h) \left(\widetilde{f}_t, \widetilde{S}_t\right)$. Then $D_t^{k+1} = D_t^k \diagdown \left(\widetilde{f}_t, \widetilde{S}_t\right)$ and repeat the Step 2 again.

The algorithm terminates when there are no remaining workers or firms. The remaining firms or workers are left unmatched.

The following example illustrates the algorithm.

[8] Note that $\left(\widetilde{f}, \widetilde{S}\right)$ may be $(\emptyset, w_1)$. In such a case, w_1 prefers to remain unemployed and is removed from the algorithm, and a new stage begins without her.

[9] The superindex k denotes the number of interactions for the Step 3 in each Stage t.

Example 5: Let $F = \{f^1,\ f^2,\ f^3\}$, and $W = \{w_1,\ w_2,\ w_3,\ w_4\}$, with preferences satisfying CBC and workers' preferences satisfying separability. Preferences are given by the following table:

w_1	w_2	w_3	w_4
$(f^1, \{w_1, w_2, w_3\})$	$(f^2, \{w_1, w_2\})$	$(f^1, \{w_1, w_2, w_3\})$	$(f^3, \{w_1, w_2, w_3, w_4\})$
$(f^2, \{w_1, w_2, w_3\})$	$(f^3, \{w_1, w_2\})$	$(f^2, \{w_1, w_2, w_3\})$	$(f^2, \{w_1, w_2, w_3, w_4\})$
$(f^3, \{w_1, w_2, w_3\})$	$(f^1, \{w_1, w_2\})$	$(f^3, \{w_1, w_2, w_3\})$	$(f^1, \{w_1, w_2, w_3, w_4\})$
$(f^1, \{w_1, w_2\})$	$\{w_2\}$	$(f^1, \{w_1, w_2, w_3, w_4\})$	$(f^3, \{w_1, w_2, w_4\})$
$(f^2, \{w_1, w_2\})$	$(f^2, \{w_1, w_2, w_3\})$	$(f^2, \{w_1, w_2, w_3, w_4\})$	$(f^2, \{w_1, w_2, w_4\})$
$(f^3, \{w_1, w_2\})$	$(f^3, \{w_1, w_2, w_3\})$	$(f^3, \{w_1, w_2, w_3, w_4\})$	$(f^1, \{w_1, w_2, w_4\})$
$(f^1, \{w_1, w_3\})$	$(f^1, \{w_1, w_2, w_3\})$	$(f^1, \{w_1, w_3\})$	$(f^3, \{w_1, w_3, w_4\})$
$(f^2, \{w_1, w_3\})$	$(f^2, \{w_1, w_2, w_3, w_4\})$	$(f^2, \{w_1, w_3\})$	$(f^2, \{w_1, w_3, w_4\})$
$(f^3, \{w_1, w_3\})$	$(f^3, \{w_1, w_2, w_3, w_4\})$	$(f^3, \{w_1, w_3\})$	$(f^1, \{w_1, w_3, w_4\})$
$\{w_1\}$	$(f^1, \{w_1, w_2, w_3, w_4\})$	$(f^1, \{w_1, w_3, w_4\})$	$(f^3, \{w_1, w_4\})$
.	$(f^2, \{w_1, w_2, w_4\})$	$(f^2, \{w_1, w_3, w_4\})$	$(f^2, \{w_1, w_4\})$
.	$(f^3, \{w_1, w_2, w_4\})$	$(f^3, \{w_1, w_3, w_4\})$	$(f^1, \{w_1, w_4\})$
.	$(f^1, \{w_1, w_2, w_4\})$	$(f^1, \{w_2, w_3\})$	$(f^3, \{w_2, w_3, w_4\})$
	$(f^2, \{w_2\})$	$(f^2, \{w_2, w_3\})$	$(f^2, \{w_2, w_3, w_4\})$
	.	$(f^3, \{w_2, w_3\})$	$(f^1, \{w_2, w_3, w_4\})$
	.	$(f^1, \{w_2, w_3, w_4\})$	$(f^3, \{w_2, w_4\})$
	.	$(f^2, \{w_2, w_3, w_4\})$	$(f^2, \{w_2, w_4\})$
		$(f^3, \{w_2, w_3, w_4\})$	$(f^1, \{w_2, w_4\})$
		$(f^1, \{w_3, w_4\})$	$(f^3, \{w_3, w_4\})$
		$(f^2, \{w_3, w_4\})$	$(f^2, \{w_3, w_4\})$
		$(f^3, \{w_3, w_4\})$	$(f^1, \{w_3, w_4\})$
		$(f^1, \{w_3\})$	$(f^3, \{w_4\})$
		$(f^2, \{w_3\})$	$(f^2, \{w_4\})$
		$(f^3, \{w_3\})$	$(f^1, \{w_4\})$
		$\{w_3\}$	$\{w_4\}$
		.	.
		.	.

f^1	f^2	f^3
$\{w_1, w_2, w_3\}$	$\{w_1, w_2\}$	$\{w_1\}$
$\{f^1\}$	$\{w_1\}$	$\{w_1, w_2\}$
$\{w_1, w_2\}$	$\{f^2\}$	$\{w_1, w_3\}$
.	.	$\{w_1, w_2, w_3\}$
.	.	$\{w_1, w_4\}$
.	.	$\{w_1, w_2, w_4\}$
		$\{w_2\}$
		$\{w_3\}$
		$\{w_2, w_3\}$
		$\{w_4\}$
		$\{w_2, w_4\}$
		$\{w_1, w_2, w_3, w_4\}$
		$\{f^3\}$
		$\{w_3, w_4\}$
		$\{w_2, w_3, w_4\}$

The CBC algorithm works as follows:

Stage 1: Let $T_1 = W$ and $F_1 = F$.

Step 1: The leader is w_1. The possible pairs set for w_1 is $D_1^1 = (F \times W^1) \cup \{w_1\}$.

Step 2: **First round:** w_1 proposes $\left(\widetilde{f}_1, \widetilde{S}_1\right) = \left(f^1, \{w_1, w_2, w_3\}\right) \in D_1^1$. The proposal is IR for w_3 and for f^1, but not for w_2. It is rejected. Let $D_1^2 = D_1^1 \diagdown \left\{\left(f^1, \{w_1, w_2, w_3\}\right)\right\}$.

Second round: w_1 proposes $\left(\widetilde{f}_1, \widetilde{S}_1\right) = \left(f^2, \{w_1, w_2, w_3\}\right) \in D_1^2$. The proposal is IR for w_3, but not for w_2 and f^2. It is rejected. Let $D_1^3 = D_1^2 \diagdown \left\{\left(f^2, \{w_1, w_2, w_3\}\right)\right\}$.

Third round: w_1 proposes $\left(\widetilde{f}_1, \widetilde{S}_1\right) = \left(f^3, \{w_1, w_2, w_3\}\right) \in D_1^3$. The proposal is IR for w_3 and f^3, but not for w_2. It is rejected. Let $D_1^4 = D_1^3 \diagdown \left\{\left(f^3, \{w_1, w_2, w_3\}\right)\right\}$.

Fourth round: w_1 proposes $\left(\widetilde{f}_1, \widetilde{S}_1\right) = \left(f^1, \{w_1, w_2\}\right) \in D_1^4$. The proposal is IR for w_2, but not for f^1. It is rejected. Let $D_1^5 = D_1^4 \diagdown \left\{\left(f^1, \{w_1, w_2\}\right)\right\}$.

Fifth round: w_1 proposes $\left(\widetilde{f}_1, \widetilde{S}_1\right) = \left(f^2, \{w_1, w_2\}\right) \in D_1^5$. The proposal is IR for w_2 and f^2. It is accepted. Then $\mu\left(f^2\right) = \{w_1, w_2\}$. The remaining agents sets are $F_2 = F_1 \diagdown \left\{f^2\right\} = \left\{f^1, f^3\right\}$ and $T_2 = T_1 \diagdown \{w_1, w_2\} = \{w_3, w_4\}$.

Stage 2: Let $F_2 = \left\{f^1, f^3\right\}$ and $T_2 = \{w_3, w_4\}$.

Step 1: The leader is w_3. The possible pairs set for w_3 is $D_2^1 = \Phi_3(F_2, T_2) = \left\{\left(f^1, \{w_3, w_4\}\right), \left(f^3, \{w_3, w_4\}\right), \left(f^1, \{w_3\}\right), \left(f^3, \{w_3\}\right), \left(f^0, \{w_3\}\right)\right\}$.

Step 2: **First round:** w_3 proposes $\left(f^1, \{w_3, w_4\}\right) \in D_2^1$. The proposal is IR for w_4, but not for f^1. It is rejected. Let $D_2^2 = D_2^1 \diagdown \left\{\left(f^1, \{w_3, w_4\}\right)\right\} = \left\{\left(f^3, \{w_3, w_4\}\right), \left(f^1, \{w_3\}\right), \left(f^3, \{w_3\}\right), \left(f^0, \{w_3\}\right)\right\}$.

Second round: w_3 proposes $\left(f^3, \{w_3, w_4\}\right) \in D_2^2$. The proposal is not IR for f^3. It is rejected. Let $D_2^3 = D_2^2 \diagdown \left\{\left(f^3, \{w_3, w_4\}\right)\right\} = = \left\{\left(f^1, \{w_3\}\right), \left(f^3, \{w_3\}\right), \left(f^0, \{w_3\}\right)\right\}$.

Third round: w_3 proposes $\left(f^1, \{w_3\}\right) \in D_2^3$. The proposal is not IR for f^1. It is rejected. Let $D_2^4 = D_2^3 \diagdown \left\{\left(f^1, \{w_3\}\right)\right\} = \left\{\left(f^3, \{w_3\}\right), \left(f^0, \{w_3\}\right)\right\}$.

Fourth round: w_3 proposes $\left(f^3, \{w_3\}\right) \in D_2^4$. The proposal is IR for f^3. It is accepted. Then $\mu\left(f^3\right) = \{w_3\}$. The remaining agents sets are $F_3 = F_2 \diagdown \left\{f^3\right\} = \left\{f^1\right\}$ and $T_3 = T_2 \diagdown \{w_3\} = \{w_4\}$.

Stage 3: Let $F_3 = \left\{f^1\right\}$ and $T_3 = \{w_4\}$.

Step 1: The leader is w_4. The possible pairs set for w_4 is $D_3^1 = \Phi_4(F_3, T_3) = \left\{\left(f^1, \{w_4\}\right), \left(f^0, \{w_4\}\right)\right\}$.

Step 2: w_4 proposes $\left(f^1, \{w_4\}\right) \in D_3^1$. The proposal is not IR for f^1. It is rejected. Let $D_3^2 = D_3^1 \diagdown \left\{\left(f^1, \{w_4\}\right)\right\} = \left(f^0, \{w_4\}\right)$. Then the leader at this moment, w_4, must be excluded. $\mu(w_4) = \{w_4\}$. And $F_4 = F_3$, $T_4 = T_3 \diagdown \{w_4\} = \emptyset$.

As the set of remaining workers is empty, the algorithm terminates in the previous stage. The remaining firms are unmatched. $\mu\left(f^1\right) = \left\{f^1\right\}$.

The matching is:

$$\mu\left(f^1\right) = \left\{f^1\right\}.$$

$$\mu\left(f^2\right) = \{w_1, w_2\}.$$

$$\mu\left(f^3\right) = \{w_3\}.$$

$$\mu(w_4) = \{w_4\}.$$

It is stable.♦

In order to prove the Theorem 2 we need the following lemmas.

Lemma 3: *The CBC algorithm always terminates in a finite number of stages. And a matching is obtained.*

As the number of workers and firms is finite, the number of possible pairs for each worker is finite too. If every IR matching is rejected by a worker, the unemployment alternative appears and it is accepted. Then each leader is assigned in a finite number of steps. As the number of workers is finite the algorithm terminates in at most n stages. As the workers are assigned to one firm or left alone, a matching is obtained.

Lemma 4: *The CBC algorithm selects an IR matching.*

Every pair of firm and coworkers, is assigned only if it is acceptable for everyone. Then the matching is IR.

Lemma 5: *There is no group of workers and firms that blocks the matchings obtained in the CBC algorithm.*

Proof. The algorithm assigns to w_1 her best pair (say $\left(f^j, S\right)$) that is acceptable for every agent included in $\left(f^j, S\right)$. Then w_1 cannot be in a blocking group. Any individual that is matched together with w_1, as his preferences satisfy CBC, does not prefer a matching without w_1. There exists an exception: to be unemployed. But this means that the matching assigned by the algorithm is not IR, and the previous lemma excludes that possibility. The firm matched with w_1, as its preferences satisfy CBC, cannot prefer a group of workers without w_1. Except if the preferred group is a subset of the group that is matched to the firm. But, in such a case, as we said before, the workers in the subset prefer to be with w_1. Then every agent that goes out from the algorithm in the first stage, does not block. As the remaining agents only can form a blocking group among them, the same argument applies to the matching problem defined in the second stage. Then there is no group that can improve the matching for every member.

The conclusions of the previous lemmas allow us to present the next Theorem, whose proof is straightforward from the above results.

Theorem 2 *If workers' and firms' preferences satisfy CBC with the same ranking and the workers' preferences are $\mathcal{W}$-lexicographic then a stable matching always exists.*

6 Conclusions

We have presented the Coalition Matching problems as a combination of two well known models: many-to-one matching models and hedonic coalition formation problems. The appropriate extensions of sufficient conditions over the preferences' domain that guarantees the existence of stable matchings in such models are not enough in this model. However there exists some sufficient conditions that have been shown in the sections 4 and 5. These positive results, although limited, can be understood as the description of the preferences in particular real-life situations.

A Appendix

We present some examples without stable matchings that satisfy some conditions over the agents' preferences.

First, it is easy to show that a stable matching may not exist if we require Tops Responsiveness instead of Essentiality. Note that this example 6 also satisfies Separability and Coalitional Substitutability.

Example 6: Suppose $F = \{f^1, f^2\}$ and $W = \{w_1, w_2, w_3\}$. Let P be the preference profile given by the following table:

f^1	f^2
$\{w_1, w_2\}$	$\{w_1, w_3\}$
$\{w_1, w_3\}$	$\{w_2, w_3\}$
$\{w_2, w_3\}$	$\{w_1, w_2\}$
$\{w_1\}$	$\{w_1\}$
$\{w_2\}$	$\{w_2\}$
$\{w_3\}$	$\{w_3\}$

$w_1 : P_1^C$	$w_2 : P_2^C$	$w_3 : P_3^C$
$\{w_1, w_2\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$
$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_3\}$
$\{w_1, w_3\}$	$\{w_1, w_2\}$	$\{w_2, w_3\}$
$\{w_1\}$	$\{w_2\}$	$\{w_3\}$

for all $w_i \in \{w_1, w_2, w_3\}$ $f^1\ P_i^F\ f^2$.

Claim: There is no stable matching in such a problem.

We can check every possible matching:

The trivial solution in which no worker is hired: It is not stable because any firm wants to hire any worker and this worker would accept. If some firm f^i hires $\{w_1, w_2, w_3\}$: then it is not IR for the firm.

If some firm f^i hires only one worker: both, worker and firm, prefer that another worker will be hired.

If f^1 hires $\{w_1, w_2\}$: $\{f^2, w_2, w_3\}$ blocks.

If f^1 hires $\{w_1, w_3\}$: $\{f^1, w_1, w_2\}$ blocks.

If f^1 hires $\{w_2, w_3\}$: $\{f^1, w_1, w_3\}$ blocks.

If f^2 hires $\{w_2, w_3\}$: $\{f^2, w_1, w_3\}$ and $\{f^1, w_1, w_3\}$ block.

If f^2 hires $\{w_1, w_3\}$: $\left\{f^1, w_1, w_3\right\}$ and $\left\{f^1, w_1, w_2\right\}$ block.
If f^2 hires $\{w_1, w_2\}$: $\left\{f^1, w_1, w_2\right\}$ blocks.
So there is no stable matching in this problem.[10]♦

Second, in the section 5 we say that if the workers' preferences satisfy CBC and we only require Coalitional Substitutability for the firms' preferences a stable matching may not exist. The following example points out this fact.

Example 7: Consider this coalition-matching problem: $F = \left\{f^1, f^2, f^3\right\}$ and $W = \{w_1, w_2, w_3\}$. Let P be the preferences profile given by the following table:

f^1	f^2	f^3
$\{w_3\}$	$\{w_1, w_3\}$	$\{w_1\}$
$\{w_1, w_2\}$	$\{w_2, w_3\}$	$\left\{f^3\right\}$
$\{w_1\}$	$\{w_1\}$	
$\{w_2\}$	$\{w_2\}$	
$\left\{f^1\right\}$	$\{w_3\}$	
	$\left\{f^2\right\}$	

$$w_1: \begin{cases} \left\{f^1\right\} \ P_1^F \ \left\{f^3\right\} \ P_1^F \ \left\{f^2\right\}. \\ \\ \{w_1, w_2\} \ P_1^C \ \{w_1\} \ P_1^C \ \{w_1, w_3\} \ P_1^C \ \{w_1, w_2, w_3\}. \end{cases}$$

$$w_2: \begin{cases} \left\{f^1\right\} \ P_2^F \ \emptyset. \\ \\ \{w_1, w_2\} \ P_2^C \ \{w_1, w_2, w_3\} \ P_2^C \ \{w_2, w_3\} \ P_2^C \ \{w_2\}. \end{cases}$$

$$w_3: \begin{cases} \left\{f^1\right\} \ P_3^F \ \left\{f^2\right\} \ P_3^F \ \left\{f^3\right\}. \\ \\ \{w_1, w_2, w_3\} \ P_3^C \ \{w_1, w_3\} \ P_3^C \ \{w_2, w_3\} \ P_3^C \ \{w_3\}. \end{cases}$$

The workers' preferences satisfy CBC and GT. The firms' preferences only satisfy Coalitional Substitutability.

<u>Claim:</u> There is no stable matching. It is easy to check that the three workers cannot be in the same firm because w_1 would prefer to work alone and f^3 will always hire her. Each worker working for a different firm is a matching that would be blocked

[10] The preference ordering for the agents would be, for w_1:

$$\left\{f^1, w_1, w_2\right\} P_1 \left\{f^2, w_1, w_2\right\} P_1 \left\{f^1, w_1, w_2, w_3\right\} P_1 \left\{f^2, w_1, w_2, w_3\right\} P_1$$
$$P_1 \left\{f^1, w_1, w_3\right\} P_1 \left\{f^2, w_1, w_3\right\} P_1 \left\{f^1, w_1\right\} P_1 \left\{f^2, w_1\right\} P_1 ...$$

Analogously for w_2 and w_3.

by $\{f^1, w_1, w_2\}$.

Only matchings with two workers hired by a firm are allowed. There exists only three kinds of IR matchings that fulfill that condition. And the possible blocking groups for each type are:

all matching that includes:	are blocked by
$\{f^1, w_1, w_2\}$	$\{f^1, w_3\}$
$\{f^2, w_1, w_3\}$	$\{f^1, w_1, w_2\}$ and $\{f^3, w_1\}$
$\{f^2, w_2, w_3\}$	$\{f^1, w_1, w_2\}$

♦

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