Many-to-One Matching without Substitutability

Marek Pycia[1]
MIT

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Abstract

This paper studies many-to-one matching such as matching between students and colleges, interns and hospitals, and workers and firms. A major question that arises in such settings is the stability of matchings. A matching is stable if no agent or pair of agents can profitably deviate. The paper provides a novel sufficient and, in a certain sense, necessary condition for stability that may be used even when there are complementarities and peer effects. The condition is particularly suited to study settings in which agents are unable to enter binding agreements. In these settings, the agents are matched and then their payoffs are determined via mechanisms such as various games, bargaining, and sharing rules. A stable matching exists for all preference profiles induced by the mechanisms if, and only if, the preferences are pairwise aligned. Agents' preferences are pairwise aligned if any two agents in the intersection of any two coalitions prefer the same one of the two coalitions. For example, a stable matching exists if agents' payoffs are determined after the matching in Nash bargaining.

[1]Department of Economics, Massachusetts Institute of Technology, E52-391, 50 Memorial Dr, Cambridge MA 02142. Email: pycia@mit.edu. I would like to thank Bengt Holmström, Glenn Ellison, Robert Gibbons, Haluk Ergin, Sergei Izmalkov, Abhijit Banerjee, Michael Piore, and Jean Tirole for their advice and support. For their comments, I am also grateful to Hannu Vartiainen, Birger Wernerfelt, Catherine Tucker, Muhamet Yildiz, David McAdams, and Eric Van den Steen as well as the participants of seminars at Harvard, MIT, Penn State, UCL, and UCLA. Special thanks are due to William Hawkins, Guy Michaels, Zaki Wahhaj, and, in particular, Anna Myjak-Pycia. This project was supported in part by grants from the MIT Industrial Performance Center and Leon Koźmiński Academy of Entrepreneurship and Management in Warsaw.

1. Introduction

This paper studies many-to-one matching such as matching between students and colleges, interns and hospitals, and workers and firms.[1] An agent on one side, say a firm, can hire as many workers as it needs, and an agent on the other side, a worker, can be employed by one firm only or remain unemployed. In this way, the agents form coalitions. The class of feasible coalitions is exogenously given. An unemployed worker is considered a coalition. All other coalitions consist of a firm and its workforce.

Gale and Shapley (1962) raised the question of stability of such matchings.[2] Each agent has preferences over the coalitions that contain this agent. A matching is stable if (i) no worker prefers to be unemployed rather than to work for the matched firm, (ii) no firm wants to keep some positions vacant rather than filling them with a group of matched workers, and (iii) no worker-firm pair that is presently unmatched prefers to match.

The most general known sufficient conditions for stability are substitutability conditions, which are derived from the Kelso and Crawford (1982) gross-substitutes condition.[3] In a formulation of Roth and Sotomayor (1990), the substitutability condition is as follows: if a firm wants to employ a worker w from a large pool of workers, then the firm wants to employ w from any smaller pool containing w. Kelso and Crawford (1982) show that if firms' preferences satisfy the substitutability condition and there are no peer effects – that is, workers' preferences depend only on the firm they apply to and not on who their peers will be – then there exists a stable many-to-one matching.

There are matching settings that do not satisfy the standard assumptions of substitutability and lack of peer effects. The substitutability condition fails if there are

[1] The college admission problem was introduced by Gale and Shapley (1962). A recent example from the realm of education is the design of a new high school admissions system in New York City, which allows both schools and students to influence the matching (Abdulkadiroğlu, Pathak, and Roth 2005). Medical labor markets are studied for example in Roth (1984), Roth (1991), Roth and Peranson (1999), Niederle and Roth (2004), and McKinney, Niederle, and Roth (forthcoming). Roth (2002) provides a survey. Roth and Sotomayor (1990) is a classic survey of theory, empirical evidence, and design applications of the many-to-one matching models satisfying the above assumption.

[2] Starting with the work of Roth (1984) on US matching between interns and hospitals, substantial empirical evidence links the lack of stability in matching with market failures. The evidence is surveyed in Roth and Sotomayor (1990) and Roth (2002).

[3] Cf. Roth and Sotomayor (1990), Echenique and Oviedo (2004b), Hatfield and Milgrom (2005), and Ostrovsky (2005). Roth (1985)'s responsiveness condition is also a variant of substitutability.

non-trivial complementarities between workers. It also fails when there are fixed costs. The complementarities are non-trivial if, for example, a firm's production process is profitable only when adequately staffed. For instance, a biotech firm may not open a new R&D lab if it is unable to hire experts in all complementary areas required for the lab's work. Substitutability fails for firms with fixed costs if their operations must be of some minimal size to ensure profitability. Peer effects are present if workers care about interactions in the workplace or if the identity of other workers non-trivially influences workloads or other day-to-day bargaining between workers.

This paper provides a novel sufficient and, in a certain sense, necessary condition for stability that may be used to analyze settings with complementarities and peer effects such as those mentioned above. The paper also shows that the condition is satisfied in several settings of economic importance that have not previously been recognized as admitting stable matchings.

The main component of the proposed condition is the pairwise alignment of preferences. Agents' preferences are *pairwise aligned* if any two agents in the intersection of any two coalitions prefer the same one of the two coalitions. For instance, a firm and a worker either both prefer to form a firm-and-one-employee coalition or both prefer a larger coalition that includes the firm, the worker, and some other workers.

The sufficient and, in a certain sense, necessary condition is developed in three stages, from specific to more general environments. Stage 1 (Section 2) presents an example of matching with payoffs determined by Nash bargaining. Stage 2 (Section 4) generalizes this example by replacing Nash bargaining with a broad class of mechanisms. This intermediate stage is of independent interest as directly applicable to a range of matching situations in which agents are unable to enter binding agreements. Stage 3 (Section 5) addresses the general problem with agents' preferences as primitives.

The setting of Stage 1 (Section 2) is as follows. There are two dates. On date 1, firms and workers match, that is, form coalitions. On this date, firms and workers cannot enter binding employment contracts. In effect, on date 1, the agents' preferences over coalitions result from the agents' expectations of the payoffs that will be determined on date 2. On date 2, each coalition creates a value and its members divide the value according to the Nash bargaining solution. This bargaining determines the agents' payoffs. Since each preference profile induced by Nash bargaining is pairwise aligned, the pairwise alignment condition is embedded in this setting.

Stage 1 (Section 2) shows that there is a stable matching in this setting.[4] This

[4]In this and other settings discussed, there exists a matching that is group stable and not only

stage also proves a stronger property of this matching setting, namely the existence of a metaranking. A metaranking is a transitive relation on all coalitions; its defining property is that, restricted to coalitions containing an agent, the transitive relation agrees with preferences of this agent.[5]

Stage 2 (Section 4) discusses matching when payoffs are determined by mechanisms. This setting preserves the timing and other elements of the setting from Stage 1, except that Nash bargaining is replaced by a mechanism from a broad class of games, bargaining protocols, and sharing rules. As in the setting of Stage 1, each coalition has a value. The mechanism takes the values of coalitions, that is the value function, and generates agents' payoffs and preferences over coalitions.

Stage 2 (Section 4) establishes a sufficient and, in a certain sense, necessary condition for stability. It is sufficient for the existence of a stable matching that agents' preferences are pairwise aligned for all value functions. It is necessary for the existence of a stable matching for all value functions that agents' preferences are pairwise aligned.

Stage 3 (Section 5) addresses the general problem with agents' preferences as primitives. At this stage, in contrast to Stage 2, there are no mechanisms. The sufficient condition imposes pairwise alignment on agents' preferences from a rich domain of preference profiles as it is not sufficient for stability to impose pairwise alignment on a single preference profile.[6] An example of a matching situation with pairwise-aligned preferences and no stable matching is included in Section 4 to explain why we need mechanisms.[7]

In the general preference framework of Stage 3 (Section 5), the pairwise alignment remains a necessary condition for the existence of stable matchings for all preference profiles from large domains of profiles.

The sufficiency and necessity results proved in this paper allow one to determine which sharing rules and games induce the existence of stable matchings. For instance,

stable. A matching is group stable if no worker prefers to be unemployed rather than to work for the matched firm, and no firm may replace some (or no) workers, with some (or no) additional workers so that the firm and all the additional workers strictly increase their payoffs.

[5]The idea of metarankings was introduced by Farrell and Scotchmer (1988). See the following discussion of literature.

[6]Section 5 also discusses the sufficient condition in a form in which the condition does not refer to a rich domain of preference profiles.

[7]As a heuristic argument consider the roommate problem, in which agents match in pairs, and any two agents may form a pair. Preferences are always pairwise aligned, but the existence of a stable matching is not assured.

Section 6 determines the class of linear sharing rules and the class of welfare maximization mechanisms that induce the existence of stable matchings. Section 6 also shows that there is always a stable matching if agents' preferences are induced by Tullock's (1980) rent-seeking game.

The idea of using pairwise alignment to study stability seems to be new. As discussed above, the paper proves that the pairwise alignment is related to the idea of a metaranking introduced by Farrell and Scotchmer (1988).[8] Farrell and Scotchmer primarily study the formation of partnerships. They show that the one-sided core is non-empty in a coalition formation game followed by an equal division of value. Banerjee, Konishi, and Sönmez (2001) relax the Farrell and Scotchmer metaranking property[9] and notice that the equal division may be replaced by some other linear sharing rules in Farrell and Scotchmer's analysis. Echenique and Yenmez (2005) use metarankings to analyze the one-sided core of many-to-one matching. They construct an algorithm that finds matchings in the one-sided core if they exist. This algorithm does not rely on either substitutability or the lack of peer effects.[10] They also verify that their algorithm efficiently finds matchings in the one-sided core if the Banerjee, Konishi, and Sönmez (2001) metaranking-type property is satisfied.

As a companion paper, Pycia (2005) studies the relation among pairwise alignment, metarankings, and coalition formation. The results on stability presented here are independent of the results of the companion paper because this paper studies many-to-one matching, while the companion paper studies one-sided coalition formation. The two papers employ independent solution concepts. This paper studies stability, while the companion paper studies the one-sided core.[11] The papers provide pairwise-alignment-based sufficient and necessary conditions for stability, and non-emptiness of the one-sided core, respectively. The conditions, however, are different.

[8] If a metaranking exists, then preferences are pairwise aligned. The converse implication is true in the special case studied in Section 4 but not in the general setting of Section 5.

[9] The relaxed metaranking property, called the top coalition property, says that each subgroup of agents contains a coalition that is weakly preferred by all its members to any other coalition of agents in the subgroup.

[10] Echenique and Oviedo (2004a) construct an algorithm that finds group stable matchings in many-to-one settings if they exist and if there are no peer effects.

[11] The main difference between these two concepts is that stability presumes that a firm can sever or establish a match with a worker without taking into account the preferences of other workers the firm matches with. The one-sided core presumes that the workers have veto power over the actions of the firm. Consequently, a many-to-one matching in the one-sided core need not be stable, and a stable matching need not belong to the one-sided core. For details, see the discussion in Section 3.

The paper proceeds as follows. Section 2 presents the Nash bargaining example. Section 3 introduces the model. Section 4 presents the theory of stability in matching with mechanisms. Section 5 presents the preference formulation of the results. Section 6 presents new settings in which stable matchings exist. The last section concludes.

2. Matching and Nash Bargaining – an Example

Let us consider the following many-to-one matching situation. On date 1, firms and workers match, that is, form coalitions. On this date, firms and workers cannot enter binding employment contracts. In effect, on date 1, the agents' preferences over coalitions reflect the agents' expectations of the payoffs that will be determined on date 2. On date 2, each resultant coalition, C, creates value $v(C) \geq 0$, and its members divide $v(C)$ according to the Nash bargaining solution. That is, each agent i is endowed with an increasing and concave utility function U_i, and agents' payoffs s_i maximize

$$\max_{s_i \geq 0, i \in C} \prod_{i \in C} \left(U_i(s_i) - U_i(0)\right)$$

subject to

$$\sum_{i \in C} s_i \leq v(C).$$

Thus, agents' preferences over coalitions are induced by Nash bargaining.

Recall that a matching is stable if no worker prefers to be unemployed rather than to work for the matched firm, no firm wants to lay off any group of its workers, and no worker-firm pair that is presently unmatched would prefer to match.[12]

Theorem 2.1. If preferences during matching are induced by Nash bargaining, then there exists a stable matching.

Proof. To construct a stable matching, let us first observe that $\frac{U_i(s_i) - U_i(0)}{U_i'(s_i)}$, called the fear of ruin coefficient,[13] is the same for every agent in a coalition that divides value in Nash bargaining. Indeed, the Lagrange multiplier in the Nash bargaining maximization equals the inverse of the fear of ruin, $\frac{U_i'(s_i)}{U_i(s_i) - U_i(0)}$. Additionally, the larger the fear of ruin

[12] The formal definition is presented in Section 3.

[13] See Aumann and Kurz (1977a, 1977b) and Roth (1979).

of an agent is, the more the agent gains in a given coalition. Thus, no agents would ever want to change a coalition that maximizes their fear of ruin. Therefore, the coalition with maximal fear of ruin may be treated as if its members did not participate in the matching between the remaining agents. In this way, one can recursively construct a stable matching. This completes the proof.[14]

The above proof may be separated into two steps. The first step constructs an index on coalitions — the fear of ruin – such that all relevant agents compare two coalitions by looking at this index only. The second step uses the index to recursively construct a stable matching.

The idea for the second step comes from Farrell and Scotchmer (1988). They study partnerships that share the surplus equally among their members. That is, if a partnership of size $\#C$ creates value $v(C)$, then each member obtains $\frac{v(C)}{\#C}$. They use the index $\frac{v(C)}{\#C}$ to recursively construct a partnership structure that belongs to the one-sided core. Except for the difference in solution concepts, their use of the index $\frac{v(C)}{\#C}$ is the same as our use of the fear of ruin in the second step of the above proof.

The above two indices, the fear of ruin and $\frac{v(C)}{\#C}$, determine metarankings. A metaranking is a transitive relation on all coalitions that, restricted to coalitions containing any particular agent, agrees with preferences of this agent. As in the above proof, if there is a metaranking, then there is a matching that is stable.

The existence of a metaranking is a strong and desirable property of a matching setting. For instance, Proposition 4.11 in the appendix shows that if there is a metaranking,

[14] Three remarks about the Nash bargaining example might be of interest. The above argument, with a small modification, may be used to show that a stable matching exists when preferences come from an asymmetric Nash bargaining where agent i has bargaining power λ_i and the division of value $v(C)$ in coalition C maximizes $\prod_{i \in C} (U_i(s_i) - U_i(0))^{\lambda_i}$ over $s_i \geq 0$, $i \in C$, subject to $\sum_{i \in C} s_i \leq v(C)$. In this extension, the bargaining powers λ_i are agent-specific but are not coalition-specific.

Furthermore, the above argument shows that the matching is group stable and not only stable. A formal definition definition of group stability is given in Section 3. Informally, a matching is group stable if no worker prefers to be unemployed rather than to work for the matched firm, and no firm may fire some (or no) workers, and employ some (or no) additional workers so that the firm and all the additional workers strictly increase their payoffs.

Finally, the values $v(C)$ may either accrue to the entire coalition or be composed of parts that accrue to individual members. In the latter case, the existence of a stable matching relies on the assumptions that agents' utilities are quasi-linear in a numeraire, and that, after the coalitions are determined, the agents can contract. Then, $v(C)$ is the sum of values that accrue to members in an optimal contract.

then group stable matchings are obtained as Strong Nash Equilibria[15] of a broad class of non-cooperative matching games.

Despite the attractiveness of the existence of metarankings as a property of matching situations, it is difficult to use metarankings as a sufficient condition for stability. To use metarankings to verify stability requires one to construct an index – such as the fear of ruin index above – for each matching setting.

Sections 4 and 5 solve this problem by connecting the existence of metarankings with the pairwise alignment, which is readily verifiable in a variety of settings.[16] For instance, in Nash bargaining, the pairwise alignment is an immediate consequence of the Independence of Irrelevant Alternatives axiom.[17]

3. Model

A finite set of agents I is divided into two non-empty disjoint sets, $I = F \cup W$. We will refer to agents from F as firms, and to agents from W as workers. Each worker seeks a firm, and each firm $f \in F$ seeks up to M_f workers, where $M_f \geq 1$. A matching is a function μ from $F \cup W$ into subsets of $F \cup W$, such that

- $\mu(w) = \{f\}$ if the worker w is employed by the firm f, and $\mu(w) = \{w\}$ if w is unemployed,
- $\mu(f) \subset W$ and the size $\#\mu(f) \leq M_f$ for every firm f, and
- $\mu(w) = \{f\}$ iff $w \in \mu(f)$, for every worker w and firm f.

Let us use the term coalition to refer to a firm f and all workers matched to f in some matching, or to refer to an unemployed worker. Thus, a coalition may consist of a firm f and any subset of workers $S \subseteq W$ of size $\#S \leq M_f$ (including $S = \emptyset$) or of an unemployed worker. Let us denote the set of all coalitions by $\mathcal{C}$. Thus,

$$\mathcal{C} = \{\{f\} \cup S : f \in F, S \subseteq W, \#S \leq M_f\} \cup \{\{w\} : w \in W\}.$$

[15] Cf. Aumann (1959), Rubinstein (1980). We may alternatively use the solution concept of Coalition-Proof Nash Equilibrium of Bernheim, Peleg, Whinston (1987).

[16] Section 5 also defines relaxed metarankings and study their connection to stability and pairwise alignment. Relaxed metarankings, unlike metarankings, always exist in one-to-one matching.

[17] Cf. Harsanyi (1959).

Note that there is a one-to-one correspondence between matchings and partitions of I into coalitions. In particular, in any matching each agent is associated with exactly one coalition.

Each agent $i \in I$ has a preference relation $\precsim_i$ over all coalitions that contain i. The profile of preferences $(\precsim_i)_{i \in I}$ is denoted by $\precsim_I$. This formulation embodies the standard assumption that each agent's preferences between two matchings are fully determined by members of the coalitions containing this agent in the two matchings.

We are interested in the existence of stable matchings in the above environment. The role of stability – most notably in preventing the unravelling of markets – has been elucidated in the empirical work started by Roth (1984). In the following definitions of stability and group stability, $C^{\mu}(i)$ denotes the coalition containing an agent i in matching μ. Specifically, the coalition containing a firm f is $C^{\mu}(f) = \{f\} \cup \mu(f)$, and the coalition containing a worker w is $C^{\mu}(w) = \mu(w) \cup \mu(\mu(w))$.

Definition 3.1 (Stability).[18] A matching μ is blocked by a firm f if there exists a subset of workers $S \subsetneq \mu(f)$ such that $\{f\} \cup S \succ_f C^{\mu}(f)$.

A matching μ is blocked by a worker w if $\{w\} \succ_w C^{\mu}(w)$.

A matching μ is blocked by firm f and worker $w \notin \mu(f)$ if there exists $S \subseteq \mu(f)$ such that

- $\#(\{w\} \cup S) \leq M_f$,
- $\{f\} \cup \{w\} \cup S \succ_f C^{\mu}(f)$, and
- $\{f\} \cup \{w\} \cup S \succ_w C^{\mu}(w)$.

A matching is **stable** if it is not blocked by any individual agent or any worker-firm pair.

Definition 3.2 (Group Stability).[19] A matching μ is blocked by a group of workers and firms if there exists another matching μ' and a group A consisting of multiple workers and/or firms, such that for all workers w in A and for all firms f in A,

- $\mu'(w) \in A$ (i.e., every student in A is matched to a college in A);

[18] Cf. Roth and Sotomayor (1990) Definition 5.3.
[19] Cf. Roth and Sotomayor (1990) Definition 5.4.

- $C^{\mu'}(w) \succ_w C^{\mu}(w)$ (i.e., every student in A prefers the new matching to the old one);

- $\omega \in \mu'(f)$ implies $\omega \in A \cup \mu(f)$ (i.e., every firm in A is matched to new workers only from A, although it may continue to be matched to some of its "old" workers from $\mu(f)$); and

- $C^{\mu'}(f) \succ_f C^{\mu}(f)$ (i.e., every firm in A prefers its new set of workers to its old one).

A matching is **group stable** if it is not blocked by any group of agents.

The stability concepts presuppose that a match is between a worker and a firm. Both the firm and the worker can unilaterally sever the match, and together they can establish the match irrespective of other agents' preferences. In particular, even though the worker and the firm are members of a coalition composed of the firm and all its employees, other coalition members – i.e., other workers – have no veto power over the creation or severance of the firm-worker match. This lack of workers' veto power is a major difference between the stability in two-sided matching and the one-sided core in coalition formation. A matching μ is in the one-sided core if there is no coalition A such that $A \succ_a C^{\mu}(a)$ for each $a \in A$. A stable matching need not belong to the one-sided core, and a matching in the one-sided core need not be stable. Group stability is a stronger property than both stability and the non-emptiness of the one-sided core.[20]

4. Mechanisms and Stability of Matching

The basic structure of the matching problems studied in this section is similar to the Nash bargaining example discussed in Section 2. The structure is as follows. There are two dates. On date 1, firms and workers match, that is, form coalitions. On this

[20] The following example illustrates the difference. There is one firm f and two workers w_1 and w_2. The firm would most like to hire both workers. A second best option for the firm would be to hire w_1, the more productive worker, only. The third best would be to hire w_2 only. The productive worker, w_1, does not like to work with w_2, and so w_1's preferences are $\{f, w_1\} \succ_{w_1} \{w_1\} \succ_{w_1} \{f, w_1, w_2\}$. Worker w_2 wants to work for firm f irrespective of whether w_1 is working there, too. The matching in which worker w_1 works for firm f, and worker w_2 is unemployed, is in the one-sided core. This matching, however, is not stable. In fact, in this example, there is no stable matching.

date, firms and workers cannot enter binding employment contracts. Consequently, the agents form their preferences by foreseeing what will happen on date 2. On date 2, each resultant coalition C realizes a payoff profile from the set of feasible payoffs

$$\left\{ (u_i)_{i\in C} \in R_+^{\#C} : \sum_{i\in C} u_i \leq v(C) \right\},$$

where $v(C)$ is the value of coalition C and $v : \mathcal{C} \to R_+$ is the value function. We allow the payoffs u_i to represent expected payoffs from lotteries over a larger space of outcomes. Coalition C realizes a feasible payoff profile by playing some game, following some bargaining protocol, or using some sharing rule. For instance, in the example of Section 2, the payoff profile was chosen via Nash bargaining. Other examples – such as Tullock's (1980) rent-seeking game or linear sharing rules – are discussed in Section 6.

A post-matching mechanism (or, mechanism) is a game or a choice rule that players use to decide which profile of payoffs will be realized. The following definition of a post-matching mechanism identifies each such game or rule with resulting agents' payoffs because ultimately the stability properties of any matching problem are determined by these payoffs alone.

Definition 4.1 (Mechanism). A post-matching mechanism is a function G that for every coalition C and value $v(C)$ determines nonnegative payoffs $G(i, C, v(C))$ for all members $i \in C$ so that

$$\sum_{i\in C} G(i, C, v(C)) \leq v(C).$$

For example, an equal division rule operating on a coalition C with value $v(C)$ produces payoffs $G(i, C, v(C)) = \frac{v(C)}{\#C}$.

This section discusses mechanisms that do not discriminate against any worker w in any coalition C in the sense defined below. For the sake of the definition, let us denote the set of payoffs that agent i may receive in coalition C for various values $v(C)$ by

$$U(i, C) = \{G(i, C, v(C)) : v(C) \geq 0\}.$$

Using this notation, we may state the following

Definition 4.2 (Non-discrimination). A post-matching mechanism is *non-discriminatory* if for any worker w, and coalitions $C, C' \ni w$ the sets of payoffs are equal $U(i, C) = U(i, C')$.

All above-mentioned mechanisms – Nash bargaining, equal division, the Tullock rent-seeking – are non-discriminatory.[21]

We are further assuming that the mechanism is monotonic and continuous, i.e., an increase in the value of a coalition continuously improves the payoffs of all agents in the coalition.

Definition 4.3 (Monotonicity and Continuity). A mechanism is *monotonic* if for any agent i and coalition $C \ni i$ the payoff $G(i, C, \tilde{v})$ is increasing in $\tilde{v} \geq 0$. A mechanism is *continuous* if $G(i, C, \tilde{v})$ is continuous in $\tilde{v} \geq 0$.

All above-mentioned mechanisms are monotonic and continuous. Any monotonic mechanism that produces Pareto optimal payoffs[22] is continuous.

This section provides a sufficient and necessary condition for the existence of stable matchings for all preference profiles induced by a non-discriminating and monotonic mechanism. These conditions build on the notion of pairwise aligned preferences. Recall that preferences are pairwise aligned if all agents in an intersection of two coalitions prefer the same coalition of the two.

Definition 4.4 (Pairwise Alignment). Preferences are pairwise aligned if for all $i, j \in I$ and coalitions $C, C' \ni i, j$, we have

$$C \precsim_i C' \Longleftrightarrow C \precsim_j C'.$$

In particular, then $C \sim_i C'$ iff $C \sim_j C'$, and $C \succ_i C'$ iff $C \succ_j C'$. Preferences generated by Nash bargaining in the setting of Section 2 are pairwise aligned.

The sufficient and necessary condition for stability is given by the following.

Theorem 4.5 (Sufficiency and Necessity). Suppose that there are at least two firms and that all firms are able to employ at least two workers. A non-discriminatory, monotonic, and continuous post-matching mechanism induces pairwise-aligned preference profiles if, and only if, there is a stable matching for each induced preference profile.

[21] A mechanism that chooses payoffs $(u_i)_{i \in C}$ that maximize a welfare functional $\sum_{i \in C} W_i(u_i)$ is non-discriminatory if the welfare components W_i satisfy an Inada type condition $W_i'(u) \to 0$ as $u \to \infty$. If this condition fails, the welfare maximization mechanism may be discriminatory, for instance, if $W_1'(u)$ and $W_2'(u)$ tend to 0 as $u \to \infty$ but $W_3'(u) > 1$ for all u.

[22] Given the set of feasible payoffs, the payoffs are Pareto optimal if $\sum_{i \in C} G(i, C, \tilde{v}) = \tilde{v}$.

Moreover, if the mechanism generates pairwise-aligned preferences, then there is a group stable matching for each induced preference profile.

We first prove the sufficiency part, then comment on the proof of the necessity part, and end this section with a discussion of which assumptions may be dropped and which assumptions may be relaxed.

The proof of the sufficiency part is in two steps. The first step shows that under the assumptions of the theorem there is a metaranking. The second step is identical to the second step in the proof of Theorem 2.1, and hence is skipped. Recall that a metaranking is defined as follows.

Definition 4.6 (Metaranking). A metaranking is a transitive relation $\preccurlyeq$ on all coalitions such that for any $i \in I$ and $C, C' \ni i$,

$$C \precsim_i C' \Longleftrightarrow C \preccurlyeq C'.$$

Two examples of metarankings determined by indices were discussed in Section 2: the fear of ruin coefficient in a matching followed by Nash bargaining and the per-head value of a coalition in a matching followed by equal division of value. The appendix discusses non-cooperative implementation of matching when there is a metaranking.

We reduced the proof of the sufficiency part of Theorem 4.5 to the following.

Proposition 4.7 (Existence of a Metaranking). Suppose that all firms are able to employ at least two workers. If a non-discriminatory and monotonic post-matching mechanism induces pairwise-aligned preference profiles, then for each induced preference profile there is a metaranking.

Proof. Because of monotonicity, $G(a, C, v'(C)) = G(a, C, v(C))$ implies $G(b, C, v'(C)) = G(b, C, v(C))$ for any values $v(C), v'(C)$. Thus, we can define the payoff translation functions $t^C_{b,a} : U(a, C) \to U(a, C)$ for each coalition C and agents $a, b \in C$ by the condition

$$t^C_{b,a}(G(a, C, \tilde{v})) = G(b, C, \tilde{v}), \ \ \tilde{v} \geq 0.$$

The non-discrimination implies that $U(a, C) = U(a, C')$ for $C, C' \ni a$, and the pairwise alignment guarantees that $t^C_{b,a} = t^{C'}_{b,a}$. Since there is a firm able to employ two workers, so $t_{b,a}$ is defined whenever at least one of the agents a and b is a worker.

Choose an arbitrary reference worker w^* and fix the value function $v : \mathcal{C} \to R_+$. Because of the non-discrimination assumption, $t_{w^*,a}(G(a, C, v(C)))$ is well defined for any agent a and coalition $C \ni a$ even when $w^* \notin C$. By pairwise consistency,

$$t_{w^*,a}(G(a, C, v(C))) = t_{w^*,a'}(G(a', C, v(C)))$$

for any different $a, a' \in C$. Indeed, if $w^* \in C$ then the claim follows straightforwardly from the pairwise consistency. If $w^* \notin C$, then first consider the case when a is a firm and a' is a worker. Then a is able to employ two workers and $\{a, a', w^*\}$ is a coalition. By the non-discrimination, there is a value function $v' : \mathcal{C} \to R_+$ such that

$$\begin{aligned} G(a', C, v'(C)) &= G(a', \{a, a', w^*\}, v'(\{a, a', w^*\})), \text{ and} \\ v'(C) &= v(C). \end{aligned}$$

Then, the pairwise alignment implies that also

$$G(a, C, v'(C)) = G(a, \{a, a', w^*\}, v'(\{a, a', w^*\})).$$

Since $w^* \in \{a, a', w^*\}$, we have

$$\begin{aligned} t_{w^*,a}(G(a, C, v(C))) &= t_{w^*,a}(G(a, C, v'(C))) \\ &= t_{w^*,a}(G(a, \{a, a', w^*\}, v'(\{a, a', w^*\}))) \\ &= t_{w^*,a'}(G(a', \{a, a', w^*\}, v'(\{a', a', w^*\}))) \\ &= t_{w^*,a'}(G(a', C, v'(C))) \\ &= t_{w^*,a'}(G(a', C, v(C))). \end{aligned}$$

In the remaining case, both a and a' are workers. Then C contains also a firm f, and by the preceding argument

$$t_{w^*,a}(G(a, C, v(C))) = t_{w^*,f}(G(f, C, v(C))) = t_{w^*,a'}(G(a', C, v(C))).$$

Consequently,

$$\chi(C) = t_{w^*,a}(G(a, C, V(C)))$$

does not depend on a if C is fixed. Monotonicity of the mechanism implies that $\chi(C)$ determines a metaranking. This completes the proof.

The necessity part of Theorem 4.5 will be proved when we prove a stronger Theorem 5.12. The proof is in the appendix to Section 5, and makes two steps. A first step considers certain configurations of coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ such that there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., 3$ (we adopt the convention that subscripts are modulo 3 that is $C_{i,i+1} = C_{3,1}$ if $i = 3$ and $C_{i-1,i} = C_{3,1}$ if $i = 1$). In these configurations, if $C_{1,2} \sim_{a_2} C_{2,3}$ and $C_{2,3} \sim_{a_3} C_{3,1}$ then $C_{1,2} \sim_{a_1} C_{3,1}$. The second steps shows then this property implies pairwise alignment.[23]

Let us finish this section with the discussion of assumptions. First notice, that for monotonic non-discriminatory mechanisms the pairwise alignment assumption may be relaxed.

Lemma 4.9. If a non-discriminatory monotonic mechanism induces preferences such that

$$C \sim_i C' \Longleftrightarrow C \sim_j C'$$

for all $i, j \in C, C' \in \mathcal{C}$, then preferences generated by the mechanism are pairwise aligned.

Proof. Fix $i, j \in I$ and $C, C' \ni i, j$. It is enough to consider the case $i \neq j$ and $C \neq C'$. Assume that the value function v is such that $C \precsim_i C'$ in the induced preference profile $\precsim_I$. Use the non-discrimination to find a value $v'(C)$ such that $C \sim_i' C'$ in the induced preference profile $\precsim_I'$. Then, $v'(C) \geq v(C)$ and $C \sim_j' C'$. The monotonicity of the mechanism implies that $C \precsim_j C'$. This completes the proof.

The pairwise alignment assumption may also be relaxed in other ways. Consider for example the asymmetric Nash bargaining model presented in Section 2. When the bargaining power of a worker becomes 0, this worker becomes a wage taker indifferent to all employment options, and a stable matching still exists.[24]

[23]The necessity part of Theorem 4.5 provides some guidance for a social planner that wants to ensure the existence of a stable matching, intervenes to influence the game or rule that dictates the division of value, and does not know the set of payoffs that coalitions are able to create. Cf. Roth (1984) and other papers on the matching in medical labor markets cited in the introduction. These authors provide empirical evidence that lack of stability is related to the unravelling of markets. They also discuss efforts of medical associations to design the matching environment in such a way as to ensure stability.

[24]In fact, if there is a metaranking in a matching problem, and the preferences are modified so that some agents become indifferent between some of the coalitions and their outside option, then the modified problem still admits a stable matching.

The assumptions about the mechanism may be considerably relaxed. Before discussing how they are relaxed in Section 5, let us notice that even for the sufficiency part, it is not enough to assume that a single preference profile is pairwise aligned. The following situation illustrates the problem.

Example 4.10. There are three workers w_1, w_2, w_3 and three firms $f_{1,2}, f_{2,3}, f_{3,1}$. Let us adopt the convention that the subscripts are modulo 3, that is, $w_{i+1} = w_1$ if $i = 3$. Assume that only three firm-worker coalitions $\{f_{i,i+1}, w_i, w_{i+1}\}$, $i = 1, 2, 3$, create positive payoffs for their members. Let the payoffs in coalition $\{f_{i,i+1}, w_i, w_{i+1}\}$ be such that w_i obtains 2 and w_{i+1} obtains 1.

In this example, the resultant preferences of agents are pairwise aligned. At the same time, there is no group stable matching. There are stable matchings given by the partitions $\{\{f_{i,i+1}, w_i, w_{i+1}\}, \{f_{i+1,i+2}\}, \{f_{i+2,i}\}, \{w_{i+2}\}\}$, $i = 1, 2, 3$. It is easy to modify the example so that there is no stable matching. It is enough to assume that agents' payoffs in coalitions $\{f_{i+1,i+2}, w_{i+2}\}$ are negligible, but positive.

The next section relaxes Theorems 4.5 and Proposition 4.7 in several ways.

First, the monotonicity and continuity assumptions, as well as the assumption that there are at least two firms, are not needed in the sufficiency part of Theorem 4.5 and Proposition 4.7 (cf. Theorems 5.2 and 5.10).

Second, the result may be presented in terms of preference profiles without reference to a post-matching mechanism. Section 5 replaces the presence of a non-discriminatory mechanism with another, substantially weaker but more technical, condition that the preference profile belongs to a rich domain of pairwise-aligned profiles. Each domain of preference profiles generated by a non-discriminatory mechanism is rich; there are, however, rich domains that cannot be rationalized as coming from a non-discriminatory mechanism. Notice that, stated directly in terms of preference profiles, the results of Section 5 may be more readily applied to settings where agents' preferences are determined before the matching by institutional constraints.

Third, the sufficient conditions for stability in Section 5 are applicable also to settings that do not admit a metaranking.

Fourth, Section 5 removes the restriction that all firms are able to employ at least two workers. Theorem 5.8 replaces this restriction with a weak assumption on one-worker firms, that is, firms that can employ at most one worker. As a consequence, the sufficient condition of Theorem 5.8 is satisfied, for instance, in the Gale and Shapley (1962) marriage markets.

5. Preference Formulation of Stability Conditions

This section presents sufficient and necessary conditions for stability in a preference formulation. Unlike the results of Section 4, the stronger results of this section do not rely on the preferences being induced by a post-matching mechanism. As such, they are more directly applicable to the college admission problem.

Recall that Example 4.10 shows that the pairwise alignment of preference alone does not guarantee that a stable matching exists. As is shown in the present section, it is enough to assume pairwise consistency on the preference profile in question, and on some related profiles. In Section 4, the domain of profiles generated by a mechanism played this role. In the present section we will assume the existence of these other profiles directly — by imposing a pairwise alignment restriction on a domain of preference profiles.

To introduce our results, let us consider a simple matching problem with payoffs determined in Nash bargaining. Suppose that two firms f_1, f_2 and two workers w_1, w_2 match on date 1. On this date, they are not able to commit to terms of employment. On date 2, each coalition creates a value and divides it according to the Nash bargaining solution. As we know from Theorem 2.1, a stable matching exists in this setting.

Let us consider a heuristic for an alternative proof of Theorem 2.1. This proof, while more complex than the proof offered in Section 2, introduces the ideas used in the proofs of the stronger counterparts of Theorem 4.5 discussed in the present section.

If a stable matching does not exist, then there would be a cycle of coalitions such that each coalition contains an agent who strictly prefers the next coalition in the cycle. For example, worker w_1 would prefer $\{f_2, w_1, w_2\}$ to $\{f_1, w_1\}$, firm f_1 would prefer $\{f_1, w_1\}$ to $\{f_1, w_2\}$, and worker w_2 would prefer $\{f_1, w_2\}$ to $\{f_2, w_1, w_2\}$.

To show that this cannot happen, let us consider an auxiliary matching situation between firms f_1, f_2 and workers w_1, w_2 in which (i) the agents still divide the values according to the Nash bargaining solution, (ii) the values created by all coalitions except for $C = \{f_1, w_1, w_2\}$ are the same as in the original matching situation, and (iii) the value created by coalition C is such that worker w_2 is indifferent between C and $\{f_2, w_1, w_2\}$. In this auxiliary situation, the preferences of agents between coalitions from the above cycle are unchanged. The preferences are pairwise aligned because they are induced by Nash bargaining. Because of the pairwise alignment of preferences between w_2 and w_1, worker w_1 would be indifferent between C and $\{f_2, w_1, w_2\}$, and hence w_1 would prefer C to $\{f_1, w_1\}$. Again, because of the pairwise alignment of preferences between w_1 and

f_1, firm f_1 would prefer C to $\{f_1, w_1\}$, and hence to $\{f_1, w_2\}$. Firm f_1's strict preference for C over $\{f_1, w_2\}$ would contradict the pairwise alignment of preferences of f_1 and w_2 over coalitions C and $\{f_1, w_2\}$.

This contradiction proves that the cycle we started with cannot occur in the auxiliary situation, and hence it cannot occur in our example. So far, we have analyzed an illustrative cycle. To complete the proof and conclude that a stable matching exists, we need to show that there are no other cycles. The argument that there are no other cycles builds on the above analysis and is further developed following the statement of Theorem 5.2, and is completed in the appendix.

The role of Nash bargaining in the above heuristic argument is to ensure that there is an auxiliary situation in which the preferences are pairwise aligned, worker w_2 is indifferent between C and $\{f_2, w_1, w_2\}$, and preferences between coalitions other than C are inherited from the original preference profile. Nash bargaining may be replaced in the above example by any other non-discriminatory post-matching mechanism. Thus, the argument whose main thrust is presented above may be used to prove the sufficiency part of Theorem 4.5 even if we drop the monotonicity and continuity assumptions.

In fact, the above heuristic argument requires only that the preference profile whose stability we analyze is embedded in a domain of pairwise-aligned profiles that is rich in the following sense. For any preference profile in the domain, any worker, and any two coalitions (of size 3 or more) containing the worker, there exists a profile in the domain in which the worker is indifferent between the two coalitions and, save for one coalition, agents' preferences over coalitions are intact. More informally, the rich domain of preference profiles allows us to make any worker indifferent between two coalitions (of size 3 or more), while keeping preferences between all but one coalition intact.

Definition 5.1 (Rich Domain). A domain of preference profiles $\mathbf{R}$ is rich if for any worker $w \in W$, coalitions $C, C' \ni w$ such that $\#C, \#C' \geq 3$, and any $\precsim_I \in \mathbf{R}$, there exists a profile $\precsim'_I \in \mathbf{R}$ such that $C \sim'_w C'$ and all agents' $\precsim'_I$ preferences between coalitions other than C are the same as in $\precsim_I$.

A domain of all preference profiles that might be generated in the Nash bargaining of Section 2 for different value functions $v : \mathcal{C} \to R_+$ is rich. Any non-discriminatory mechanism induces a rich domain of preference profiles when applied to different configurations of coalitions' payoff profile sets.[25] The domain of all profiles in any matching problem is also rich.

[25]Denoting by $u_i(C)$ agent i utility from joining coalition C, and by u_I the profile of utilities of

The main result of the paper is that if a preference profile belongs to a rich domain of pairwise-aligned profiles, then there exists a stable matching. This result contains Theorem 4.5.

Theorem 5.2 (Sufficiency). Suppose that all firms are able to employ at least two workers. If a preference profile $\precsim_I$ belongs to a rich domain of pairwise aligned preference profiles, then $\precsim_I$ admits a matching that is stable and group stable.

A heuristic argument for why we may expect Theorem 5.2 to be true was presented at the beginning of this section. Let us develop it here. The proof of the theorem has two main steps. The first step shows that there are no cycles of coalitions $C_{1,2}, C_{2,3}, ..., C_{m,1}$ for some $m \geq 2$ such that

(a) There exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) For at least one i the preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$ and at least one of $C_{i-1,i}, C_{i,i+1}$ has three or more members.

Let us refer to such cycles as blocking cycles. The second step shows that if there are no blocking cycles then there exists a group stable matching. Let us first discuss, the more difficult first step, and then the easier second step.

A blocking cycle cannot have length 2. Indeed, $C_{2,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,1}$ and the pairwise alignment imply that $C_{2,1} \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,1}$. A blocking cycle cannot have length 3 when one of the agents a_1, a_2, a_3 is a firm. Indeed, assume that there is a cycle

$$C_{3,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}$$

and $C_{3,1}$ has three or more members. If two or three of the agents a_1, a_2, a_3 are firms, then this is the same firm, and one can use the transitivity of this firm's preferences and pairwise alignment of preference to show that all agents are indifferent on the cycle. If exactly one of the agents a_1, a_2, a_3 is a firm, then there is a coalition $C = \{a_1, a_2, a_3\}$ and we may use a slightly modified argument from the opening of this section.

agents $i \in I$, we may express a utility counterpart of the rich domain condition as follows. For any worker $w \in W$, coalitions $C, C' \ni i$, and any utility profile u_I there exists utility profile u'_I such that $u'_w(C) = u'_w(C')$ and $u'_j\left(\tilde{C}\right) = u_j\left(\tilde{C}\right)$ for all $j \in I$ and coalitions $\tilde{C} \neq C$. A natural question one may ask is whether on any rich domain of preference profiles one may impute utilities so that the above utility counterpart of richness is satisfied. In general, the answer is no. A counterexample is presented in the appendix.

If C is different from the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then there exists a pairwise-aligned preference profile $\precsim_I' \in \mathbf{R}$ such that

$$C \sim_{a_3}' C_{3,1}$$

and

$$C_{3,1} \precsim_{a_1}' C_{1,2} \precsim_{a_2}' C_{2,3} \precsim_{a_3}' C_{3,1}$$

with indifference if there was an $\precsim_I$ indifference in the cycle. A repeated application of the pairwise-alignment property of $\precsim_I'$, shows that

- a_1 is $\precsim_I'$ indifferent between C and $C_{3,1}$, and thus prefers C to $C_{1,2}$;
- a_2 prefers C to $C_{1,2}$, and thus to $C_{2,3}$; and
- a_3 prefers C to $C_{2,3}$, and thus to $C_{3,1}$.

None of the preferences on the cycle may be strict, as otherwise a_3 would strictly prefer C to $C_{3,1}$, contrary to a_3's indifference between these two coalitions.

If C equals one of the coalitions $C_{3,1}, C_{1,2}, C_{2,3}$, then we can repeat the above argument without the need to refer to the rich domain.

To show that there are no other blocking cycles requires overcoming some obstacles. The main obstacle is the lack of a single coalition containing all agents $a_1, ..., a_m$. In fact, such a coalition does not exist if two of the agents are firms. Even when the cycle has length 3 and all agents a_1, a_2, a_3 are workers, there may not exist a coalition containing all three agents if all firms are able to employ at most two workers. How to overcome this obstacle is shown in the proof presented in the appendix.[26]

The second step in the proof of Theorem 5.2 is easier. It requires us to show that the lack of blocking cycles is a sufficient condition for stability. One could show it directly. Let us take, however, a longer route, in order to re-express this sufficient condition in a more informative way, and highlight the connection with the existence of metarankings. First let us define.

Definition 5.3 (Relaxed Metaranking). A relaxed metaranking is a transitive relation $\preccurlyeq$ on all coalitions such that

[26]Theorem 5.2 is proved as a corollary of more general Theorem 5.8, which relaxes the assumption that all firms are able to employ at least two workers. The proof of Theorem 5.8 is in the appendix.

(1) For each agent $i \in I$, and coalitions $C, C' \ni i$,

$$C \precsim_i C' \text{ implies } C \preccurlyeq C'.$$

(2) For each agent $i \in I$, and coalitions $C, C' \ni i$ such that at least one of C,C' has three or more members,

$$C \preccurlyeq C' \text{ implies } C \precsim_i C'.$$

Each metaranking is also a relaxed metaranking. An identity relation on coalitions in the marriage problem is a relaxed metaranking for any profile of agents' preferences. Roughly speaking, a relaxed metaranking has two properties: (i) the coalitions higher in the ranking are preferred to the coalitions lower in the ranking by all relevant agents, and (ii) if two coalitions share the same level in the ranking, then either all relevant agents are indifferent between them, or both coalitions have at most two members.

Lemma 5.4. There exists a relaxed metaranking if and only if there are no blocking cycles.

Proof. ($\Longrightarrow$) For an indirect proof, consider coalitions $C_{12}, C_{23}..., C_{m1}$ such that $a_i \in C_{i-1,i} \cap C_{i,i+1}$, $i \in \{1, ..., m\}$, satisfy conditions (a) and (b) of the definition of a blocking cycle. By symmetry, we can assume that $\#(C_{m,1}) \geq 3$ and $C_{m,1} \prec_{a_1} C_{1,2}$. Then $C_{1,2} \preccurlyeq C_{2,3}$, $C_{2,3} \preccurlyeq C_{3,4}$, etc., and by transitivity $C_{1,2} \preccurlyeq C_{m,1}$. Thus $C_{1,2} \precsim_{a_1} C_{m,1}$, contradicting $C_{m,1} \prec_{a_1} C_{1,2}$.

($\Longleftarrow$) Define relation $\preccurlyeq$ so that $C \preccurlyeq C'$ whenever there exists a sequence of coalitions $C_{i,i+1} \in \mathcal{C}$ such that

- $C = C_{1,2}$,
- $C' = C_{m,m+1}$, and
- there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

Then $\preccurlyeq$ is transitive. It remains to verify conditions (1) and (2). To prove (1) take $C_{1,2} = C$, $C_{2,3} = C'$ and $a_1 = i$. To prove (2), assume that C or C' has three or more members, that $i \in C \cap C'$, and that $C \preccurlyeq C'$. Now, if $C \succ_i C'$, then there would exist a blocking cycle; hence $C \precsim_i C'$. This completes the proof.

Given the equivalence between the lack of blocking cycles and the existence of relaxed metarankings, to complete the second step in the proof of Theorem 5.2 it is enough to show the following.

Proposition 5.5 (Sufficiency). If there exists a relaxed metaranking, then there is a group stable matching.

Proof. The theorem is true if I contains only one agent. Let us assume that the theorem is true on any subset of I to prove the general case by induction.

Let $\preccurlyeq$ be the relaxed metaranking. Consider the family of coalitions

$$\mathcal{C}^{\max} = \{C : \text{there does not exist coalition } C' \text{ such that } C \prec C'\},$$

which is non-empty since there is only a finite number of coalitions and $\preccurlyeq$ is transitive.

If there is $C_0 \in \mathcal{C}^{\max}$ such that $\#(C_0) \geq 3$, then notice that $C_0 \succsim_i C$ for any $i \in C_0$ and $C \ni i$. By the inductive assumption, there exists a partition $\{C_1, ..., C_k\}$ that corresponds to a group stable matching on $I - C_0$. Then $\{C_0, C_1, ..., C_k\}$ is a partition of I that determines a group stable matching.

In the remaining case, all $C \in \mathcal{C}^{\max}$ have two or fewer members. Consider a one-to-one matching between firms from F and workers from W with preferences inherited from $\precsim_I$. By Gale and Shapley's (1962) result, there exists a group stable matching in this new problem; let

$$Q = \{C'_1, ..., C'_K\}$$

be a partition of I that corresponds to such group stable matching. We can assume that $C'_1, ..., C'_k \in \mathcal{C}^{\max}$ and $C'_{k+1}, ..., C'_K \notin \mathcal{C}^{\max}$ for some $k \geq 0$. Notice that for any $C' \in \mathcal{C}^{\max}$, any agent $i \in C'$ strictly prefers C' to any $C \notin \mathcal{C}^{\max}$ containing i. Indeed, if $C' \precsim_i C$ then $C' \preccurlyeq C$ and hence $C \in \mathcal{C}^{\max}$. Thus, $k \geq 1$.

By the inductive assumption, there is a group stable many-to-one matching on $I - C'_1 - ... - C'_k$. Let

$$\{C''_1, ..., C''_m\}$$

be the corresponding partition of $I - C'_1 - ... - C'_k$.

Now, it is enough to notice that $C'_1, ..., C'_k, C''_1, ..., C''_m$ is a group stable many-to-one matching on I. Indeed, if it is not group stable then there would exist a blocking group A that includes an agent $a \in C'_i$ for some $i \in \{1, ..., k\}$. Agent i would prefer a coalition C to C'_i. There would be two options. If $C \in \mathcal{C}^{\max}$, then matching Q would not be group stable, contrary to its construction. If $C \notin \mathcal{C}^{\max}$, then $C'_i \succ_a C$ (by the same argument that we used above to show that $k \geq 1$). This strict preference would contradict the assumption that $C'_i \precsim_a C$. This completes the proof.[27]

Theorem 5.2 presumed that each firm is able to employ at least two agents. If there are firms that cannot employ more than one worker, then the pairwise alignment condition is no longer sufficient for stability,[28] as the following example demonstrates.

Example 5.6. Let $F = \{f_1, f_2\}$ and $W = \{w_1, w_2\}$. Let the firms' employment capacities equal $M_{f_1} = 1$ and $M_{f_2} = 2$. Let the preference profile $\precsim_I$ be such that

$$\begin{aligned} \{f_1, w_1\} \succ_{w_1} \{f_2, w_1, w_2\} \succ_{w_1} \{f_2, w_1\} \succ_{w_1} \{w_1\}, \\ \{f_2, w_1, w_2\} \succ_{w_2} \{f_1, w_2\} \succ_{w_2} \{f_2, w_2\} \succ_{w_2} \{w_2\}, \\ \{f_1, w_2\} \succ_{f_1} \{f_1, w_1\} \succ_{f_1} \{f_1\}, \text{ and} \\ \{f_2, w_1, w_2\} \succ_{f_2} \{f_2, w_2\} \succ_{w_2} \{f_2, w_1\} \succ_{w_2} \{f_2\}. \end{aligned}$$

There does not exist a stable matching, the main reason being that

$$\{f_1, w_1\} \succ_{w_1} \{f_2, w_1, w_2\} \succ_{w_2} \{f_1, w_2\} \succ_{f_1} \{f_1, w_1\}.$$

On the other hand, $\precsim_I$ is pairwise aligned. Moreover, the domain of all pairwise-aligned preference profiles is rich.

Thus, in order to extend Theorem 5.2 to cases of many-to-one matching with one-worker firms, i.e., firms with employment capacity $M_f = 1$, we need an additional assumption. The assumption is based on the idea of a blocking one-worker firm, i.e., a one-worker firm that belongs to a blocking-like cycle of three coalitions.

Definition 5.7 (Blocking One-Worker Firm). A firm f unable to employ more than one worker is a blocking one-worker firm if there exist workers $w, w\prime \in W$ and a coalition $C \ni w, w'$ such that

$$\{f, w\} \succsim_w C \succsim_{w'} \{f, w'\} \succsim_f \{f, w\},$$

[27] In fact, this proof demonstrates that a slightly weaker condition is sufficient for group stability. This condition says that in any subset of agents either there is a coalition that is weakly preferred by all its members to all other coalitions in the subset, or there is a group of one- and two-member coalitions that are weakly preferred by all its members to any coalition not in the group. This condition is weaker than both the existence of a relaxed metaranking and the Banerjee, Konishi, and Sönmez (2001) top coalition property mentioned in the introduction.

[28] One-to-one matching is an exception. If the matching is one-to-one then all profiles are pairwise aligned and admit stable matchings.

with one preference strict.

Using this notion we may state the following.

Theorem 5.8 (Sufficiency). If a preference profile belongs to a rich domain of pairwise-aligned preference profiles and there are no blocking one-worker firms, then there is a matching that is stable and group stable. Moreover, there exists a relaxed metaranking.

This result contains Theorem 5.2 because in the latter there are no one-worker firms. This strengthened result covers the Gale and Shapley marriage market in which all preference profiles are pairwise aligned and no one-worker firm can be blocking because there are no cycles of three coalitions. There are no cycles of three coalitions because there are no firms able to employ two workers.

The heuristic for Theorem 5.8 is identical to the one for Theorem 5.2. The proof is presented in the appendix.

Let us finish this section with two results connecting pairwise alignment, relaxed metarankings, and metarankings. The first result is an observation that every preference profile that admits a relaxed metaranking may be embedded in a rich domain of pairwise aligned preference profiles.

Proposition 5.9. **(a)** If a preference profile admits a relaxed metaranking then it is pairwise aligned and there are no blocking one-worker firms.

(b) The domain of profiles admitting a relaxed metaranking is rich.

The proof of (a) is straightforward. The proof of (b) is in the appendix.

The second result says when pairwise alignment on a domain of preferences implies that there exists a metaranking.

Theorem 5.10 (Existence of a Metaranking). Suppose that there is a firm able to employ two or more workers and that a domain of preference profiles $\mathbf{R}$ satisfies the following condition. For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$, there exists a profile $\precsim'_I \in \mathbf{R}$ such that $C \sim'_w C'$ and all agents' $\precsim'_I$-preferences between coalitions other than C are the same as in $\precsim_I$. If preference profiles in domain $\mathbf{R}$ are pairwise aligned and are such that there are no blocking one-worker firms, then each preference profile in $\mathbf{R}$ admits a metaranking.

The proof relies on the same ideas as the proofs of Theorems 5.2 and 5.8, and is presented in the appendix. It is easy to modify the proof of Proposition 5.9 to show that the domain of preference profiles admitting a metaranking satisfies the domain condition of Theorem 5.10.

Let us finish with a necessity counterpart of our results. The assumptions are formulated using the following notion of a perturbation of preference profile that (i) keeps all preferences between coalitions except for a reference coalition C, and (ii) perturbs agents' preferences over C in a co-monotonic way.

Definition 5.11 (Monotonic C-Perturbation). Given a coalition C, we say that a preference profile $\precsim'_I$ is a monotonic C-perturbation of a profile $\precsim_I$ if:

- For any agent $j \in I$ and coalitions $C_1, C_2 \neq C$ containing j we have

$$C_1 \precsim'_j C_2 \Longleftrightarrow C_1 \precsim_j C_2.$$

- If there is $i \in C$ and $C'' \ni i$ such that $C \succsim_i C''$ and $C \prec'_i C''$, then for any $j \in I$ and $C' \ni j$, if $C \precsim_j C'$, then $C \prec'_j C'$.
- If there is $i \in C$ and $C' \ni i$ such that $C \precsim_i C'$ and $C \succ'_i C'$, then for any $j \in I$ and $C' \ni j$, if $C \succsim_j C'$ then $C \succ'_j C'$.

For instance, if a preference profile belongs to the domain of preferences generated by a monotonic non-discriminatory mechanism, then the domain also contains its monotonic C-perturbations.

Theorem 5.12 (Necessity). Suppose that either there are at least two firms able to employ two or more workers each, or that there are no such firms. Suppose also that a domain of preferences $\mathbf{R}$ satisfies the following conditions:

(1) For any agent $i \in I$, coalitions $C, C' \ni i$ such that $\#C' \geq 3$, and any $\precsim_I \in \mathbf{R}$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that $C \sim'_i C'$.

(2) For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that $C \precsim'_i C'$.

(3) For any agent $i \in I$, coalitions $C, C' \ni i$, and any $\precsim_I \in \mathbf{R}$ such that $C \sim_i C'$, there exists a monotonic C-perturbation $\precsim'_I \in \mathbf{R}$ such that

- $C \succ'_i C'$.
- for any $j \in C$ if $C'' \succ_j C$ then $C'' \succ'_j C$.
- for any $j \in C$ if $C'' \prec_j C$ then $C'' \prec'_j C$.

Then, if all profiles from $\mathbf{R}$ admit a stable matching, then all profiles from $\mathbf{R}$ are pairwise aligned and are such that there are no blocking one-worker firms.[29]

This theorem generalizes the necessity part of Theorem 4.5 and is proved in the appendix. The two main steps of the proof are discussed in Section 4. The final step makes use of the following.

Remark 5.13. As in Lemma 4.9, if a domain of preference profiles $\mathbf{R}$ satisfies (1), and for all $i, j \in C, C' \in \mathcal{C}$,

$$C \sim_i C' \Longleftrightarrow C \sim_j C',$$

then preferences in $\mathbf{R}$ are pairwise aligned.

The next section applies the theoretical results of the paper to some examples.

6. Applications and Examples

This section adds to the Nash bargaining example of Section 2 three further examples of settings in which our results on mechanisms of Section 4 are applicable. The mechanisms considered are linear sharing rules, maximization of a welfare objective, and Tullock's (1980) rent-seeking game. The section also determines the class of non-discriminatory, monotonic, and Pareto optimal mechanisms that induce pairwise aligned profiles, and hence stable matchings.

We consider the setting of Section 4. Recall that there are two dates. On date 1, firms and workers match but do not contract. Agents' preferences are determined by their payoffs on date 2. On date 2, each coalition C realizes a payoff profile from the set of feasible payoffs

$$\left\{ (u_i)_{i \in C} \in R_+^{\#C} : \sum_{i \in C} u_i \leq v(C) \right\},$$

[29] The domain of all preference profiles that admit a relaxed metaranking satisfies the assumptions (1)-(3).

where $v(C)$ is the value of coalition C and $v : \mathcal{C} \rightarrow R_+$ is the value function. We allow the payoffs u_i to represent expected payoffs from lotteries over a larger space of outcomes. Coalition C realizes a payoff profile by playing some game, following some bargaining protocol, or using some sharing rule.

Linear sharing rules. On date 2, agents divide the value using a coalition-specific linear sharing rule. The share of agent i in the value created by coalition C is $k_{i,C}$. This agent obtains

$$u_i = k_{i,C} v(C).$$

The shares $k_{i,C} > 0$ are coalition-specific, $\sum_{i \in C} k_{i,C} = 1$, and $k_{i,C}$ do not depend on the realization of $v(C)$.

In this case, the pairwise-alignment requirement takes the following simple form.

Corollary 6.1 (Sufficiency). If agents divide the values using a linear sharing rule with shares $k_{i,C}$, then there exists a stable matching if

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all C, C' and $i, j \in C \cap C'$.[30]

This corollary is an immediate consequence of Theorem 4.5 because linear sharing rules with $k_{i,C} > 0$ are nondiscriminatory, monotonic, and continuous. This corollary follows from Theorem 4.5 even if there are firms that can employ only one worker. We need, then, to reinterpret each such firm as being able to employ two workers, but generating the value 0 if employing two workers.[31]

The condition on shares is also necessary, in the following sense.

Corollary 6.2 (Necessity). Suppose that there are at least two firms able to employ two or more workers each. If agents divide the values using a linear sharing rule

[30]Banarjee, Konishi, and Sönmez (2001) showed that this class of linear sharing rules leads to non-empty one-sided core in coalition formation. Pycia (2005) constructs a slightly larger class of linear sharing rules that guarantees non-emptiness of the one-sided core in coalition formation. Only the linear sharing rules from this larger class guarantee that the one-sided core is non-empty for all value functions v.

[31]By the remark following Lemma 4.9, we can also extend the result to allow for $k_{i,C} = 0$.

with shares $k_{i,C}$, and there exists a stable matching for all value functions $v : \mathcal{C} \to R_+$, then

$$\frac{k_{i,C}}{k_{j,C}} = \frac{k_{i,C'}}{k_{j,C'}}$$

for all C, C' and $i, j \in C \cap C'$.

This corollary is an immediate consequence of Theorem 5.12.

Notice, that if agents' utilities are $U_i(s) = s^{\lambda_i}$, then the Nash bargaining will lead to linear division of value, and the resultant sharing rule will satisfy the above condition. Corollary 6.2 implies a partial converse of this statement. If there are firms able to employ two workers, and a profile of shares $k_{i,C}$ guarantees an existence of stable matching for all $v : \mathcal{C} \to R_+$ then the shares $k_{i,C}$ may be rationalized as coming from a Nash bargaining.

Welfare maximization and Pareto optimal mechanisms. The agents are risk-neutral. On date 2, the members of each formed coalition C choose a utility profile $\left(u_i^C\right)_{i\in C} \in R_+^{\#C}$ that maximizes the Bergson-Samuelson separable welfare functional

$$\max_{\left(u_i^C\right)_{i\in C}} \sum_{i\in C} W_i(u_i).$$

subject to $\sum_{i\in C} u_i \leq v(C)$. The welfare components W_i, $i \in I$, are increasing and concave. They are agent-specific, but not coalition-specific.

Lensberg's (1987) results imply that payoffs $\left(u_i^C\right)_{i\in C}$ are pairwise aligned.[32] Indeed, $\chi(C) = W_i'(u_i)$, for some $i \in C$, determine a metaranking. Hence, we obtain the following.

Corollary 6.3 (Sufficiency). If payoffs are determined by the maximization of a Bergson-Samuelson separable welfare functional, then there is a stable matching.

Lensberg's (1987) results also suggest that all Pareto optimal and continuous choice rules that produce pairwise-aligned profiles may be interpreted as maximization of a

[32]Lensberg (1987) studies the consistency of solution concepts. Pairwise alignment of preference profiles is related to the consistency requirement as, in many environments, a consistent solution concept generates pairwise aligned preferences. The idea of consistency of solution concepts was introduced by Harsanyi (1959) in his analysis of the independence of irrelevant alternatives in Nash bargaining. Lensberg (1987,1988), Thomson (1988), Lensberg and Thomson (1989), Hart and Mas-Collel (1989), and Young (1994) analyzed consistency in the context of Nash bargaining, welfare functions, Walrasian trade, the Shapley value, and sharing rules. Thomson (2004) gives an up-to-date survey of these results.

Bergson-Samuelson separable welfare functional. His results cannot be directly applied in the present context, both because he considers a one-sided problem[33] and because he assumes pairwise alignment of preferences for a much larger space of applications of the choice rule than is available in our context. The appendix provides a simple proof of the following many-to-one result inspired by Lensberg (1987).

Proposition 6.4. Suppose that all firms are able to employ at least two workers. Suppose also that a post-matching mechanism G is non-discriminatory and monotonic, and the payoffs $(G(i, C, v(C)))_{i \in C}$ are Pareto optimal in

$$V(C) = \left\{ (u_i)_{i \in C} \in R_+^{\# C} : \sum_{i \in C} u_i \leq v(C) \right\}$$

for all value functions $v : \mathcal{C} \to R_+$. If the mechanism induces pairwise-aligned preference profiles, then there exist increasing strictly concave differentiable functions $W_i : U_i \to R$ for $i \in I$ such that $W_i'(0) = +\infty$, and

$$(G(i, C, v(C)))_{i \in C} = \arg \max_{\sum_{i \in C} u_i \in V(C)} \sum_{i \in C} W_i(u_i).$$

This proposition,[34] implies the following.

Corollary 6.5 (Necessity). Suppose that there are at least two firms and that all firms are able to employ at least two workers. Suppose also that a post-matching mechanism G is non-discriminatory and monotonic, and the payoffs $(G(i, C, v(C)))_{i \in C}$ are Pareto optimal in

$$V(C) = \left\{ (u_i)_{i \in C} \in R_+^{\# C} : \sum_{i \in C} u_i \leq v(C) \right\}$$

for all value functions $v : \mathcal{C} \to R_+$. If the mechanism induces preference profiles that admit stable matchings, then there exist increasing strictly concave differentiable functions $W_i : U_i \to R$ for $i \in I$ such that $W_i'(0) = +\infty$, and

$$(G(i, C, v(C)))_{i \in C} = \arg \max_{\sum_{i \in C} u_i \in V(C)} \sum_{i \in C} W_i(u_i).$$

[33]For instance, Lensberg assumes that any collection of agents can form a coalition, while in many-to-one matching two firms cannot form a coalition.

[34]Both in Proposition 6.4 and Corollary 6.5, it is enough to assume that agents' payoff are Pareto optimal in a subset $V'(C)$ of the quasi-linear set $V(C)$ as long as the Pareto frontier of each $V'(C)$ is continuous in the value $v(C)$.

Rent-seeking. On date 2, agents in each formed coalition $C = \{a_1, ..., a_k\}$ engage in Tullock's (1980) rent-seeking game over a prize $v(C)$. Each $a_i \in C$ will be able to lobby at cost c_i to capture the prize $v(C)$ with probability $\frac{c_i}{c_1+...+c_k}$. Thus, if agents expand resources $c_1, ..., c_k$ then agent a_i obtains in expectation

$$\frac{c_i}{c_1 + ... + c_k} v(C) - c_i.$$

The agents play the Nash equilibrium of this rent-seeking game; every agent lobbies at cost $\frac{k-1}{k^2} v(C)$ and has expected payoff $\frac{v(C)}{k^2}$. Theorem 4.5 applies and there is a stable matching in any matching problem with payoffs determined by the Tullock rent-seeking.

7. Conclusion

This paper proposes a novel sufficient condition for stability of matchings that may be used to study matching with complementarities and peer effects. The main component of this condition is the pairwise alignment of preferences. The condition is particularly useful in the study of stability of matchings when preferences are induced by post-matching mechanisms. There exist stable and group stable matchings if a non-discriminatory mechanism generates pairwise aligned preferences. For monotonic, continuous, and non-discriminatory mechanisms, pairwise alignment is also a necessary condition for stability.

The sufficiency and necessity results allow one to determine which sharing rules or games induce the existence of stable matchings. There is always a stable matching if agents' preferences are induced by Nash bargaining or Tullock's (1980) rent-seeking game. The paper also applies the sufficiency and necessity results to (i) determine the class of linear sharing rules that always induce agents' preferences such that a stable matching exists, and (ii) determine the class of monotonic, non-discriminatory, Pareto optimal mechanisms – such as welfare maximization – that induce the existence of stable matchings.

A natural direction to extend the results of the present paper would be to generalize them to the Hatfield and Milgrom (2005) model of matching with contracts. This model incorporates as special cases the college admission setting, in which agents have preferences over coalitions, the setting in which wages are determined during matching, and the ascending package auctions. Under certain conditions,[35] such an extension of the results of the present paper is possible if there are two categories of workers. The first category encompasses the workers, such as crucial researchers in a biotech R&D lab, with whom it is not possible to write contracts because of the inherent complexity of the relationship with these workers and incompleteness of the contractual environment. These workers might provide complementary inputs to the firm production process. The second category includes workers, such as lab assistants, with whom the firm may contract but whose inputs are substitutable.

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[35]This extension requires an assumption similar to the lack of blocking one-worker firms assumed in Theorem 5.8.

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Appendices to Sections 4, 5, and 6

Appendix to Section 4. A Result on Non-Cooperative Implementation

The following results show that if there is a metaranking then the non-cooperative implementations of matching will result in a group stable matching.[36] Recall that in a game a profile of players' strategies σ is in a Strong Nash Equilibrium if there does not exist a subset of players that can improve the payoffs of all its members by a coordinated deviation, while players not in the subset continue to play strategies from σ.[37]

Proposition 4.11. Consider a non-cooperative game between workers and firms that has the following properties

(a) the game ends with a matching μ,

(b) the payoff of each agent i is determined by the coalition $C^{\mu}(i)$ that the agent belongs to in the matching, and

[36] There is substantial empirical evidence that stability of matching is related to well functioning matching markets. The group stability by itself, however, is not a strategic concept. Roth and Sotomayor (1990) survey the theoretical results about manipulation of the matching process via misrepresentation of preferences. Sönmez (1997,1999) illustrates the theoretical problems with agents' trying to manipulate the matching process via capacity restrictions or pre-arranged matches.

[37] Cf. Aumann (1959), Rubinstein (1980). We may alternatively use the solution concept of Coalition-Proof Nash Equilibrium of Bernheim, Peleg, Whinston (1987).

(c) for each coalition C, there is a profile of strategies of agents in C such that $C^{\mu}(i) = C$ for all $i \in C$, irrespective of strategies of agents not in C.

If agents' payoffs are such that there exists a metaranking of coalitions, then there is a Strong Nash Equilibrium of this game, all strong perfect equilibria correspond to group stable matchings, and any group stable matching corresponds to a strong perfect equilibrium.

An example of a game satisfying conditions (a)-(c) is the Gale and Shapley (1962) deferred acceptance algorithm. Another example is a game in which each worker applies for one or no jobs, and then each firm selects its workforce from among its applicants.

Proof. The proof or Theorem 2.1[38] shows that there is a group stable matching. Let us first show that any group stable matching is implementable as a Strong Nash Equilibrium of the game, and then show that each Strong Nash Equilibrium results in a group stable matching.

Consider a group stable matching. Let $\{C_1, ..., C_k\}$ be the corresponding coalition structure. One of the coalitions, C_{i_1}, is a maximal coalition in the metaranking, another coalition, C_{i_2}, is a maximal coalition among coalitions of agents from $W \cup F - C_{i_1}$, and we can recursively find coalitions $C_{i_3}, ..., C_{i_k}$ in this way. By (c), there is a profile of strategies of agents from C_i that enforces the formation of C_i. These profiles are in a Strong Nash Equilibrium.

For the remaining implication, consider a Strong Nash Equilibrium and the resulting matching with corresponding coalition structure $\{C_1, ..., C_k\}$. Notice that there is a coalition C_{i_1} that is maximal in the metaranking. Indeed, otherwise the assumption (c) would imply that the members of C_{i_1} would have a coordinated profitable deviation in the game. Recursively, we can find a coalition C_{i_2} that is maximal among coalitions of agents from $W \cup F - C_1$, and so on. An inspection of these coalitions show that the matching is group stable. This completes the proof.

Appendix to Section 5.

A counterexample showing that the class of rich domains is larger than its utility counterpart (cf. the footnote to the definition of the rich domain).

[38]The claim is also proved in a stronger form of Proposition 5.5.

Let $u_i(C)$ denotes agent i's utility from joining coalition C, and u_I the profile of utilities of agents $i \in I$. We may express a utility counterpart of the rich domain condition as follows.

For any $w \in W$, $C, C' \in \mathcal{C}_i$, $\#C, \#C' \geq 3$, and any $u_I \in P$ there exists $u'_I \in P$ such that

- $u'_i(C) = u'_i(C')$.
- $u'_j\left(\tilde{C}\right) = u_j\left(\tilde{C}\right)$ for all $j \in I$ and $\tilde{C} \in \mathcal{C} - \{C\}$.

The following counterexample will show that there are rich domains of preference profiles that are not representable by ordinary utilities that satisfy the above utility counterpart of richness.

Consider a firm f and three workers w_1, w_2, w_3. Let P be a domain of preference profiles consisting of the following three subdomains.

- The first subdomain of profiles contains all profiles $\precsim_I^1$ with the following properties

$$\begin{aligned}\{f, w_1, w_3\} &\sim_{w_1}^1 \{f, w_1, w_2\}, \\ \{f, w_1, w_2\} &\succ_{w_2}^1 \{f, w_2, w_3\}, \\ \{f, w_2, w_3\} &\succ_{w_3}^1 \{f, w_1, w_3\}.\end{aligned}$$

- The second subdomain of profiles contains all profiles $\precsim_I^2$ with the following properties

$$\begin{aligned}\{f, w_1, w_3\} &\succ_{w_1}^2 \{f, w_1, w_2\}, \\ \{f, w_1, w_2\} &\sim_{w_2}^2 \{f, w_2, w_3\}, \\ \{f, w_2, w_3\} &\succ_{w_3}^2 \{f, w_1, w_3\}.\end{aligned}$$

- The third subdomain of profiles contains all profiles $\precsim_I^3$ with the following properties

$$\begin{aligned}\{f, w_1, w_3\} &\succ_{w_1}^3 \{f, w_1, w_2\}, \\ \{f, w_1, w_2\} &\succ_{w_2}^3 \{f, w_2, w_3\}, \\ \{f, w_2, w_3\} &\sim_{w_3}^3 \{f, w_1, w_3\}.\end{aligned}$$

This domain of preference profiles is rich and it is not possible to represent the preferences by ordinary utilities that satisfy the utility counterpart of richness. Indeed, assume that each profile in P is represented by a utility profile u_I and that the resultant domain of utility profiles satisfies the above utility counterpart of richness. Take a utility profile u_I^1 representing a preference profile from the first subdomain with minimal $u_{w_1}^1(\{f, w_1, w_3\})$. Find a utility profile u_I^2 identical with u_I^1 except on $\{f, w_1, w_2\}$ and such that $u_{w_2}^2(\{f, w_1, w_2\}) = u_{w_2}^2(\{f, w_2, w_3\})$. Then, find a profile u_I^3 identical with u_I^2 except on $\{f, w_2, w_3\}$ and such that $u_{w_3}^3(\{f, w_2, w_3\}) = u_{w_3}^3(\{f, w_1, w_3\})$. Finally, notice that there cannot exist a profile u_I^4 identical with u_I^3 except on $\{f, w_1, w_3\}$ and such that $u_{w_1}^4(\{f, w_1, w_3\}) = u_{w_1}^4(\{f, w_1, w_2\})$. Indeed, such a profile would have to represent a preference profile from the first subdomain. However,

$$\begin{aligned} u_{w_1}^4(\{f, w_1, w_3\}) &= u_{w_1}^4(\{f, w_1, w_2\}) = u_{w_1}^3(\{f, w_1, w_2\}) < u_{w_1}^3(\{f, w_1, w_3\}) \\ &= u_{w_1}^2(\{f, w_1, w_3\}) = u_{w_1}^1(\{f, w_1, w_3\}) \end{aligned}$$

contradicting the selection of u_I^1 so that $u_{w_1}^1(\{f, w_1, w_3\})$ is minimal. This completes the proof.

Proof of Theorem 5.2. This theorem follows from Theorem 5.8 proved next.

A lemma for the proof of Theorem 5.8. Let us precede the proof of Theorem 5.8 with a preparatory lemma.

Lemma 5.8.1. Let the profile $\precsim_I$ belong to a rich domain $\mathbf{R}$ of pairwise-aligned preference profiles. Assume that there are no blocking one-worker firms. Then there are no cycles of three coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ such that

(a) there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$,

(b) $C_{3,1} \succsim_{a_3} C_{2,3} \succsim_{a_2} C_{1,2} \succsim_{a_1} C_{3,1}$ with at least one strict preference .

Proof. For an indirect proof, assume that there are coalitions $C_{1,2}, C_{2,3}, C_{3,1} \in \mathcal{C}$ such that

(a) there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$,

(b) $C_{3,1} \succsim_{a_3} C_{2,3} \succsim_{a_2} C_{1,2} \succsim_{a_1} C_{3,1}$ with at least one strict preference.

Consider the following four cases

Case 1: $a_1, a_2, a_3 \in F$. Then $a_1 = a_2 = a_3$ is a firm whose preferences are circular.

Case 2: $a_1, a_2 \in F, a_3 \in W$. Then $a_1 = a_2$ and we can shorten the cycle to $m = 2$, and use the argument from the discussion in Section 5.

Case 3: $a_3 \in F, a_1, a_2 \in W$. The case of firm a_3 able to employ two workers was discussed in Section 5. If a_3 is able to employ at most one worker, then $C_{3,1} = \{a_1, a_3\}$ and $C_{2,3} = \{a_2, a_3\}$ and the result follows from the lack of blocking one-worker firms. (Notabene, this is the only place in the proof that uses the lack of blocking one-worker firms).

Case 4: $a_1, a_2, a_3 \in W$. Then, either $a_i = a_{i+1}$ for some $i = 1, 2, 3$ and the pairwise alignment directly proves the claim, or all a_i are different and each $C_{k,k+1}$ has three members and contains a firm able to employ two workers. Take a firm $f_0 \in F$ able to employ two workers; then $\{a_1, f_0, a_2\}, \{a_2, f_0, a_3\}, \{a_3, f_0, a_1\} \in \mathcal{C}$.

If $C_{1,2} = \{a_1, f_0, a_2\}$ then

$$C_{1,2} \sim_{a_1} \{a_1, f_0, a_2\};$$

if $C_{1,2} \neq \{a_2, f_0, a_3\}$ then use the rich domain assumption to find $\precsim_I$ such that the above indifference is true and all preferences not involving $\{a_2, f_0, a_3\}$ are preserved. Abusing notation, we will continue to denote the new preference profile by $\precsim_I$. Similarly, if $C_{1,2} = \{a_2, f_0, a_3\}$ then

$$C_{1,2} \sim_{a_2} \{a_2, f_0, a_3\};$$

if $C_{1,2} \neq \{a_2, f_0, a_3\}$ then use the rich domain assumption to find $\precsim_I$ such that the above indifference is true and all preferences not involving $\{a_2, f_0, a_3\}$ are preserved. If $\{a_2, f_0, a_3\} \nsim_{f_0} \{a_1, f_0, a_2\}$, then

$$C_{1,2} \sim_{a_1} \{a_1, f_0, a_2\} \nsim_{f_0} \{a_2, f_0, a_3\} \sim_{a_2} C_{1,2}$$

contrary to what we proved in Case 3. Thus

$$\{a_2, f_0, a_3\} \sim_{f_0} \{a_1, f_0, a_2\}.$$

Now, if $C_{2,3} = \{a_3, f_0, a_1\}$ then

$$C_{2,3} \sim_{a_3} \{a_3, f_0, a_1\};$$

if $C_{2,3} \neq \{a_3, f_0, a_1\}$ then use the rich domain assumption to find $\precsim_I$ such that the above indifference is true and all preferences not involving $\{a_3, f_0, a_1\}$ are preserved. Then $C_{2,3} \succ_{a_2} C_{1,2} \sim_{a_2} \{a_2, f_0, a_3\}$ and

$$\{a_2, f_0, a_3\} \prec_{f_0} \{a_3, f_0, a_1\}.$$

Finally, on $C_{3,1}, \{a_3, f_0\}, \{a_1, f_0\}$ we have

$$\begin{aligned} C_{3,1} &\succsim_{a_3} C_{2,3} \sim_{a_3} \{a_3, f_0, a_1\} \succ_{f_0} \{a_2, f_0, a_3\} \sim_{f_0} \{a_1, f_0, a_2\} \\ &\sim_{a_1} C_{1,2} \succsim_{a_1} C_{3,1}, \end{aligned}$$

contrary to what we proved in Case 3. This completes the proof.

Proof of Theorems 5.8. For an indirect proof, assume that $\precsim_I$ does not admit a stable matching. In particular, a relaxed metaranking does not exist. By Lemma 5.4, the lack of a relaxed metaranking means that there exists a blocking cycle of coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ for some $m \geq 2$ such that

(a) There exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) For at least one i the preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$ and at least one of $C_{i-1,i}, C_{i,i+1}$ has three or more members.

We will proceed by induction. Notice that the case $m = 2$ follows directly from the pairwise alignment, and the case $m = 3$ follows from Lemma 5.8.1. For an inductive step, fix $m \geq 4$, and assume that there are no blocking cycles of strictly fewer than m coalitions.

Step 1. First let us demonstrate that there exists k such that

- $C_{k,k+1}$ has three or more members, and
- a_{k+1}, a_{k+3} or a_k,a_{k-2} are workers.

To prove this claim take $C_{i,i+1}$ with three or more members and consider two cases.

Case 1: either a_i or a_{i+1} is a worker or both are. By symmetry we can assume that a_{i+1} is a worker. If a_{i+3} is also a worker then the claim is proved, so assume that a_{i+3} is a firm. If a_{i+2} or a_{i+4} is a firm, then it is the same firm as a_{i+3}. Then however, there would exist a blocking cycle of $m-1$ coalitions, either $C_{1,2}, ..., C_{i+1,i+2}, C_{i+3,i+4}, ..., C_{m,1}$ or $C_{1,2}, ..., C_{i+3,i+3}, C_{i+4,i+5}, ..., C_{m,1}$, contrary to the inductive assumption. So, assume that both a_{i+2} and a_{i+4} are workers. If $a_{i+1} = a_{i+2}$ then again there would be a blocking cycle of $m-1$ coalitions contrary to the inductive assumption. Finally, if $a_{i+1} \neq a_{i+2}$ then $C_{i+1,i+2}$ contains two workers and hence $\#C_{i+1,i+2} \geq 3$, a_{i+2} and a_{i+4} are workers, and hence the claim is true.

Case 2: both a_i and a_{i+1} are firms. Then in fact $a_i = a_{i+1}$. Look at a_{i-1} and a_{i+2}. If one of them is a firm, then it is the same firm as $a_i = a_{i+1}$, and we could shorten the cycle, contrary to the inductive assumption. So, assume that a_{i-1} and a_{i+2} are workers. Notice that a_i is able to employ two workers because $\#C_{i,i+1} \geq 3$ and consider two subcases depending on whether $\{a_{i-1}, a_i, a_{i+2}\}$ is identical to one of $C_{j,j+1}$.

- If $\{a_{i-1}, a_i, a_{i+2}\} = C_{j,j+1}$, then either at least one agent a_j,a_{j+1} is a worker, and we can reduce the problem to Case 1, or both a_j and a_{j+1} are firms. If a_j and a_{j+1} are firms then $a_j = a_{j+1} = a_i$, and hence we can without loss of generality assume that $C_{i,i+1} = \{a_{i-1}, a_i, a_{i+2}\}$. The pairwise alignment then implies that $C_{i-1,i} \precsim_{a_{i-1}} C_{i,i+1}$ or $C_{i-1,i} \prec_{a_{i-1}} C_{i,i+1}$ depending on whether $C_{i-1,i} \prec_{a_i} C_{i,i+1}$. Thus, we can substitute a_{i-1} for a_i to form the blocking cycle

$$C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} ... \precsim_{a_{i-1}} C_{i-1,i} \precsim_{a_{i-1}} C_{i,i+1} \precsim_{a_{i+1}} ... \precsim_{a_m} C_{m,1}$$

  with at least one strict preference, and reduce the problem to Case 1.

- If $\{a_{i-1}, a_i, a_{i+2}\} \neq C_{j,j+1}$ for all $j = 1, ..., m$, then we can use the rich domain assumption to find a preference profile such that $\{a_{i-1}, a_i, a_{i+2}\} \sim_{a_i} C_{i,i+1}$ and all preferences on the blocking cycle are preserved. Since $a_i = a_{i+1}$, we can replace $C_{i,i+1}$ with $\{a_{i-1}, a_i, a_{i+2}\}$, and argue as above. This completes the proof of the claim.

In view of the above claim, and the symmetry of the problem, we can assume that a_1 and a_3 are workers and $C_{m,1}$ has three or more members. Set $C = \{a_1, a_3, f\}$ where f is a firm that can employ two workers (such a firm exists if there exists a blocking cycle).

Step 2. First consider the case when $C = C_{i,i+1}$, for some $i = 1, ..., m$. Look at $C_{1,2}, C_{2,3}, C$ and conclude from Lemma 5.8.1 that either $C_{1,2} \prec_{a_1} C$, or $C_{2,3} \succ_{a_3} C$, or $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$.

- If $C = C_{i,i+1}$ and $C_{1,2} \prec_{a_1} C$ then $i \neq 1$ and the shorter cycle

$$C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1} \prec_{a_1} C_{i,i+1}$$

  satisfies (a) and (b) because $C_{m,1} \precsim_{a_1} C_{1,2} \prec_{a_1} C = C_{i,i+1}$ and $\#(C) \geq 3$. This is impossible, however, by the inductive assumption.

- If $C = C_{i,i+1}$ and $C_{2,3} \succ_{a_3} C$ then $i \neq 2$ and the shorter cycle

$$C_{i,i+1} \prec_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_i} C_{i,i+1}$$

satisfies (a) and (b) because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$ and $\#(C) \geq 3$. Again, this is impossible by the inductive assumption.

- If $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$ then the cycle $C, C_{3,4}..., C_{m,1}$ is blocking contrary to the inductive assumption.

Step 3. Finally consider the case $C \neq C_{i,i+1}$ for all i. Because $\#(C_{m,1}) \geq 3$, we can use the rich domain assumption to find a pairwise-aligned preference profile $\precsim_I$ such that there are no blocking one-worker firms, and all preferences along the blocking cycle are preserved and $C \sim_{a_1} C_{m,1}$. Abusing notation let us refer to the new profile as $\precsim_I$. Consider two subcases depending on preference of a_3 between C and $C_{2,3}$.

- If $C \prec_{a_3} C_{2,3}$, then consider the collection of $m-1$ coalitions $C, C_{3,4}, C_{4,5}, ..., C_{m,1}$. This is a blocking cycle of length $m-1$ because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$ and $\#(C) \geq 3$.

- If $C \succsim_{a_3} C_{2,3}$, then consider the collection of three coalitions $C_{1,2}, C_{2,3}, C$. Since $C \sim_{a_1} C_{m,1}$, we have $C \precsim_{a_1} C_{1,2}$. Thus the collection C_1, C_2, C satisfies

$$C \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C.$$

By Lemma 5.8.1 all agents are then indifferent. But then $C, C_{3,4}..., C_{m,1}$ is a blocking cycle of $m-1$ coalitions, contrary to the inductive assumption. This completes the proof.

Proof of Proposition 5.9(b). It is enough to show that for any $w \in W$ and any $C, C' \ni w$ such that $\#C, \#C' \geq 3$; if a profile $\precsim_I$ admits a relaxed metaranking then there exists a profile $\precsim'_I$ that admits a relaxed metaranking, agrees with $\precsim_I$ except for coalition C, and satisfies $C \sim'_w C'$. Denote by $\preccurlyeq$ the relaxed metaranking of $\precsim_I$ and fix C, C' and w. Consider $\precsim'_I$ that agrees with $\precsim_I$ except for coalition C. Furthermore, for any $j \in C$ and any coalition $C'' \ni j$, set $C \precsim_j C''$ iff $C' \preccurlyeq C''$ and $C'' \precsim_j C$ iff $C'' \preccurlyeq C'$. Now, consider the candidate relaxed metaranking $\preccurlyeq'$ identical to $\preccurlyeq$ except on C, and such that $C \preccurlyeq' C''$ iff $C' \preccurlyeq' C''$, and $C' \preccurlyeq' C$ iff $C'' \preccurlyeq' C'$.

Notice that $\preccurlyeq'$ is transitive. To verify that $\preccurlyeq'$ is indeed a relaxed metaranking, it is enough to verify conditions (1) and (2) defining the relaxed metaranking in case of comparisons of C and some other coalition C''.

Condition (1) is satisfied because $C \precsim'_i C''$ means that $C' \preccurlyeq C''$, and hence $C' \preccurlyeq' C''$. A similar argument works for $C'' \precsim'_i C$.

Condition (2) is satisfied for C, irrespective of whether C or C'' has three or more members. Indeed, if $C \preccurlyeq' C''$ and the claim of the implication is false, that is, $C \succ'_j C''$, then $C' \succ C''$; and thus $C \succ' C''$, which would be a contradiction. A similar argument works for $C'' \preccurlyeq' C$. This completes the proof.

A lemma for the proof of Theorem 5.10. Let us precede the proof of Theorem 5.10 with a preparatory lemma.

Lemma 5.10.1. Fix preference profile $\precsim_I$. If there are no cycles of coalitions $C_{12}, C_{23}..., C_{m1} \in \mathcal{C}$ for any $m \geq 2$ such that

(a) there exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$,

(b) at least one preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$,

then $\precsim_I$ admits a metaranking.

Proof. Define relation $\preccurlyeq$ so that $C \preccurlyeq C'$ whenever there exists a sequence of coalitions $C_{i,i+1} \in \mathcal{C}'$ such that

* $C = C_{1,2}$,

* $C' = C_{m,m+1}$,

* there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ such that $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

This is a transitive relation on coalitions, and it is straightforward to verify that this relation is a metaranking. This completes the proof.

Proof of Theorem 5.10. For an indirect proof, assume that $\precsim_I$ does not admit a metaranking. By Lemma 5.10.1 this means that there exists a cycle of coalitions $C_{12}, C_{23}..., C_{m,1} \in \mathcal{C}$ for some $m \geq 2$ such that

(a) There exists $a_i \in C_{i-1,i} \cap C_{i,i+1}$ for $i = 1, ..., m$ and $C_{i-1,i} \precsim_{a_i} C_{i,i+1}$.

(b) At least one preference is strict $C_{i-1,i} \prec_{a_i} C_{i,i+1}$.

We will proceed by induction. The case $m = 2$ follows directly from the pairwise alignment. The case $m = 3$ was proved in Lemma 5.8.1. For an inductive step fix $m \geq 4$ and assume that the claim is true for all collections of strictly fewer than m coalitions.

As a preparatory step, let us demonstrate that there exists a_k and a_{k+2} that are both workers. Indeed, first notice that if both a_i and a_{i+1} are firms, then $a_i = a_{i+1}$ and we can shorten the cycle and invoke the inductive assumption to find a contradiction. Hence, there exists a_i who is a worker. If now a_{i+2} is a firm, then both a_{i+1} and a_{i+3} are workers, or we can shorten the cycle and invoke the inductive assumption. Without loss of generality assume that a_1 and a_3 are workers. Take a firm f able to employ two or more workers, and set $C = \{a_1, a_3, f\}$.

First, consider the case when $C = C_{i,i+1}$, for some $i \in \{1, ..., m\}$. Look at $C_{1,2}, C_{2,3}, C$ and conclude from Lemma 5.8.1 that either $C_{1,2} \prec_{a_1} C$, or $C_{2,3} \succ_{a_3} C$, or $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$.

If $C = C_{i,i+1}$ and $C_{1,2} \prec_{a_1} C$ then $i \neq 1$ and then the shorter cycle

$$C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1} \prec_{a_1} C_{i,i+1}$$

satisfies conditions (a) and (b) contrary to the inductive assumption. If $C = C_{i,i+1}$ and $C_{2,3} \succ_{a_3} C$ then $i \neq 2$ and the shorter cycle

$$C_{i,i+1} \prec_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_i} C_{i,i+1}$$

satisfies conditions (a) and (b) contrary to the inductive assumption. Finally, if $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$ then the shorter cycle $C, C_{3,4}..., C_{m,1}$ satisfies conditions (a)-(b), contrary to the inductive assumption.

Finally, consider the remaining case $C \neq C_{i,i+1}$ for all i. Use the assumption on the domain from the theorem to find a pairwise aligned profile $\precsim_I$ such that there are no blocking one-worker firms, and $C \sim_{a_1} C_{m,1}$ and all preferences along the cycle are preserved. Let us refer to the new profile as $\precsim_I$. Consider two cases depending on the preference of a_3 between C and $C_{2,3}$.

If $C \prec_{a_3} C_{2,3}$, then the collection of $m - 1$ coalitions $C, C_{3,4}, C_{4,5}, ..., C_{m,1}$ satisfies (a)-(b) since $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$, and we can invoke the inductive assumption.

If $C \succsim_{a_3} C_{2,3}$, then consider the collection of three coalitions $C_{1,2}, C_{2,3}, C$. Since $C \sim_{a_1} C_{m,1}$, we have $C \precsim_{a_1} C_{1,2}$. Thus the collection C_1, C_2, C satisfies

$$C \precsim_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C.$$

By Lemma 5.8.1 all agents are then indifferent. But then $C, C_{3,4} ..., C_{m,1}$ satisfies (a) and (b) and consists of $m-1$ coalitions, contrary to the inductive assumption. This completes the proof.

Lemmas for the proof of Theorem 5.12. Let us precede the proof of Theorem 5.12 with two lemmas.

Lemma 5.12.1. Assume that a domain $\mathbf{R}$ of preference profiles satisfies the conditions (2)-(3) of Theorem 5.12 and that all profiles in $\mathbf{R}$ admit stable matchings. Assume that $C_{1,2}, ..., C_{3,1}, a_1, ..., a_3$ are such that $\{a_i\} \subseteq C_{i-1,i} \cap C_{i,i+1}$ (all subscripts modulo 3), and that

(a) if $a_i \in W$ then $\{a_i\} = C_{i-1,i} \cap C_{i,i+1}$, and

(b) if $a_i \in F$ then $C_{i,i+1} = \{a_i\} \cup S \cup \{a_{i+1}\}$ for some $S \subset C_{i-1,i}$.

Then, if $C_{3,1} \sim_{a_1} C_{1,2,}$ and $C_{1,2} \sim_{a_2} C_{2,3}$, then $C_{2,3} \succsim_{a_3} C_{3,1}$.

Proof. For an indirect proof assume that there exists a cycle $C_{1,2}, ..., C_{3,1}$ that satisfies (a), (b), and $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \sim_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Use (3) with $C = C_{2,3}$ and $i = a_2$ to find a preference profile $\precsim_I \in \mathbf{R}$ such that $C_{3,1} \sim_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$ (we continue to denote the new profile by the same symbol). Then, use (3) with $C = C_{1,2}$ and $i = a_1$ to find $\precsim_I$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Then, for all $i \in C_{1,2} \cup ... \cup C_{3,1}$, and $C \ni i$ different from $C_{1,2}, C_{2,3}, C_{3,1}$, use (2) to find $\precsim_i \in \mathbf{R}$ such that $C \precsim_i C_{k,k+1}$ for $k = 1, ..., 3$. Use (3) to find $\precsim_I \in \mathbf{R}$ such that $C \prec_i C_{k,k+1}$ for $k = 1, ..., 3$ and all $i \in I$, and $C \ni i$ different from $C_{1,2}, C_{2,3}, C_{3,1}$.

Recursively for $i = 1, 2, 3$, use (2) to modify the preference profile – while preserving all the above mentioned strict preferences – so that there exists a sequence of subsets

$$C^1_{i,i+1} \subset C^2_{i,i+1} \subset ... \subset C^{m_i}_{i,i+1} = C_{i,i+1}$$

for some $m_i \in \{1, 2, ...\}$ such that

- $C^1_{i,i+1} = \{f_i\}$ for some $f_i \in F$,
- $C^{k+1}_{i,i+1} = C^k_{i,i+1} \cup \left\{a^k_i\right\}$ for some $a^k_i \in W$,
- $a^{m_i}_i = a_i, a^{m_i-1}_i = a_{i+1}$,

- $C_{i,i+1}^{k} \precsim_{f_i} C_{i,i+1}^{k+1}$, and
- $C \precsim_a C_{i,i+1}^{k}$ for any $a \in C_{i,i+1}^{k}$ and $C \ni a$ different from $C_{i,i+1}^{k+1}, ..., C_{i,i+1}^{m_i-1}, C_{1,2}, C_{2,3}, C_{3,1}, C_{1,2}^{m_1}, ..., C_{i-1,i}^{m_{i-1}}$.

Use (3) to modify the preferences and strengthen the last two of the above properties:

- $C_{i,i+1}^{k} \prec_{f_i} C_{i,i+1}^{k+1}$, and
- $C \prec_a C_{i,i+1}^{k}$ for any $a \in C_{i,i+1}^{k}$ and $C \ni a$ different from $C_{i,i+1}^{k+1}, ..., C_{i,i+1}^{m_i-1}, C_{1,2}, C_{2,3}, C_{3,1}, C_{1,2}^{m_1}, ..., C_{i-1,i}^{m_{i-1}}$,

while maintaining the preferences $C_{3,1} \prec_{a_1} C_{1,2} \prec_{a_2} C_{2,3} \prec_{a_3} C_{3,1}$,and $C \prec_a C_{i,i+1}$for all $a \in C \cap C_{i,i+1}$.

The resultant profile of preferences does not admit a stable matching. This completes the proof.

Lemma 5.12.2. Suppose that there are at least two firms able to employ two or more workers each. Let $\mathbf{R}$ be a rich domain of preference profiles. Assume that each profile $\precsim_I \in \mathbf{R}$ satisfies the claim of Lemma 5.12.1: for every cycle $C_{1,2}, ..., C_{3,1}, a_1, ..., a_3$ such that $\{a_i\} \subseteq C_{i-1,i} \cap C_{i,i+1}$ and the conditions (a) and (b) are true we have

$$C_{3,1} \sim_{a_1} C_{1,2,} \text{ and } C_{1,2} \sim_{a_2} C_{2,3} \text{ imply } C_{2,3} \succsim_{a_3} C_{3,1}.$$

Then, if $A, B \in \mathcal{C}$, $B \subset A$, $\#(A-B)=1$, and $a, b \in B$, then $A \sim_a B$ implies $A \sim_b B$.

Proof. Take $A, B \in \mathcal{C}$ such that $B \subset A$, $\#(A-B)=1$, and take $a, b \in B$. If $a=b$ then the claim is true. If $a \neq b$, then $\#B \geq 2$ and $\#A \geq 3$. Moreover, then $A \cap B$ contains a firm that can hire two or more workers. Consider three cases.

Case 1: $a, b \in W$.

There are at least two firms, so there exists $c \in F - A - B$. Consider the cycle $A, \{b,c\}, \{a,c\}$. Change $\precsim_I$ so that $\{a,c\} \sim_a A$ and $\{b,c\} \sim_b A$ while preferences between coalitions different than $\{b,c\}, \{a,c\}$ are preserved. Let us denote the new profile by $\precsim_I$. Then, Lemma 5.12.1 implies that $\{a,c\} \sim_c \{b,c\}$. If $B \sim_a A$ then $B \sim_a \{a,c\}$, and Lemma 5.12.1 applied to the cycle $B, \{a,c\}, \{b,c\}$ implies that $B \sim_b \{b,c\}$. Hence, $B \sim_b A$.

Case 2: $a \in F, b \in W$.

Take $c \in A - B \subset W$ and $f \in F_2 - \{a\}$; f exists since there are at least two firms able to employ two or more workers each. Let

$$C = A - \{b\} = B - \{b\} \cup \{c\}$$

and

$$C' = \{b, c, f\}.$$

We will repeat the Case 1 argument with some modifications. Note that $C \cap C' = \{c\}$ and $A \cap C' = \{b\}$, so condition (a) is satisfied for the cycle C, C', A and all its permutations. Moreover, firm $a \in A \cap C$, and both $A - C$ and $C - A$ are singletons or empty. Hence also condition (b) is satisfied. Similar relations are true for the cycle C, C', B and all its permutations. Thus, the claim of Lemma 5.12.1 is satisfied for cycles C, C', A and C, C', B.

Using the rich domain assumption, we can find a preference profile that preserves preferences between coalitions other than C' and such that

$$C' \sim_b A.$$

Using the rich domain assumption again, we can find a profile that preserves preferences between coalitions other than C and such that

$$C \sim_c C'.$$

Now, Lemma 5.12.1 implies that $C \sim_a A$.

Since $A \sim_a B$ was preserved in the above changes of the preference profile, we have

$$B \sim_a C.$$

Furthermore, c is indifferent between C and C'. Thus, Lemma 5.12.1 applied to B, C, C' gives

$$C' \sim_b B.$$

Since b was also shown to be indifferent between C' and A, we have $B \sim_b A$ as required.

Case 3: $a \in W, b \in F$.

After renaming the agents, we can assume that $a \in F, b \in W$ and $A \sim_b B$, and use virtually the same argument as in Case 2. This completes the proof.

Proof of Theorem 5.12. If there are no firms able to employ two or more workers each, then all preference profiles are consistent and there are no blocking one-worker firms.

If there are at least two firms able to employ two or more workers each, then apply Lemmas 5.12.1 and 5.12.2 to show that for all $i, j \in C, C' \in \mathcal{C}$, all profiles satisfy the condition

$$C \sim_i C' \Longrightarrow C \sim_j C'.$$

Remark 5.13 then shows that all profiles are pairwise aligned. The lack of blocking one-worker firms follows directly from Lemma 5.12.1. This completes the proof.

Appendix to Section 6

Proof Proposition 6.4. The proof of Proposition 4.7 for monotonic mechanisms, presented in Section 4, constructs the payoff translation functions $t_{b,a} : U_a \to U_b$ for any agents a, b such that one of them is a worker. Recall that for each coalition $C \ni a, b$, we have

$$t_{b,a}\left(G\left(a, C, V\right)\right) = G\left(b, C, V\right).$$

By the monotonicity of mechanism G, functions $t_{b,a}$ are strictly increasing. Since G generates Pareto optimal profiles, functions $t_{b,a}$ are continuous.

Choose an arbitrary reference worker w^*, notice that $0 \in U_{w^*}$, and define

$$\psi_a\left(u\right) = f\left(t_{w^*,a}\left(u\right)\right), \ a \in I$$

where $f : U_{w^*} \to R$ is decreasing, $f\left(s\right) \to +\infty$ as $s \to 0+$, and such that all ψ_a are right hand side integrable at 0. Notice that there exists a function f that satisfies these conditions. Indeed, there is a finite number k of functions $t_{w^*,a}$ which are all continuous, increasing, and have value 0 at 0. Take

$$t^{\min} = \min_a \left\{t_{w^*,a}\right\}$$

and notice that it is also continuous and increasing, and has value 0 at 0. The functions ψ_a are integrable if $f \circ s^{\min}$ is. This will be so if, for example,

$$f\left(t\right) = \left[\frac{1}{\left(s^{\min}\right)^{-1}\left(t\right)}\right]^2.$$

Moreover, f is decreasing (since $s^{\min}$ is increasing), and $f\left(s\right) \to +\infty$ as $s \to 0+$ (because $s^{\min}\left(t\right) \to 0$ as $t \to 0$). Notice that ψ_a are positive and strictly decreasing and define,

$$W_a\left(s\right) = \int_0^u \psi_a\left(\tau\right) d\tau.$$

Now, W_a are concave and increasing.

It remains to be shown that the solution to

$$\max_{\sum_{a\in C}\tilde{u}_a\in V}\sum_{a\in C}W_a\left(\tilde{u}_a\right)=\sum_{a\in C}\int_0^{\tilde{u}_a}\psi_a\left(\tau\right)d\tau$$

coincides with $G\left(a,C,V\right)$. Concavity of the problem implies that there is a solution. Since the slope at 0 for each $\int_0^{\tilde{u}_a}\psi_a\left(\tau\right)d\tau$ is infinite, so the solution is internal. The differentiability of the objective function implies that the internal solution is given by the first order Lagrange conditions

$$\psi_a\left(\tilde{u}_a\right)=\lambda$$

and the possibility constraint $\left(\tilde{u}_a\right)|_{a\in C}\in V$. The first order condition can be rewritten as

$$t_{w^*,a}\left(\tilde{u}_a\right)=f^{-1}\left(\lambda\right)$$

or

$$\tilde{u}_a=t_{a,w^*}\left(f^{-1}\left(\lambda\right)\right).$$

If there is no worker in C, then $C=\{f\}$ for some $f\in F$ and the claim we are proving is true. Otherwise, fix a worker $w\in C$ and notice that for agents $a\in C$

$$G\left(a,C,V\right)=t_{a,w}\left(G\left(a,C,V\right)\right)$$

Lemma 5.8.1 from the appendix to section 5 shows that

$$t_{a,w^*}\circ t_{w^*,w}=t_{a,w}.$$

Hence,

$$G\left(a,C,V\right)=t_{a,w^*}\left(t_{w^*,w}\left(G\left(a,C,V\right)\right)\right)=t_{a,w^*}\left(x\right)$$

for some $x\in R$.

This equation, the analogous equation for $\tilde{u}_a$ above, the monotonicity of t_{a,w^*}, the Pareto optimality of the mechanism, and the possibility constraint $\left(\tilde{u}_a\right)|_{a\in C}\in V$ imply that

$$\tilde{u}_a=G\left(a,C,V\right).$$

This completes the proof.