An Analytical Solution for Networks of Oldest Friends

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Abstract

Sequentially Nash Credible Joint Plans (SN) as in Nieva (February 2006) are shown to exist *also* whenever actions sets are infinite in a modification of *all* three-player Aumann-Myerson (1988) (A-M) bilateral link formation games. In contrast to A-M, binding transfers can occur if pairs match pairs of non negative payoff proposals out of the sum of their Myerson values (1977) in the prospective network. Pairs can also enunciate simultaneous negotiation statements about payoff-relevant play and bargain cooperatively over payoffs induced by tenable and reliable joint plans where the disagreement one suggests link rejection. A SN is for the most the one that suggests credibly−so followed through−the Nash solution in the bargaining game. In contrast to the bargaining network literature and the transfer game in Bloch and Jackson (2005), the one here is bilateral, sequential and has unique payoff predictions. In strictly superadditive cooperative games the complete graph never forms. The simple majority game yields the nucleolus in coalition structure.

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1 Introduction

Network structures, represented by sets of *links,* play an important role in the outcome of many social interactions (See Jackson (2005b)). In particular, its effect on how payoffs are allocated within a network is important not only in terms of fairness considerations but because it determines players' incentives to form networks. Hence, it is relevant to have a theory that explains not only how they form but how *payoffs* are allocated and how this depends on relevant circumstances.

The present paper focuses on payoff division in communication network formation games by studying the role played by bilateral sequential *bargaining*, aimed at influencing *cooperation*, coordination of actions, and *focal simultaneous negotiation*, simultaneous message exchange to influence equilibrium outcomes, according to a finite rule of order.

I do so by proposing a coalitional−as players can cooperate within a coalition of players provided they have *communication links*−network sequential bilateral bargaining procedure that entails *modifying all* three-player Aumann-Myerson (1988) (A-M) network games, where pairs are allowed to propose not simply bilateral links but *also* "transfers" out of their "static" *Myerson* (1977) *values.* The latter are an important *payoff allocation rule* for such games also called transferable utility (TU) *cooperative games* with network effects where the *worth* of what a coalition of players can achieve is given by the *characteristic function*.

The focal theory of equilibrium selection used is an extension of neologism proofness as in Farrel (1993) and a refinement of publicly correlated subgame perfect equilibrium and it is due to Nieva (February 2006). The author defines an *almost non cooperative solution* (ANC), "Sequentially Nash Credible Joint plans" (SN), for a *friendships' environment* that induces a coalitional "strategic" network bilateral focal sequential bargaining game. A version of SN that assumes a "last-mover advantage" exists for *strategic network games,* where link choice and *actions* are strategic variables, whenever action sets are finite. Another version that instead assumes an "oldest-friend focal effect" exists also for my modification of A-M whenever the underlying cooperative game is a three-player simple majority game provided one introduces focal negotiation and infinite action sets are interpreted as the infinite proposals sets in my transfer game.

The present paper extends existence of SN to my modified A-M game for *all* normalized three-player cooperative games. Thus, the claim in that paper that SN is a unifying single valued solution concept for network games is supported as I proof that SN exist for more general *trivial* strategic network games where besides proposals and link choices there is no other strategic variable. More importantly, the interpretation of a friendships' environment as my modified A-M makes clear that any network game with a given *value function,* that gives what a network can achieve, *with* or without externalities could be interpreted as a trivial strategic network game and the existence of SN could be assessed.

As the "smoothed Nash demand game" was given in that paper as a "novel" non cooperative foundation for this ANC solution for the friendships' environment that predicts the nucleolus in coalition structure for the simple majority game, the present paper is a contribution to the bargaining literature that looks for noncooperative foundations for reasonable cooperative solutions to TU cooperative games.

Before an informal presentation of the model, I want to situate the present paper within the related literature. With respect to the network literature, this has relied in solution concepts where players evaluate prospective networks according to analytical payoff allocation rules that are *static* in the sense that they depend only on the fixed network structure (see for example Jackson and Wolinksky (1996)). As bargaining and transfers in the process of network formation are observed empirically (See Bloch and Jackson (2005)), network payoffs should depend on the possibilities of players forming other networks. The network bargaining literature (See Jackson (2005)) has addressed that problem by allowing link formation and payoff division to happen simultaneously. However, the emphasis has been on network formation and the tension between equilibrium outcomes and efficiency rather than on payoff division. The reason may be found in the well known difficulty of bargaining games yielding unique outcomes whenever players can cooperate.

More specifically, some papers address the problem of static network payoffs by disregarding static payoff allocation rules while allowing non cooperative bargaining over the total payoffs a network can achieve.[1] Bargaining in the form of payoff and link proposals, occur multilaterally and simultaneously in Slikker and Van de Noweland (2001). Currarini and Morelli (2000) have instead a sequential model and still multilateral model. In the present paper bargaining occurs instead sequentially and bilaterally. Navarro and Perea (2001) use a bilateral sequential model, however, the latter authors' goal objective is to *implement* the Myerson value.

Within the same strand of literature, the model in this paper is closer to Jackson and Bloch (2005) as the authors assume to begin with that payoffs are received individually−in the present paper players would receive their Myerson values if my transfers are zero. They consider the possibility of different types of binding transfers schemes at the time of link formation in a multilateral simultaneous bargaining model and find its implications on the efficiency in network outcomes. The difference with these authors is that "implied implicit transfers" here are only direct and the bargaining process occurs sequentially and bilaterally.

[1] The reader may be interested in a related paper by Jackson (April 2005) which addresses the problem instead axiomatically by proposing payoff allocation rules that account for simultaneous possibilities of extra link formation.

More importantly, in neither of the positive models above payoff predictions are unique.

With respect to the literature that looks for foundations for cooperative solutions, SN can be implemented with an appropriate smoothed Nash demand game that contrasts with most foundations as these are based on variations of the Rubinstein model where players are restricted to use stationary strategies. The reader is referred to Nieva (February 2006) for relating SN to the specific related bargaining literature and the literature on strategic transmission as an equilibrium selection device.

In A-M, pairs of players propose permanent bilateral communication links and evaluate induced communication structures, represented by *graphs*−sets of links−using their Myerson (1977) values. The Myerson values are reasonable as more payoff is assigned to players with more links. Links are formed if the pair agrees. As in bridge, after the last link has been formed, each of the pairs must have a last chance to form an additional link. If then every pair rejects, the game *ends*. This game is of perfect information. Hence, it has subgame perfect equilibria in pure strategies. Each equilibrium has a unique graph formed at the end of play.

Consider the following modification, at each stage of the A-M game, a link, say ij, "may" form if both players i and j play *choice* y in the *simultaneous link choice "formation" game*. If at least one of them plays a *unilateral rejection*, n, the link does not form.

Following any outcome of the link "formation" game a *current simultaneous action game* takes place. Actions are interpreted as *proposals pairs.* Each player proposes a non negative payoff for player i and for player j. Proposals pairs are *feasible* if they add to the sum of the pair Myerson values in the *immediate prospective graph*, the one that would form if the link ij forms. Proposals pairs *match* if they are feasible and coincide. If the compete graph is the immediate prospective graph then only the Shapley values that coincide with the associated Myerson values are feasible proposals pairs.

A link forms and a given transfer scheme is binding for players i and j, if and only if both choose y and match proposals pairs. Otherwise, the link does not form. With respect to payoff outcomes, if the immediate prospective graph does not form and the game ends, payoffs in the *last proposal match*−the one that led to the formation of the last graph−are realized. The third player gets her Myerson value in such last graph. Otherwise stage payoffs are zero unless the complete graph forms, in which case the Shapley values are realized. Note that whenever a pair of players did not form its link, the underlying two-player strategic form game has the same action profile set but play of any action profile is payoff-irrelevant. This is needed for my modification of the A-M model to fit the model in Nieva (February 2006). Also, in contrast, in the present paper, the link forms provided both choose y and the pair matches proposals.

Pairs can also engage in *preliminary negotiations* and enunciate before play payoff-irrelevant simultaneous negotiation statements represented by a correlated strategy, "promise-requests", in the link formation game and the current payoff proposal game and similar future games for future preliminary negotiators, "future-requests", even if they don't have a communication link. I assume that they have a temporary communication technology. To solve the natural equilibrium selection problem in my payoff-relevant bilateral transfer games which persists even with pre-play communication (See Rabin (1994)), I use criteria in Nieva (February 2006) to select through "endogenous O-F Nash effective cooperative negotiation" a *joint plan*−which consists of two similar negotiation statements−that is "credible". Informally, any pair bargains cooperatively over payoffs induced by "tenable and reliable" joint plans about the *payoff-relevant game* to follow−these joint plans suggest subgameperfect publicly correlated equilibria. The "disagreement joint plan" suggests, in particular, link rejection. A *credible* joint plan is the one consistent "for the most" with the Nash Bargaining Rule (NBR), and is defined as an *Oldest-Friends (O-F) Joint Plan*.

An assumption−the *Oldest-Friends focal effect*−that in Nieva's (February 2006) theory is claimed to be sufficient for his "joint plan bargaining problems" to be well defined whenever actions sets are finite are not so whenever action sets are infinite−in the present paper, sets of proposals pairs−in the general case. This assumption is that only oldest pairs of "successful" preliminary negotiators that formed their link may influence future play. Whenever the worth of a two-player coalition is zero, the possibility of open bargaining *feasible sets* does not disappear−the O-F focal effect ensures that the simple majority game given in the paper in question does not have that problem. Hence, I adopt a second natural assumption not to be found in that paper: pairs suggest to future pairs of players in a way to induce well defined rational play, specifically, to induce closed feasible sets in bargaining games "whenever appropriate". Informally, if you want people do what you want, you suggest them to play well defined games; "you may not want to confuse them".

I then prove by construction that all possible joint plan bargaining problems at any stage of the *multistage game with communication* associated to the modified A-M are well defined and O-F Joint Plans exist at the beginning of the game; these have been defined in Nieva (February 2006) as *Sequentially Nash Credible Joint Plans*.

In section two, I solve a three-player simple majority game with A-M and then I illustrate how the O-F focal effect induces the nucleolus by finding SN using my modification of the game. In section three, I define O-F Joint Plans. In section four, notation for graphs is given, Myerson values are described and notation in the A-M model is defined. In section five, the A-M model is modified by setting up a multistage-payoff relevant game. Next, endogenous O-F Nash effective cooperative negotiation is assumed at each history of the associated multistage game with communication game. In section six, existence of SN is proved by constructing recursively well defined joint plan bargaining problems. My predictions are partially characterized for strictly superadditive games. Conclusions follow.

2 An Example

Consider the three-player simple majority game with characteristic function:

$v(1) = 0,$ $v(2) = 0,$ $v(3) = 0,$
$v(13) = 1$ $v(23) = 1$ $v(12) = 1,$
$v(123) = 1.$

where, for example, $v(13)$ is the total wealth players 1 and 3 can assure if they collude and cooperate.

Graph g^{ij} is the one that only has a link between player i and j, ij. Graph g^{ij+jl} is the one that would result if links jl is added to graph g^{ij} for $i \neq j \neq l$, where $i, j, l \in \{1, 2, 3\}$. Graph g^N denotes the complete graph where all players are linked. Also, if I write that some values for player i and j are (x, y), the first (second) value component refers to player i (j). Myerson values for different graphs are given in the following table (the first, second third component in the triplet corresponds to player 1,2, and 3 respectively):

One-link	Values	Two-Link	Values	Complete	Values
g^{13}	$(\frac{3}{6}, 0, \frac{3}{6})$	g^{13+32}	$(\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$	g^N	$(\frac{2}{6}, \frac{2}{6}, \frac{2}{6})$
g^{23}	$(0, \frac{3}{6}, \frac{3}{6})$	g^{12+23}	$(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$		
g^{12}	$(\frac{3}{6}, \frac{3}{6}, 0)$	g^{21+13}	$(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$		

Note how the player who has relatively more links or friends gets more.

In the rest of the paper, I assume that links 12, 23 and 13 are proposed in that order.

Claim 1 *The A-M solution has three subgame perfect equilibrium outcomes in which either of the one link graph is the last to form.*

Proof. From any two link graph the complete graph follows as the players not linked get more if they link, $\frac{2}{6}$ instead of $\frac{1}{6}$. A one link graph is last to form as any player in that link would reject a second link as the complete graph would follow next in which case her payoff would go down from $\frac{3}{6}$ to $\frac{2}{6}$.

Suppose links 12 and 23 have been rejected. Link 13 would form as players 1 and 3 would expect to get half instead of zero payoffs in case the game would end after rejection. One stage backwards, player 3 is indifferent between linking or not with player 2. One more stage backwards, player 2 is indifferent between linking or not with player 1 if players expect link 23 to form. On the other hand, player 1 is indifferent between linking or not with 2 if players expect link 23 not to form and instead link 13 to form. Thus, depending on the decision of the indifferent player, there are several subgame perfect equilibria outcomes in which either of the one link graph forms. ■

Claim 2 *In the three-player modified simple majority game, a Sequentially Nash Credible Joint Plan has the first pair suggesting "half-each" payoffs and future-requesting joint plans that suggest consecutive rejection of the next two links in the order.*

See appendix for proof.

3 Simultaneous Negotiation Problems

3.1 A Two-Player Negotiation Problem

I consider the problem of two players i and j, the *negotiators*, when they have the opportunity to make simultaneous negotiation statements to players i, j and l in *preliminary negotiations,* say with a temporary communication technology, about a *payoff-relevant game* with finite horizon to follow. Suppose first that past statements that negotiators i and j may know about at the time they negotiate are not influential.

The payoff-relevant game begins with a simultaneous communication *link "formation" game* where the *choice* sets that negotiators i and j have available are denoted by the sets $A_i = A_j = \{y, n\}$. The communication link is assumed to be permanent. Such a set A_l for player l has a trivial unique payoff-irrelevant choice, "move nothing". Denote by $A = A_i \times A_j \times A_l$ the associated *choice profile set* in the payoff-relevant game Also, the two-player choice profile set for i and j is denoted by A_{ij}. A bilateral link ij between players i and j forms "only if" both players play y :(note, this is a necessary condition in contrast to Nieva (February 2006)). Hence n is considered a *unilateral rejection.*

Play of a choice $a = (a_i, a_j, a_l) \in A$ can be identified with an *immediate contingency* a, the one that occurs right after a is played, or alternatively a *current contingency* a in the payoff-relevant game, at which a *current simultaneous game* a takes place where the *action* sets that negotiators i and j have available are denoted by $B_{i,a}$ and $B_{j,a}$. Such a set $B_{l,a}$ for player l has as trivial unique payoff-irrelevant action. Denote by $B_a = B_{i,a} \times B_{j,a} \times B_{l,a}$ the associated *action profile set.* Individual sets are assumed to be the same regardless of a, i.e., $B_a = B$ for all a. The set of *current joint strategies* $\times B$ in the payoff-relevant game is the Cartesian product of B_a for all a, that is,

$$\times B = \prod_a B_a = B^4,$$

An element of $\times B$ is denoted by $\times b$ and b_a, is the *a-th component* of $\times b$.

In any immediate contingency (a, b), the one that can be identified with play of choice a followed by action profile $b \in B$, a *future* game (a, b) takes place. The set of joint strategies in this game are denoted by $\times Z_{p/(a,b)}$, a Cartesian product of $Z_{p(a,b)}$ sets, that is,

$$\times Z_{p/(a,b)} = \prod_{p(a,b)} Z_{p(a,b)} \tag{1}$$

. Each $Z_{p(a,b)}$ stands for the choice or action profile set in any contingency of the payoff-relevant game that may follow the (a, b) occurrence including immediate contingency (a, b). Any such a contingency is denoted by $p(a, b)$. It is assumed that $\times Z_{p/(a,b)}$ only depends on the link ij forming or not; as it will turn out, a link will form, if and only if both players choose "yes" and play "identical feasible" actions.

The set of *future joint strategies* in the payoff-relevant game is

$$\times Z = \prod_{(a,b)} \prod_{p(a,b)} \times Z_{p(a,b)} \tag{2}$$

, or, using Eq. (1),

$$\times Z = \prod_{(a,b)} \times Z_{p/(a,b)} \tag{3}$$

, the Cartesian product of sets of joint strategies in all possible future games (a, b). Any contingency in any future game (a, b) is defined as a *future contingency* p of the payoff-relevant game.

For any $(a, \times b, \times z)$, where $a \in A$, $\times b \in \times B$ and $\times z \in \times Z$, $U_m(a, \times b, \times z)$ denotes the expected utility payoff outcome for player $m = i, j, l$ if a, $\times b$ and $\times z$ are played in the payoff-relevant game.

For simplicity and, wlg., I define a *correlated strategy* on a strategy profile set T as a function τ from T to the Real interval $[0, 1]$ such that $(\tau(t))_{t \in T^{\subset}} \in \Delta T^{\subset}$ is a probability distribution over some <u>finite</u> strategy profile subset $T^{\subset}$ of T, and $\tau(t) = 0$ if $t \notin T^{\subset}$. The set of correlated strategies on T is denoted by $\mho^{T}$. A given correlated strategy τ may be implemented with a mediator that randomly chooses a profile t of pure strategies in $T^{\subset}$ with probability $\tau(t)$. Then the mediator would *recommend* each player, say i, j and l, *publicly* to implement the strategy t_i, t_j and t_l respectively. If such mediation is possible one has the equivalent of *direct unmediated communication possibilities.*

A vector of correlated strategies ϑ on a Cartesian product of action profile sets that depend on events $e \in E$ and denoted by $\times T = \prod_{e} T_e$ is defined as:

$$\vartheta = \prod_{e} \vartheta_e \tag{4}$$

, where ϑ_e is a correlated strategy on T_e. The interpretation is that if event $e \in E$ occurs, the mediator would implement correlated strategy ϑ_e, the e-th component of ϑ. The set of all vectors of correlated strategies on $\times T$ is denoted by $\mho^{\times T}$.

A *negotiation statement* for player i μ_i *suggests* play in the communication game *corresponding* to the payoff-relevant game. Any contingency in the communication game corresponds to a given contingency in the payoff-relevant game in the sense that besides past negotiation statements and recommendations by different negotiators and mediators respectively any such corresponding contingency includes the same sequence of choices and actions that led to the given contingency in the payoff-relevant game.

A negotiation statement μ_i is represented, abusing notation, by three components. The first component is a correlated strategy on action profile set A in the simultaneous link formation game, a *link promise-request* $\alpha_i \in \mho^{A}$. The second component is a correlated strategy on action profile sets in current contingencies a, a current *promise-request* $\beta_i \in \mho^{\times B}$. The third component is a correlated strategy on choice or action profile sets in future contingencies p, a *future-request* $\zeta_i \in \mho^{\times Z}$, using Eqs. (2) and (4). It is implicit that these suggestions are, wlg., the *same* regardless of specific recommendations that may have occurred as in the theory proposed in this paper credibility of negotiation statements will depend on past sequences of choices and actions and later on even on past negotiation statements but not on specific recommendations. For simplicity in the notation, we thus abstract from recommendations (See also 4.2.2).

Using Eqs. (3) and (4), it will be useful to express ζ_i as follows:

$$\zeta_i = \prod_{(a,b)} \zeta_{i,p/(a,b)} \tag{5}$$

, where $\zeta_{i,p/(a,b)} \in \mho^{\times Z_{p/(a,b)}}$.

In particular, if the negotiator announces $\beta_{i,a}$ $(\zeta_{i,p})$,for the corresponding current (future) contingency in the communication game to the current (future) contingency a (p) of the payoff-relevant game, she is *requesting* player j to obey player i's mediator according to $\beta_{i,a}$ $(\zeta_{i,p})$. She is *promising* to obey her own mediator according to $\beta_{i,a}$ $(\zeta_{i,p})$. She is requesting player j to obey player i's mediator according to $\beta_{i,a}$ $(\zeta_{i,p})$. The request to player l is trivial in this particular case.

A negotiation statement for player i is thus an element of $\mho = \mho^A \times \mho^{\times B} \times \mho^{\times Z}$ and it is denoted by $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \mho$.

A negotiation statement for player j is defined analogously and her negotiation statement $\mu_j \in \mho$.

To formalize the credibility, reliability and tenability of a negotiation statement whenever there are two simultaneous negotiators, one needs to deal first with the problem of conflicting simultaneous negotiation statements. To set up this problem precisely, I will define first a tenable and reliable statement for a player when she is the sole negotiator.

Let player i be the *sole* negotiator with negotiation statement $\mu_i = (\alpha_i, \beta_i, \zeta_i)$ given player $j's$ statement $\mu_j = (\alpha_j, \beta_j, \zeta_j)$, where the latter is to be regarded as noise. I assume in this section that there exists a well defined non empty *tenability correspondence* $Q : \mho \to\to \mho^{\times Z}$, where $Q(\mu_i)$ represents the set of all vectors of correlated strategies that could be rationally implemented by the players in future contingencies of the communication game p_c−that correspond to future contingencies p−following the negotiator's statement if they would believe negotiation statement μ_i. A negotiation statement μ_i is *future tenable* iff $\zeta_i \in Q(\mu_i)$. One writes then $\mu_i \in \underline{\mho} \subset \mho$.

Let $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \mho$ and, wlg., noise $\mu_j = (\alpha_j, \beta_j, \zeta_j)$ be given. If $a' \in A$ was played following μ_i in the communication game, consider the following *a'-concatenated*[2] *strategic form game* $\left(B_i \times B_j, \pi_{ij}^{\mu_i/a'}\right)$, where payoffs are given by

[2] The term concatenated is taken from Gibbons (1992) that uses the Nash equilibria of a one shot

$$\pi_{ij}^{\mu_i/a'}(b_i, b_j) = \left[\sum_z U_i(.)\Pr[U_i(.)], \sum_z U_j(.)\Pr[U_j(.)]\right] \quad (6)$$

if (b_i, b_j) is played, $U_m(.) = U_i(a', \times b, \times z)$, the a'-th component of $\times b$ is such that $b_{a'} = (b_i, b_j, b_l)$ and $\Pr[U_m(.)]$ is the probability that $U_m(.)$ results given that contingency $(a', b_{i,a'})$ occurred and play is consistent with ζ_i thereafter, for $m = i, j$.

Note that $\pi_l^{\mu_i/a'}(b_i, b_j)$, the associated payoff to player l can be computed analogously and $\pi^{\mu_i/a'}(b_i, b_j)$ would then refer to a payoff triplet for all players. Recall, b_l is trivial.

Let μ_i and the a'-concatenated game be given and hence players are expected to obey future-request ζ_i. A request in $\beta''_{i,a'}$ by player i is tenable if it is optimal for player j to obey player i's mediator given that player i is believed to fulfill his promise to obey the mediator. A promise in $\beta''_{i,a'}$ by player i is reliable if it is optimal for player i to obey the mediator given that player j is expected to obey the mediator. Equivalently, I will say that a promise-request $\beta''_{ia'}$ by player i is reliable and tenable given μ_i if $\beta''_{ia'}$ is a *publicly correlated equilibrium* of $\left(B_i \times B_j, \pi_{ij}^{\mu_i/a'}\right)$. A statement $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \mho$ is *current reliable and tenable* iff any promise-request $\beta_{i,a'}$ is reliable and tenable for all $a' \in A$ given μ_i.

Given $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \mho$, α_i is tenable and reliable iff α_i is a publicly correlated equilibrium of $\left(A_i \times A_j, \pi_{ij}^{\mu_i}\right)$, where payoffs for $\left(a'_i, a'_j\right) \in A_i \times A_j$ are

$$\pi_{ij}^{\mu_i}\left(a'_i, a'_j\right) = \sum_b \beta_{i,a'}(b)\, \pi_{ij}^{\mu_i/a'}(b_i, b_j) \quad (7)$$

, the expected payoffs for players i and j if current contingency a' occurs and play is consistent with β_i and ζ_i thereafter. A statement $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \mho$ is *link reliable and tenable* iff α_i is reliable and tenable.

A statement μ_i is reliable and tenable iff it is future tenable, link and current reliable and tenable. Such a statement will be said to belong to $\widetilde{\mho}$.

As for the Aumann (1990) critique, one should consider only $\mu_i = (\alpha_i, \beta_i, \zeta_i) \in \widetilde{\mho}$ where any $\beta_{i,a'}$ implies putting positive probability only on *self-signaling* Nash equilibria−that is player i wants to suggest any Nash equilibrium if and only if it is true (See Farrel (1993) for a detailed explanation of the term)−of $\left(B_i \times B_j, \pi_{ij}^{\mu_i/a'}\right)$ for all $a' \in A$. In my modification of A-M all Nash equilibria turn out to be self-signalling.

Analogously, one defines reliability and tenability of μ_j for player j whenever she is the sole negotiator and has her own mediator. Note that $\mu_i, \mu_j \in \mho$, so μ_i is tenable and reliable whenever player i is the sole negotiator if and only if μ_j is tenable and reliable whenever player j is the sole negotiator.

concatenated strategic form game (see figure 2.3.4) to find the subgame perfect equilibria of the two period repeated game in figure 2.3.3.

In case neither of the negotiation statements by players i and j are noise, the tenability of one player's statement$-$link, current and future tenability$-$depends on the statement of the other one. If one has conflicting requests, who would players obey if they are willing to obey either of the negotiators, or equivalently, if both negotiators' statements are tenable whenever they are the sole negotiators? The subsections that follow address this problem.

A simultaneous negotiation problem for players i and j as just described is denoted by $\Phi_{ij} = (A, B, \times Z, U, Q)_{ij}$.

3.2 O-F and Nash Coherent Joint Plans

3.2.1 Preliminary Definitions

We define for any two vectors x and y in $\mathbb{R}^2$

$x \geq y$ (x is *as least as good as* y) iff $x_i \geq y_i$ and $x_j \geq y_j$, and

$x > y$ (x is *strictly better* than y) iff $x_i > y_i$ and $x_j > y_j$, $i \neq j$.

A *bargaining problem* for agents i and j consists of a pair (F, ψ), where F is a closed convex subset of $\mathbb{R}^2$, $\psi = (\psi_i, \psi_j)$ is a vector in $\mathbb{R}^2$ and the set of *individually rational feasible allocations* (IRF *set*)

$F \cap \{(x_i, x_j) \,|x_i \geq \psi_i \text{ and } x_j \geq \psi_j \text{ or } x_{ij} \geq \psi_{ij}\}$

is non-empty and bounded. Here F represents the set of feasible payoff allocations or the *feasible set*, and ψ represents the disagreement payoff allocation or the *outside options*.

A bargaining game (F, ψ) is *essential* iff there exists at least one allocation x in F that is strictly better for agents than the disagreement allocation ψ, i.e., $x > \psi$.

A point x in F is *strongly (Pareto) efficient* iff there is no other point y in F such that $y \geq x$ and $x_w > y_w$ for at least one player $w \in \{i, j\}$. A point x in F is *weakly (Pareto) efficient* iff there is no other point y in F such that $y > x$. The *feasible frontier* is the set of feasible payoffs allocations that are strongly Pareto efficient in F. The IRF *frontier* is the set of points in F that are strongly Pareto efficient in the IRF set.

3.2.2 A Joint Plan Bargaining Problem

Before I develop a notion of credibility whenever negotiation statements are simultaneous by adding "Nash or O-F Nash effective cooperative negotiation", necessary conditions for simultaneous statements to be reliable and tenable *in this context* have to be given for these eventually to be credible.

Negotiation statements for both players are *similar* if $\mu_i = \mu_j$. A *joint plan* is a negotiation statement $\mu \in \mho$ such that there exists similar statements for players 1 and 2 and $\mu_1 = \mu_2 = \mu$. Abusing notation, μ will also refer to (μ_1, μ_2), where it may seldom be the case that $\mu_1 \neq \mu_2$, in which case there will be no confusion as the term joint plan will not be implicit! Such a joint plan μ is tenable and reliable iff

μ is tenable and reliable for player i or j whenever any of them is the sole negotiator. Finally, only joint plans can be tenable and reliable *in this context*.

If one would not restrict players to enunciate only tenable and reliable joint plans then tenable and reliable statements that happen to coincide could not be focal because it can be shown (at the reader's request) that there are both Nash equilibria of the communication game associated, say, to a payoff-relevant simultaneous game where tenable and reliable individual statements conflict and ones where such statements don't conflict. Outside players or the two players themselves would ask, does the pair mean what it says? Is the pair really agreeing? In a different way, how could the pair be agreeing if there is the possibility that the pair could enunciate such conflicting statements. For a pair of tenable and reliable statements to be focal this restriction is necessary.

Next, a *joint plan bargaining problem* (F, ψ, Φ_{ij}) for players i and j derived from a simultaneous negotiation problem $\Phi_{ij} = (A, B, \times Z, U, Q)$ is a bargaining problem (F, ψ) with two characteristics:

1. For each payoff pair $(x_i, x_j) \in F$, there exists an associated tenable and reliable joint plan
   $\mu = (\alpha, \beta, \zeta) \in \widetilde{\mho}$ such that $(x_i, x_j) = \sum_a \alpha(a)\, \pi^{\mu}_{ij}(a_i, a_j)$, where $\pi^{\mu}_{ij}(a_i, a_j)$ is as defined in Eq. (7).

2. The disagreement payoff allocation is $\psi = (x_i, x_j) = \pi^{\widehat{\mu}}_{ij}(\widehat{a}_i, \widehat{a}_j)$ the payoff associated to the *disagreement joint plan* $\widehat{\mu} = \left(\widehat{\alpha}_{\widehat{a}}, \widehat{\beta}, \widehat{\zeta}\right) \in \widetilde{\mho}$ that is tenable and reliable and derived as follows: Let the *disagreement future-request* for now be given by $\widehat{\zeta}$ (See 5.2.2 for derivation). The disagreement current promise-request $\widehat{\beta}$ is such that $\widehat{\beta}_{a'}$ for all $a'_{ij} \neq (y, y)$, corresponds to a unique Nash equilibrium strategy profile of $\left(B_i \times B_j, \pi^{\widehat{\mu}/a'}_{ij}\right)$; wlg., $\widehat{\beta}_{a'}$. if $a'_{ij} = (y, y)$ is arbitrary fixed to any value. Finally, wlg., $\widehat{\alpha}_{\widehat{a}}$ is a *degenerate* correlated strategy that puts probability one on unilateral rejections, that is $\widehat{\alpha}_{\widehat{a}}(\widehat{a}) = 1$, where $\widehat{a}_{ij} = (n, n)$.

Denote the $\widehat{a}$-*disagreement concatenated strategic form game without communication*—in the sense that players think that the communication link won't form—by $\left(B_i \times B_j, \pi^{\widehat{\mu}/\widehat{a}}_{ij}\right)$. Note that play suggested by this joint plan $\widehat{\mu}$ is self-enforcing in part because (n, n) is always a Nash equilibrium of any $\left(A_i \times A_j, \pi^{\mu_i}_{ij}\right)$.

Definition 3 *A tenable and reliable joint plan $\mu = (\alpha, \beta, \zeta)$ such that its link promise-request puts positive probability on link formation, that is, $\alpha(a) > 0$ where $a_{ij} = (y, y)$, and its current promise-request following a suggests with positive probability "identical and feasible" actions is called* ***successful*** *otherwise it is* ***unsuccessful*** *and one says preliminary negotiations or negotiators are successful or otherwise* ***unsuccessful***.

If $\alpha(a) = 1$, and current promise-request following a only suggests randomizations among "identical and feasible" actions, it is ***fully successful****.*

Note that the associated communication link can form only if $a_{ij} = (y, y)$ is played; recall that the link forms if in addition "identical and feasible" actions are played. Also $\widehat{\mu}$ is unsuccessful.

Definition 4 *The technology of communication implicit in definition 3 is characterized as* ***apparent and contingent*** *in a sense explained in 3.2.4, remark 5.*

Any such plan bargaining game will be denoted by (F, ψ, Φ_{ij}).

3.2.3 Nash Coherent Joint Plans

Define a *solution* of the joint plan bargaining problem (F, ψ, Φ_{ij}) to be a payoff pair $(x_i, x_j) \in F$ and an associated tenable and reliable joint plan $\mu \in \widetilde{\mho}$.

Players i and j can carry out negotiations *endogenously, Nash effectively* and *cooperatively* if given the simultaneous negotiation problem Φ_{ij}, they can construct and solve (F, ψ, Φ_{ij}) as follows:

1. The solution is derived from the *non transferable utility* (NTU) *Nash Bargaining Rule* (NBR) applied to the associated (F, ψ). The NTU NBR solution solves: $\arg\max_{x \in F(h),\, x \geq \psi} (x_i - \psi_i)\left(x_j - \psi_j\right)$.

2. If the IRF set is a singleton, i.e., $\nexists (x_i, x_j) \in IRF$ s.t. $x > \psi$ and the disagreement point is identical to any individually rational feasible payoff associated to any fully successful tenable and reliable joint plan, a *last-mover advantage* is assumed in the sense that the solution is required to consist of a fully successful tenable and reliable joint plan $\mu \in \widetilde{\mho}$ and hence the link would form if μ is followed through.

There is *endogenous cooperation* in the sense that failed cooperation is possible and *meaningful* as both successful and unsuccessful preliminary negotiations are possible Nash bargaining outcomes.[3]

A joint plan μ is *Nash Coherent* and hence a *credible joint plan* if it is the solution component of a joint plan bargaining problem (F, Φ_{ij}, ψ) where players can negotiate Nash effectively and cooperatively[4]. Whenever I want to refer to players i and j's set of Nash Coherent Joint Plans in Φ_{ij} given ψ, I write $\eta(\Phi_{ij}, \psi) \subset \widetilde{\mho}$.

[3] In standard bargaining problems disagreement or failed cooperation is not meaningful in the sense that in general it does not occur and if it "would occur" only the disagreement payoffs pair is obtained. In this paper disagreement is in contrast meaningful as different payoffs for the third player may occur after disagreement and an opportunity to form a permanent link has been not used.

[4] One can think of pairs having possibilities to set up a smooth Nash demand game that yields in the limint as unique equilibrium outcome the NTU NBR payoff. Then the unique Rabin's (1994)

3.2.4 Oldest-Friends Joint Plans

I will be interested in developing credibility criteria for simultaneous statements assuming instead that past joint plans by successful negotiators may influence a negotiation problem in a future contingency of the communication game p_c corresponding to p.

So let pairs of players, out of a total of three, take turns to conduct preliminary negotiations and then play a respective payoff-relevant game according to a finite rule of order in stages k of a corresponding multistage game with communication, where $k = 1, ...K + 1$. Also, let statements enunciated by different past pairs that were involved in preliminary negotiations be denoted by μ^{k^-}, i.e., $\mu^{k^-} = (\mu^1, ..., \mu^{k-1})$, where $\mu^t = \left(\mu^t_j, \mu^t_l\right)$ with $\mu^t_j = \mu^t_l$ as for restriction on 3.2.2, for $t = 1, ...k - 1$, $k > 1$ and $i \neq j \neq l$. To allow for such influence, it will be useful to think of contingencies in the communication game p_c having not just a past sequence of choices $a^{k^-}$and actions b^{k^-}but, in addition, a sequence of joint plans μ^{k^-}. The current negotiation problem at contingency p_c at stage $k-$history (defined in 4.2.2) of the corresponding multistage game with communication is then denoted by $\Phi_{ij,\mu^{k-}}$ and the tenability correspondence by $Q_{\mu^{k-}}$.

To formulate these criteria, I make the following assumption:

Oldest-Friends Focal Effect: Let one or both players in the pair of negotiators be indifferent between joint plans with payoffs in the IRF set of a joint plan bargaining problem $\left(F^k, \psi^k, \Phi_{il,\mu^{k-}}\right)$. If $k > 1$, the solution to $\left(F^k, \psi^k, \Phi_{il,\mu^{k-}}\right)$ involves the payoff in the IRF that is future-requested in the tenable and reliable joint plan by the *oldest pair* of successful negotiators that formed their link$-$according to the rule of order$-$among the past preliminary negotiators that included one of the indifferent players. Otherwise $\eta\left(\Phi_{il,\mu^{k-}}, \psi^k\right)$ is used.

The credible joint plans that are predicted under this assumption will be defined as *Oldest-Friends Joint Plans (O-F Joint Plans)* and its set is denoted by $\eta^f\left(\Phi_{ij,\mu^{k-}}, \psi^k\right)$. One then says that pairs can carry out negotiations *endogenously, O-F Nash effectively* and *cooperatively.* For existence purposes, specially when action sets are infinite, note that the outside options depend on past sequences of choices and actions in the multistage game with communication.

Remark 5 *It is said that there is an apparent and contingent technology of communication because even though a link may form if a successful, reliable and tenable joint*

negotiated equilibrium in a game with preplay communication where there is such a payoff-relevant bargaining game would be associated to the tenable and reliable joint plan that yields the NTU NBR. Alternatively and without any *cooperative transformation* (See Myerson (1991)) of the original payoff-relevant game, one could assume that pairs focus in the Rabin's negotiated equilibrium equilibrium that yields the NBR prediction. Without any of these assumptions, unique payoff predictions are only possible if negotiations are lengthy and one has pure coordination payoff relevant games.

plan is enunciated, link formation does not occur in equilibrium of the communication game unless such joint plan is in addition Nash Coherent or O-F <u>*and*</u> *it is not the case that both choose a unilateral rejection or no "identical and feasible" action suggested with positive probability is played.*

Remark 6 *Disagreement joint plans enunciated by older pairs of negotiators are trivially followed as any* $\widehat{\mu} = \left(\widehat{\alpha}_{\widehat{a}}, \widehat{\beta}, \widehat{\zeta}\right)$ *future-requests in* $\widehat{\zeta}$ *optimal future play in case of disagreement (See Remark 8).*

Remark 7 *If a pair can carry out negotiations endogenously, either Nash or O-F Nash effectively and cooperatively, contingencies of the communication game,* p_c*, include* <u>*only*</u> *tenable and reliable joint plans, that is,* μ^{k^-} *consists of* $\mu^t = \left(\mu_j^t, \mu_l^t\right)$*, such that* $\mu_j^t = \mu_l^t \in \widetilde{\mho}^t$ *for all* $t = 1, ...k-1$*,* $k > 1$*. Hence, abusing notation,* μ^t *could be regarded as a tenable and reliable joint plan with no risk of confusion whatsoever from now on, i.e.,* $\mu^t = \mu_j^t = \mu_l^t$

Remark 8 *It is implicit in Remark 6 that if any tenable and reliable joint plan that is not part of a solution to an older joint plan bargaining problem is enunciated, this joint plan still may*[5] *influence future play in the relevant subgames of the communication game*[6]*.*

The situation in last Remark 8 is analogous to the case where say only the male enunciates a negotiation statement a day before the battle of the sexes game is played. Suggesting both going to the Ballet concert influences play in the subgame that follows this statement where the battle of the sexes game is played (because it suggests a Nash equilibrium in that subgame) in the second day. However, it is not even a Nash equilibrium in the whole game that begins the day before. when the male statement is enunciated, because "its not the best the male can say" (See Myerson (1991), pp. 110-111).

O-F Joint Plans Formal Definition Let $i, j, l \in \{1, 2, 3\}$, and $i \neq j \neq l$. Suppose players i and j had successful preliminary negotiations, link ij formed and have enunciated, as part of their future-request, the tenable and reliable joint plan $\gamma \in \widetilde{\mho}_{il}$ and only then j and l successfully negotiated, formed link jl and future-requested $\delta \in \widetilde{\mho}_{il}$ where it maybe that $\gamma \neq \delta$. Schematically, as the bargaining problem for i and l follows, one has the following physical apparent and contingent order of link formation:

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[5] For that to happen the plan has to be succesful, the link has to form and the indiference cases have to occur.

[6] These are "credible neologisms" in the sense of Farrel (1993). That is they would be understood and would signal bilateral cooperation and believed if the pair of negotiators would enunciate such tenable and reliable joint plans. However, these are not "the best joint plan" the pair can enunciate because these are not associated to the NBR.

$(i,j) \qquad (j,l) \qquad (i,l)$.

For all essential bargaining problems for i and l, set

$\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \eta\left(\Phi_{il,\mu^{k-}}, \psi^k\right)$.

Otherwise:

<u>Case 1.</u> If $\nexists (x_i, x_l) \in IRF^k$ s.t. $x_i^k > \psi_i^k$, however $\exists (x_i, x_l) \in IRF^k$ s.t. $x_l^k > \psi_l$, set $\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \gamma$;

<u>Case 2.</u> If $\nexists (x_i, x_l) \in IRF^k$ s.t. $x_l^k > \psi_l^k$, however $\exists (x_i, x_l) \in IRF^k$ s.t. $x_i^k > \psi_i^k$ set $\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \delta$;

graphically, in the plane (x_i, x_l), the IRF^k set for $\left(F^k, \psi^k, \Phi_{il,\mu^{k-}}\right)$ is a straight closed vertical and horizontal closed segment respectively.

<u>Case 3.</u> If $\nexists (x_i, x_l) \in IRF^k$ s.t. $x^k > \psi^k$

set $\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \gamma$.

In words, there are three cases in which the assumption turns out to imply a not essential $\left(F^k, \psi^k, \Phi_{il,\mu^{k-}}\right)$ to be "effectively" a singleton. As oldest friends' tenable and reliable statements are the only ones that are credibly understood by their literal meanings, the only possible payoff $(x_i, x_l) \in IRF^k$ and associated joint plan to be bargained about by players i and l is the one that confirms the joint plan by the oldest successful pair of friends that has one of its member, i or l, indifferent between any payoff in IRF^k.

In addition, if one only has pair (i,j) enunciating as part of its future-request $\gamma \in \widetilde{\mho}_{il}$ and thus one has schematically,

$(i,j) \qquad (i,l)$,

For all essential bargaining problems for i and l, set

$\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \eta\left(\Phi_{il,\mu^{k-}}, \psi^k\right)$.

Otherwise

<u>Case 1.</u> $\nexists \left(x_i^k, x_l^k\right) \in IRF^k$ s.t. $x_i^k > \psi_i^k$, however $\exists (x_i, x_l) \in IRF^k$ s.t. . $x_l^k > \psi_l^k$ set $\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \gamma$;

<u>Case 2.</u> $\nexists (x_i, x_l) \in IRF^k$ s.t. $x^k > \psi^k$

set $\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \gamma$.

If there are no past successful negotiators, that is, no apparent and contingent links have formed,

$\eta^f\left(\Phi_{il,\mu^{k-}}, \psi^k\right) = \eta\left(\Phi_{il,\mu^{k-}}, \psi^k\right)$.

In contrast to the case with a last-mover advantage, in *all* non essential bargaining games in this bilateral sequential negotiation and bargaining environment an O-F Joint Plan may be the disagreement one, in which case, unsuccessful preliminary negotiations occurs and the link does not form.

It is clear that if the tenability correspondence is non-empty and the joint plan bargaining game is well defined then, depending on the assumption, either Nash

Coherent or O-F Joint Plans exist in any corresponding contingency p_c (corresponding to p should be understood) of the communication game. In what follows, I develop a theory of rational behavior in corresponding future contingencies in the latter game and formalize the idea of a history whenever the underlying payoff-relevant game is my modification of the A-M game. This theory will be relevant for the construction of the tenability correspondence that has been assumed so far as given, in particular, the disagreement future-request $\widehat{\zeta}$. More importantly, it will make it possible to define Sequentially Nash Credible Joint Plans.

4 Preliminary Definitions

4.1 Notation for Graphs

Denote by $N = \{1, 2, 3\}$ the set of players. A graph g is a set of unordered pairs of distinct agents belonging to N. Each pair is represented by a link (non-directed segment) between the two players (nodes). Thus, g stands also for the set of links for graph g.

We denote by ij ,or equivalently ji, the link that joins agents i and j, where $i \neq j \neq l,\ i, j, l \in N$. If $ij \in g$, we say that i and j are *directly linked* in graph g. Iff ij, $jl \in g$, we say that i and l are *indirectly* linked by j.

We use often ij as a superscript for referring to the graph g that contains only link ij, say g^{ij}. In turn, the superscript ijl would refer to the graph where only player j is directly linked to two agents. Later on, we will distinguish among different orderings of ijl representing the order in which links have been formed.

The graph where every pair is directly linked, or linked from now on, is called the *complete* graph, and is denoted by g^N. The empty graph where no pair is linked is represented by $g^{\varnothing}$.The set G of all possible graphs on N is $\{g : g \subseteq g^N\}$. We use, $g^{\theta+ij}$ when referring to the graph that results to adding link ij to graph g^{θ}, where $\theta \in \{\varnothing, il, ilj\}$ $i \neq j \neq l$, $i, j, l \in \{1, 2, 3\}$.

Let $B \subseteq N$, $g \subseteq G$, $i \in B$, $j \in B$ be given. Agents i and j are *connected* in B by g iff there is a *path* in g from i to j and stays within B. That is, iff i and j are directly or indirectly linked under some g', where g' is such that $g' \subseteq g$ and $g' \subseteq G'$, and G' is the set of all graphs of B.

4.2 Payoffs in Communication Structures as Graphs

Let a *cooperative game* v be given with N as the player set. Given N, let CL be the set of all coalitions (non-empty subsets) of N, $CL = \{B \subseteq N, B \neq \varnothing\}$. A *characteristic function* $v : CL \to \mathbb{R}$ associates the maximum wealth or transferable utility (TU) *payoff* achievable if the coalition $B \in CL$ forms and coordinates effectively.

There are intermediate cases between N-player games that are played cooperatively and non-cooperatively. For predicting payoff outcomes in these cases, Myerson

(1977) assumes that effective coordination can occur if pairs of players establish at least bilateral agreements or friendship relationships that are represented by links of communication. For example a link between two agents lets them get all the benefits of effective coordination. In this context a set of links is denoted equivalently as a cooperation, communication or *cooperation structure*.

Let a coalitional game v be given with N as player set and g as the cooperation structure. For each player i and given the graph g and the characteristic function v, the Myerson value for player i is denoted by $\phi_i^g = \phi_i^g(v)$.

Define $B|g$ as the unique *partition* of B in which groups of players are together iff they are connected in B by g. Loosely speaking, it is the collection of smaller coalitions, or connected *components* of $B|g$, into which B would break up, if players could only coordinate along the links in g.

I founded this practical method by Myerson (1977) to be useful to give intuition and to derive the Myerson values: Given v and g, define a coalitional game v^g by

$v^g(S) := \sum v^g(S_j)$,

where the sum ranges over the connected components S_j^g of $S|g$. Then

$\phi_i^g(v) = \phi_i(v^g)$

where ϕ_i denotes the ordinary Shapley (1953) value for player i.

In words the Myerson value is the Shapley value of an auxiliary cooperative game where any given coalition gets all its worth provided all players in that coalition are at least indirectly linked. Otherwise the payoffs in that coalition are the sum of the worth of its subcoalitions that in contrast get all their worth (including possible trivial singleton coalitions).

I normalize three-player cooperative games by focusing in characteristic functions $v : CL \to [0, d]$ with

$v(1) = 0, \quad v(2) = 0, \quad v(3) = 0, \quad a$

$v(13) = a, \quad v(23) = b, \quad v(12) = c, \quad v(123) = d,$

where $d =$. These are defined as *normalized cooperative games.*

4.3 Adding Useful Notation in the A-M Model

Consider g^N, where $N = 3$, i.e., $g = \{(1,2), (2,3), (1,3)\}$. The rule of order according to which pairs of players propose links in A-M can be represented by the function $\rho_\emptyset : g^3 \to \{1, 2, 3\}$. Wlg., I will assume a fixed $\rho_\emptyset$, where

$\rho_\emptyset(12) = 1 \quad \rho_\emptyset(23) = 2 \quad \rho_\emptyset(13) = 3.$

The interpretation is that pair $(1, 2)$ in the initial history as of stage 1 discusses the *first* link 12 in the game. If 12 is rejected, 23 follows, and if 23 is in turn rejected, 13 follows. If 13 is rejected the game ends

If a first link ij has just been accepted I will write that a *first round of play* has been completed. Suppose that is the case. The rule of order for the *left out* pairs to propose a *second* link in the game,

$\rho_{ij} : g^3 \backslash g^{ij} \to \{1, 2\}$, for $i, j \in \{1, 2, 3\}$, $i \neq j$,

is derived from $\rho_\emptyset$ and one has:
$\rho_{12}(23) = 1, \rho_{12}(13) = 2,$
$\rho_{23}(13) = 1, \rho_{23}(12) = 2$ or
$\rho_{13}(12) = 1, \rho_{13}(23) = 2$
depending on either link 12, 23 or 13 being the first to form respectively. The interpretation is analogous as before. In particular, if all left out pairs reject the game ends.

If two links have just been accepted, and thus a *second* round of play has been completed, the pair not linked yet is next. If the left out pair rejects, the game ends. If the *third* round of play has been completed (and thus, three links have formed) the game ends.

Given $\rho_\emptyset$, an *A-M-history* is a sequence of links acceptances and rejections. If the game ends, then an A-M *final* history is reached. Except for the latter, each history has an *immediate prospective graph*$-$the one that would result if the associated link being proposed forms. The immediate prospective graph that *may* result after link decisions have been made is defined as the *next* prospective graph. Unless otherwise stated let, *from now on*, $\theta \in \{\varnothing, il, ilj\}$ $i \neq j \neq l$, $i, j, l \in \{1, 2, 3\}$. Also, let k be the stage of the game one is at and $\rho_\emptyset$ be given. An immediate prospective graph will be denoted by $g^{\theta+ij}$.

I assume that the order of ijl matters. Non final histories are then denoted uniquely by $h^k_{AM}\left(g^{\theta+ij}\right)$. For example, only $h^1_{AM}\left(g^{\varnothing+12}\right)$ stands for the initial history. If link 12 is rejected, the next history is denoted uniquely by $h^2_{AM}\left(g^{\varnothing+23}\right)$ and so on. History $h^5_{AM}\left(g^{13+32}\right)$, or equivalently $h^5_{AM}\left(g^{132}\right)$, corresponds to link 13 being the first link to form, following ρ_{13}, link 12 being rejected so that link 32 is next to be discussed in stage 5. Analogously, $h^3_{AM}\left(g^{123+13}\right)$ has third link 13 next to be proposed in stage 3 after link 12 formed in stage 1 and link 23 was accepted in stage 2.

With respect to payoff outcomes, let g^θ be the last graph to form at the end of the game. Then each player gets her Myerson value in graph g^θ. In particular, if in history $h^k_{AM}\left(g^{ilj+ij}\right)$ link ij is accepted then players get their Myerson value in the complete graph. Otherwise payoffs are zero.

5 A Modification of the A-M Game

I want to add endogenous, O-F Nash effective cooperative negotiation, as defined in section 3.2.4, each time pairs of players have preliminary negotiations in my modification of the A-M model. Next I derive the tenability correspondence. Hence, I first describe my modification.

5.1 The Abstract Model

I will consider a $K+1$-*multistage game with payoff-relevant observed actions* M based in Fudemberg and Tirole (1991), however with substages.

5.1.1 Choice and Actions Sets and Histories

At the beginning of the first stage 1, all players $m = 1, 2, 3$ select simultaneously from choice sets A_{m,h^1}, where A_{m,h^1} for the player m that is not associated to the link being proposed has a trivial unique payoff-irrelevant action "move nothing". For the other two players $A_{m,h^1} = \{y, n\}$. A permanent communication link may forms, if both latter players play y. I let the initial history be $h^1 = \varnothing$ at the start of play. At the end of the first substage, all players observe the substage 1's choice profile. Let $a^1 = (a_1^1, a_2^1, a_3^1)$ be the first substage's choice profile. At the beginning of the second substage players know history $h^{1.2}$ that can be identified with a^1 given that h^1 is trivial. In the second substage, regardless of a^1, all players $m = 1, 2, 3$ choose simultaneously from the same action sets, that is, $B_{m,h^{1.2}} = B_{m,h^1}$, $m = 1, 2, 3$, where B_{m,h^1} for the player m that is not associated to the link being proposed has a trivial unique payoff-irrelevant action. At the end of the second substage, all players observe the second substage's action. Let $b^1 = (b_1^1, b_2^1, b_3^1)$ be the second substage action profile. At the beginning of stage 2 players know history h^2 that can be identified with (a^1, b^1) or, equivalently, $(h^{1.2}, b^1)$.

In general, choices and actions for player m will depend on previous choices and actions, so I let A_{m,h^2} denote the action set for player m at history h^2 and B_{m,h^2} denote the action set for player m at history $h^{2.2}$. By iteration, histories in general are

$h^k = \left(a^1, b^1, a^2, b^2, ..., a^{k-1}, b^{k-1}\right)$

and

$h^{k.2} = \left(a^1, b^1, a^2, b^2, ..., a^{k-1}, b^{k-1}, a^k\right)$

and B_{m,h^k} is the action set for player m at stage k when the history is $h^{k.2}$ and A_{m,h^k} is the action set for player m at stage k when the history is h^k. I let $K + 1$ be the total number of stages in the game. By definition each h^{K+1} describes an entire sequence of choices and actions from the start of the game on. I denote H^{K+1} as the set of all terminal histories that can be identified with the set of possible *outcomes* when the game is played.

Note that this is a model where pairs of players have non trivial stage action sets whenever they follow−depending on a link being formed or not and according to the rule of order−to propose a link. The third player has a trivial unique payoff-irrelevant choice or action. If the last pair in the rule of order played the associated stage games all players move nothing there after, that is choice and action profile sets are singletons there after. Let $K + 1$ be the total number of stages in the game. By definition each h^{K+1} describes an entire sequence of actions from the start of the game on. I denote H^{K+1} as the set of all terminal histories that can be identified with the set of possible *outcomes* when the game is played.

5.1.2 Pure Strategies and Payoff Outcomes

A pure strategy for player i is a contingent plan on how to play in the first and second substage at stage k of the game for respective possible histories h^k and $h^{k.2}$. I let H^k or $H^{k.2}$ denote the set of all substage k-histories, and

$A_{i,H^k} = \cup_{h^k \in H^k} A_{i,h^k}$ and

$B_{i,H^{k.2}} = \cup_{h^{k.2} \in H^{k.2}} B_{i,h^{k.2}}$.

A pure strategy for player i is a sequence of maps $\{s_i^k\}_{k=1}^K$, where each s_i^k maps $H^k \cup H^{k.2}$ to the set of player i's feasible choices A_{i,H^k} and actions $B_{i,H^{k.2}}$ (i.e., satisfies $s_i^k(h^k) \in A_{i,h^k}$ and $s_i^k(h^{k.2}) \in B_{i,h^{k.2}}$ for all $h^k \in H^k$ and $h^{k.2} \in H^{k.2}$). The set of all pure strategies for player i in the payoff-relevant multistage game is denoted by S_i.

A sequence of choices and actions for a profile for such strategies $s \in S$ is called the *path* of the strategy profile, where S is the set of all strategy profiles: the first substage choices are $a^1 = s^1(h^1)$. Second substage actions are $b^1 = s^1(a^1)$. The first substage choices in stage 2 are $a^2 = s^2(a^1, b^1)$. The second substage actions in stage 2 are $b^2 = s^2(a^1, b^1, a^2)$ and so on. Since the terminal histories represent an entire sequence of play or path associated with a given strategy profile, one can represent each players' corresponding *overall's* payoff as a function $u_i : H^{K+1} \to \mathbb{R}$. Abusing notation, I denote the payoff vector to profile $s \in S$ as $u(s) = u(h^{K+1})$, as one can assign an outcome in H^{K+1} to each strategy profile $s \in S$.

5.1.3 Nash Equilibrium

A pure-strategy *Nash equilibrium* in this context is a strategy profile s such that no player i can do better with a different strategy or, using standard Fudemberg and Tirole's (1991) notation, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

5.1.4 Subgameperfect Equilibrium

Since all players know the history h^k or $h^{k.2}$, one can view respectively the game from stage k on with history h^k or $h^{k.2}$ as an extensive form game in its own and denote it by $M(h^k)$ or $M(h^{k.2})$. To define the payoff functions in this game, note that if the sequence of choices and actions or path in stages k through K are a^k or b^k through b^K, the final history will be $h^{K+1} = (h^k, a^k, b^k, ..., b^K)$ or $h^{K+1} = (h^{k.2}, b^k, ..., b^K)$. The payoffs for player i will be $u_i(h^{K+1})$.

Strategies in $M(h^k)$ or $M(h^{k.2})$ respectively are defined in a way where the only histories one needs consider are those consistent with h^k or $h^{k.2}$. Precisely, any strategy profile s of the whole game induces a strategy profile $s|h^k$ on any $M(h^k)$ or $s|h^{k.2}$ on any $M(h^{k.2})$. For each i, $s_i|h^k$ or $s_i|h^{k.2}$ is the *restriction* of s_i to the histories consistent with h^k or $h^{k.2}$. One denotes the restriction profile set by $S|h^k$ or $S|h^{k.2}$.

Let histories h^{K+1} be such that $h^{K+1} = (h^k, a^k, b^k, ..., b^K)$ or $h^{K+1} = (h^{k.2}, b^k, ..., b^K)$ and the associated subset of H^{K+1} be denoted by $H^{K+1}(h^k)$ or $H^{K+1}(h^{k.2})$. As one can assign respectively an outcome in $H^{K+1}(h^k)$ or $H^{K+1}(h^{k.2})$ to each restriction profile $s|h^k$ or $s|h^{k.2}$ where $s \in S$, the overall payoff vector to the restriction $s|h^k$or $s|h^{k.2}$, will be denoted abusing notation by $u(s|h^k)$ or $u(s|h^{k.2})$. Thus, one can speak of Nash equilibria of $M\left(h^k\right)$ or $M\left(h^{k.2}\right)$.

It will be useful to express $S|h^k$ as a set of points rather than a set of mappings. Hence, consider the following Cartesian product derived recursively for all h^k:

$$S|h^k = A_{h^k} \times B^4_{h^k} \times \prod_{\left(a^k,b^k\right)} S|\left[h^k, a^k, b^k\right] \tag{8}$$

. A strategy profile s of a multi-stage payoff-relevant game with observed actions M is a *subgame-perfect equilibrium* if, for every h^k and $h^{k.2}$, the restriction $s|h^k$ and $s|h^{k.2}$ to $M\left(h^k\right)$ and $M\left(h^{k.2}\right)$ respectively is a Nash equilibrium of $M\left(h^k\right)$ and $M\left(h^{k.2}\right)$.

5.1.5 Interpretation

In history h^1, I define a next prospective graph−as defined in the A-M model−stage outcome function that depends on an element of the link choice profile set A_{h^1} and the initial immediate prospective graph $g^{\varnothing+12}$ as follows: Link 12 may form if a^1 is such that $a^1_{12} = (y, y)$ and so would graph $g^{\varnothing+12}$. The next prospective graph would be g^{12+23} following the A-M rule of order. If $a_{12} \neq (y, y)$, link 12 is rejected and the next pair in the rule of order $\rho_\emptyset$ follows, that is link 23 is proposed. The next prospective graph in this case is $g^{\varnothing+23}$.

After any outcome in the first substage the second substage action set for agent $m = 1, 2$ consist of payoff *proposals* pairs and it is the same. Formally,

$B_{m,h^{1.2}} = B_{m,h^1} = \{b_m = (b(1), b(2))\, |b(1) \geq 0, b(2) \geq 0\}$,

i.e., player m, proposes a payoff for player 1, $b(1)$, and one for player 2, $b(2)$; both are restricted to be non-negative.

For player 3, the choice set $B_{3,h^{1.2}} = B_{3,h^1}$ is the singleton "do nothing".

Player m's payoff pair proposal is *feasible,* iff

$b_m(1) + b_m(2) = \phi_1^{g^{\varnothing+12}} + \phi_2^{g^{\varnothing+12}}$, $m = 1, 2$.

In words, a proposals pair by player m is feasible, iff its components add up to the sum of both agents' (1 and 2) Myerson values in the immediate prospective graph $g^{\varnothing+12}$.

Proposals pairs *coincide* iff $b_1 = b_2$. Proposals pairs *match* for player's 1 and 2, iff their proposals pairs are feasible and coincide. A proposals pair by player m is called a *unilateral proposals pair rejection* if b_m is not feasible. I define b to be a *proposal match* iff proposals pairs for player's i and j match. Otherwise b is not a proposal match. A link forms, if and only if after $a^1_{12} = (y, y)$, proposal pairs match. Otherwise the link does not form. For a given transfer scheme to be binding players i and j have to match proposals pairs.

It will be useful to index a history in the payoff-relevant multistage game by its immediate prospective graph. The initial history is then arbitrarily indexed as $h^1_{g^{\emptyset+12}(\varnothing)}$.

A generic history in any stage k that had the sequence .. $\left(a^1, b^1, .., a^{(k-1)}, b^{(k-1)}\right)$ and led to immediate prospective graph $g^{\theta+ij}$ is denoted by $h^k_{g^{\theta+ij}\left(a^1,b^1,..,a^{(k-1)},b^{(k-1)}\right)}$. Whenever much specificity is not necessary, one writes $h^k_{g^{\theta+ij}\left(a^{(k-1)},\beta^{(k-1)}\right)}$, $h^k_{g^{\theta+ij}(.)}$ or even $h^k_{(.)}$.

In general, the stage k choice pair set in history $h^k_{g^{\theta+ij}(.)}$ for player $m = i, j$ is equal $A_{m,h^k} = \{y, n\}$. Such a set A_l for player l has a trivial unique payoff-irrelevant choice, "move nothing".

The payoff action profile set for players i, j and l is denoted by

$B_{h^k_{g^{\theta+ij}(.)}} = \left\{B_{m,h^k_{g^{\theta+ij}(.)}}\right\}_{m=i,j,l}$

where $B_{l,h^k_{g^{\theta+ij}(.)}}$ is trivial and $B_{m,h^k_{g^{\theta+ij}(.)}} = B_{m,h^1}$ for $m = i, j$.

But for $g^{\theta+ij} = g^N$, player m's proposals pair is feasible, iff

$b_m(i) + b_m(j) = \phi_i^{g^{\theta+ij}} + \phi_j^{g^{\theta+ij}}$,

for $m = i, j$.

If $g^{\theta+ij} = g^N$, I define there to be only one feasible proposals pair, that associated to the Myerson values in the complete graph. This is given by

$b_m = (b(i), b(j)) = \left(\phi_i^{g^N}, \phi_j^{g^N}\right)$, for $m = i, j$.

As before, b is a proposal match iff proposals pairs for player's i and j match. Sometimes, I refer to a proposal match and its components by simply b and $(b(i), b(j))$ instead of b and

$[(b_i(i), b_i(j)), (b_j(i), b_j(j)), b_l]$ respectively.

In history $h^k_{g^{\theta+ij}(.)}$, link ij forms if forms if and only if a^k is such that $a^k_{ij} = (y, y)$ and a proposal match occurs, and so does graph $g^{\theta+ij}$. Also for a given transfer scheme to be binding players i and j have to match proposals pairs. The next prospective graph follows according to the A-M rule of order. If $a_{12} \neq (y, y)$ link ij is rejected. If that is the case only the next pair in the rule of order $\rho_\emptyset$ may follow. Analogously, a next prospective graph may follow or not.

Following the A-M game, if the game ends, all agents move "nothing" thereafter until stage K. Outcomes are non existent or trivial as links cannot be formed anymore. After stage K the final history $K + 1$ follows.

Payoffs outcomes are realized at the end of stage k whenever the game ends in A-M. In that case, the last pair that formed a link receives its payoff proposal match and the third player receives her Myerson value in the resulting last graph.

Formally, the overall's payoff function $u = (u_1, u_2, u_3)$ is constructed from payoff functions ν in all possible non final histories as follows:

If the immediate prospective graph is the complete graph, that is, the associated history $h^{k.2}_{g^{\theta+ij}\left(a^{-k},b^{-k},a^k\right)}$ is such that $g^{\theta+ij} = g^N$, *and* link ij forms, that is the case only provided $a^k_{ij} = (y, y)$ and proposals match, then the three players get their Myerson value in the complete graph, i.e., player m gets $\nu_{m,h^{k.2}_{g^{\theta+ij}(.)}}\left(b^k\right) = \phi_m^{g^N}$ for $m = i, j, l$, where b^k is the proposal match that leads to g^N.

Suppose $i \neq j \neq l$, $i, j, l \in \{1, 2, 3\}$.

Let $\theta = ilj$. Suppose one is at history $h^{k.2}_{g^{\theta+ij}\left(a^{-k},b^{-k},a^k\right)}$, where $a^k_{ij} \neq (y, y)$, and regardless of any b^k being played, link ij does not form (thus, it is assumed that play of any b^k is payoff irrelevant), and hence, the game ends in the sense of A-M, then the stage payoffs for players i, j and l are given by

$\nu_{i,h^{k.2}_{g^{\theta+ij}\left(b^{k-1}\right)}}\left(b^k\right) = \phi^{g^\theta}_i$,

$\nu_{j,h^{k.2}_{g^{\theta+ij}\left(b^{k-1}\right)}}\left(b^k\right) = b^{k-1}(j)$ and

$\nu_{l,h^{k.2}_{g^{\theta+ij}\left(b^{k-1}\right)}}\left(b^k\right) = b^{k-1}(l)$,

where b^{k-1} is the *last proposal match* that occurred in stage $k-1$ where the last link lj was accepted and thus g^θ formed.

Analogously, let $\theta = il$. Suppose that $a^k_{ij} \neq (y, y)$ and that the game would end in the sense of A-M in such a case, then payoffs regardless of b^k are

$\nu_{i,h^{k.2}_{g^{\theta+ij}\left(b^{k-2}\right)}}\left(b^k\right) = b^{k-2}(i)$,

$\nu_{j,h^{k.2}_{g^{\theta+ij}\left(b^{k-2}\right)}}\left(b^k\right) = \phi^{g^\theta}_j$ and

$\nu_{l,h^{k.2}_{g^{\theta+ij}\left(b^{k-2}\right)}}\left(b^k\right) = b^{k-2}(l)$,

where $k-2$ is the stage where the last proposal match, b^{k-2}, occurred.

Let $\theta = \varnothing$. Suppose that $a^k_{ij} \neq (y, y)$ and that the game would end in the sense of A-M in such a case, then payoffs regardless of $b^k$are

$\nu_{h^{3.2}_{g^\varnothing+13(.)}}\left(b^3\right) = \left(\phi^{g^\varnothing}_1, \phi^{g^\varnothing}_2, \phi^{g^\varnothing}_3\right) = (0, 0, 0)$,

In any other $k = 1, ...K$ substage history payoffs are zero.

There is no discounting. Thus, player $m's$, for $m = 1, 2, 3$, overall payoff at the h^{K+1} terminal history that has as past history the outcome where the game "ends" at stage $k \leq K$ with a graph g^θ, where $\theta = ilj$, is given by

$u_m\left(h^{K+1}\right) = \nu_{m,h^{k.2}_{g^{\theta+ij}\left(b^{k-1}\right)}}\left(b^k\right)$.

For example, $u_j\left(h^{K+1}\right) = b^{k-1}(j)$ (See above).

Analogously, if the game ends with a graph g^θ, where $\theta = il$, one has

$u_m\left(h^{K+1}\right) = \nu_{m,h^{k.2}_{g^{\theta+ij}\left(b^{k-2}\right)}}\left(b^k\right)$.

If the game ends with a graph g^θ, where $\theta = \varnothing$, one has

$u\left(h^{K+1}\right) = \nu_{h^{3.2}_{g^\varnothing+13(.)}}\left(b^3\right) = \left(\phi^{g^\varnothing}_1, \phi^{g^\varnothing}_2, \phi^{g^\varnothing}_3\right) = (0, 0, 0)$.

If the game ends with a graph $g^{\theta+ij} = g^N$ then

$u_m\left(h^{K+1}\right) = \nu_{m,h^{k.2}_{g^{\theta+ij}(.)}}\left(b^k\right) = \phi^{g^N}_m$.

5.1.6 Vectors of Correlated Strategies

I will be interested in defining negotiation statements at any history of an associated multistage game with communication represented as a vector of correlated strategies in a "corresponding" history of the payoff-relevant multistage game. Because mixed payoff proposals would have zero probability of inducing any payoff proposal match, as I claimed earlier on in section 3, the following formalization that uses my definition of correlated strategies is wlg.:

A vector of correlated strategies is a sequence of maps $\{\omega^k\}_{k=1}^K$, where each ω^k maps H^k and $H^{k.2}$ to the set of correlated strategies on elements of A_{H^k} and $B_{H^{k.2}}$ (i.e., $\omega^k(h^k)$ is a correlated strategy on A_{h^k} for all $h^k \in H^k$ and $\omega^k(h^{k.2})$ is a correlated strategy on B_{h^k} for all $h^{k.2} \in H^{k.2}$). I denote by $W|h^1$ the set of all vectors of correlated strategies in history h^1.

Given $\omega|h^1 \in W|h^1$, I am interested in the probability $\Pr[s/\omega|h^1]$ of the path $\left(a^1, b^1, a^2, ..., a^K, b^K\right)$ corresponding to strategy profile $s \in S$. This will be given by the expression

$\Pr[s/\omega|h^1] = \omega^1_{h^1}(a^1) * \omega^1_{(a^1)}(b^1) * \omega^2_{(a^1,b^1)}(a^2) *, ..., *\omega^K_{(a^1,b^1,a^2,...,a^K)}\left(b^K\right).$

Let $\omega|h^k \in W|h^k$ be the set of all vectors of correlated strategies in the subgame that begins in history h^k. It will be also of interest to know the probability $\Pr\left[\left(s|h^k\right)/\omega|h^1\right]$ of the path $\left(h^k, a^k, b^k, ..., a^K, b^K\right)$ corresponding to the restriction $s|h^k$ of $s \in S$ on $M\left(h^k\right)$ for any $h^k \in H^k$ for all k. This will be given by the expression

$$\Pr\left[\left(s|h^k\right)/\omega|h^k\right] = \omega^k_{h^k}\left(a^k\right) * \omega^k_{\left(h^k,a^k\right)}\left(b^k\right) *, ..., *\omega^K_{\left(h^k,a^k,b^k,...,a^K\right)}\left(b^K\right) \qquad (9)$$

.

5.2 Credibility in the Communication Game Histories

I want to add O-F Nash effective cooperative negotiation as defined in 3.2.4 to the multistage payoff-relevant game. In order to do so, at every relevant "history" of the associated multistage game with communication game a player $m = i, j, l$ that moves non trivially can engage in preliminary negotiations. To set up negotiation problems as in section 3, future joint strategies in the associated payoff-relevant games to follow are defined in this context. Note that different vectors of correlated strategies enunciated at different stages of the communication game by the same player should be implemented by having respectively different mediators that, at each stage, make a public announcement or recommendation observed by all players. For simplicity, the associated notation in the multistage game with communication will be abstracted from for the most!

5.2.1 Future Joint Strategies

I denote the set of future joint payoff relevant strategies at stage $k \leq K$ as $\times Z_{h^k}$. Using Eq. (3), one can recursively derive $\times Z_{h^k}$ for all h^k as a Cartesian product of joint strategies in future games $\left[h^k, a^k, b^k\right]$:

$$\times Z_{h^k} = \prod_{(a^k,b^k)} \times Z_{h'^k_{(.)}/[h^k,a^k,b^k]} \tag{10}$$

, where $h^{k'}_{(.)}$ should be interpreted as a future contingency in the payoff-relevant game to follow at h^k. For any restriction of strategy profiles S can be expressed as a Cartesian product from Eq. (8), at each recursion, when obtaining $\times Z_{h^k}$,

$$\times Z_{h'^k_{(.)}/[h^k,a^k,b^k]} = S| \left[h^k, a^k, b^k\right] \tag{11}$$

Hence, at each recursion

$$\times Z_{h^k} = \prod_{(a^k,b^k)} S| \left[h^k, a^k, b^k\right] \tag{12}$$

.

5.2.2 Negotiation Problems and The Tenability Correspondence

Now one can define utility functions at histories $h^k_{(.)}$ where the arguments are link choices, current actions and future joint strategies by using:

$U_{h^k_{(.)}}\left(a^k, \times b^k, \times z^k\right) = u(s|h^k)$,

where $s|h^k = \left(a^k, \times b^k, \times z^k\right)$ after using Eq. (8) and (12). This expression refers to the expected utilities for the three players if $a^k \in A_{h^k}$, $\times b^k \in \times B^4_{h^k}$ and $\times z^k \in \times Z_{h^k_{(.)}}$ are played following $h^k_{(.)}$.

To formulate negotiation problems and joint plan bargaining problems in the notation of section 3, I assume that a *history of the multistage game with communication* corresponding to the multistage payoff-relevant game, $\mathring{h}^k_{(.)} \neq h^k_{(.)}$, includes in the subscript (.), in addition to a sequence of past choices $a^{k^-}$and actions, b^{k^-}, a sequence of past tenable and reliable joint plans $\left(\mu^1, ..., \mu^{k-1}\right) = \mu^{k^-}$ (See Remark 5) and past recommendations by different mediators. Abstracting from recommendations, for each negotiation problem in $\mathring{h}^k_{\left(\mu^{k-}, a^{k-}, b^{k-}\right)}$, a *corresponding history* to the unique $h^k_{\left(a^{k-}, \beta^{k-}\right)}$, one sets $B = B_{h^k_{(.)}}$ and $\times Z = \times Z_{h^k_{(.)}}$ and $U = U_{h^k_{(.)}}$.

The negotiation problem is trivial in histories where players move nothing. Wlg., and as a way of illustration, assume link ij is accepted, and the rule of order has next links il, and jl being proposed in that order, link il is rejected and link jl is accepted. One defines O-F Joint Plans in histories $\mathring{h}^k_{g^{ijl+il}(.)}$.

The set of future joint strategies $\times Z_{\mathring{h}^k_{g^{ijl+il}(.)}} = \times Z_{h^k_{g^{ijl+il}(.)}}$, or $\times Z^k$, if <u>no</u> confusion arises, is the Cartesian product of singleton action profile sets. So the tenability correspondence in a history $\mathring{h}^k_{g^{ijl+il}(.)}$ is trivially defined as

$Q_{\mathring{h}^k_{g^{ijl+il}(.)}}\left(\mu^k\right) = \mho^{\times Z_{\mathring{h}^k_{g^{ijl+il}(.)}}}$.

If no confusion arises, I will write only $\mho^{\times Z^k}$, also a singleton, a Cartesian product of functions that put probability one on the unique element of the trivial action set profiles at each future history of the payoff-relevant game to follow $\mathring{h}^k_{g^{ijl+il}(.)}$. Set the tenability correspondence in section 3, $Q_{\mu^{k-}}(\mu) = Q_{\mathring{h}^k_{(\mu^{k-})}}(\mu^k)$.

For any trivial (as a trivial future game follows) a'^k-concatenated strategic form game $\left(B^k_i \times B^k_j, \pi^{\mu^k_m/a'^k}\right)$, where $\mu^k_m = \left(\alpha^k_m, \beta^k_m, \zeta^k_m\right) \in \mho^k$, to be well defined, one sets for any $b^k \in B_{h^k_{g^{ijl+il}(.)}}$ and the unique trivial $z^k \in \times Z^k$

$$\Pr\left[U_{m,h^k_{g^{ijl+il}(.)}}(.)\right] = 1 \tag{13}$$

, where as for Eq.(6), $U_{m,h^k_{g^{ijl+il}(.)}}(.) = U_{m,h^k_{g^{ijl+il}(.)}}\left(a'^k, \times b^k, \times z^k\right)$, the a'^k-th component of $\times b^k$ $b^k_{a'^k} = \left(b^k_i, b^k_j, b^k_l\right)$ and $\Pr\left[U_{m,h^k_{g^{ijl+il}(.)}}(.)\right]$ is the probability that $U_{m,h^k_{g^{ijl+il}(.)}}(.)$ results given that $\left(a'^k, b^k_{i,a'^k}\right)$ occurred and play is consistent with ζ^k_m thereafter, for $m = i, j$.

The outside options in the joint plan bargaining problem are $\psi^k = \left(x^k_i, x^k_j\right)$ with disagreement plan $\widehat{\mu}^k = \left(\widehat{\alpha}^k_{\widehat{a}^k}, \widehat{\beta}^k, \widehat{\zeta}^k\right)$ where the disagreement future-request $\widehat{\zeta}^k$ is trivial. The disagreement plan suggests unilateral link rejections, that is, $\widehat{\mu}^k$ is such that $\widehat{\alpha}^k_{\widehat{a}^k}$ has $\widehat{a}^k_{ij} = (n, n)$ and $\widehat{\beta}^k_{a'^k}$ for any $a'^k \neq (y, y)$ is any same fixed correlated strategy for in any a'-concatenated game, where $a'^k_{ij} \neq (y, y)$, any pair of proposals pairs is a Nash equilibrium; wlg., $\widehat{\beta}_{a'}$ if $a' = (y, y)$ is arbitrary fixed to any given unilateral proposals pair rejection pair, (unfeasible proposals pairs) as any is a Nash equilibrium of any concatenated game. So $\widehat{\mu}^k$ is tenable and reliable.

One completes the formulation of the negotiation problem in the notation of section 3 in history $\mathring{h}^k_{g^{ijl+il}(.)}$, if the sequence of past statements is given by μ^{k^-}, by setting $\Phi_{il,\mu^{k-}} = \Phi_{\mathring{h}^k_{g^{ijl+il}\left(\mu^{k-}\right)},}$ or simply Φ^k.

Recall that to each such history in the communication game $\mathring{h}^k_{(.)}$, there are associated future-requests by pairs il or jl that may have formed in some order. Suppose that $\left(F, \Phi^k_{ij}, \psi\right)_{\mathring{h}^k_{(.)}}$ is well defined, then $\eta^f_{\mathring{h}^k_{(.)}}\left(\Phi^k_{ij}, \psi^k\right)$, the credible joint plan set,.can be defined and exists for any such possible history.

In general, suppose that one has inductively defined a non empty O-F Joint Plan set in any $\mathring{h}^k_{g^{\theta+ij}(.)}$, that is, for all $\theta \in \{\varnothing, il, ilj\}$ $i \neq j \neq l$, $i, j, l \in \{1, 2, 3\}$ suppose.

$$\eta^f_{\mathring{h}^k_{g^{\theta+ij}(.)}}\left(\Phi^k, \psi^k\right) \neq \varnothing \tag{14}$$

. As for Eqs. (5) and (12) the vector of correlated strategies ζ^k of section 3 in future histories of the future game can be expressed in terms of a vector of correlated strategies of the payoff-relevant game to follow history h^k in the multistage payoff-relevant game of section 5 for

$$\zeta^k = \prod_{\left(a^k,b^k\right)} \zeta_{h^{k\prime}_{(.)}/\left[h^k_{g^{\theta+ij}(.)},a^k,b^k\right]} \in \prod_{\left(a^k,b^k\right)} W|\left[h^k_{g^{\theta+ij}(.)},a^k,b^k\right] \tag{15}$$

. Let $\mu^k = \left(\alpha^k,\beta^k,\zeta^k\right)$ have $\zeta^k \in Q_{\mathring{h}^k_{(.)}}\left(\mu^k\right)$, that is, μ^k is future tenable. The future-request ζ^k should be such that for any $\left(a^k,b^k\right) \in A_{h^k_{(.)}} \times B_{h^k_{(.)}}$

$\zeta_{h'^k_{(.)}/\left[h^k_{g^{\theta+ij}(.)},a^k,b^k\right]} \in \eta^f_{\mathring{h}^{k+1}_{(.)}}\left(\Phi^{k+1},\psi^{k+1}\right)$,

where $\mathring{h}^{k+1}_{(.)} = \left[\mathring{h}^k_{g^{\theta+ij}(.)},r^k_a,a^k,r^k_b,b^k,\mu^k\right]$, for all link choice recommendations $r^k_a \in A_{h^k_{(.)}}$ and payoff proposal recommendations $r^k_b \in B_{h^k_{(.)}}$. That is, any $\zeta_{h'^k_{(.)}/\left[h^k_{g^{\theta+ij}(.)},a^k,b^k\right]}$, should equal the identical Credible Joint Plans in the histories that follow $\mathring{h}^k_{(.)}$ after players i and j enunciated μ^k, $\left(a^k,b^k\right)$ was played and any pair of recommendations occurred; for all possible recommendations belong to the support of α^k and β^k in the given $\mu^k = \left(\alpha^k,\beta^k,\zeta^k\right)$. Recall from section 3, that credibility of joint plans depend only on past choices and actions in the last-mover advantage case and, under the O-F focal effect, credibility depends in addition on past successful joint plans and not on its specific recommendations. Hence, for simplicity, I will ignore recommendations and write instead $\left[\mathring{h}^k_{g^{\theta+ij}(.)},\beta^k,\mu^k\right]$, provided indexing by μ^{k^-} is not relevant.

It is implicit that if $\mu^k = \left(\alpha^k,\beta^k,\zeta^k\right)$ is such that histories $\mathring{h}^{k+1}_{(.)}$ have players move nothing, $\eta^f_{\mathring{h}^{k+1}_{(.)}}\left(\Phi^{k+1},\psi^{k+1}\right)$ is a trivial joint plan, as actions profile sets there and thereafter are singletons.

Remark 9 *If one assumes the O-F focal effect, tenable future-requests ζ^k in $\mu^k = \left(\alpha^k,\beta^k,\zeta^k\right)$, i.e., $\zeta^k \in Q_{\mathring{h}^k_{(.)}}\left(\mu^k\right)$, may be different depending on the μ^{k^-} associated to $\mathring{h}^k_{\left(\mu^{k^-}\right)}$, as different past successful joint plans may influence play in each history in a different way.*

By the inductive assumption in Eq. (13) $Q_{\mathring{h}^k_{g^{\theta+ij}(.)}}\left(\mu^k\right) \neq \varnothing$. Next, for any a'^k-concatenated strategic form game $\left(B^k_i \times B^k_j, \pi^{\mu^k_m/a'^k}_{ij}\right)$, where $\mu^k_m = \left(\alpha^k,\beta^k,\zeta^k\right) \in \mho^k$, to be well defined, one sets for any $b^k \in B_{h^k_{(.)}}$ and $\times z^k \in \times Z^k$

$\Pr\left[U_{m,h^k_{g^{\theta+ij}(.)}}(.)\right] = \Pr\left[\left(s|h^k\right)/\omega|h^k\right]$,

where $U_{m,h^k_{(.)}}(.)$ is defined as in Eq.(13) and $\Pr\left[U_{m,h^k_{(.)}}(.)\right]$ equals to $\Pr\left[\left(s|h^k\right)/\omega|h^k\right]$ in Eq. (9); the latter is the probability of the path corresponding to the restriction

$s|h^k = \left(a'^k, \times b^k, \times z^k\right)$, from Eqs. (8) and (12), given the vector of correlated strategies $\omega|h^k$ that can be expressed as $\ddot{\mu}^k = \left(\ddot{\alpha}^k_{a'^k}, \ddot{\beta}^k, \zeta^k\right)$ for Eq. (15); $\ddot{\mu}^k$ puts probability 1 on both a'^k and on b^k after a'^k occurred, i.e., $\ddot{\beta}^k_{a'^k}\left(b^k\right) = 1$ and is consistent with ζ^k_m thereafter, for $m = i, j$.

The outside options in the associated joint plan bargaining problem are derived as before and are $\psi^k = \left(x^k_i, x^k_j\right)$ with tenable and reliable $\widehat{\mu}^k = \left(\widehat{\alpha}^k_{\widehat{a}^k}, \widehat{\beta}^k, \widehat{\zeta}^k\right)$, i.e., $\widehat{\mu}^k \in \widetilde{\mho}$. Note that the disagreement future-request $\widehat{\zeta}^k$ can be derived given the finite rule of order and has the peculiar feature that it depends only on statements of pairs that have had or will have successful preliminary negotiations; of course, it "depends" on the current pair's disagreement future-request trivially (See Remark 6). Also, recall, it was fixed when outside options were defined in 3.2.2.

In general, history $\mathring{h}^k_{g^{\theta+ij}(.)}$ has future-requests by pairs that have successfully negotiated before in a given order. The ones of pairs that were unsuccessful are "basically" ignored. Assume that $(F, \Phi_{ij}, \psi)_{\mathring{h}^k_{g^{\theta+ij}(.)}}$ is well defined, then $\eta^f_{\mathring{h}^k_{g^{\theta+ij}(.)}}(\Phi_{ij}, \psi)$ exist for any possible history.

5.3 Sequentially Nash Credible Joint Plans

Suppose that O-F Joint Plans exist for all histories, only then the inductive assumption in Eq. (14) is justified. Then, the ones at the beginning of play are defined as *Sequentially Nash Credible Joint Plans (SN)*. Formally,

$$SN = \eta^f\left(\Phi_{ij}, \psi\right) \tag{16}$$

, where $\eta^f\left(\Phi_{ij}, \psi\right) = \eta^f_{\mathring{h}^1}\left(\Phi_{ij}, \psi\right)$ was defined in 3.2.4.

Clearly SN suggest subgameperfect publicly correlated equilibria in the multistage game with communication.

6 Existence Theorem

In each history $\mathring{h}^k_{g^{\theta+ij}(.)}$, it will be useful to have a term for expected payoffs associated to future tenable joint plans that are possible by matching proposals pairs provided a link forms. Formally, $\left(x^k_i, x^k_j\right) \in PMF^k$, the *proposal match payoff feasible set* in $\mathring{h}^k_{g^{\theta+ij}(.)}$, if there exists a proposal match $b^k \in B_{h^k_{g^{\theta+ij}}}$ and $\mu^k = \left(\alpha^k_{a^k}, \beta^k, \zeta^k\right) \in \underline{\mho}^k$, i.e., μ^k is future tenable, and a^k is such that $a^k_{ij} = (y, y)$, β^k is such that $\beta^k_{a^k}\left(b^k\right) = 1$, $b^k \in B^k$ and $\pi^{\mu^k/a^k}_{ij}\left(b^k_{ij}\right) = \left(x^k_i, x^k_j\right)$ (See Eq. 6). The set of strong Pareto efficient points of PMF^k is the frontier of PMF^k.

Joint plan bargaining problems $(F, \Phi, \psi)_{\mathring{h}^k_{g^{\theta+ij}(.)}}$ are classified into two types. In *type 1*, there exists a better payoff proposal match.

There exists a *better payoff proposal match* if some element of PMF^k is as least as good as the outside options, that is, $\left(x_i^k, x_j^k\right) \geq \pi_{ij}^{\widehat{\mu}^k/\widehat{a}^k}\left(\widehat{b}_{ij}^k\right)$, for some $\left(x_i^k, x_j^k\right) \in PMF^k$.

Note that, in particular, the β^k associated to a better payoff proposal match is current reliable and tenable as in the $a^k = (y, y)$-concatenated game $\left(B_{ij}^k \times B_j^k, \pi^{\mu^k/a^k}\right)$ it implies a Nash equilibrium. Also the a_{ij}^k associated to a better payoff proposal match is a Nash equilibrium of $\left(A_i \times A_j, \pi_{ij}^{\mu_i}\right)$ and so the associated joint plan $\mu^k = \left(\alpha_{a^k}^k, \beta^k, \zeta^k\right)$ is reliable and tenable. By definition, such $\left(x_i^k, x_j^k\right)$ belongs to the feasible set. Thus, feasible sets are convex combinations of outside options and payoffs associated to better payoff proposal matches. Such convex combinations may have corresponding non degenerate tenable and reliable joint plans in the sense that the latter entail non degenerate link or current promise-requests.

In *type 2,* there are not better payoff proposal matches and the only feasible payoff pair is the one associated to the disagreement joint plan, hence the associated link will not form. Joint plans associated to elements in PMF^k don't induce Nash equilibria in their associated $\left(B_i^k \times B_j^k, \pi^{\mu^k}\right)$ as it is always better to unilaterally reject. Neither in $\left(B_{ij}^k \times B_j^k, \pi^{\mu^k/a^k}\right)$, where $a^k = (y, y)$, as it is always better to propose something unfeasible.

As it will become clearer in the construction proof, the existence of these two types imply that feasible and IRF sets in each history coincide.

Theorem 10 *Sequentially Nash Credible Joint Plans exist for three-player normalized cooperative games with the Myerson value as a payoff allocation rule.*

Proof. As feasible and IRF sets coincide, for the joint plan bargaining game to be well defined at the initial history it suffices to show that the IRF sets are closed in any possible future history of the multistage game with communication.

<u>Part 1.</u> The Joint Plan Bargaining Problem in $\mathring{h}^6_{g^{132+12}(.)}$

Outside options in histories with the same last proposal match b^5, $\mathring{h}^6_{g^{132+12}(b^5)}$ (that is equivalent to $\mathring{h}^6_{g^{132+12}(a^5,b^5)}$, where $a^5_{32} = (y, y)$ as link 32 formed in stage 5), are $\left(\psi_1^6, \psi_2^6\right) = \left(\phi_1^{132}, b^5(2)\right)$. The PMF^6 consists only of payoffs in the complete graph $\left(\phi_1^{g^N}, \phi_2^{g^N}\right)$.

Note that player 1 can for the most do better in the complete graph because $\phi_1^{132} \leq \phi_1^{g^N} \Leftrightarrow 0 \leq c$, (See diagram in Appendix). Recall that $b^5(3) + b^5(2) = \phi_3^{g^{132}} + \phi_2^{g^{132}}$. Denote $\phi_3^{g^{132}} + \phi_2^{g^{132}} - \phi_2^{g^N} = \frac{2d+a+b-c}{6} > 0$ by $\overline{b}^5(3)$ and $\phi_2^{g^N}$ by $\overline{b}^5(2)$.

As player 2 would loose in the complete graph, the bargaining game is of type 2 iff $b^5(2) > \overline{b}^5(2) = \phi_2^{g^N}$. The IRF^6 consists just of ψ_{12}^6. Otherwise, the IRF^6 contains the unique element of the PMF^6, $\left(\phi_1^{g^N}, \phi_2^{g^N}\right)$, that now is associated to a better proposal match, the one that leads to link 12 forming and hence the complete graph. The IRF^6 consists of convex combinations of the outside options $\left(\psi_1^6, \psi_2^6\right)$ and $\left(\phi_1^{g^N}, \phi_2^{g^N}\right)$.

In any case, the IRF^6 is closed, thus, the O-F focal effect ensures that for any $\mathring{h}^6_{g^{132+12}(.)}$ one can compute $\eta^f_{\mathring{h}^6_{g^{132+12}(.)}}\left(\Phi^6, \psi^6\right)$.

<u>Part 2.</u> The Joint Plan Bargaining Problem in $\mathring{h}^5_{g^{13+32}(.)}$

Outside options in histories $\mathring{h}^5_{g^{13+32}(b^3)}$ are $\left(\psi_3^5, \psi_2^5\right) = \left(b^3(3), \phi_2^{g^{13}} = 0\right)$.

It suffices to check that the IRF^5 is closed in bargaining games of type 1. In what follows of Part 2, I assume that $b^3(3)$ induces such type. Let $\breve{\mu}^5 = \left(\alpha^5_{\breve{a}^5}, \breve{\beta}^5, \breve{\zeta}^5\right)$ be a *fully successful degenerate joint plan*; that is, $\breve{a}^5_{32} = (y, y)$, $\breve{\beta}^5_{\breve{a}^5}\left(\breve{b}^5\right) = 1$, where $\breve{b}^5$ is a proposal match. Assume history $\mathring{h}^6_{g^{132+12}\left(b^3, \breve{\mu}^5, \breve{b}^5\right)}$ is reached.

From Part 1, whenever $\breve{b}^5(2) > \overline{b}^5(2)$, $\mu^6 \in \eta^f_{\mathring{h}^6_{g^{132+12}\left(b^3, \breve{\mu}^5, \breve{b}^5\right)}}\left(\Phi^6, \psi^6\right)$ is a disagreement joint plan, i.e., $\mu^6 = \widehat{\mu}^6$. As players 3 and 2's $\breve{\mu}^5 = \left(\alpha^5_{\breve{a}^5}, \breve{\beta}^5, \breve{\zeta}^5\right)$ is future tenable, it has to future-request $\widehat{\mu}^6$, formally, $\breve{\zeta}^5_{h^{k'}_{(.)}/\left[h^5_{g^{13+32}(b^3)}, \breve{a}^5, \breve{b}^5\right]} = \widehat{\mu}^6$. Associated payoffs $\pi^{\breve{\mu}^5}_{32, \mathring{h}^5_{g^{13+32}(b^3)}}\left(\breve{a}^5_{32}\right) = \breve{b}^5_{32}$ are illustrated in figure 1 by the segment in bold not including $\overline{b}^5_{32}$ for the simple majority game where $c > 0$ and $\overline{b}^5(2) = \phi_2^{g^N} = \frac{2}{6}$.

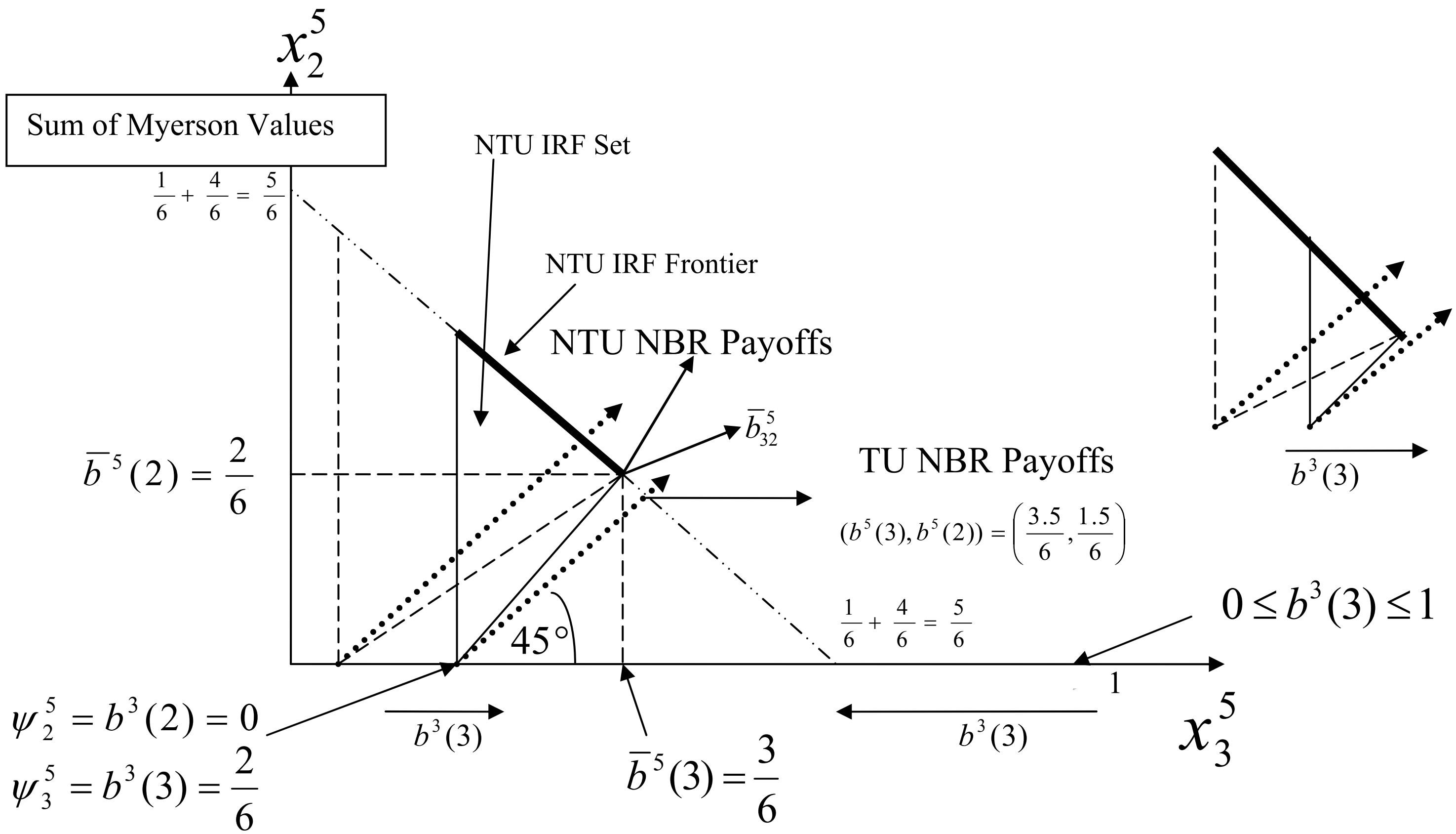
Assume that $c = 0$ (See lemma 11 for the case $c > 0$) and hence player 1 in the induced $\mathring{h}^6$ is indifferent to further linking. If $\breve{\mu}^5$ has $\breve{b}^5(2) \leq \overline{b}^5(2)$, the bargaining game in $\mathring{h}^6_{g^{132+12}\left(b^3, \breve{\mu}^5, \breve{b}^5\right)}$ is of type 1, however, it is not essential. As for the O-F focal effect, future-requests in $\breve{\mu}^5$ of O-F Joint Plans in $\mathring{h}^6_{g^{132+12}\left(b^3, \breve{\mu}^5, \breve{b}^5\right)}$ depend on μ^3, so it is useful to write $\mathring{h}^6_{g^{132+12}\left(\mu^3, b^3, \breve{\mu}^5, \breve{b}^5\right)}$. Let $a^6_{12} = (y, y)$; if μ^6 is an O-F Joint Plan, then its link promise-request α^6 may entail either link formation, that is, $\alpha^6\left(a^6\right) = 1$,.and proposal matches after a^6, or the given unilateral rejection and so $\alpha^6\left(a^6\right) = 0$ or a "mix", in which case $\alpha^6\left(a^6\right) < 1$.

One may think there will be a jump in payoffs whenever $\breve{b}^5(2) = \overline{b}^5(2)$ depending on μ^3. However, as $\overline{b}^5(3) = \phi_3^{132}$ if $c = 0$, payoffs are always $\pi^{\breve{\mu}^5}_{32, \mathring{h}^5_{g^{13+32}(\mu^3, b^3)}}\left(\breve{a}^5_{32}\right) = \left(\overline{b}^5(3), \overline{b}^5(2)\right)$ (See in contrast Remark 13).

If $\breve{b}^5(2) < \overline{b}^5(2)$ payoffs $\pi^{\breve{\mu}^5}_{32, \mathring{h}^5_{g^{13+32}(\mu^3, b^3)}}\left(\breve{a}^5_{32}\right)$ are equal to convex combinations between $\left(\overline{b}^5(3), \overline{b}^5(2)\right)$ and $\left(\breve{b}^5(3), \breve{b}^5(2)\right)$ depending on μ^3.

<u>Assumption 2:</u> Plans μ^3 are such that the latter convex combinations are con-

Players 3 and 2's Bargaining Game-Figure 1



Link 12 and 23 were rejected in stage 1 and 2 respectively. In stage 3, link 13 formed with payoff proposal match b^3 with $\left(b^3(1), b^3(3)\right)=\left(\frac{4}{6},\frac{2}{6}\right)$. Link 12 was rejected. In stage 5, link 32 is proposed with induced outside options (ψ_3^5, ψ_2^5). Link 32 is last to form with payoffs (x_3^5, x_2^5) equal to an associated proposal match with $\left(b^5(3), b^5(2)\right)=\left(\frac{3}{6},\frac{2}{6}\right)=\bar{b}_{32}^5$.

tinuous on $\left[0,\overline{b}^5(2)\right[$.[7]

It follows that in any $\mathring{h}^5_{g^{13+32}(\mu^3,b^3)}$, the IRF^5 is closed. Also, the IRF^5 frontier has right side endpoint $\left(\overline{\overline{x}}^5_3,\overline{\overline{x}}^5_2\right)$ in the plane (x^5_3,x^5_2) equal to or to the southeast of $\overline{b}^5_{32}$ (depending on μ^3). Thus, $\eta^f_{\mathring{h}^5_{g^{13+32}(\mu^3,b^3)}}\left(\Phi^5,\psi^5\right)$ can be computed.

Part 3. Joint Plan Bargaining Problem in $\mathring{h}^4_{g^{13+12}(.)}$

Let $\mu^5=\left(\alpha^5,\beta^5,\zeta^5\right)$. The outside options in any $\mathring{h}^4_{g^{13+12}(.)}$ depend on b^3 and μ^3 (See Part 2) as follows:

$$\psi^4_{12}=\pi^{\widehat{\mu}^4}_{12,\mathring{h}^4_{g^{13+12}(\mu^3,b^3)}}\left(\widehat{b}^4_{12}\right)=\sum_{a^5}\alpha^5\left(a^5\right)\pi^{\mu^5}_{12,\mathring{h}^5_{g^{13+32}(\mu^3,b^3,\widehat{\mu}^4,\widehat{b}^4)}}\left(a^5_{32}\right) \tag{17}$$

where $\mu^5=\widehat{\zeta}^4_{h^{k'}_{(.)}/\left[h^4_{g^{13+12}(b^3)},\widehat{b}^4\right]}$, the group component of $\widehat{\zeta}^4$ that contains correlated strategies in histories that follow and include $\left[h^4_{g^{13+12}(b^3)},\widehat{b}^4\right]$. Also, as $\widehat{\mu}^4$ is future tenable, μ^5 is O-F in $\mathring{h}^5_{g^{13+32}\left(\mu^3,b^3,\widehat{\mu}^4,\widehat{b}^4\right)}$, i.e.,

$\mu^5\in\eta^f_{\mathring{h}^5_{g^{13+32}(\mu^3,b^3,\widehat{\mu}^4,\widehat{b}^4)}}\left(\Phi^5,\psi^5\right)$.

Analogously, as in $\mathring{h}^5_{g^{13+32}(.)}$, one can prove that the IRF^4 set is always closed in any $\mathring{h}^4_{g^{13+12}(.)}$, assuming now $b=0$. Hence, $\eta^f_{\mathring{h}^4_{g^{13+12}(.)}}\left(\Phi^4,\psi^4\right)$ can be computed.

Part 4. The Joint Plan Bargaining Problem in $\mathring{h}^3_{g^{\varnothing}+13(.)}$

Players 1 and 2s' outside options are $\psi^3_{13}=\left(\phi^{g^{\varnothing}}_1,\phi^{g^{\varnothing}}_3\right)=(0,0)$. As for Parts 1, 2 and 3, $\widetilde{\mho}^3$ can be derived. I argue that the IRF^3 set and frontier are closed if payoffs in the IRF^3 set are continuous on "appropriate subsets" of the tenable and reliable set $\widetilde{\mho}^3$ composed of degenerate fully successful joint plans $\breve{\mu}^3=\left(\breve{\alpha}^3_{\breve{a}},\breve{\beta}^3,\breve{\zeta}^3\right)$. It can be shown that all associated payoffs in such subsets correspond to all what is achievable by such degenerate elements of $\widetilde{\mho}^3$. Assume $\widetilde{\mho}^3$ is known.

Back to $\mathring{h}^5_{g^{13+32}\left(\breve{\mu}^3,\breve{b}^3,\widehat{\mu}^4,\widehat{b}^4\right)}$

As $\overline{\overline{x}}^5_3\geq\overline{b}^5(3)>0$ and $\breve{b}^3(1)+\breve{b}^3(3)=\phi^{g^{13}}_1+\phi^{g^{13}}_3=a\geq0$, different $\breve{\mu}^3\in\widetilde{\mho}^3$ that differ in $\breve{b}^3$ induce a total of three *classes* of bargaining games (for an approximate graphical representation see figure 1as it assumes $c>0$):

Class 1: If $\breve{b}^3$ is such that $\psi^5_3=\breve{b}^3(3)=\overline{\overline{x}}^5_3$, player 3 will be indifferent between forming or not link 32. The bargaining game in $\mathring{h}^5_{g^{13+32}\left(\breve{\mu}^3,\breve{b}^3,\widehat{\mu}^4,\widehat{b}^4\right)}$ will be not essential but it is of type 1.

[7] Actually, players 3 and 2 don't loose anything by always assigning probability 1 to $\left(\breve{b}^5(3),\breve{b}^5(2)\right)$. In the latter sense, assumption 2 is not even necessary!

Class 2: If $\breve{b}^3$ is such that $\breve{b}^3(3) < \overline{\overline{x}}_3^5$, then the bargaining game is of type 1 and agent 3 is better off if link 32 forms.

Class 3: If $\breve{b}^3$ is such that $\breve{b}^3(3) > \overline{\overline{x}}_3^5$, then the bargaining game is of type 2.

The case $a = \overline{\overline{x}}_3^5$ exhibits the first two classes of bargaining games. If $a < \overline{\overline{x}}_3^5$ then only the second class results. The case $a > \overline{\overline{x}}_3^5$ exhibits the three classes.

Depending on these three ranges of a, one needs to consider at most three "types of families" of subsets of $\widetilde{\mho}^3$. Wlg., I focus on the case $a > \overline{\overline{x}}_3^5$ where one can distinguish three types of families.

Consider the expected payoff function associated to the O-F Joint Plan
$\mu^5 = \left(\alpha^5, \beta^5, \zeta^5\right) \in \eta^f_{\mathring{h}^5_{g^{13+32}(\breve{\mu}^3,\breve{b}^3,\widehat{\mu}^4,\widehat{b}^4)}}\left(\Phi^5, \psi^5\right).$

This is given by

$$f_{32}^5\left(\breve{\mu}^3, \breve{b}^3, \widehat{\mu}^4, \widehat{b}^4\right) = \sum_{a^5} \alpha^5\left(a^5\right) \pi^{\mu^5}_{32,\mathring{h}^5_{g^{13+32}(\breve{\mu}^3,\breve{b}^3,\widehat{\mu}^4,\widehat{b}^4)}}\left(a_{32}^5\right) \tag{18}$$

As $\widetilde{\mho}^3$ is known, one can redefine $\breve{\mu}^5$ (including $\breve{\alpha}^5$ and $\breve{\beta}^5$), $\breve{b}^3$, $\widehat{\mu}^4$and, $\widehat{b}^4$ wlg., as some given *auxiliary* functions of degenerate fully successful joint plans $\breve{\mu}^3 = \left(\breve{\alpha}^3_{\breve{a}}, \breve{\beta}^3, \breve{\zeta}^3\right) \in \widetilde{\mho}^3$, where $\breve{\zeta}^3$ future-requests O-F μ^5 after players 1 and 2 suggested the disagreement $\widehat{\mu}^4$and played $\widehat{b}^4$. Hence, one can reinterpret from now on this payoff function as a function of only $\breve{\mu}^3$; so one writes simply $f_{32}^5\left(\breve{\mu}^3\right)$.

Denote a subset belonging to *a family of disjoint subsets of* $\widetilde{\mho}^3$ *of a first type* (type "$\rightarrow$") by $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$, i.e., the "last name" of any subset in this family is $\rightarrow$ and the "first name" is Q^5. For any subset $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$, if $\acute{\mu}^3 \neq \breve{\mu}^3$ and $\acute{\mu}^3, \breve{\mu}^3 \in \overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$, the respective induced tenability correspondences in histories $\mathring{h}^5_{g^{13+32}\left(\acute{\mu}^3\right)}$ and $\mathring{h}^5_{g^{13+32}\left(\breve{\mu}^3\right)}$ are such that $\acute{Q}^5 = \breve{Q}^5 = Q^5$ (recall from Part 2 that Q^5 depends non trivially on any $\breve{\mu}^3$ in cases of indifference of player 1). Respective future-requests $\acute{\zeta}^3$ and $\breve{\zeta}^3$ suggest fully successful O-F Joint Plans μ'^5 and μ''^5 with $\zeta'^5 = \zeta''^5 = \zeta^5\left(Q^5\right)$, a function of Q^5. Also, $\acute{b}^3(3) \neq \breve{b}^3(3)$ and both belong to the closed interval $\left[0, \overline{\overline{x}}_3^5\right]$. Assume for now that "appropriate subsets" of $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$ exist with an "appropriate metric". One can show (See below) that the reinterpreted payoff function in Eq. (18), $f_{32}^5\left(\breve{\mu}^3\right)$, is continuous on any such appropriate subset of $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$.

As payoff for player 1 is constant regardless of link 32 being last to form or not

$$f^5\left(\breve{\mu}^3\right) = f^5\left(\breve{\mu}^3, \breve{b}^3, \widehat{\mu}^4, \widehat{b}^4\right) = \sum_{a^5} \alpha^5\left(a^5\right) \pi^{\mu^5}_{\mathring{h}^5_{g^{13+32}(\breve{\mu}^3,\breve{b}^3,\widehat{\mu}^4,\widehat{b}^4)}}\left(a_{32}^5\right) \tag{19}$$

,

with range on $\mathbb{R}^3$, i.e., $f^5\left(\breve{\mu}^3\right) = \left[f_1^5\left(\breve{\mu}^3\right), f_2^5\left(\breve{\mu}^3\right), f_3^5\left(\breve{\mu}^3\right)\right]$ is continuous on any appropriate subset of $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$.

By analogous arguments, $f^5(\breve{\mu}^3)$ is continuous on any appropriate subset of $\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)$ which belongs to a *second type* $\leftarrow$ of family that is derived almost identically as before: Any such $\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)$ is such that future-requests in $\acute{\mu}^3, \breve{\mu}^3 \in \overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)$ suggest in $\mathring{h}^5_{g^{13+32}(.)}$ the disagreement joint plan as O-F Joint Plans, that is, $\mu'^5 = \mu''^5 = \widehat{\mu}^5$. As before, $\breve{b}^3(3) \neq \grave{b}^3(3)$, however they belong to $\left[\overline{\overline{x}}^5_3, a\right]$.

As for the O-F focal effect, there is a third type of family where in contrast to the first two types $\breve{b}^3(3) = \overline{\overline{x}}^5_3$ and future-requested $\mu^5 = (\alpha^5, \beta^5, \zeta^5)$, abusing language, *mixes* over the disagreement joint plan and the fully successful joint plan that has an immediate promise-request that puts probability one on the unique (not easy to see) proposal match that yields $\left(\overline{\overline{x}}^5_3, \overline{\overline{x}}^5_2\right)$, the standard Nash bargaining solution in $\mathring{h}^5_{g^{13+32}\left(\breve{\mu}^3, \breve{b}^3, \widehat{\mu}^4, \widehat{b}^4\right)}$. Formally, such μ^5 differs from $\widehat{\mu}^5$ in that α^5 is not always degenerate, i.e., $\alpha^5(a^5) \leq 1$, and $\beta^5_{a^5}\left(\overline{\overline{x}}^5\right) = 1$, where $a^5 = (y, y)$ (See in simple majority example future-requests under type 3 joint plans). For simplicity of exposition, we analyze the first two types but the arguments apply to the third type too.

Graphically, as long as $\breve{b}^3(3) \neq \acute{b}^3(3)$ different plans, $\acute{\mu}^3$ and $\breve{\mu}^3$, in any given appropriate subset of $\overrightarrow{\widetilde{\mho}}^3(Q^5)\left(\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)\right)$ induce bargaining games with the same PMF^5 frontier but with different outside options that move along the horizontal axis to the right towards $\overline{\overline{x}}^5_3$ ($\rightarrow$) (to the left ($\leftarrow$) towards $\overline{\overline{x}}^5_3$) in the plane (x^5_3, x^5_2). (See figure 1, albeit $c > 0$ and hence$\left(\overline{\overline{x}}^5_3, \overline{\overline{x}}^5_2\right) = \overline{b}^5_{32}$; see lemma 11)

Back to $\mathring{h}^4_{g^{13+12}\left(\breve{\mu}^3, \breve{b}^3\right)}$

Using Eq. (17), outside options are

$$\psi^4_{12} = \pi^{\widehat{\mu}^4}_{12, \mathring{h}^4_{g^{13+12}\left(\breve{\mu}^3, \breve{b}^3\right)}}\left(\widehat{a}^4_{12}\right) = \sum_{a^5} \breve{\alpha}^5\left(a^5\right) \pi^{\breve{\mu}^5}_{12, \mathring{h}^5_{g^{13+32}\left(\breve{\mu}^3, \breve{b}^3, \widehat{\mu}^4, \widehat{b}^4\right)}}\left(a^5_{32}\right) \tag{20}$$

As the last expression in this equation is $f^5_{12}(\breve{\mu}^3)$ from equation (19), a function solely of $\breve{\mu}^3$, outside options $\psi^4_{12}(\breve{\mu}^3)$ are continuous on any appropriate subset of $\overrightarrow{\widetilde{\mho}}^3(Q^5)\left(\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)\right)$.

I proceed by constructing appropriate subsets of $\overrightarrow{\widetilde{\mho}}^3(Q^5)$. Player 2's outside option, $\psi^4_2(\breve{\mu}^3)$ weakly increases while $\psi^4_1(\breve{\mu}^3) \leq \overline{b}^4(1)$ is constant (where $\overline{b}^4$ is defined analogously as $\overline{b}^5$ is) whenever $\breve{\mu}^3 \in \overrightarrow{\widetilde{\mho}}^3(Q^5)$ has a lower $\breve{b}^3(3) \in \left[0, \overline{b}^5(3)\right]$. Thus, there may exist some $\breve{b}^3(3)$ where player 1 is indifferent between linking or not with agent 2. <u>As before</u> one may distinguish 3 classes of bargaining games depending on parameter cases. Also, one may have to distinguish two *types* (not three as before) *of conditional* (on $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$) *families of disjoint subsets of* $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$, where subsets in these two types of conditional families are denoted by either $\overrightarrow{\widetilde{\mho}}^{3\uparrow}\left(Q^5,Q^4\right)$ or $\overrightarrow{\widetilde{\mho}}^{3\downarrow}\left(Q^5,Q^4\right)$. These subsets will have elements $\breve{\mu}^3\in\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$ that future-request on players 1 and 2 a fully successful joint plan and a disagreement joint plan respectively and are defined as *appropriate* subsets of $\overrightarrow{\widetilde{\mho}}^3\left(Q^5\right)$.

Consider payoffs associated to O-F Joint Plan $\mu^4\in\eta^f_{\mathring{h}^4_{g^{13+12}\left(\breve{\mu}^3,\breve{b}^3\right)}}\left(\Phi^4,\psi^4\right)$

This is given by

$$f^4_{12}\left(\breve{\mu}^3,\breve{b}^3\right)=\sum_{a^4}\alpha^4\left(a^4\right)\pi^{\mu^4}_{12,\mathring{h}^4_{g^{13+12}\left(\breve{\mu}^3,\breve{b}^3\right)}}\left(a^4_{12}\right)\tag{21}$$

.

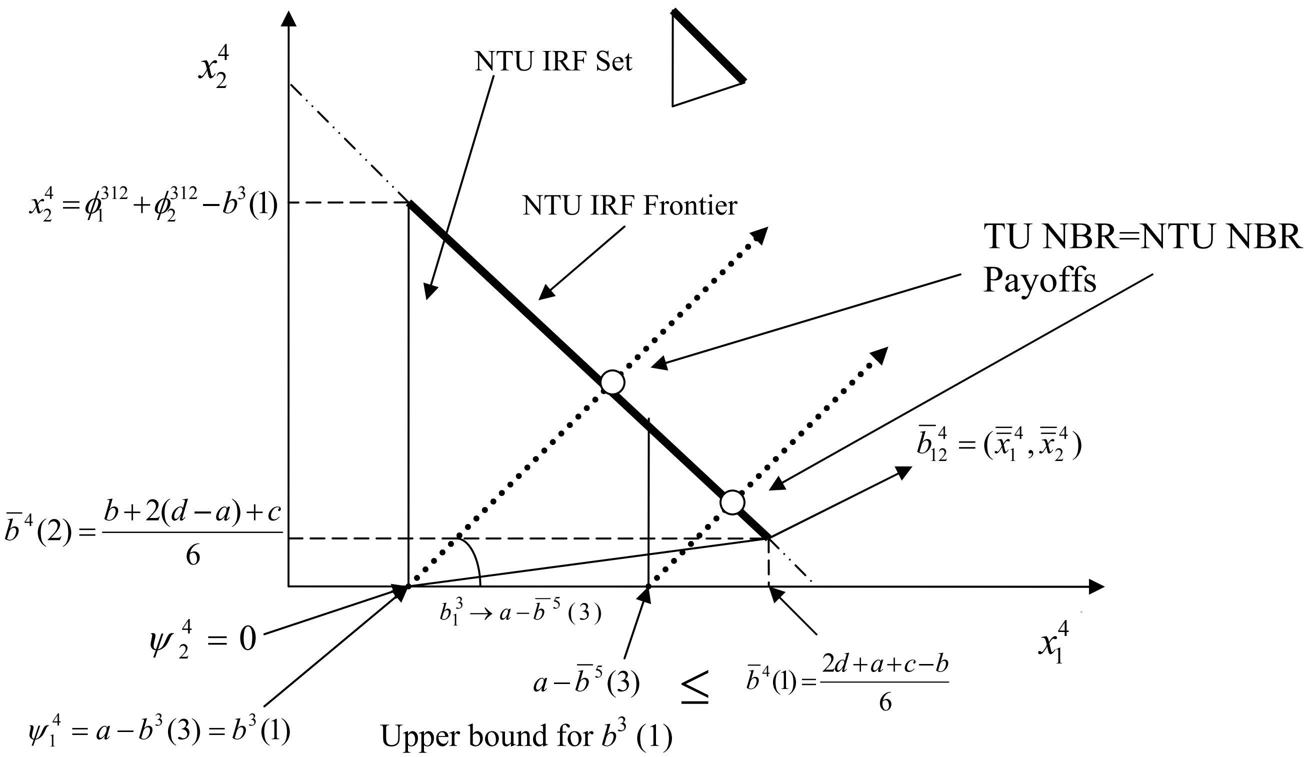
As before, after reinterpreting the payoff expression as a function of only $\breve{\mu}^3$, i.e., $f^4_{12}\left(\breve{\mu}^3\right)$, one can show that this is continuous on any appropriate $\overrightarrow{\widetilde{\mho}}^{3\uparrow}\left(Q^5,Q^4\right)$ or $\overrightarrow{\widetilde{\mho}}^{3\downarrow}\left(Q^5,Q^4\right)$. Informally, the outside options $\psi^4_{12}\left(\breve{\mu}^3\right)$ are continuous on the latter appropriate sets and the NBR payoffs in the associated bargaining games are continuous on the outside options for a fix PMF^4 frontier (Recall, composition of continuous functions are continuous). So is $f^4\left(\breve{\mu}^3\right)$ with range in $\mathbb{R}^3$.

Analogously, $f^4\left(\breve{\mu}^3\right)$ is continuous, if necessary (depending on parameter values), on any element of two (as before, actually three but we analyze two) types of conditional families of subsets of $\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)$, denoted either by $\overleftarrow{\widetilde{\mho}}^{3\rightarrow}\left(\ddot{Q}^5,\ddot{Q}^4\right)$ (where an element $\breve{\mu}^3$ is such that $\breve{b}^3(1)\rightarrow a-\overline{\overline{x}}^5_3$ and suggests a fully successful joint plan to players 1 and 2 in $\mathring{h}^4_{g^{13+12}\left(\breve{\mu}^3,\breve{b}^3\right)}$; recall $\breve{b}^3(3)\in\left[\overline{\overline{x}}^5_3,a\right]$) or $\overleftarrow{\widetilde{\mho}}^{3\leftarrow}\left(\ddot{Q}^5,\ddot{Q}^4\right)$ (this latter subset if exists due to specific parameter values it is a singleton in A-M). Figure 2, where in particular $\left(\overline{\overline{x}}^4_1,\overline{\overline{x}}^4_2\right)=\overline{b}^4$ and $\left(\overline{\overline{x}}^5_3,\overline{\overline{x}}^5_2\right)=\overline{b}^5$ is somewhat useful to illustrate this claim's proof.

It follows that $\pi^{\breve{\mu}^3}_{13,\mathring{h}^3_{g^{\varnothing+13}(.)}}\left(\breve{a}^3\right)$, a function of $\breve{\mu}^3$, is continuous if necessary on any element of these four (or more) conditional families of subsets, for any possible Q^5 $\left(\ddot{Q}^5\right)$.

Note that as $\breve{b}^3(3)$ varies along a closed interval associated with any given such appropriate subset, the only components of $\breve{\mu}^3=\left(\breve{\alpha}^3_{\breve{a}^3},\breve{\beta}^3,\breve{\zeta}^3\right)$ that may vary are the degenerate correlated strategy in current contingency $\breve{a}$, $\breve{\beta}^3_{\breve{a}^3}$, ($\breve{\zeta}^3_{h^4_{g^{13+12}\left(\breve{b}^3\right)}}=\alpha^4$, the correlated strategy in contingency $h^4_{g^{13+12}\left(\breve{b}^3\right)}$ in the language of section 3, varies only if considering a third type of conditional family not analyzed) $\breve{\zeta}^3_{h^{4.2}_{g^{13+12}\left(\breve{b}^3,a^4\right)}}=\beta^4_{a^4}$, a correlated strategy in contingency $h^{4.2}_{g^{13+12}\left(\breve{b}^3,a^4\right)}$ provided $\breve{\zeta}^3_{h^4_{g^{13+12}\left(\breve{b}^3\right)}}=\alpha^4_{a^4}$, where

Players 1 and 2's Bargaining Game-Figure 2



$a^4 = (y, y)$ (analogously $\breve{\zeta}^3_{h^5_{g^{13+32}(\breve{b}^3,\widehat{b}^4)}}$ varies if considering a third type of family not analyzed) and $\breve{\zeta}^3_{h^{5.2}_{g^{13+12}(\breve{b}^3,\widehat{b}^4,a^5)}}$ provided $\breve{\zeta}^3_{h^5_{g^{13+32}(\breve{b}^3,\widehat{b}^4)}} = \alpha^5_{a^5}$, where $a^5 = (y, y)$. It can be shown that any such appropriate subset, now completely characterized, is a metric space (See a metric in the Appendix) and my earlier claims on continuity can be justified.

In any $\mathring{h}^3_{g^{\varnothing}+13(.)}$, convex combinations over the payoffs associated to joint plans in any given appropriate subset and the outside options ψ^3_{13} yield a closed IRF^3 set and frontier. So, O-F Joint Plans exist.

In turn, the outside options in any $\mathring{h}^2_{g^{\varnothing}+23(.)}$ can be derived and the IRF^2 set and frontier are closed by similar arguments−now assuming, whenever appropriate and in that order $a = 0$ and $c = 0$. O-F Joint Plans exist. The same is the case for the unique (as there are no past statements) $\mathring{h}^1_{g^{\varnothing}+12(.)}$, assuming whenever appropriate and in that order $b = 0$ or $a = 0$. After using lemma 11, the theorem follows for all parameter values. ■

Lemma 11 *If $a, b, c > 0$, the complete graph never forms.*

Proof. In part 2 of theorem 10, in contrast, if $\breve{b}^5$ has $\breve{b}^5(2) \leq \overline{b}^5(2)$, player 1 gains by forming link 12 and O-F joint plans don't depend on μ^3 anymore. Bargaining games in $\mathring{h}^6_{g^{132+12}\left(b^3,\mu^5,\breve{\beta}^5\right)}$ are of type 1.

In particular, if $\breve{b}^5(2) = \overline{b}^5(2)$ the bargaining game in such $\mathring{h}^6$ is not essential and given that player 2 is indifferent, the bargaining outcome depends on $\breve{\mu}^5$. As for the O-F focal effect a future tenable, $\breve{\mu}^5$ may have players 2 and 3 future-request an O-F Joint Plan that is a disagreement joint plan, a fully successful joint plan in $\mathring{h}^6_{g^{132+12}\left(b^3,\breve{\mu}^5,\breve{b}^5\right)}$ or a mix. In the first case, payoffs $\pi^{\breve{\mu}^5}_{32,\mathring{h}^5_{g^{13+32}(b^3)}}(\breve{a}^5_{32})$ would be "assured" to be $\left(\overline{b}^5(3), \overline{b}^5(2)\right)$. In the second case payoffs are $\left(\phi^N_3, \phi^N_2\right) \neq \left(\overline{b}^5(3), \overline{b}^5(2)\right)$. Note that as $\phi^N_3 < \overline{b}^5(3)$ and $\phi^N_2 = \overline{b}^5(2)$, this payoff pair is not in the PMF^5 frontier. These payoffs are not strongly Pareto efficient. The same is the case if a mix would be future-requested.

If $\breve{b}^5(2) < \overline{b}^5(2)$, the bargaining game is essential and $\breve{\mu}^5$ future-requests in $\mathring{h}^6_{g^{132+12}\left(b^3,\breve{\mu}^5,\breve{b}^5\right)}$ a fully successful joint plan, in which case payoffs are again $\left(\phi^N_3, \phi^N_2\right)$.

Thus, the IRF^5 set and frontier are closed in the bargaining game in any $\mathring{h}^5_{g^{13+32}(b^3)}$. Moreover, an O-F Joint Plan $\breve{\mu}^5$ cannot future-request something different than a disagreement joint plan in $\mathring{h}^6_{g^{132+12}\left(b^3,\mu^5,\breve{\beta}^5\right)}$ as an O-F Joint Plan in $\mathring{h}^5_{g^{13+32}(b^3)}$ (depending on b^3) suggests with probability one a proposal match in the IRF^5 frontier or, in cases of indifference of player 3, mixes between fully successful joint plans or the disagreement one. In no such a case link 12 formation is suggested. Lemma 11 follows as the analysis for the cases $a, b > 0$ are similar. ■

Corollary 12 *The complete graph never forms in strictly superadditive games.*

Remark 13 *If whenever* $\breve{b}^5(2) = \overline{b}^5(2)$ *one would apply the standard Nash Bargaining solution to the bargaining game in* $\mathring{h}^6$ *in question whenever* $c > 0$, *the complete graph would form and payoffs would be* $\left(\phi_3^N, \phi_2^N\right) \neq \left(\overline{b}^5(3), \overline{b}^5(2)\right)$, *a point inside the IRF*^{5} *set. Hence this set would not be closed. Tenable and reliable joint plans by players 3 and 2 solve that problem naturally by inducing either* $\left(\phi_3^N, \phi_2^N\right)$, $\left(\overline{b}^5(3), \overline{b}^5(2)\right)$ *or a mix.*

7 Conclusions

This paper adds O-F Nash effective endogenous cooperative negotiation (See 3.2.4) to my modification of the A-M model, where pairs of players play a transfer game over the sum of their Myerson values in the prospective network. Negotiation statements at each history of the associated multistage game with communication are credible, in most cases, if they are the outcome of a joint plan bargaining problem where feasible payoffs are those induced by tenable and reliable joint plans. The disagreement tenable and reliable joint plan promise-requests link rejection. Sequentially Nash Credible Joint plans exist and analytical payoffs predicted are unique if players listen to oldest friends provided the latter suggest rational play, in particular, induce closed feasible bargaining sets whenever appropriate. In particular, the simple majority game yields the nucleolus in coalition structure.

It would be important to see if the nucleolus is always obtained for the three-player general case. In a slightly different communication environment, in a preliminary version of this paper, among other results, it is shown that all payoff predictions in that model are efficient. I conjecture that the same results hold in the model of this paper.

It would be relevant to use payoff allocation rules different than the Myerson value and/or allow for N players. The issue of externalities and efficiency could be then studied.

8 Appendix

8.1 An Appropriate Metric

Let $\breve{\mu}^3 = \left(\breve{\alpha}^3_{\breve{a}^3}, \breve{\beta}^3, \breve{\zeta}^3\right)$ be an element of <u>any given appropriate</u> subset of $\overrightarrow{\widetilde{\mho}}^3(Q^5)$ $\left(\overleftarrow{\widetilde{\mho}}^3\left(\ddot{Q}^5\right)\right)$. Given $\breve{\beta}^3_{\breve{a}}$ define $\gamma \in [0,1]$ so as to satisfy

$$\gamma\left(\phi_1^{13} + \phi_3^{13}, 0\right) + (1-\gamma)\left(0, \phi_1^{13} + \phi_3^{13}\right) = \breve{b}^3_{13} \tag{22}$$

.

Let $\breve{\zeta}^{3}_{h^{k.2}_{g^{\theta+ij}(a^k)}} = \beta$, where $k > 3$, be the component of $\breve{\zeta}^{3}$ that corresponds to a future contingency p or history $h^{k.2}_{g^{\theta+ij}(a^k)}$ where $a^k = (y, y)$ or, better yet, $\breve{\zeta}^{3}_{h^{k.2}}$, that follows $h^3_{g^{13}(.)}$. For each correlated strategy $\breve{\zeta}^{3}_{h^{k.2}}$ define $\gamma \in [0, 1]$ so as to satisfy

$$\gamma \left(\phi_i^{g^{\theta+ij}} + \phi_j^{g^{\theta+ij}}, 0 \right) + (1 - \gamma) \left(0, \phi_i^{g^{\theta+ij}} + \phi_j^{g^{\theta+ij}} \right) = \pi_{ij, \mathring{h}^k}^{\breve{\zeta}^{3}_{h^{k\prime}/h^k}} \left(a^k_{ij} \right) \tag{23}$$

.

If $a^k \neq (y, y)$, set wlg. $\gamma = 0$. If future contingency p is history $h^k_{g^{\theta+ij}(.)}$ instead, future-request $\breve{\zeta}^{3}$ may suggest $\mu^k = \breve{\zeta}^{3}_{h^{k\prime}_{(.)}/h^k}$ that are mixes over the disagreement joint plan and the fully successful joint plan that has a current promise-request in current contingency $a^k = (y, y)$ that puts probability one on the unique (not easy to see) proposal match that yields $\left(\overline{\overline{x}}^k_i, \overline{\overline{x}}^k_j \right)$, and hence the standard Nash bargaining solution in $\mathring{h}^k$. Hence for $\breve{\zeta}^{3}_{h^k}$ one sets

$$\gamma = \breve{\zeta}^{3}_{h^k} \left(a^k \right) \tag{24}$$

, where wlg., $a^k = (y, y)$.

Define the vector of gammas associated to $\breve{\mu}^3$ as $\left(\gamma \left(\breve{\beta}^3_{\breve{a}} \right), \left\{ \gamma \left(\breve{\zeta}^3_p \right) \right\}_p \right)$. The distance between two different $\breve{\mu}^3$ could be given by any standard infinite dimensional distance between their associated vector of gammas. Such rare metrics are necessary specially as for the complex IRF frontiers of histories $\mathring{h}^5_{g^{13+32}(.)}$ and $\mathring{h}^4_{g^{13+12}(.)}$ whenever $c = 0$ and $b = 0$ respectively (hence equation 23 suffices for histories $h^{k.2}$). Also, because in cases the IRF set is a closed segment, oldest friends may suggest tenable and reliable plans $\breve{\mu}^k$ that induce any payoff on that segment (hence equation 24 suffices for histories h^k).

8.2 Proof of Claim 2

Proof. Let the first two links in the rule of order 12 and 23 be rejected in stage 1 and 2 of the game respectively. Next to propose in stage 3 is pair $(1, 3)$.[8]

Part 1

I. Suppose that players 1 and 3 have a fully successful joint plan (that is, it is reliable, tenable, suggests link forming, that is, $\alpha^3_{a^3}$ puts probability one on a^3 where $a^3_{13} = (y, y)$) that suggests[9] a half-each payoff proposal match, that is, it recommends

[8] Note that if pair $(1, 3)$ rejects the game ends with zero payoffs. If it accepts, pair $(1, 2)$ follows; in turn, if $(1, 2)$ rejects, pair $(2, 3)$ is next; because every not linked pair must have a last opportunity to propose (as in bridge). If link 23 does not form the game ends, and so on.

[9] In the language of section 3, this plan has a promise-request, a degenerated correlated strategy, that puts probability 1 on both proposing $(\frac{3}{6}, \frac{3}{6})$.

each one to propose $(\frac{3}{6}, \frac{3}{6})$, a payoff for player 1 and another one for player 3. Suppose after link 13 forms, link 12 is rejected in stage 4 and link 32 is being discussed in stage 5. I want to find out, to begin with, what are all the tenable future-requests for players 1 and 3 on players 3 and 2 in this contingency.

First, let's see what players 3 and 2 can achieve by enunciating a future tenable joint plan that suggests a proposal match (Note this joint plan is not necessarily reliable and tenable) such that player 2 is offered (out of the sum of their Myerson values in the immediate prospective graph g^{13+32}, $\frac{4}{6}+\frac{1}{6}$) less than what she would get in the complete graph, $\frac{2}{6}$. After link 32 forms, as a future tenable joint plan, it would have to future-request players 1 and 2 to enunciate their unique O-F Joint Plan that suggests link formation and a proposal match (both propose their Shapley values) and thus form the third link 12. This is the case as the latter players' joint plan bargaining problem would be "essential", both gain by linking. The expected payoffs for player 3 and 2 associated to their joint plan (Plan a) would be $(\frac{2}{6}, \frac{2}{6})$, their Myerson values in the complete graph.

Second, if instead players 3 and 2 can enunciate a future tenable joint plan that suggests a proposal match such that player 2 is offered strictly more than $\frac{2}{6}$, this joint plan has to future-request players 2 and 1 to enunciate the unique O-F Joint Plan that suggests both unilaterally rejecting the third link. Link 32 would be the last to form (Plan of type b). The associated expected payoffs pair (x_3^5, x_2^5) for players 3 and 2 would lie on the diagonal in figure 1 to the northwest of $\overline{b}_{32}^5 = (\frac{3}{6}, \frac{2}{6})$.

Third, if instead players 3 and 2's future tenable joint plan suggests a proposal match that offers exactly $\frac{2}{6}$ to player 2, proposal match b^5 such that $b_{32}^5 = (\frac{3}{6}, \frac{2}{6}) = \overline{b}_{32}^5$ in figure 1, player 2 would be indifferent between forming or not the third link. As players 3 and 2 could be the only relevant oldest pair of successful negotiators according to the O-F focal effect, there are three types of future tenable joint plans (provided these are tenable and reliable, there are three more types of histories in the communication game corresponding to the payoff-relevant contingency that follows $b_{32}^5 = (\frac{3}{6}, \frac{2}{6})$) if $b_{32}^5 = \overline{b}_{32}^5$. One type of joint plan would future-request an O-F Joint Plan that suggests link 12 to be formed (Plan d1). The other one would future-request an O-F Joint Plan that suggests link 12 to be rejected (Plan d2). The third one consist of mixes (Plans d3). The associated expected payoffs (x_3^5, x_2^5) for players 3 and 2 would be respectively $(\frac{2}{6}, \frac{2}{6})$, $\overline{b}_{32}^5 = (\frac{3}{6}, \frac{2}{6})$ and convex combinations of the latter pairs of payoffs. Note how the O-F focal effect prevents the IRF^5 set to be open at $\overline{b}_{32}^5 = (\frac{3}{6}, \frac{2}{6})$!

As outside options for players 3 and 2 are $(\frac{3}{6}, 0)$, joint plan d2 with payoffs $(\frac{3}{6}, \frac{2}{6})$ is the only reliable, tenable that has strong Pareto efficient payoffs (Note that the joint plan that suggests link 32 rejection is also tenable and reliable;d1 and d2 are only future tenable). Thus, d2 is the unique <u>Nash Coherent</u> Joint Plan for players 3 and 2. Moreover it is fully successful. Player 1 would get in the latter case her Myerson value in graph g^{13+32}, $\frac{1}{6}$. See figure 1, however, set the outside options for players 3 and 2 $\left(\psi_3^5, \psi_2^5\right) = (\frac{3}{6}, 0)$.

Back to players' 1 and 3's discussion, as player 3 gets the same independently of link 32 forming or not, the O-F focal effect implies that O-F Joint Plans whenever link 32 is being discussed are up to the oldest fully successful friends 1 and 3. Fully successful joint plans for players 1 and 3 vary if the O-F Joint Plan they future-request either suggest link 32 rejection (type 1 plans), link formation with proposal match $(\frac{3}{6}, \frac{2}{6})-$and thereafter link 12 rejection$-$(type 2 joint plan) or mixes (type 3 joint plans). Associated expected payoffs for players 1, 2 and 3 would be respectively $(\frac{3}{6}, 0, \frac{3}{6})$, $(\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ and convex combinations between $(\frac{3}{6}, 0, \frac{3}{6})$ and $(\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$. As it will become clear soon, the O-F focal effect ensures payoffs will be $(\frac{3}{6}, 0, \frac{3}{6})$ and hence the nucleolus will be implemented in coalition structure!

One stage backwards, as of link 12 discussions in stage 4, one can now characterize all possible type 1 fully successful joint plans for players 1 and 3. As the outside option pair for players 1 and 2 is $(\frac{3}{6}, 0)$, using analogous reasons as in bargaining among players 3 and 2 above, a fully successful joint plan for players 1 and 3 would have to future-request an O-F Joint Plan that suggests either unilaterally rejecting link 12 (type 1.1 Joint Plan) or link formation with a proposal match $(\frac{3}{6}, \frac{2}{6})$ (type 1.2 Joint Plan) or a mix.(type 1.3 Joint Plans). Expected payoffs pairs for players 1 and 3 would be respectively $(\frac{3}{6}, \frac{3}{6})$, $(\frac{3}{6}, \frac{1}{6})$ and convex combinations between $(\frac{3}{6}, \frac{3}{6})$ and $(\frac{3}{6}, \frac{1}{6})$. On the other hand, one can characterize the unique type 2 joint plan for players 1 and 3. As the outside options pair for players 1 and 2 is $(\frac{1}{6}, \frac{2}{6})$, their joint plan bargaining game is essential and such a fully successful joint plan for players 1 and 3 would have to future-request an O-F Joint Plan for players 1 and 2 that suggests link formation and a proposal match. Also, analogously as before, an O-F Joint Plan that suggests link 23 rejection after link 12 forms would be future-requested. The NTU NBR yields payoffs of $(\frac{1}{6} + \frac{1}{6}, \frac{2}{6} + \frac{1}{6})$ for players 1 and 2. Player 3 would get her Myerson value in g^{13+12}, $\frac{1}{6}$. Under any joint plan of type 3, the bargaining game for players 1 and 2 is also essential, thus player 3 would get also $\frac{1}{6}$ and player 1 could not get more than $\frac{3}{6}$!

II. Suppose that players 1 and 3 have a fully successful joint plan that suggests proposal matches where player 3 is offered less than half.

If link 12 is rejected then in any O-F Joint Plan for players 3 and 2, they would suggest link formation and a proposal match as the joint plan bargaining game is essential (See figure 1 where player 3 is offered $b^3(3) = \frac{2}{6}$ and hence outside options are $\left(\psi_3^5, \psi_2^5\right) = (\frac{2}{6}, 0)$). Based on the analysis in I, link 23 would be the last link to form. In particular, if player 3's outside option is zero (Note that player 2's outside option is, as in I, again zero) the NTU NBR would give player 2 half of the sum of their Myerson values, that is, $\frac{2.5}{6}$. That is the most she would get. The least she may get is, following I, $\frac{2}{6}$ (See figure 1 where she gets exactly that).

One stage backwards, as player 1's outside option is $\frac{1}{6}$ and that of 2's is at most $\frac{2.5}{6}$, the joint plan bargaining game as of link 12 discussions is essential (as $\frac{1}{6} + \frac{2.5}{6} < \frac{5}{6}$, the sum of players 1 and 2's Myerson values) whenever player 3 is offered less than half. Analogously as in the case of type 2 joint plan in I, it can be shown that under any fully successful joint plan by players 1 and 3 with future-requests consistent with the previous analysis, link 12 would form right after link 13 forms and then the third link 23 would be rejected.

III. Now suppose player 3 is offered more than half.

If link 12 is rejected then in any O-F Joint Plan for players 3 and 2, they suggest unilateral rejections. Note that as link 23 does not form, player 2 gets zero in g^{13}, and player 3 would get more than $\frac{3}{6}$.

One stage backwards as of link 12 discussions, as the outside option pair for players 1 and 2 is $(\psi_1^4, 0)$, where $\psi_1^4 < \frac{3}{6}$, as in II, a fully successful joint plan for players 1 and 3 consistent with the previous analysis would have to future-request on players 1 and 2 an O-F Joint Plan that suggests link formation and a proposal match. Again, link 12 would be the last link to form.

Fully successful joint plans in cases II, III and I, where in the latter case one does not include the fully successful joint plan for players 1 and 3 that future-requests unilateral rejections of links 12 and 32−in that order−after link 13 forms (type 1.1 plan), have expected payoffs for players 1 and 3 that would give at least one player (either 1 or 3) less than a half and the other one at most $\frac{3}{6}$.

Part 2 . Because the outside options are zero as of link 13 discussions, from Part 1, out of any fully successful tenable and reliable joint plan, type plan 1.1 is the only one that yields strong Pareto efficient payoffs, $(\frac{3}{6}, \frac{3}{6})$, if obeyed. Thus, it is the unique O-F Joint Plan as of link 13 preliminary negotiations. Note that as of link 13 preliminary negotiations no link has formed−as past preliminary negotiations have been unsuccessful−so the unique tenable and reliable joint plan by not linked pairs, the disagreement joint plan is basically ignored or trivially followed.

Part 3. One stage backwards, fully successful joint plans for players 2 and 3 are analogous to the one in the bargaining problem for players 1 and 3. In contrast, outside options are zero for player 2 and a half for player 3. As players 2 and 3 have no preceding oldest successful negotiators, the unique O-F Joint Plan suggests link formation and a half-half proposal match and future-requests consecutive rejection of the next two links in the order (it is a plan analogous to type 1.1 plan). At the beginning of the game, a similar argument can be applied as of link 12 discussions and the claim follows ∎

Diagram: Myerson Values for Normalized Games

One-link Graphs

$\left(\frac{a}{2}\right) 3$

/

$\left(\frac{a}{2}\right)$ 1 2 (0)

graph g^{13}

3(0)

$\left(\frac{c}{2}\right)$ 1——2 $\left(\frac{c}{2}\right)$

graph g^{12}

3 $\left(\frac{b}{2}\right)$

\

(0)1 2 $\left(\frac{b}{2}\right)$

graph g^{23}

Two-Link Graphs

$\left(\frac{2d+b+a}{6}\right)$ 3

/ \

$\left(\frac{2(d-b)+a}{6}\right)$1 2 $\left(\frac{2(d-a)+b}{6}\right)$

graph g^{132}

$\left(\frac{2(d-c)+a}{6}\right)$ 3

/

$\left(\frac{2d+a+c}{6}\right)$ 1——2 $\left(\frac{2(d-a)+c}{6}\right)$

graph g^{312}

3 $\left(\frac{2(d-c)+b}{6}\right)$

\

$\left(\frac{2(d-b)+c}{6}\right)$ 1—— 2 $\left(\frac{2d+b+c}{6}\right)$

graph g^{123}

Complete graph

3 $\left(\frac{a+2(d-c)+b}{6}\right)$

/ \

$\left(\frac{c+2(d-b)+a}{6}\right)$1——2 $\left(\frac{b+2(d-a)+c}{6}\right)$

graph g^N

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