ON THE INEFFICIENCY OF THE AMERICAN OPTION CONTRACT IN INCOMPLETE MARKETS

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1 Introduction

European options in incomplete markets have been largely studied. Not so much has been done for the American ones. For the reader not familiar on this topic, a complete analysis, in the discrete time case, on main results can be found in [5]. An interesting paper on the continuous time case is [7]. For what concerns an utility maximization approach, the only attempt considered in literature is the *neutral valuation* approach studied in [6].

In this paper we shall consider an utility maximization approach different with respect to the neutral one. We shall take into account the different approaches that the two agents, involved in an American option contract, may have in trading it.

In an incomplete market, for any contingent claim, an interval of the real line of arbitrage-free prices is defined. Since super-hedging is not a realistic solution, the seller has to bring some risk, considering only partial-hedging strategies. In the case of an American option the seller is faced also with the uncertainty of the exercise time selected by the buyer which represents another source of risk.

Classically, in literature, the exercise strategy of the buyer is studied without taking into account the price x at which the option is traded. That is, if H_τ is the pay-off of the option, if exercised at the stopping time τ, it is assumed that the aim of the buyer is to choose a payoff from the class $\{H_\tau\}_\tau$ which is optimal in the sense that it has maximal expected utility. Thus, the problem he has to solve is

$$\max_{\tau} \mathbb{E}^{\mathbb{Q}} \left[u \left(H_{\tau} \right) \right]$$

where $\mathbb{Q}$ is a probability on the underlying probability space and u a utility function.

In this case, the optimal stopping strategy $\hat{\tau}$ chosen by the buyer is independent of the price x.

Depending on the behaviour of the buyer, his exercise time may be different for different arbitrage-free prices. This happens, for example, if he takes trading strategies "explicitly" into consideration in maximizing his utility. For American contingent claims this approach has been considered just in [2] and [9].

The starting task of our study it was to analyze the influence of the eventual knowledge the seller has on the behaviur of the buyer.

To this gaol, we have studied the extreme case in which the seller knows perfectly the stopping time chosen by the buyer for each possible price. This is an extreme case, but not totally unrealistic, since in many cases it is possible to suppose that the seller is in the position to make some hypothesis on the buyer's attitude to risk and on his perception of the market direction.

In such a case the seller can restrict the set of all possible exercise times and so select an hedging portfolio which reduces his risk.

This way, the American option contract may be modeled as a hierarchical game in which the seller will select the price of the option minimizing his risk, considering, for every admissible risk-free price, the possible exercise times chosen by the buyer. The price at which the option will be traded is a *hierarchical equilibrium point*.

Modeling the American option contract in this way, we shall prove that it is convenient for the buyer too if the seller has some knowledge on his stopping strategy, in the sense that, if this happens, the option can be traded at a smaller price.

This way, we shall see that, for a certain class of agents, the classical American option contract is a not perfectly efficient contract, in the sense that his efficiency may be improved (both from the seller and the buyer point of view) if the buyer declares his stopping strategy at time 0.

Starting from this fact, the authors will introduce a new contract which is a modification of the American one. It is what they call a *pseudo-American* contract: the difference with respect to the American contract is that the buyer chooses at time 0 at which time, depending on the trading price and on the basis of all possible future evolutions of the market, he will exercise the option.

It will be shown that, for a certain class of agents, this new contract is more efficient than the American one.

The paper is organized as follows: in Section 2 we introduce the mathematical setup and we briefly resume the main results on American options in incomplete markets; in Section 3 we study the mathematical model of the possible behaviour of two agents who are interested, the one to sell the option and the other to buy it; in Section 4 we study how things change in the extreme case in which the seller has perfect knowledge of the stopping strategy of the buyer; in Section 5 we introduce a new contract which is a modification of the classical one and that, for a certain class of agents, it is more efficient than that one; in Section 6 we present a very easy incomplete market and we illustrate, by numerical computations, all the results and the considerations done. At the end, an Appendix is provided, which contains the proof of a technical result.

2 The model

In this Section we formalize the mathematical setup and, for the reader not familiar with American options in incomplete markets, we illustrate and explain the main properties needed for the sequel.

Let $T > 0$ be the horizon of the contract and $E \subseteq [0, T]$ finite (in this case, $0, T \in E$) or $E = [0, T]$. Consider a market in which the discounted prices process of the underlying assets are described as a d-dimensional semimartingale $X = (X_t)_{t \in E}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in E}$, which satisfies the usual conditions and such that $\mathcal{F}_0$ is trivial. We suppose that the market is incomplete and we assume absence of arbitrage supposing that the set of all equivalent martingale measures $\mathcal{P}$ is not empty. Let $\mathcal{T}^E_{t,T}$ be the set of all stopping times with values in $[t, T] \bigcap E$.

Definition 2.1. *A self-financing strategy (V_0, ξ) is given by an initial capital V_0 and by a d-dimensional, $\mathbb{F}$-predictable process ξ such that the resulting wealth process*

$$V_t^{V_0,\xi} = V_0 + \int_0^t \xi_s \cdot dX_s, \quad t \in E \tag{1}$$

is well defined.

Let us fix a c.a.d.l.a.g. and $\mathbb{F}$-adapted process $\Lambda = (\Lambda_t)_{t \in E}$ uniformly bounded from above by a positive constant C and such that $\inf_{\mathbb{P}^* \in \mathcal{P}, \tau \in \mathcal{T}^E_{0,T}} \mathbb{E}^* [\Lambda_\tau] < +\infty$. Let us define, for any $y \in \mathbb{R}$,

$$\mathcal{A}^\Lambda_E(y) = \left\{ \xi \text{ self-financing strategies } : V_t^{y,\xi} \geq -\Lambda_t, \forall t \in E \right\} \tag{2}$$

Obviously $\xi \in \mathcal{A}^\Lambda_E(y)$ implies that $V^{y,\xi}$ is a $\mathcal{P}$-supermartingale, in the sense that it is a $\mathbb{P}^*$-supermartingale for every $\mathbb{P}^* \in \mathcal{P}$.

Consider a contingent claim that may be exercised at every instant $t \in E$, whose discounted pay-off is given by a positive, $\mathbb{F}$-adapted and c.a.d.l.a.g. process $H = (H_t)_{t \in E}$.

Suppose

$$\sup_{\tau \in \mathcal{T}^E_{0,T}, \mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \left[H_\tau \right] < +\infty$$

where $\mathbb{E}^*$ is the expectation with respect to $\mathbb{P}^*$.

Let us now introduce the notion of arbitrage-free price for an American contingent claim in a general incomplete framework.

For any exercising stopping time $\tau \in \mathcal{T}^E_{0,T}$, let

$$\mathcal{I}^E \left(H_\tau \right) = \left\{ \mathbb{E}^* \left[H_\tau \right] : \mathbb{P}^* \in \mathcal{P} \right\}$$

We shall consider the following definition of arbitrage-free price for an American contingent claim as done in ([5]), Definition 6.31.

Definition 2.2. *A real number x is called an "arbitrage-free price" of a discounted American contingent claim H if the following two conditions are satisfied:*

1. *the price x is not too high in the sense that there exists some $\tau \in \mathcal{T}^E_{0,T}$ and $x' \in \mathcal{I}^E \left(H_\tau \right)$ such that $x \leq x'$.*
2. *the price x is not too low in the sense that there exists no $\tau' \in \mathcal{T}^E_{0,T}$ such that $x < x'$ for all $x' \in \mathcal{I}^E \left(H_{\tau'} \right)$.*

The set of all arbitrage-free prices of H is denoted by $\mathcal{I}^E(H)$.

By definition, $x \in \mathcal{I}^E(H)$ implies that there exist $\mathbb{P}^* \in \mathcal{P}$ and $\tau \in \mathcal{T}^E_{0,T}$ such that $x = \mathbb{E}^* \left[H_\tau \right]$; furthermore $\mathcal{I}^E(H)$ is an interval of the real line with endpoints

$$\underline{x} = \sup_{\tau \in \mathcal{T}^E_{0,T}} \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \left[H_\tau \right]$$

$$\overline{x} = \sup_{\mathbb{P}^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}^E_{0,T}} \mathbb{E}^* \left[H_\tau \right]$$

Definition 2.3. *Any self-financing strategy (V_0, ξ) such that*

$$V_t^{V_0, \xi} \geq H_t, \forall t \in E$$

is called a superhedging strategy.

Theorem 2.4. *There exists a superhedging strategy with initial investment* $\overline{x}$*, and this is the minimal amount needed to implement it.*

Proof. See Corollary 7.9 in [5] for E discrete; see [8] for E continuous. $\square$

Remark 2.1. *(The Bermudan option case) Let us consider a market in which the underlying assets change continuously in time, while the option may be exercised just in a discrete subset of the interval* $[0, T]$*. In this case, the superhedging problem has been exhaustively studied in [13]. In the following we will not explicitly consider this more technical case, even if all the results remain valid.*

Definition 2.5. *A discounted American claim* H *is called "attainable" if there exists a stopping time* $\tau \in \mathcal{T}^E_{0,T}$ *and an admissible strategy* $(\tilde{V}_0, \tilde{\xi})$ *whose wealth process* V *satisfies*

$$V_t \geq H_t, \text{ for all } t, \text{ and } V_\tau = H_\tau.$$

The trading strategy $(\tilde{V}_0, \tilde{\xi})$ *is called an "hedging strategy" for* H*.*

Proposition 2.1. *The following conditions are equivalent:*

1. *H is attainable;*
2. *$\mathcal{I}^E(H) = \{\overline{x}\} = \{\underline{x}\}$;*
3. *$\overline{x} \in \mathcal{I}^E(H)$.*

Proof. *3.* implies that there exist $\mathbb{P}^* \in \mathcal{P}, \hat{\tau} \in \mathcal{T}^E_{0,T}$ such that

$$\mathcal{I}^E(H_{\hat{\tau}}) \ni \mathbb{E}^*[H_{\hat{\tau}}] = \overline{x} = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}[H_{\hat{\tau}}] \tag{3}$$

and this implies that the European type contingent claim with payoff $H_{\hat{\tau}}$ is attainable; thus,

$$\overline{x} = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}[H_{\hat{\tau}}] \leq \sup_{\tau \in \mathcal{T}^E_{0,T}} \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}[H_{\hat{\tau}}] = \underline{x}$$

and *2.* follows.

2.$\Rightarrow$*3.* is trivial.

3. implies again (3). By superhedging, there exists a self-financing portfolio $V^{\overline{x},\xi}$ such that $V^{\overline{x},\xi}_\tau \geq H_\tau$ for every $\tau \in \mathcal{T}^E_{0,T}$. But, by (3), $V^{\overline{x},\xi}_{\hat{\tau}} = H_{\hat{\tau}}$, that is *1.*.

1.$\Rightarrow$*3.* is trivial, by definition. $\square$

Next result is needed to consider not empty sets of strategies of type (2).

Lemma 2.1. $\mathcal{A}_E^\Lambda(y) \neq \emptyset$ *if and only if*

$$y + \inf_{\mathbb{P}^* \in \mathcal{P}, \tau \in \mathcal{T}_{0,T}^E} \mathbb{E}^* \left[\Lambda_\tau\right] \geq 0 \tag{4}$$

Proof. For every $\xi \in \mathcal{A}_E^\Lambda(y)$, $V^{z-x,\xi}$ is $\mathcal{P}$-supermartingale bounded from below which dominates the process $-\Lambda$. Thus $y \geq \sup_{\mathbb{P}^* \in \mathcal{P}, \tau \in \mathcal{T}_{0,T}} \mathbb{E}^* \left[-\Lambda_\tau\right]$ and (4) follows.

On the other end, by (4) and the superhedging strategy, there exists a trading strategy η such that

$$C + y + \int_0^t \eta_s \cdot dX_s \geq -\Lambda_t + C.$$

Obviously $\eta \in \mathcal{A}_E^\Lambda(y)$. □

3 The strategies of the agents

In this Section, we shall present a reasonable model to simulate a realistic behaviour of the buyer and of the seller.

Let us indicate by S and B, respectively, the seller and the buyer of the contingent claim. Let us suppose H not to be attainable and x to be an arbitrage-free price for the contingent claim H.

We now take the point of view of the buyer of the American claim H.

In the literature, the optimal strategy of the buyer is studied without taking into account the price x at which he bought the option. That is, it is assumed that the aim of the buyer is to choose a payoff from the class $\{H_\tau\}_{\tau \in \mathcal{T}_{0,T}^E}$ which is optimal in the sense that it has maximal expected utility. Thus, the problem he has to solve is

$$\max_{\tau \in \mathcal{T}_{0,T}^E} \mathbb{E}^{\mathbb{Q}} \left[u\left(H_\tau\right)\right]$$

where $\mathbb{Q}$ is a probability on $(\Omega, \mathcal{F})$ and u a utility function.

This way, the optimal stopping strategy $\hat{\tau}$ chosen by B is independent of the price x.

Let us consider, instead, the following, more realistic approach.

Let $z \geq 0$ be the initial wealth of B and suppose that he will consider self-financing strategies in (2) for fixed C and Λ.

Furthermore, let $y + \inf_{\mathbb{P}^* \in \mathcal{P}, \tau \in \mathcal{T}_{0,T}^E} \mathbb{E}^* \left[\Lambda_\tau\right] \geq 0$ (see Lemma (2.1)).

If B accepts to buy the American contingent claim H at the arbitrage-free price x, then he has to invest the remaining wealth $z - x$ in the market and, for every strategy he adopts in $\mathcal{A}_E^\Lambda(z-x)$, for every stopping-time he chooses for the exercise, B is sure that his default is bounded from below by $-\Lambda$. In fact,

$$H_\tau \mathbb{I}_{\{t\geq\tau\}} + V_t^{z-x,\xi} \geq H_\tau \mathbb{I}_{\{t\geq\tau\}} - \Lambda_t \geq -\Lambda_t$$

Thus $-\Lambda$ represents the risk he is propense to accept, and obviously, B will accept a price x only if $\mathcal{A}_E^\Lambda(z-x)\neq\emptyset$. By Lemma (2.1), this is equivalent to the condition $x \leq z + \inf_{\mathbb{P}^*\in\mathcal{P},\tau\in\mathcal{T}_{0,T}^E} \mathbb{E}^*\left[\Lambda_\tau\right]$.

Let

$$\pi\left(\Lambda, z\right) = z + \inf_{\mathbb{P}^*\in\mathcal{P},\tau\in\mathcal{T}_{0,T}^E} \mathbb{E}^*\left[\Lambda_\tau\right]. \tag{5}$$

If B does not buy the American contingent claim H, then he will invest all his wealth z in a self-financing strategy $\xi\in\mathcal{A}_E^\Lambda(z)$.

If we suppose that B takes a maximizing utility criterion, then we may associate to him a utility functional.

Let $\mathbb{P}^B$ be a subjective probability equivalent to $\mathbb{P}$ and u be a utility function, i.e. a strictly increasing, concave and continuous function, defined on $[-C,+\infty)$.

If B buys the American contingent claim H, his utility at time 0 is

$$U_0(H,x) = \sup_{\tau\in\mathcal{T}_{0,T}^E}\ \sup_{\xi\in\mathcal{A}_E^\Lambda(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\xi}\right)\right], \tag{6}$$

otherwise, it is

$$U_0(z) = \sup_{\eta\in\mathcal{A}_E^\Lambda(z)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(V_T^{z,\eta}\right)\right] \tag{7}$$

Remark 3.1. *Obviously $U_0(H,x)$ is not increasing with respect to x.*

In such a context, B will buy the American contingent claim H if and only if

$$U_0\left(H,x\right) \geq U_0(z) \tag{8}$$

This way B will accept to buy the option at any price x in the interval

$$\mathcal{X} = \left\{y\in\mathcal{I}^E(H) : U_0\left(H,x\right)\geq U_0(z)\right\}\bigcap\left\{y\leq\pi\left(\Lambda,z\right)\right\}. \tag{9}$$

Let π^B be the right-end point of $\mathcal{X}$. If $\pi^B\in\left\{y\in\mathcal{I}^E(H) : U_0\left(H,x\right)\geq U_0(z)\right\}$, then it is the buyer's indifference price.

Furthermore, if B buys the American option H, then he would like to exercise it in a stopping time $\hat{\tau}(x)$ such that

$$U_0(H,x) = \sup_{\xi \in \mathcal{A}_E^{\Lambda}(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_{\hat{\tau}(x)} + V_T^{z-x,\xi}\right)\right], \tag{10}$$

if such an optimal stopping time exists (see next Remark (3.2)). Otherwise, his stopping strategy $\hat{\tau}(x)$ will, in any case, verify

$$\sup_{\xi \in \mathcal{A}_E^{\Lambda}(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_{\hat{\tau}(x)} + V_T^{z-x,\xi}\right)\right] \geq U_0(z) \tag{11}$$

Remark 3.2. *If B buys the option, that is, his expected utility is given by* $U_0(H,x)$*, then he will be looking for an exercising stopping strategy* $\hat{\tau}(x)$ *such that (10) is verified.*

Let, $t \in E \cap [0,T]$ *and, for every* $\xi \in \mathcal{A}_E^{\Lambda}(z-x)$*,*

$$\mathcal{A}_{E,t}^{\Lambda}(\xi) = \left\{\eta \in \mathcal{A}_E^{\Lambda}(z-x) : \eta_s = \xi_s, \forall s \leq t\right\}$$

and

$$U_t(H,\xi) = esssup_{\eta \in \mathcal{A}_{E,t}^{\Lambda}(\xi)} esssup_{\tau \in \mathcal{T}_{t,T}^E} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\eta}\right)\middle| \mathcal{F}_t\right].$$

From stochastic control theory (see [3]), an optimal stopping strategy $\hat{\tau}(x)$ *for B exists if and only if there exists a* $\hat{\xi}(x) \in \mathcal{A}_E^{\Lambda}(z-x)$ *such that*

1. $U_{t\wedge\hat{\tau}(x)}\left(H, \hat{\xi}(x)\right)$ *is a martingale;*
2. $U_{\hat{\tau}(x)}\left(H, \hat{\xi}(x)\right) = \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_{\hat{\tau}(x)} + V_T^{z-x,\hat{\xi}(x)}\right)\middle| \mathcal{F}_{\hat{\tau}(x)}\right]$.

The existence of an optimal stopping strategy $\hat{\tau}(x)$ *is guaranteed in the following two cases:*

1. Ω *is finite and E discrete, that is,* $E = \{0 = t_0 < t_1 < t_2 < \cdots < t_N = T\}$.
   In this case, since $\mathcal{T}_{0,T}$ *is finite, for every* $x \in \mathcal{I}^E(H)$*, there exists an optimal stopping time* $\hat{\tau}(x)$ *such that the utility* $U_0(H,x)$ *is really achieved as in(10).*
2. *B invests only in the risk free asset (see next Remark (3.4)).*
   In this case
   $$U_0(H,x) = \sup_{\tau \in \mathcal{T}_{0,T}^E} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + z - x\right)\right]$$
   and the existence of an optimal stopping time $\hat{\tau}(x)$ *is guaranteed by Snell envelope theory (see again [3]), always, in the discrete time case (see also [5]) and, if* $u(H_t + z - x)$ *is left-continuous in mean, in the continuous time case (that is* $E = [0,T]$*).*

Remark 3.3. *As shown by the numerical examples presented in Section 6, the optimal stopping strategy $\hat{\tau}(x)$ really depends on x.*

Proposition 3.1. *If the stopping strategy $\hat{\tau}(x)$ chosen by B is such that there exists a trading strategy $\hat{\xi} \in \mathcal{A}_E^{\Lambda}(z-x)$ for which*

$$\mathbb{E}^{\mathbb{P}^B}\left[u\left(H_{\hat{\tau}(x)} + V_T^{z-x,\hat{\xi}}\right)\right] \geq U_0(z),$$

then $x \in \mathcal{I}^E\left(H_{\hat{\tau}(x)}\right)$.

Proof. Let $x \notin \mathcal{I}^E\left(H_{\hat{\tau}(x)}\right)$:

1. $x < \mathbb{E}^*\left[H_{\hat{\tau}(x)}\right]$ for every $\mathbb{P}^* \in \mathcal{P}$ implies that x is not an arbitrage-free price for H, by point 2. of Definition (2.2);

2. $x > \mathbb{E}^*\left[H_{\hat{\tau}(x)}\right]$ for every $\mathbb{P}^* \in \mathcal{P}$ implies, by superhedging, that there exits $\rho \in \mathcal{A}_E^0(x)$ such that $V_{\hat{\tau}(x)}^{x,\rho} \geq H_{\hat{\tau}(x)}$ and $\mathbb{P}\left(V_{\hat{\tau}(x)}^{x,\rho} > H_{\hat{\tau}(x)}\right) > 0$.

   In this case,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^B}\left[u\left(H_{\hat{\tau}(x)} + V_T^{z-x,\hat{\xi}}\right)\right] &< \mathbb{E}^{\mathbb{P}^B}\left[u\left(V_{\hat{\tau}(x)}^{x,\rho} + V_T^{z-x,\hat{\xi}}\right)\right] = \\ &= \mathbb{E}^{\mathbb{P}^B}\left[u\left(z + \int_0^T \left(\rho_s \mathbb{I}_{s\leq\hat{\tau}(x)} + \hat{\xi}_s\right)\cdot dX_s\right)\right] \leq \\ &\leq \sup_{\eta\in\mathcal{A}_E^{\Lambda}(z)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(V_T^{z,\eta}\right)\right] = U_0(z).\end{aligned}$$

   And this is not true, by hypolthesis.

□

The existence of the trading strategy $\hat{\xi}$ required in the statement of the previous Proposition is guaranteed under the sufficient conditions of the following Lemma.

Lemma 3.1. *If the utility function u is uniformly bounded from above (e.g. the exponential utility function) or Ω is finite, for every $\tau \in \mathcal{T}_{0,T}^E$, there exists $\hat{\xi} \in \mathcal{A}_E^{\Lambda}(z-x)$ such that*

$$\sup_{\xi\in\mathcal{A}_E^{\Lambda}(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\xi}\right)\right] = \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\hat{\xi}}\right)\right].$$

Proof. The proof is standard. For completeness it will be provided in the Appendix. □

Remark 3.4. *Let us consider the case of a "small investor" who is not in the position to invest in the market via a self-financing strategy, but who is interested in maximizing his utility. In this case*

$$U_0(H,x) = \sup_{\tau \in \mathcal{T}^E_{0,T}} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + z - x\right)\right]$$

(see Remark (3.2), for the conditions on the existence of an optimal stopping time), and

$$U_0(z) = u\left(z - x\right).$$

As before, B will buy the American contingent claim H if and only if

$$U_0\left(H,x\right) \geq U_0(z),$$

but Proposition (3.1) is no more true. That is, it could happen that the stopping strategy $\hat{\tau}(x)$ *chosen by B maximizing his utility is such that* $x \notin \mathcal{I}^E\left(H_{\hat{\tau}(x)}\right)$*. In particular, since part 1. of the proof of Proposition (3.1) remains valid, it could happen that B chooses* $\hat{\tau}(x)$ *such that* $x > y$ *for every* $y \in \mathcal{I}^E\left(H_{\hat{\tau}(x)}\right)$ *(see Example (6.3)).*

Let us now take the point of view of the seller. In general, selling at $x \in \mathcal{I}^E(H)$, S assumes a risk and, obviously, he will be interested in selling at a price x as large as possible.

For example, if we suppose that S adopts a minimizing shortfall risk criterion to reduce the risk, the best hedge he can achieve is obtained by solving the following optimization problem:

$$R^E_0(x) = \inf_{\xi \in \mathcal{A}^0_E(x)} \sup_{\tau \in \mathcal{T}^E_{0,T}} \mathbb{E}^{\mathbb{P}^S}\left[l\left(\left(H_\tau - V^{x,\xi}_\tau\right)^+\right)\right] \tag{12}$$

where $\mathbb{P}^S$ is a subjective probability equivalent to $\mathbb{P}$ and l is a loss function, i.e. an increasing, convex and continuous function, defined on $[0,+\infty)$ and with $l(0) = 0$. In [10] an analysis of this functional can be found: in that paper the existence of a trading strategy $\tilde{\xi} \in \mathcal{A}^0_E(x)$ that realizes the infimum in (12) is proved and results on computable approximations are provided.

Remark 3.5. *The following are obvious:*

1. $R^E_0(x)$ *is not increasing with respect to* $x \in \mathcal{I}^E(H)$*;*
2. *if* $x \in \mathcal{I}^E(H)$*, then* $R^E_0(x) > 0$*.*

In such a context, the price at which S is looking for selling the option is as near as possible to the right-end point π^B of the interval $\mathcal{X}$ defined in (9).

Remark 3.6. *The above model of the possible behaviours of a buyer and of a seller of an American option wants to cover, for example, the case in which the buyer claims the option to make profits, while the seller trades it to provide market with liquidity. Obviously, alternative scenarios are possible, but the considerations and the conclusions of next Sections remain valid.*

4 Pricing the American option when the buyer declares his strategy: the Stackelberg game

Things change if we suppose that the seller introduced in the previous Section has some information on the behaviour of the buyer.

Let $\tau : \mathcal{I}^E(H) \to \mathcal{T}^E_{0,T}$ be the function such that, for every $x \in \mathcal{I}^E(H)$, gives the stopping time $\tau(x)$ the buyer chooses for exercising his option. Let us consider the extreme case in which S knows exactly such a function: the risk he can consider to assume is

$$r_0^E(x) = r_0^E(x, \tau(x)) = \inf_{\xi \in \mathcal{A}^0_E(x)} \mathbb{E}^{\mathbb{P}^S}\left[l\left(\left(H_{\tau(x)} - V^{x,\xi}_{\tau(x)} \right)^+ \right) \right] \leq R_0^E(x). \quad (13)$$

This way, he can propose a price $\hat{x}$ that minimizes such a risk. That is, he can consider the optimization problem

$$\inf_{x \in \mathcal{X}} r_0^E(x). \quad (14)$$

Suppose that there exists an $\hat{x}$ such that $r_0^E(\hat{x}) = \min_{x \in \mathcal{X}} r_0^E(x)$.

On the contrary to what happens if S has no information on the behaviour of B, $r_0^E(x)$ is in general no more monotone, thus $\hat{x}$ could be smaller than the right-end point of $\mathcal{X}$ and, consequently (see Remark (3.1)), $U_0(H, \hat{x})$ greater than the utility the buyer could achieve not giving any information to the seller S.

Furthermore

$$r_0^E(\hat{x}) = \inf_{x \in \mathcal{X}} r_0^E(x) \leq \inf_{x \in \mathcal{X}} R_0^E(x).$$

Thus $\hat{x}$ is a better price both for the buyer and the seller.

Remark 4.1. *If an $\hat{x}$ that solves (14) does not exist, the seller can propose a $\tilde{x}$ such that $r_0^E(\tilde{x})$ is as near as possible to the infimum defined in (14) and the considerations done for $\hat{x}$ remain valid for $\tilde{x}$.*

Remark 4.2. *In the "small investor" case considered in Remark (3.4), as already observed, it could happen that F chooses $\hat{\tau}(x)$ such that $x > y$ for every $y \in \mathcal{I}^E\left(H_{\hat{\tau}(x)}\right)$ (see Example (6.3)). In this case $r_0(x) = 0$.*

Remark 4.3. *Let us consider the case in which S knows the utility functional considered by B and that he uses this information to evaluate his risk. It could happen, and this is confirmed by concrete examples (see next Section), that for some prices x there is a set of stopping times $\Gamma(x)$ that are indifferent for the buyer, in the sense that all of them maximize his utility functional. In this case the seller may consider the risk functional*

$$r_0^E(x) = \inf_{\xi \in \mathcal{A}_E^0(x)} \sup_{\tau \in \Gamma(x)} \mathbb{E}^{\mathbb{P}^S}\left[l\left(\left(H_\tau - V_\tau^{x,\xi}\right)^+\right)\right] \tag{15}$$

and all done considerations remain valid, that is the efficiency of the contract growth for both agents point of view.

Announcing the exercise strategy at time 0, the buyer does not take into account the possibility to decide it in the future, during the life of the option. But, considering the model presented in Section 3 and Remark 3.2, it is obvious that the stopping strategies only depend on $\mathbb{P}^B$ and on the utility function u. Thus, the buyer does not loose any opportunity if they do not change during the life time of the option. This way, the model presented makes sense only for a buyer who is in the position not to change his utility functional and not to reach new information until time T.

This model may be seen as a negotiation model too: the buyer has to value if it is convenient to him to reveal something on his behaviour (and then to act coherently with it) or not to reveal anything and so, eventually, to accept to trade the option at a higher price.

The case presented is obviously an extreme case.

In such a context, the American contingent claim contract may be modeled as a Stackelberg game (see [1], [11]).

In the setting of Stackelberg games, the relationship between the seller and the buyer of the American option may be modeled as follows.

The first level agent, called Leader, is the seller: he has to minimizes his risk which is a function of the trading price of the option x and of the stopping time τ.

The second level agent, called Follower, is the buyer: he has to maximize his utility, which is a function of the trading price of the option x and of the stooping time τ.

The Leader announces his strategy, that is, a price $x \in \mathcal{I}^E(H)$. The Follower reacts with a stopping time $\tau(x)$ that maximizes his utility for fixed x.

Formally (see (13) and (6)):

I Level:

$$\inf_{x\in\mathcal{X}} r_0^E\left(x,\tau\right)$$

where $\tau = \tau(x)$ solves

II Level:

$$\sup_{\tau\in\mathcal{T}_{0,T}^E} U_0^E\left(x,H,\tau\right)$$

where $U_0^E\left(x,H,\tau\right) = \sup_{\xi\in\mathcal{A}_E^\Lambda(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\xi}\right)\right]$.

A couple $(\hat{x},\tau(\hat{x}))$ that solves the two levels problem is called a *Stachelberg equilibrium point.*

This way, the price at which the option is effectively traded is a Stackelberg equilibrium point.

5 An alternative contract

On the basis of what seen in all previous Sections, it is possible, in an incomplete market, to define a pseudo-American type option contract that allows B to optimize his utility as for the American contract, to S to reduce his risk and to trade the option at a smaller price.

Let us consider the following alternative contract.

Fixed an expiration time $T > 0$ and a contingent claim whose payoff at time $t \leq T$ is the random variable H_t, a *pseudo-American option* is a contract that gives the right to the buyer to choose, at time 0, the possible stopping time at which he will exercise the option: such a stopping time may depend on the price at which the option is traded and on the future states of the world.

This way, a pseudo-American contract is given fixing a couple $(H,\overline{\tau})$, where $H = (H_t)_{t\in[0,T]}$ represents the collections of all the possible payoffs and $\overline{\tau} : \mathbb{R}^+ \to \mathcal{T}_{0,T}^E \cup \{-\infty\}$ is a function that, for every price $x > 0$:

- gives the stopping time $\overline{\tau}(x)$ at which the buyer will necessarily exercise the option, if $\overline{\tau}(x) \in \mathcal{T}_{0,T}^E$;
- indicates that B does not accept to buy the option at the price x, if $\overline{\tau}(x) = -\infty$.

In other words, it is the contract that fixes, for every future state of the world, at which date B will exercise the option depending on the price x. Furthermore, we suppose that S proposes H, while B proposes $\overline{\tau}$.

S has to decide at which price he is intentioned to trade the option.

Let us indicate by $\mathcal{I}^E(H,\overline{\tau}) \subseteq \mathbb{R}^+$ the set of all the arbitrage-free prices of the pseudo-American option.

Proposition 5.1. *The following are true:*

1. $x \in \mathcal{I}^E(H,\overline{\tau})$ *if and only if* $\overline{\tau}(x) \neq -\infty$ *and* $x \in \mathcal{I}\left(H_{\overline{\tau}(x)}\right)$.
2. $x \in \mathcal{I}^E(H,\overline{\tau})$ *implies that there exists* $y \in \mathcal{I}^E(H)$ *such that* $x \leq y$.

Proof.

1. is true, since if $\overline{\tau}(x) = -\infty$ the option is not traded, that is x is not a candidate to price the option; if, instead $x \notin \mathcal{I}\left(H_{\overline{\tau}(x)}\right)$, since $\overline{\tau}(x)$ is declared at time 0, obviously one of the agents has an arbitrage opportunity.
2. is a consequence of point 1.

□

Remark 5.1. *If $x < y$ for every $y \in \mathcal{I}^E(H)$, x may be an arbitrage-free price for the pseudo-American option providing that $x \in \mathcal{I}\left(H_{\overline{\tau}(x)}\right)$. This way, the lower bound to the set of all the arbitrage-free prices may be smaller than that of the classical American case, that is it may be smaller then $\underline{x}(H)$ (see all the Examples in Section (6)).*

Considering the model presented in Section 3, we may suppose that B chooses the function $\overline{\tau}$, maximizing his utility $U_0(H,x)$ (see (6)) when (8) is verified.

In such a case $\overline{\tau}(x) \neq -\infty$ if and only if there exists $\tau \in \mathcal{T}^E_{0,T}$ such that $\sup_{\xi \in \mathcal{A}^{\Lambda}_E(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\xi}\right)\right] \geq U_0(z)$.

Furthermore, if the hypothesis of Proposition (3.1) are verified, $\overline{\tau}(x) \neq -\infty$ and $x \in \mathcal{I}^E(H)$, then $x \in \mathcal{I}\left(H_{\overline{\tau}(x)}\right)$, that is B optimizes as for the classical American option contract.

Remark 5.2. *As already noticed in Section 4, announcing the exercise strategy at time 0 the buyer looses the possibility to decide it in the future during the life of the option. But, as seen in Section 4, considering the model presented in Section (3) and Remark 3.2, the stopping strategies only depend on $\mathbb{P}^B$ and on the utility function u. Thus, the buyer does not loose any opportunity if they do not change during the life time of the option. This way, the pseudo-American contract is obviously more convenient than the classical American one, for a buyer who has no opportunity to reach new information until time T.*

6 Examples

In this Section we shall consider an incomplete market to illustrate the model and the considerations presented above. The market is the easiest to construct. In spite of his simplicity, this model puts the same in evidence the importance of the information the seller has on the behaviour of the buyer and the inefficiency of the classical American option contract.

Consider the following market (see Example 4.10 [12]) $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\omega_5\}$, $T = 2$, $E = \{0, 1, 2\}$ and the filtration $\mathcal{F}_t$ given by the discounted price process S, and S such that

- $S_0(\omega_i) = 5$ $(i = 1, 2, 3, 4, 5)$
- $S_1(\omega_i) = 8$ $(i = 1, 2, 3)$; $S_1(\omega_i) = 4$ $(i = 4, 5)$
- $S_2(\omega_1) = 9$, $S_2(\omega_2) = 7$, $S_2(\omega_3) = 6$, $S_2(\omega_4) = 6$, $S_2(\omega_5) = 3$

Let us consider the probability Q^q given by: $Q^q(\omega_1) = \frac{q}{4}$, $Q^q(\omega_2) = \frac{2-3q}{4}$, $Q^q(\omega_3) = \frac{2q-1}{4}$, $Q^q(\omega_4) = \frac{1}{4}$, $Q^q(\omega_5) = \frac{1}{2}$. It is easy to show that $\mathcal{P} = \left\{Q^q : \frac{1}{2} < q < \frac{2}{3}\right\}$.

The set of all stopping-times is finite and given by $\mathcal{T}^E_{0,2} = \{0, 1, 2, \tau, \sigma\}$, where $\tau = \mathbb{I}_{\{S_1=8\}} + 2\mathbb{I}_{\{S_1=4\}}$ and $\sigma = 2\mathbb{I}_{\{S_1=8\}} + \mathbb{I}_{\{S_1=4\}}$.

An admissible portfolio $V^{y,\eta}$ is of the form

- $V_0^{y,\eta}(\omega_i) = y$ $(i = 1, 2, 3, 4, 5)$
- $V_1^{y,\eta}(\omega_i) = y + 3\eta_0$ $(i = 1, 2, 3)$; $V_1^{y,\eta}(\omega_i) = y - \eta_0$ $(i = 4, 5)$
- $V_2^{y,\eta}(\omega_1) = y + 3\eta_0 + \eta_{1,1}$, $V_2^{y,\eta}(\omega_2) = y + 3\eta_0 - \eta_{1,1}$, $V_2^{y,\eta}(\omega_3) = y + 3\eta_0 - 2\eta_{1,1}$, $V_2^{y,\eta}(\omega_4) = y - \eta_0 + 2\eta_{1,2}$, $V_2^{y,\eta}(\omega_5) = y - \eta_0 - \eta_{1,2}$

and the strategy η, to be in $\mathcal{A}^{\Lambda}_E(y)$, must satisfy

- $\frac{-\Lambda_1(\omega_1)-y}{3} \leq \eta_0 \leq y + \Lambda_1(\omega_4)$
- $-\Lambda_2(\omega_1) - y - 3\eta_0 \leq \eta_{1,1} \leq (y + 3\eta_0 + \Lambda_2(\omega_2)) \wedge \left(\frac{y+3\eta_0+\Lambda_2(\omega_3)}{2}\right)$
- $\frac{-\Lambda_2(\omega_4)-y+\eta_0}{2} \leq \eta_{1,2} \leq y - \eta_0 + \Lambda_2(\omega_5)$

Example 6.1. *In the market introduced above, let us consider an American contingent claim with pay-offs:*

- $H_0(\omega_i) = 0$ *$(i = 1, 2, 3, 4, 5)$*
- $H_1(\omega_i) = 1$ *$(i = 1, 2, 3)$; $H_1(\omega_i) = 0.093$ $(i = 4, 5)$*

- $H_2(\omega_1) = 0$, $H_2(\omega_2) = 1.7$, $H_2(\omega_3) = 4$, $H_2(\omega_4) = 0.3$, $H_2(\omega_5) = 0$

It is easy to verify that $I(H_0) = \{0\}$, $I(H_1) = \{0.31975\}$, $I(H_\tau) = \{0.325\}$, $I(H_2) = \big(0.2875, 0.408\overline{3}\big)$, $I(H_\sigma) = \big(0.28225, 0.403108\overline{3}\big)$ *and* $\mathcal{I}^E(H) = \big[0.325, 0.408\overline{3}\big)$.

Consider an agent B, interested in making profits buying H, whose utility is exponential, $u(y) = -e^{-\beta y}$, *with* $\beta > 0$, *and whose subjective probability* $\mathbb{P}^B$ *is:* $\mathbb{P}^B(\omega_1) = 0.1$, $\mathbb{P}^B(\omega_2) = 0.1$, $\mathbb{P}^B(\omega_3) = 0.4$, $\mathbb{P}^B(\omega_4) = 0.12$, $\mathbb{P}^B(\omega_5) = 0.28$. *We suppose that he fixes* $\Lambda = 0.5$ *as a lower bound to the shortfall.*

If $\beta = 1$ *and if* $z = 0$, *then B is not interested in buying the option if* $x > 0.3466$ *While, if* $x \leq 0.3466$, *then his stopping strategy is*

$$\tau(x) = \begin{cases} \sigma, & \textit{if } 0.33647 \leq x \leq 0.3466, \\ 2, & \textit{if } 0.325 \leq x \leq 0.33647 \end{cases} \tag{16}$$

Let us observe that for $x = 0.33647$ *the stopping times* 2 *and* σ *are indifferent for B. Firstly, consider the case* $\tau(0.33647) = 2$.

Let us now consider another agent who is interested in selling the American contingent claim H, who measures his risk by the expected shortfall and whose subjective probability $\mathbb{P}^S$ *is:* $\mathbb{P}^S(\omega_1) = 0.1$, $\mathbb{P}^S(\omega_2) = 0.1$, $\mathbb{P}^S(\omega_3) = 0.4$, $\mathbb{P}^S(\omega_4) = 0.10$, $\mathbb{P}^S(\omega_5) = 0.30$.

Considering the optimal stopping strategy (16) chosen by B, his shortfall risk is, by (13),

$$r_0^E(x) = \begin{cases} \inf_{\xi \in \mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_\sigma - V_\sigma^{x,\xi}\right)^+\right], & \textit{if } 0.33647 < x \leq 0.3466, \\ \inf_{\xi \in \mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_2 - V_2^{x,\xi}\right)^+\right], & \textit{if } 0.325 \leq x \leq 0.33647 \end{cases}$$

Since $r_0^E(x)$ *is decreasing in both intervals* $[0.325, 0.33647]$ *and* $[0.33647, 0.3466)$, *the minimum of* $r_0^E(x)$ *is achieved in* $x = 0.33647$ *or* $x = 0.3466$. *But, by numerical computations,* $r_0^E(0.33647) = 0.03$ *and* $r_0(0.3466) = 0.032673$ *and thus the optimal price for the option in* $\hat{x} = 0.33647$.

If, instead $\tau(0.33647) = \sigma$, *then the minimum of the shortfall risk is not achieved and, as explained in Remark (4.1), the seller shall propose a* $\tilde{x} < 0.33647$ *as near as possible to it.*

If one considers the corresponding pseudo-American option, then

$$\overline{\tau}(x) = \begin{cases} -\infty, & \textit{if } x > 0.3466, \\ \sigma, & \textit{if } 0.33647 < x \leq 0.3466, \\ 2, & \textit{if } 0 < x \leq 0.33647 \end{cases}$$

and then $\mathcal{I}^E(H,\overline{\tau}) = (0.2875, 0.3466]$. *Thus,*

$$r_0^E(x) = \begin{cases} \inf_{\xi\in\mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_\sigma - V_\sigma^{x,\xi}\right)^+\right], & \text{if } 0.33647 < x \le 0.3466, \\ \inf_{\xi\in\mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_2 - V_2^{x,\xi}\right)^+\right], & \text{if } 0.2875 \le x \le 0.33647 \end{cases}$$

and the minimum is again in $\hat{x} = 0.33647$.

Example 6.2. *Consider the same contingent claim introduced in Example (6.1) and an agent B interested in making profits buying H, whose utility is again exponential,* $u(y) = -e^{-\beta y}$, *with* $\beta = 1$, *and whose subjective probability* $\mathbb{P}^B$ *is:* $\mathbb{P}^B(\omega_1) = 0.1$, $\mathbb{P}^B(\omega_2) = 0.1$, $\mathbb{P}^B(\omega_3) = 0.4$, $\mathbb{P}^B(\omega_4) = 0.115$, $\mathbb{P}^B(\omega_5) = 0.285$. *We suppose that he fixes, again,* $\Lambda = 0.5$ *as a lower bound to the shortfall and* $z = 0$.

From numerical computations it follows that B is not interested in buying the option if $x > 0.3492$ *While, if* $x \le 0.3492$, *his stopping strategy is*

$$\tau(x) = \sigma, \ \text{for } 0.325 \le x \le 0.3492,$$

Let us now consider another agent who is interested in selling the American contingent claim H, who measures his risk by the expected shortfall. For any choice of his subjective probability $\mathbb{P}^S$,

$$r_0^E(x) = inf_{\xi\in\mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_\sigma - V_\sigma^{x,\xi}\right)^+\right], \ \text{for } 0.325 \le x < 0.3492,$$

which is strictly decreasing. Thus, even if the seller can use the knowledge of the stopping strategy of the buyer in quantifying his risk, the price at which the option is traded is $\pi^B = 0.3492$ *(see (9)).*

If one considers the corresponding pseudo-American option, then

$$\overline{\tau}(x) = \begin{cases} -\infty, & \text{if } x > 0.3492, \\ \sigma, & \text{if } 0.29857 \le x \le 0.3492, \\ 2, & \text{if } 0 < x \le 0.29857. \end{cases}$$

Thus $\mathcal{I}^E(H,\overline{\tau}) = (0.2875, 0.3492]$. *Suppose* $\overline{\tau}(0.29857) = 2$ *(otherwise, considering a value as near as possible from below to* 0.29857, *the conclusions are the same).*

Considering, now, a seller with a subjective probability $\mathbb{P}^S$*:* $\mathbb{P}^S(\omega_1) = 0.389$, $\mathbb{P}^S(\omega_2) = 0.1$, $\mathbb{P}^S(\omega_3) = 0.01$, $\mathbb{P}^S(\omega_4) = 0.001$, $\mathbb{P}^S(\omega_5) = 0.5$, *his risk is*

$$r_0^E(x) = \begin{cases} \inf_{\xi\in\mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_\sigma - V_\sigma^{x,\xi}\right)^+\right], & \text{if } 0.29857 < x \le 0.3492, \\ \inf_{\xi\in\mathcal{A}_E^0(x)} \mathbb{E}^{\mathbb{P}^S}\left[\left(H_2 - V_2^{x,\xi}\right)^+\right], & \text{if } 0.2875 < x \le 0.29857 \end{cases}$$

Since $r_0^E(x)$ is decreasing in both intervals $[0.2875, 0.29857]$ and $[0.29857, 0.3492)$, the minimum of $r_0^E(x)$ is achieved in $x = 0.29857$ or $x = 0.3492$. But, by numerical computations, $r_0^E(0.29857) = 0.0044$ and $r_0(0.3492) = 0.0064$ and thus the optimal price for the option in $\hat{x} = 0.29857$.

Example 6.3. *Let us consider again the contingent claim of Example (6.1), but consider a buyer which is a "small investor" in the sense that he does not invest in a self-financing portfolio and that is not in the position to make some hypothesis on the future state of the world. More specifically, we suppose that his subjective probability $\mathbb{P}^B$ is: $\mathbb{P}^B(\omega_1) = \mathbb{P}^B(\omega_2) = \mathbb{P}^B(\omega_3) = \mathbb{P}^B(\omega_4) = \mathbb{P}^B(\omega_5) = 0.2$, and, considering a logarithmic utility function, that is $u(x) = ln(x + c)$ for a certain constant c,*

$$U_0(H, x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathbb{P}^B}\left[ln\left(H_\tau + z - x + c\right)\right]$$

and

$$U_0(z) = ln(z + c).$$

If $c = 0.41$ and $z = 0$, from numerical computations it follows that B is not interested in buying the option if $x > 0.3978$ While, if $x \leq 0.3978$, his stopping strategy is

$$\tau(x) = \begin{cases} 1, & \text{if } 0.335 \leq x \leq 0.3978, \\ \tau, & \text{if } 0.325 \leq x \leq 0.335 \end{cases}$$

Since $I(H_1) = \{0.31975\}$ and $I(H_\tau) = \{0.325\}$, $r_0^E(x) = 0$ always.

7 Appendix

In order to prove Lemma (3.1), we recall the concept of Fatou convergence in the setting of stochastic processes (see [4]).

Definition 7.1. *Let X^n be a sequence of stochastic processes defined on a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ and $\mathbb{T}$ a dense countable subset of $[0, T]$ such that $T \in \mathbb{T}$. The sequence X^n is Fatou convergent on $\mathbb{T}$ to a process X, if X^n is uniformly bounded from below,*

$$\begin{aligned} X_t &= \limsup_{s\downarrow t, s\in\mathbb{T}} \limsup_{n\to+\infty} X_s^n \\ &= \liminf_{s\downarrow t, s\in\mathbb{T}} \liminf_{n\to+\infty} X_s^n \end{aligned}$$

almost surely for all $t \in [0,T)$*, and*

$$X_T = \lim_{n \to +\infty} X_T^n$$

almost surely.

The following lemma is essentially Lemma 5.2 in [4]. For completeness we present the proof.

Lemma 7.1. *Let* V^n *be a sequence of* $\mathcal{P}$*-supermartingales bounded from below by an adapted and r.c.l.l. process* Υ *with* $\sup_{\tau \in \mathcal{T}_{0,T}^E, \mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\Upsilon_\tau] < +\infty$*. Let* $V_0^n \le y$*, for every* n *and, if* $E = [0,T]$ *let* $\mathbb{T}$ *be a dense countable subset of* $[0,T]$ *with* $T \in \mathbb{T}$*. There is a sequence* $J^n \in conv\left(V^n, V^{n+1}, \ldots\right)$*, and a* $\mathcal{P}$*-supermartingale* J *such that,* $J_0 \le y$ *and* $J_t \ge \Upsilon_t$*,* $\forall t \in E$*. Furthermore,* J_t^n *converges* $\mathbb{P}$*- a.s. to* J_t*,* $\forall t \in E$*, if* E *is discrete; otherwise, that is* $E = [0,T]$*, the sequence* J^n *is Fatou convergent on* $\mathbb{T}$ *to* J*.*

Proof. The proof follows that of Lemma 5.2 in [4]. We have only to take care of the fact that we require that all our $\mathcal{P}$-supermartingares need to be bounded from below by the process Υ.

Let $X_t^n = V_t^n - \Upsilon_t \ge 0$ and let us consider the sequence of sets $conv\,(X^m : m \ge n)$. It is easy to convince oneself that every element $Z \in conv\,(X^m : m \ge n)$ is of the form $Z = J - \Upsilon$ where $J \in conv\,(V^m : m \ge n)$ is a $\mathcal{P}$-supermartingale with $J_0 \le y$.

By a diagonal procedure and applying Lemma 5.1 in [4], it is possible to construct a sequence $Y^n \in conv\,(X^m : m \ge n)$ such that Y_t^n converges almost surely to a random variable Y'_t for every $t \in E$, if E is discrete, or for every $t \in \mathbb{T}$, if $E = [0,T]$. But $Y^n = J^n - \Upsilon$ where $J^n \in conv\,(V^m : m \ge n)$ is a $\mathcal{P}$-supermartingale and $J_0^n \le y$. Thus, if $J'_t = Y'_t + \Upsilon_t$, for $s < t$, $s,t \in \mathbb{T}$, if $E = [0,T]$, or for $s < t$, $s,t \in E$, if E is discrete, we have

$$\begin{aligned}\mathbb{E}^*\left[J'_t \middle| \mathcal{F}_t\right] &\le \liminf_{n \to +\infty} \mathbb{E}^*\left[J_t^n \middle| \mathcal{F}_t\right] \le \\ &\le \liminf_{n \to +\infty} J_s^n = \liminf_{n \to +\infty} Y_s^n + \Upsilon_s = \\ &= Y'_s + \Upsilon = J'_s\end{aligned}$$

that is, if E is finite, J' is a $\mathcal{P}$-supermartingale with $J'_t \ge \Upsilon_t$, $\forall t \in E$ and $J'_0 \le y$. Considering $J = J'$, the Lemma is proved in the finite time case.

If $E = [0,T]$, using a standard construction based on Doob's Upcrossing lemma, we have that the process

$$J_t = \lim_{s \downarrow t, s \in \mathbb{T}} J'_s$$

is a c.a.d.l.a.g. $\mathcal{P}$-supermartingale . Furthermore

$$J_t - \Upsilon_t = \lim_{s \downarrow t, s \in \mathbb{T}} J'_s - \Upsilon_s \ge 0,$$

$$J_0 - \Upsilon_0 \le \liminf_{s \downarrow 0, s \in \mathbb{T}} \liminf_{n \to +\infty} \mathbb{E}^*\left[J_s^n - \Upsilon_s\right] \le y - \Upsilon_0$$

and J^n is Fatou-convergent to J. □

Proof. Proof of Lemma (3.1).

Let $\xi^n \in \mathcal{A}_E^\Lambda(z-x)$ be maximizing. By Lemma (7.1), there exits a sequence $J^n \in conv\left(V^{z-x,\xi^m} : m \geq n\right)$ and a $\mathcal{P}$-supermartingale J, such that $J_0 \leq z-x$, $J_t \geq -\Lambda_t$, $\forall t \in E$, such that J_T^n converges $\mathbb{P}$-a.s. to J_T.

But, by the concavity of u, J_T^n is again maximazing and, if u is bounded from above or Ω is finite,

$$\mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + J_T\right)\right] \geq \limsup_{n\to+\infty} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + J_T^n\right)\right].$$

But, by superhedging, there exists a trading strategy $\hat{\xi}$ such that

$$C + z - x + \int_0^t \hat{\xi}_s dX_s \geq J_t + C.$$

Thus, $\hat{\xi} \in \mathcal{A}_E^\Lambda(z-x)$ and

$$\sup_{\xi\in\mathcal{A}_E^\Lambda(z-x)} \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\xi}\right)\right] \geq \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + V_T^{z-x,\hat{\xi}}\right)\right] \geq$$

$$\geq \mathbb{E}^{\mathbb{P}^B}\left[u\left(H_\tau + J_T\right)\right]$$

and the thesis follows. □

8 Conclusions

With an American option, the traders buy or sell, respectively, the right to claim the underlying payoff at any time between 0 and the expiration time T. But there are some buyers who are not in the position to take advantage as much as possible of this right, in the sense that their information on the market direction and their attitude to risk have no possibility to change in a relevant way during the life time of the option. This way, as shown in the paper in Section 3, they can determine at time 0 the stopping times at which it will be optimal for them to exercise the option.

This way, such a class of agents, buying an American option, pay for a right that are not in the position to exercise as much as possible and for them the American option is not en efficient contract.

From the seller point of view to trade an American option with such a type of buyer can be inefficient too, since, as shown in Section 4, if he knows at time 0 the stopping times at which the option will be exercised, he may be in the position to sell the option at a lower price reducing his risk at the same time.

For such a class of agents, a pseudo-American option (see Section 5) is a more efficient contract.

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