The article explores joint consumption equilibrium environments. It illustrates network formation through one-to-one directional synapses. Family (couple) arrangements, spontaneously generated under a decentralized general equilibrium price system are suggested - involving link and direction-specific transfer prices along with standard resource one. The research also inspects preference characteristics able to generate monogamous choices and assortative matching and mating. Assortative mating (and income pooling) is clarified, related to exclusivity or taste-for unicity at the utility level with respect to shared good, with optimal assignment connected to equalization of the marginal benefit of the match - adequately defined - across individuals in the economy.

Contrast with a multiple external effect good - one-to-many communication; (or) shared by a fixed number of, more than two, individuals; common property - and with a pure public good is also provided. If paired consumption with end-point specificity generates (or may generate), under reasonable assumptions, a unique decentralized equilibrium solution, supporting an efficient allocation, multiple agent sharing among more than two individuals and individual types requires, along with excludability, perfect differentiation of a larger number of consumption - partnership - roles.

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Calls and Couples: Communication, Connections, Joint – Consumption and Transfer Prices †

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ABSTRACT

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This research proceeds to the formal characterization of general equilibrium and efficient allocation of an exchange economy where individuals value a pure private good and mixed one(s) the fractions of which must be shared wholly and unilaterally with one and only one other individual in the community. Such "shared" good – not necessarily attached to an externality: both individuals may have to pay or spend resources to enjoy it – involves joint consumption and reproduces private calls, one-to-one communication or information sharing. The initiating – "proposing" - party is identified, (potentially) not irrelevantly valued by individuals, and there is continuous veto power at the end-side of a match. A decentralized equilibrium requires two general prices – adding up to a uniquely determined full-price -, and pair-(and direction-)specific transfer prices between intervening consumers for the shared good. Efficiency requires the Samuelson condition over marginal utilities.

Agent multiplicity – utility patterns and corner solutions - sheds light on endogenous match rank pricing, making and mating. Specific functional forms (two and three-stage CES special cases, allowing for taste for variety as for unicity) generate interpretable conclusions, namely, regarding the qualification of assortative mating.

Contrast with a multiple external effect good – one-to-many communication; (or) shared by a fixed number of, more than two, individuals; common property - and with a pure public good is also provided. If paired consumption with end-point specificity generates, under reasonable assumptions, a unique decentralized equilibrium solution, supporting an efficient allocation, multiple agent sharing among more than two individuals and individual types requires, along with excludability, perfect differentiation of a larger number of consumption – partnership - roles.

Principles behind the theory are also applicable to input and cost sharing and pricing in partnerships, co-operative societies and joint-ventures.

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Calls and Couples: Communication, Connections, Joint – Consumption and Transfer Prices

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Calls and Couples: Communication, Connections, Joint – Consumption and Transfer Prices

"[2] and as they were drinking wine on that second day, the king again asked, "Queen Esther, what is your petition? It will be given you. What is your request? Even up to half the kingdom, it will be granted." [3] Then Queen Esther answered, "If I have found favor with you, O king, and if it pleases your majesty, grant me my life - this is my petition. And spare my people - this is my request. [4] For I and my people have been sold for destruction and slaughter and annihilation. If we had merely been sold as male and female slaves, I would have kept quiet, because no such distress would justify disturbing the king." In *Book of Esther* 7: 2-4.

Introduction.

Mutual agreement is required for a large number of everyday transactions. Some are over a pure private good or service, and standard marginal pricing insures efficient allocations. Others, generate partial externalities or are even totally public, requiring superseding judgement. A fringe (…) are social in nature, its consumption implying benefits for two - or a given number of - affected agents. They may or may not require direct costs from those traders (e.g., time) – they may or may not involve an externality -, they are identifiable both by the initiating and ending side of the transaction and require complete consensus regarding its consumption/expenditure level.

The requirement of mutual agreement – involving excludability - allows a decentralized price system to insure an efficient allocation, provided discrimination between the two consumption sides is perfect: then, effectively, it is as if the two roles would distinguish themselves as two (times the number of individual types in the economy) different goods but not sold separately. The argument resembles the one applied to club goods – yet, here, the externality status is minor to qualify equilibrium properties [1], confined to a given or fixed number of people [2], and stresses the requirement of equal consumption of a total common "property" or durable; optimal pricing is (can be) achieved through transfers – or implicit consumption price discrimination –, which are due even if agents are homogeneous as long as they value differently the two roles (making and attending calls) in the "call society".

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[1] Or we could say that we would fall under Coase's theorem…

[2] Say, total congestion is achieved with a fixed or maximum number of partners.

Understandably, a similar modelling framework has been applied in the economics of family and family formation: early examples [3] are Manser and Brown (1980) and McElroy and Horney (1981), suggesting marriage for allowing joint consumption by two agents - that bargain with each other while possessing, maintaining well-defined, “selfish” [4], individual preferences and budget constraints [5] - of special - household - public goods. Even if similar, our formalization presents a crucial difference: excludability by either side, and “family role” definition for each potential match; then, under the usual ideal assumptions [6], a decentralized general equilibrium can be expected to promote efficient mating.

In family economics, two agent bargaining – interaction - is generally assumed. One can propose functional forms that are able to generate monogamy as polygamy – the later reproducing multi-(even if one-to-one)-connections. Assortative matching and mating can be studied with reference to the properties of the uncompensated individual demands and indirect utility functions [7] – which now also depend on partner(s) income and preferences - generated under exclusivity conditions. Then, transferable utility, or income – this mimicking, or effectively originating, budget pooling by the couple -, leads to the emergence of dowry systems.

The framework can also encompass more complex societies – allow common property to be shared by more than two agents. In principle, network formation could be simulated by assuming that each connection between any two nodes is unique, with a node – as a neuron – having a life of its own. In the limit, joint-consumption by more than two individuals leads to a similar environment as that in the presence of a public good. With excludability, the only difficulty for a decentralized equilibrium arises from lack of competition and the leading (as others) role definition.

Also, productive factors – as outputs – can be shared by different divisions or plants of a firm… The theory suggests the adequate properties of an internal pricing scheme able to generate an efficient decentralized system management.

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[3] That also include, more recently, Lam (1988) and Lundberg and Pollak (1993) – see Bergstrom (1996) and Weiss (1997) for a recent survey.

[4] Even if we can argue that some degree of altruism – and partner-specific inclination - can always be reflected in preferences for goods that are or must be shared with other individuals.

[5] Most of these family models end up by assuming pooled income.

[6] Which, of course, rule out imperfect information or foresight, ex-post contract default, etc… The absence of the ideal conditions is what makes bargaining models of the family so appealing.

[7] See Becker (1973), Lam (1988). The analysis here differs both because budget constraints are never pooled, nor objective functions altered by connection establishment.

The exposition proceeds as follows: notation and individuals' utility functions are defined in section I. Section II states the properties of an efficient allocation, and section III those of a decentralized equilibrium. In section IV, we proceed to the derivation of demands, indirect utilities and equilibrium configurations for specific functional forms and in section V, assortative mating is qualified under different transferability environments. Contrast with multiple emission entities is dealt with in section VI. In section VII, input sharing is modelled according to the same principles. The exposition ends with a brief summary in section VIII.

I. Notation: Preferences and Shared Goods.

. There are n consumers in the economy. Each consumer, i, enjoys utility from the consumption of a private good, the quantity of which is denoted by x_i, from the quantity of "calls" he makes to individual j, z_i^j – the consumption of z proposed by i and accepted by j - and from those he receives from that same individual, y_i^j – the consumption of z proposed by j and accepted by i:

$$(1) \quad U^i(x_i, z_i^1, z_i^2, \ldots, z_i^{i-1}, z_i^{i+1}, ..., z_i^n, y_i^1, y_i^2, \ldots, y_i^{i-1}, y_i^{i+1}, ..., y_i^n), \quad i = 1, 2, \ldots, n$$

For simplicity, we will denote it by $U^i(x_i, z_i^j, y_i^j)$. Also, $\frac{\partial U^i(x_i, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial x_i} = U_x^i$, $\frac{\partial U^i(x_i, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial z_i^j} = U_{z_j}^i$ and $\frac{\partial U^i(x_i, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial y_i^j} = U_{y_j}^i$. $U^i(x_i, z_i^j, y_i^j)$ is assumed to exhibit the usual properties – continuity, twice-differentiability and quasi-concavity.

The consumption of z requires feedback: it implies that:

$$(2) \qquad z_i^j = y_j^i, \quad i \neq j, i, j = 1, 2, \ldots, n$$

The distinction between z_i^j and y_j^i has two purposes: on the one hand, it represents the fact that there is perfect discrimination of the two consumption roles, and that i (may) faces a different net price for z_i^j than that charged to j for y_j^i; (but…) as we assume that there is *mutual excludability* between the i and j in the consumption of (both) z_i^j and y_j^i (z_j^i and y_i^j), i has the ability to control both z_i^j and y_i^j. These two conditions will allow for an efficient price system to develop. It would appear to apply well to calls, and it suggests the natural arising of gender differentiation – further stressed in economic dwelling by the requirement of definition of "head of household" status, of individual responsible for the child education…

On the other, it allows us to explore and understand similarities and differences between a pure externality (i.e., z_i^j and y_j^i are completely *non-rival*) and mere joint-consumption at equal levels - suggesting generalizations reproducing economies of scale in joint-consumption.

If i gets the same satisfaction from calling as from getting a call from j, then the utility has the special form:

(3) $U^i(x_i, z_i^1 + y_i^1, z_i^2 + y_i^2, \ldots, z_i^n + y_i^n) = U^i(x_i, z_i^j + y_i^j)$, i = 1, 2, …, n

Also, if calls to and from any individual type are valued similarly, even if receiving and answering calls differentiated:

(4) $U^i(x_i, z_i^1 + z_i^2 + \ldots + z_i^{i-1} + z_i^{i+1} + \ldots + z_i^n, y_i^1 + y_i^2 + \ldots + y_i^{i-1} + y_i^{i+1} + \ldots + y_i^n)$
$= U^i(x_i, z_i + y_i)$

i = 1, 2, …, n

Of course, such additivity may occur in sets, with individual types arising distinctively for each i at the utility level.

. Each individual is endowed with amount W_x^i of good x and W_z^i of good z. We will consider two scenarios:

- one in which only z_i^j requires W_z – on a one-to-one basis -, with y_i^j being a (almost) complete externality

- another in which both z_i^j as y_i^j require the use of W_z.

Yet, (2) – i.e., agreement from interlocutor -, must always be insured. And, of course, whether an externality or pure joint-consumption at the same level for both sides applies (or other – see below), it must recognized by every individual in the economy.

A link between i and j requires no "fixed" costs, i.e., independent from the amount of z_i^j (or y_i^j) traded [8]. Network access (or set-up) costs – pure access to the markets where z and y are trade - are also assumed negligible [9].

. A complex decentralized price system is proposed: p_x is the unit price of good x. The price of a call from i to j is composed of three parts: a general "call tariff" p_z, an

[8] These could justify the emergence of monogamous couples even with preferences exhibiting taste for variety... And of dowries and bequests in the market independent of household quantity.

[9] They would not affect the general conclusions in what concerns marginal properties of interior solutions, provided that they are independent of network quantities aggregation… They would then justify an access pricing fixed fee independent of the use intensity.

answering tariff p_y, and a specific unit transfer from i to consumer j for attending the call, t_i^j. I.e., the consumption of z_i^j by i requires an additional "service" from j, priced at t_i^j.

Then the (exhausted) budget constraint of individual i is:

$$(5) \quad p_x x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j)\, z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)\, y_i^j = p_x W_x^i + p_z' W_z^i$$

or

$$(6) \quad p_x x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j)\, z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)\, z_j^i = p_x W_x^i + p_z' W_z^i$$

Summing (5) over i, as $\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i$, we conclude that the general tariffs must add up to the operating cost of a call p_z', at which W_z is traded.

$$(7) \quad p_z' = p_z + p_y$$

Notice that once we allow for transfers, payment can be collected on one-side of the call – charging ($p_z + p_y$) to z – only: in practice, the actual individual transfers would also include the recovery of p_y.

For example, for common calls, $p_z = p_z'$ and $p_y = 0$. Child allowance schemes - see Lundberg and Pollak (1993), p. 1001 –, or merely nature's assignment of child-bearing and rearing costs, illustrate other unbalanced arrangements.

. If y is non-rival with respect to z, p_z' is split between both sides of the call according to (7). Off-springs would appear to work as such. But a diner in a restaurant by a couple would involve twice the resources a solitary diner would – and (but) just require the same level of expenditure by the two individuals, the leveling of the quantity purchased by each of the two partners. In this type of cases, because now $\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j = 2 \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i$, (aggregating (5)) $p_z' = \frac{p_z + p_y}{2}$ would price W_z – the average price paid by both i and j[10] - or rather $p_z + p_y$ would price one double unit of z_i^j-cum-y_j^i – given

[10] Allowing the price to still differ in both ends…

that it involves consumption duplication, nobody would want to buy or sell one of the two sides of the match separately. With joint-consumption, there will be a sort of sale complementarity; p_z' will then be the average price of the unit of W_z^i, sold in pairs.

A straight-forward generalization would allow for an intermediate state where $(z_i^j + y_j^i)\frac{1+d}{2}$ of W_z, $0 \leq \delta \leq 1$, is required to produce the "consumable" pair z_i^j-cum-y_j^i – $z_i^j \frac{1+d}{2}$ purchased by i, $y_j^i \frac{1+d}{2}$ by j - a value of δ smaller than 1 representing economies of scale in household consumption; then $\frac{p_z + p_y}{1+d}$ would price W_z [11]. Or – allowing z_i^j to stand for half the total joint purchase so that $p_z' = \frac{p_z + p_y}{2}$ - assume utility functions are of the form $U^i(x_i, \frac{2\,z_i^j}{1+d}, \frac{2\,y_i^j}{1+d})$, requiring $z_i^j = y_j^i$, allowing or not differentiated pricing of z_i^j and y_j^i – hypothetically, δ could be pair specific, δ_i^j; such formulation would certainly be useful in the study of labor supply – if x_i denotes leisure, priced at W_i, $I^i = V^i + W_i T^i$ - full-income - where V^i and T^i are exogenous non-labor earnings and time endowment of i respectively, and pure private goods using W_z, g_{ij}, $j \neq i$, are also allowed such that we can write anybody's utility function as $U^i(x_i, g_{ij} + \frac{2\,z_i^j}{1+d}, \frac{2\,y_i^j}{1+d})$ or $U^i(x_i, \frac{g_{ij}}{2} + \frac{2\,z_i^j}{1+d}, \frac{g_{ij}}{2} + \frac{2\,y_i^j}{1+d})$ (and corner solutions naturally arise).

[11] Of course, δ is assumed to be known by all market characters.

II. Efficient Allocation.

. Admit an efficient allocation is sought. Then, one wants to maximize an individual's, say i, utility, subject to the existing endowments and limiting utility levels of all other consumers. Assume first that the receiver actually gets an externality. Then:

$$\text{(8)} \qquad \underset{x_i, z_i^j, y_i^j, x_j, z_j^l, y_j^l}{Max} \quad U^i(x_i, z_i^j, y_i^j)$$

s.t.: (8a) $\quad U^j(x_j, z_j^l, y_j^l) \geq \overline{U}^j \;, \; j \neq i, j = 1, 2, \ldots, n$

(8b) $\quad z_i^j = y_j^i \;, \; i \neq j, i, j = 1, 2, \ldots, n$

(8c) $\quad \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} W_x^i$

(8d) $\quad \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j \leq \sum_{i=1}^{n} W_z^i$

In lagrangean form and replacing (8b):

$$\text{(9)} \qquad \underset{x_i, z_i^j, x_j, z_j^l, l_j, m_x, m_z}{Max} \quad U^i(x_i, z_i^j, z_j^i) + \sum_{\substack{j \neq i \\ j=1}}^{n} l_j \left[\overline{U}^j - U^j(x_j, z_j^l, z_l^j)\right] +$$

$$+ \; m_x \left(\sum_{i=1}^{n} W_x^i - \sum_{i=1}^{n} x_i\right) + m_z \left(\sum_{i=1}^{n} W_z^i - \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j\right)$$

Interior FOC require:

(10) $\quad U_x^i - m_x = 0$ (1 equation)

(11) $\quad - l_j \, U_x^j - m_x = 0 \;, \; j \neq i, j = 1, 2, \ldots, n$ (n-1 eqs.)

(12) $\quad U_{z_j}^i - l_j \, U_{y_i}^j - m_z = 0 \;, \; j \neq i, j = 1, 2, \ldots, n$ (n-1 eqs.)

(13) $\quad U_{y_j}^i - l_j \, U_{z_i}^j - m_z = 0 \;, \; j \neq i, j = 1, 2, \ldots, n$ (n-1 eqs.)

(14) $\quad - l_j \, U_{z_l}^j - l_l \, U_{y_j}^l - m_z = 0 \;, \; j \neq i, l \neq j, l = 1, 2, \ldots, n$

(15) $\quad - l_j \, U_{y_l}^j - l_l \, U_{z_j}^l - m_z = 0 \;, \; j \neq i, l \neq j, l = 1, 2, \ldots, n$

along with (8a) (8c) and (8d) in equality. (12) to (15) include n x (n – 1) different equations – the number of existing z_i^j's.

(10) and (11) imply the usual

(16) $$l_j = -\frac{U_x^i}{U_x^j}, \quad j \neq i, j = 1, 2, \ldots, n$$

Replacing in (12) and (13) and equating the two (and (10)):

(17) $$\frac{U_{z_j}^i}{U_x^i} + \frac{U_{y_i}^j}{U_x^j} = \frac{U_{y_j}^i}{U_x^i} + \frac{U_{z_i}^j}{U_x^j} \quad (= \frac{m_z}{U_x^i}) = \frac{m_z}{m_x}, \quad j \neq i, j = 1, 2, \ldots, n$$

Finally, from (14) and (15):

(18) $$\frac{U_{z_l}^j}{U_x^j} + \frac{U_{y_j}^l}{U_x^l} = \frac{U_{y_l}^j}{U_x^j} + \frac{U_{z_j}^l}{U_x^l} \quad (= \frac{m_z}{U_x^i}) = \frac{m_z}{m_x}, \; j \neq i, l \neq j, l = 1, 2, \ldots, n$$

. If the second consumer does not obtain an externality, then (8d) is replaced by

(19) $$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j \geq \sum_{i=1}^{n} W_z^i$$

The last term of the lagrangean (9) becomes $m_z \, (\sum_{i=1}^{n} W_z^i - 2 \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j)$. Then (17) and (18) are replaced respectively by:

(20) $$\frac{U_{z_j}^i}{U_x^i} + \frac{U_{y_i}^j}{U_x^j} = \frac{U_{y_j}^i}{U_x^i} + \frac{U_{z_i}^j}{U_x^j} \quad (= 2\,\frac{m_z}{U_x^i}) = 2\,\frac{m_z}{m_x}, \quad j \neq i, j = 1, 2, \ldots, n$$

and

(21) $$\frac{U_{z_l}^j}{U_x^j} + \frac{U_{y_j}^l}{U_x^l} = \frac{U_{y_l}^j}{U_x^j} + \frac{U_{z_j}^l}{U_x^l} \quad (= 2\,\frac{m_z}{U_x^i}) = 2\,\frac{m_z}{m_x}, \; j \neq i, l \neq j, l = 1, 2, \ldots, n$$

(17) and (18) reproduce the well-known condition that the sums of the marginal rates of substitution of consumption partners must equate the marginal rate of transformation in the economy. (20) and (21) – in absence of externality - require that the *average* of those marginal rates of substitution equals the marginal rate of transformation.

Notice that the efficiency (Samuelson-type) condition, implying equalization of the sum (or averages if just joint-consumption) of the marginal rates of substitution between the shared and private good at the two consumption ends across the economy, is immune to mating or transferability considerations: it applies to any given welfare – ex-ante or ex-post transfers, as appropriate – utility levels of other individuals, $j \neq i$, we supply to the generic problem.

III. Supporting General Equilibrium.

. Let each individual be subject to the general linear price conditions stated in section I: in the economy, one unit of x costs p_x; one unit of z costs p_z' being jointly purchased and split between a caller and a receiver, accompanied by a consumer set/couple-specific unit transfer t_i^j. Any individual, i, solves:

$$(22) \qquad \underset{x_i, z_i^j, y_i^j}{Max} \quad U^i(x_i, z_i^j, y_i^j)$$

$$\text{s.t.: } (23) \qquad p_x x_i + \sum_{\substack{j\neq i \\ j=1}}^{n} (p_z + t_i^j)\, z_i^j + \sum_{\substack{j\neq i \\ j=1}}^{n} (p_y - t_j^i)\, y_i^j = p_x W_x^i + p_z' W_z^i = I^i$$

The lagrangean will take the form:

$$(24) \qquad \underset{x_i, z_i^j, y_i^j, m}{Max} \quad U^i(x_i, z_i^j, y_i^j) +$$

$$+ \; m \; [p_x W_x^i + p_z' W_z^i - p_x x_i - \sum_{\substack{j\neq i \\ j=1}}^{n} (p_z + t_i^j)\, z_i^j - \sum_{\substack{j\neq i \\ j=1}}^{n} (p_y - t_j^i)\, y_i^j \,]$$

and FOC for i = 1,2,…, n:

$$(25) \qquad U_x^i - m\, p_x = 0$$

$$(26) \qquad U_{z_j}^i - m\, (p_z + t_i^j) = 0 \;, \quad j \neq i, j = 1, 2, \ldots, n$$

$$(27) \qquad U_{y_j}^i - m\, (p_y - t_j^i) = 0 \;, \quad j \neq i, j = 1, 2, \ldots, n$$

with the budget constraint. Notice that as i can veto and ends up paying for y_j^i, optimization in it is due – and (27) arises - whether its consumption by i and j is completely non-rival (i.e., works as a complete “externality”) or not: there is *mutual excludability* between the i and j in the consumption of (both) z_i^j and y_i^j. For a perfect externality, (27) would not take place – case that will be contrasted with the current one in section VI…

Then:

$$(28) \qquad \frac{U_{z_j}^i}{U_x^i} = \frac{p_z + t_i^j}{p_x} \;, \quad j \neq i, j = 1, 2, \ldots, n \text{ (n – 1 eqs. for each i)}$$

and

(29) $$\frac{U^i_{y_j}}{U^i_x} = \frac{p_y - t^i_j}{p_x}, \quad j \neq i,\ j = 1, 2, \ldots, n \text{ (n – 1 eqs. for each i)}$$

The conditions are valid for any consumer. Equilibrium requires additionally mutual consent on the call, (8b), with the price share, (7), that supplies and demands equate, i.e., (8c) and (8d) in equality.

(30) $$z^j_i = y^i_j, \quad i \neq j,\ i, j = 1, 2, \ldots, n \text{ (n x (n – 1) eqs.)}$$

(31) $$p_z' = p_z + p_y$$

(32) $$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} W^i_x$$

(33) $$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z^j_i = \sum_{i=1}^{n} W^i_z$$

It is straightforward to conclude that under common assumptions, provided we fix either $\frac{p_z}{p_x}$ or $\frac{p_y}{p_x}$, there will be an and a unique equilibrium relative price vector, $\left(\frac{p_z}{p_x}, \frac{p_y}{p_x}, \frac{p_z'}{p_x}, \frac{t^2_1}{p_x}, \ldots, \frac{t^n_1}{p_x}, \ldots, \frac{t^2_n}{p_x}, \ldots, \frac{t^{n-1}_n}{p_x}\right)$ – with n x (n – 1) + 3 elements: we have 2 (n – 1) equations of form (28) and (29) and the budget constraint per consumer (generating the n + 2 n (n – 1) = n (2 n – 1) individual demands), and the n (n – 1) + 3 composed of (30), (31) and aggregate market equilibrium ones – n (3 n – 2) + 3 equations – yet, the sum of the budget constraints together with (32) and (33) imply (31) and only n (3 n – 2) + 2 would be independent; on the other hand, the relative prices and the allocations z^j_i and y^i_j together include the same number of unknowns: n x (n – 1) + 3 relative prices and n (2 n – 1) quantities.

In other words, the price system has now two degrees of freedom: not only (and as usual) may p_x be supplied, or x fixed as numeraire, as an exogenous convention about the splitting of the full price p_z' between the two "end-sides" of the deal – proposing and accepting parties - must also be agreed upon and supplied by society – usually taking the form $p_y = 0$…

. One can show that such system supports an efficient solution. Every consumer j will solve a similar problem and choose baskets such that

(34) $$\frac{U^{j}_{z_l}}{U^{j}_{x}} = \frac{p_z + t^{l}_{j}}{p_x} \quad , \quad l \neq j,\ l = 1, 2, \ldots, n$$

and

(35) $$\frac{U^{j}_{y_l}}{U^{j}_{x}} = \frac{p_y - t^{j}_{l}}{p_x} \quad , \quad l \neq j,\ l = 1, 2, \ldots, n$$

Considering the relations towards l = i: (28) plus (35), and (29) plus (34) generate:

(36) $$\frac{U^{i}_{z_j}}{U^{i}_{x}} + \frac{U^{j}_{y_i}}{U^{j}_{x}} = \frac{U^{i}_{y_j}}{U^{i}_{x}} + \frac{U^{j}_{z_i}}{U^{j}_{x}} = \frac{p_z + p_y}{p_x} \quad , \quad j \neq i,\ j = 1, 2, \ldots, n$$

which reproduces (17), with $\frac{p_z + p_y}{p_x}$ having correspondence with $\frac{m_z}{m_x}$. As it must be valid for any consumer pair, it encompasses (18).

Then, effectively, unit transfers are set such that:

(37) $$\frac{t^{j}_{i}}{p_x} = \frac{U^{i}_{z_j}}{U^{i}_{x}} - \frac{p_z}{p_x} = \frac{p_y}{p_x} - \frac{U^{j}_{y_i}}{U^{j}_{x}}$$

Notice that $t^{j}_{i} > 0$ and a transfer is due from i to j for the former's call if i appreciates (relative to consuming x) making calls to j more than its direct payment (i.e., $\frac{p_z}{p_x}$); and if j appreciates (relative to consuming x) receiving calls from i less than people have to pay to receive calls (i.e., $\frac{p_y}{p_x}$).

No "lump-sum" transfers from i to j, are required or fit to insure equilibrium - a "dowry" would be here proportional to the bridal value: each link is free and everybody expected to be linked with everybody… They would be if there were (physical, i.e., in terms of the available resources, W_x and W_z) "fixed costs" associated with the establishment of each particular link.

However, once linkages are person-specific, the described equilibrium may be difficult to emerge due to lack of competition in unit transfer price formation; then, the exogeneity and constancy of the net of transfers prices as faced by individuals – required for (28) and (29) to apply - becomes questionable. One can claim that links are interchangeable, and/or that other links provide interpersonal-link comparisons – nevertheless, the argument remains…

. Let us explore a little more deeply the demand formation in the economy.

Problem (24) generates conventional individual demands $x_i(I^i, p_x, p_z+t_i^1, p_z+t_i^2, \ldots, p_z+t_i^n, p_y-t_1^i, p_y-t_2^i, \ldots, p_y-t_n^i) = x_i(\frac{I^i}{p_x}, 1, \frac{p_z+t_i^1}{p_x}, \frac{p_z+t_i^2}{p_x}, \ldots, \frac{p_z+t_i^n}{p_x}, \frac{p_y-t_1^i}{p_x}, \frac{p_y-t_2^i}{p_x}, \ldots, \frac{p_y-t_n^i}{p_x})$ and $z_i^j(I^i, p_x, p_z+t_i^1, p_z+t_i^2, \ldots, p_z+t_i^n, p_y-t_1^i, p_y-t_2^i, \ldots, p_y-t_n^i) = z_i^j(\frac{I^i}{p_x}, 1, \frac{p_z+t_i^1}{p_x}, \frac{p_z+t_i^2}{p_x}, \ldots, \frac{p_z+t_i^n}{p_x}, \frac{p_y-t_1^i}{p_x}, \frac{p_y-t_2^i}{p_x}, \ldots, \frac{p_y-t_n^i}{p_x})$ – where $I^i = p_x W_x^i + p_z' W_z^i$ - enjoy standard properties. And z_i^j must equal $y_j^i(I^j, p_x, p_z+t_j^1, p_z+t_j^2, \ldots, p_z+t_j^n, p_y-t_1^j, p_y-t_2^j, \ldots, p_y-t_n^j) = y_j^i(\frac{I^j}{p_x}, 1, \frac{p_z+t_j^1}{p_x}, \frac{p_z+t_j^2}{p_x}, \ldots, \frac{p_z+t_j^n}{p_x}, \frac{p_y-t_1^j}{p_x}, \frac{p_y-t_2^j}{p_x}, \ldots, \frac{p_y-t_n^j}{p_x})$, which is also a consumer demand, but of another individual.

Systems of Marshallian or uncompensated demands $x_i(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z + p_y)$ and $z_i^j(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z + p_y)$ independent of transfer prices can be derived from (36) and, replacing (34) and (35) in the budget constraint, from:

$$(38)\ x_i + \sum_{\substack{j\neq i\\ j=1}}^{n} \frac{U_{z_j}^i}{U_x^i} z_i^j + \sum_{\substack{j\neq i\\ j=1}}^{n} \frac{U_{y_j}^i}{U_x^i} y_i^j = \frac{I^i}{p_x} = W_x^i + \frac{p_z+p_y}{p_x} W_z^i,\ i = 1,2,\ldots,n$$

Those demand functions would be homogeneous of degree 0 in $I^1, I^2, \ldots, I^i, \ldots, I^n$, p_x and $p_z + p_y$ but would not exhibit all of the other usual properties. They are independent of transfer prices because they already internalized its formation (rule). Moreover, each individual's demand – including that of the purely private good – is expected to be a function of everybody else's income, and not independent of its particular distribution, the same being true for indirect utility functions.

Compensated effects of an individual i's demand can be derived at fixed utility of all individuals, $x_i(U^1, U^2, \ldots, U^i, \ldots, U^n, p_x, p_z + p_y)$ – obeying (36) and $U^j(x_j, z_j^l, z_l^j) = U^j$, j =,1,2…,n -, and at fixed utility of i and fixed income of all others, $x_i(I^1, I^2, \ldots, U^i, \ldots, I^n, p_x, p_z + p_y)$.

Of equal relevance for private goods, demands conditional on the common purchases, $x_i(I^i, p_x, z_j^l, y_j^l) = x_i(I^i, p_x, z_j^l, z_l^j)$ would come from solving (38) with respect to x_i

(with more private goods, it would also imbed equality of their common marginal rate of substitution to their relative prices) for individual i. For compensated demands, $x_i(U^i, p_x, z_j^l, y_j^l) = x_i(U^i, p_x, z_j^l, z_l^j)$ would arise then from the traditional conditions (here, just inverting the utility function; with more private goods, MRS between them should equal the corresponding price ratio), yet i's conditional expenditure function would be generated according to the left hand-side of (38).

Requiring the sum (over all i) of Marshallian demands $x_i(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z + p_y) = x_i(\frac{I^1}{p_x}, \frac{I^2}{p_x}, \ldots, \frac{I^i}{p_x}, \ldots, \frac{I^n}{p_x}, 1, \frac{p_z + p_y}{p_x})$ to equalize available resource endowment (supply) in the economy – and replacing the I^i's by the corresponding definition - would allow us to infer the general equilibrium relative full price, $\frac{p_z + p_y}{p_x}$.

. If the second consumer does not obtain an "externality", then (33) is replaced by

$$\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} z_i^j + \sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} y_i^j = \sum_{i=1}^{n} W_z^i \tag{39}$$

or, given (31):

$$2\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i$$

With the same preferences and endowments, the equilibrium allocation will differ from the one before, but share all other mathematical properties except for the optimal endowment price: now, $(p_z + p_y)$ is the price of a pair of units of W_z^i and (31) is (also) replaced by:

$$p_z' = \frac{1}{2}(p_z + p_y) \tag{40}$$

. Finally, if consumers are homogeneous (have the same preferences *and* endowments) but receiving and making calls are valued differently so that the typical utility function is of type (4), there is only a need for two prices – potentially, p_z and p_y - to characterize equilibrium, yet z_i^j is sold to (in if there is no externality) pairs.

If form (3) is applicable – and there were indifference (perfect substitutability) between z_i^j and y_i^j at the utility level and at both consumption sides, as the marginal utility for i of consuming one extra unit of z_i^j is equal to that of consuming y_i^j, the net price he will pay for either, say p_i^j, would equalize in an interior solution; then, simply adjusting z_i^j by not answering some, or prolonging a call by calling after a hang-up would insure an adequate distribution of expenses: choosing then z_i^j such that $p_i^j(z_i^j + y_i^j) = p_z' z_i^j$, would also insure that $p_z' z_j^i = p_j^i(z_j^i + y_j^i)$, both adding the full expenditure on the resource. Then, again, unit transfers are really redundant – the argument of potential lack of competition in unit transfer price formation removed - but, in general, not otherwise…

With agent types multiplicity and some set additivity of form (4) at the utility level, the exogenous splitting rule of the total p_z' and perfect individual type identification – discrimination – and consumer replication, a uniquely decentralized equilibrium can arise, produce a unique equilibrium relative full price(s), a type-to-type specific transfer, and it is efficient. Then, it would be as if i buys z_i^j for $p_z + p_y$ and then j buys y_i^j from (individuals of type) i for $(p_y - t_j^i)$; replication – for competition – implies that some t_i^j's equalize.

Or, in a different light but representing the same structure, if we assume that n is a fixed number of possible connections, coinciding with the number of agent types in the economy, provided that calls with each type may accumulate – i.e., an individual of type i can receive calls from more than (as a fraction of those made by) one individual of type j –, the previous price system is sufficient. If they cannot, and only one individual of each type (that is, income and preferences, identifying i and j) can be connected to another to allow z_i^j, a lump-sum transfer system for each connection – with i receiving net $(K_i - K_j)$ from a connection with an individual of type j, $j \neq i$, j = 1,2,..,n -, may emerge, leaving identical individuals indifferent in equilibrium.

Likewise, in family couples, (4) would hardly imply monogamy; if we allow for (3) and assume that there are fixed – n – individual types (characterized both by preferences and income level) in the economy and z_i^j represents a potential joint consumption of an individual of type i with another of type j, partner selection and stable family establishment could arise from extensive corner solutions, multiple marriages from less extensive ones. Gender (or "head of household" status) naturally distinguishes each side of the partnership and provides the required end-side discrimination – type identification should also be perfect -, and conditions for an efficient decentralized equilibrium are therefore staged.

A corner solution for $z_i^j = 0$ will require that also $y_i^j = 0$; it will occur iff, at the prevailing relative price level, $\frac{U^i_{z_j}}{U^i_x} + \frac{U^j_{y_i}}{U^j_x} < \frac{p_z + p_y}{p_x}$ [12] at $z_i^j = y_j^i = 0$ at positive consumption of the other goods (and budget constraint multipliers in the appropriate lagrangean – according to Khun – Tucker conditions). If i and j are not connected, in the optimal solution, $z_i^j = y_j^i = 0$ and also $z_j^i = y_i^j = 0$. The equilibrium relative full price may be expected to go down while the inequality condition is not met as long as demand and supply allow, and exclusion – as in a purely private good does – would (could) occur spontaneously. For any interior solution, $U^i(x_i*, z_i^j*, y_i^j*) > U^i(\frac{I^i}{p_x}, 0, 0)$; it must also supersede the utility that the individual can obtain paying in full any of the arguments other than x_i, say r, – consuming zero of the others – if shared consumption is allowed but not a psychological *sine qua non*. That is, for the solution for which (28), for j = r, is replaced by:

(41) $$\frac{U^i_{z_r}}{U^i_x} = \frac{p_z + p_y}{p_x}$$

Or (29) by

(42) $$\frac{U^i_{y_r}}{U^i_x} = \frac{p_z + p_y}{p_x}$$

(or both...) If marginal utilities are non-negative, these are the maximum individual net prices ever observed – a potential adoption by i of r's offspring.

If we impose exclusivity – or other exogenous discrete congestion threshold -, yet interchangeable connectivity (one can have but one mate, but any pair is possible… Again, this may solve for the lack of competition in what transfer price formation is concerned…), a more complex price exchange is required to insure equilibrium, now at the matching stage – which or may not feedback to the relative full price level of the shared resource in the economy. (Dowries are a type of transfer known in history, off-springs – involving expenditure - an obvious common good to parents.) Its study is deferred to section V.

[12] Corner solutions are commonly generated with linear functional forms – a special case of the CES.

IV. Specific Functional Forms: Multi-Level CES Utility Functions

. In this section, we want to illustrate the impact of preferences on the network equilibrium formation. This is determined by utility function shapes and their, along with income, distribution; we therefore assume a general nested CES technology but allow individual specific characteristic coefficients.

We shall assume that individuals maximize utility subject to prices and an exogenous income $I^i = p_x W_x^i + p_z' W_z^i$. I^i, p_x, p_z, and p_y are externally fixed – replacing, for simplicity, the fixed individual endowments, p_x, and p_y (or p_z) of the previous section. An equilibrium will consist of individual allocations, a relative equilibrium full price, $\frac{p_z + p_y}{p_x}$, and net of unit (relative) transfer prices. For later convenience, we will present the marshallian demands and indirect utilities as a function of I^i, i = 1,2,…,n, p_x and p_z' – say, applicable to a small economy that interconnects internally but takes international prices as given -, along with the autarky equilibrium price level – then replaced in demands and indirect utility.

Allocations can be determined from (36), $y_i^j = z_j^i$, and individual budget constraints (replaced by):

$$(43) \qquad x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U^i_{z_j}}{U^i_x} z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U^i_{y_j}}{U^i_x} y_i^j = \frac{I^i}{p_x}, \quad i = 1,2,..., n$$

Unit transfers can later be inferred from (37) – and net-of-transfers prices from (34) and (35) - but redundant to determine equilibrium.

. For simplicity, let us consider an economy with a small number of consumers – let n = 3 [13]. Utilities – that we assume separable in the set $[(x_i), (z_i^j, y_i^j), (z_i^{j'}, y_i^{j'})]$ - take the form:

$$(44) \qquad U^i(x_i, z_i^j, y_i^j, z_i^{j'}, y_i^{j'}) =$$

$$= A \{a_i \, x_i^{r_i} + a_{ij} \, [b_{ij} z_i^{j\,l_{ij}} + (1-b_{ij}) y_i^{j\,l_{ij}}]^{\frac{r_i}{l_{ij}}} + a_{ij'} \, [b_{ij'} z_i^{j'\,l_{ij'}} + (1-b_{ij'}) y_i^{j'\,l_{ij'}}]^{\frac{r_i}{l_{ij'}}} \}^{\frac{m_i}{r_i}}$$

[13] A competitive equilibrium would hardly be expected; but it allows us to derive explicit solutions highlighting the impact of preferences and income on the equilibrium.

$$a_i + a_{ij} + a_{ij'} = 1, \quad a_i, a_{ij}, a_{ij'} > 0, \quad 0 < b_{ij}, b_{ij'} < 1, \quad \rho_i, \lambda_{ij}, \lambda_{ij'} \leq 1$$

Then, $\sigma_i = \dfrac{1}{1-\rho_i}$ denotes the elasticity of substitution between (among…) x_i and the two composites, $[b_{ik} z_i^{k\,\lambda_{ik}} + (1-b_{ik}) y_i^{k\,\lambda_{ik}}]^{\frac{1}{\lambda_{ik}}}$, in each of which $\sigma_{ik} = \dfrac{1}{1-\lambda_{ik}}$ is the elasticity of substitution between z_i^k and y_i^k within the composite k = j, j'.

(Even if we depart from this general functional form, we will only derive the full equilibrium for special cases. Features implied by some of the first-order optimization conditions are, nevertheless, inspected in general…)

The relevant ratios in the economy are then for i = 1,2,3 and k = j, j':

$$(45) \qquad \frac{U^i_{z_k}}{U^i_x} = \frac{a_{ik}\, b_{ik}\, [b_{ik} z_i^{k\,\lambda_{ik}} + (1-b_{ik}) y_i^{k\,\lambda_{ik}}]^{\frac{\rho_i}{\lambda_{ik}}-1}\, z_i^{k(\lambda_{ik}-1)}}{a_i\, x_i^{(\rho_i-1)}}$$

and

$$(46) \qquad \frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}\, (1-b_{ik})\, [b_{ik} z_i^{k\,\lambda_{ik}} + (1-b_{ik}) y_i^{j\,\lambda_{ik}}]^{\frac{\rho_i}{\lambda_{ik}}-1}\, y_i^{k(\lambda_{ik}-1)}}{a_i\, x_i^{(\rho_i-1)}}$$

Given the strong separability, the ratios of marginal utilities of i with respect to j are independent of goods other than x_i and (z_i^j, y_i^j), i.e., of $(z_i^{j'}, y_i^{j'})$. Yet, the general equilibrium system remains highly nonlinear; special cases for the link consumption sub-utility allow us to derive some conclusions:

i) $\lambda_{ik} = \rho_i$, k = j, j': the sub-function embeds in the second-stage general CES formulation.

$$(47) \qquad \frac{U^i_{z_k}}{U^i_x} = \frac{a_{ik}\, b_{ik}\, z_i^{k(\rho_i-1)}}{a_i\, x_i^{(\rho_i-1)}}$$

and

$$(48) \qquad \frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}\, (1-b_{ik})\, y_i^{k(\rho_i-1)}}{a_i\, x_i^{(\rho_i-1)}}$$

Then:

$$(49) \qquad \frac{a_{ik}\, b_{ik}\, z_i^{k(\rho_i-1)}}{a_i\, x_i^{(\rho_i-1)}} + \frac{a_{ki}\, (1-b_{ki})\, z_i^{k(\rho_k-1)}}{a_k\, x_k^{(\rho_k-1)}} = \frac{p_z + p_y}{p_x}, \text{ i = 1,2,3; k = j, j'}$$

A solution would be obtained combining the last expressions with the three budget constraints, leading to a nonlinear system:

$$(50)\ p_x x_i + p_x \left[\frac{a_{ij}\, b_{ij}\, z_i^{j\, r_i}}{a_i\, x_i^{(r_i-1)}} + \frac{a_{ij'}\, b_{ij'}\, z_i^{j'\, r_i}}{a_i\, x_i^{(r_i-1)}} + \frac{a_{ij}\,(1-b_{ij})\, z_j^{i\, r_i}}{a_i\, x_i^{(r_i-1)}} + \frac{a_{ij'}\,(1-b_{ij'})\, z_{j'}^{i\ r_i}}{a_i\, x_i^{(r_i-1)}} \right] = I^i$$

Reciprocity of some sort requires $a_{ik}\, b_{ik} = a_{ki}\,(1-b_{ki})$. With reciprocity and constant ρ_i, (50) simplifies to:

$$(51)\ p_x x_i + p_x \left\{ \left[1+\left(\frac{a_{ji}\, b_{ji}}{a_{ij}\, b_{ij}}\right)^{\frac{1}{1-r}}\right] \frac{a_{ij}\, b_{ij}\, z_i^{j\, r}}{a_i\, x_i^{(r-1)}} + \left[1+\left(\frac{a_{j'i}\, b_{j'i}}{a_{ij'}\, b_{ij'}}\right)^{\frac{1}{1-r}}\right] \frac{a_{ij'}\, b_{ij'}\, z_i^{j'\, r}}{a_i\, x_i^{(r-1)}} \right\} = I^i$$

Allow:
1) $\rho_3 = 1$; $\rho_1 = \rho_2 = \rho$ (but otherwise free parameters. Then:

$$\frac{x_1}{z_3^1} = \left[\left(\frac{p_z+p_y}{p_x} - \frac{a_{31}\, b_{31}}{a_3}\right)\frac{a_1}{a_{13}\,(1-b_{13})}\right]^{\frac{1}{1-r}}$$

$$\frac{x_1}{z_1^3} = \left[\left(\frac{p_z+p_y}{p_x} - \frac{a_{31}\,(1-b_{31})}{a_3}\right)\frac{a_1}{a_{13}\, b_{13}}\right]^{\frac{1}{1-r}}$$

$$\frac{x_2}{z_3^2} = \left[\left(\frac{p_z+p_y}{p_x} - \frac{a_{32}\, b_{32}}{a_3}\right)\frac{a_2}{a_{23}\,(1-b_{23})}\right]^{\frac{1}{1-r}}$$

$$\frac{x_2}{z_2^3} = \left[\left(\frac{p_z+p_y}{p_x} - \frac{a_{32}\,(1-b_{32})}{a_3}\right)\frac{a_2}{a_{23}\, b_{23}}\right]^{\frac{1}{1-r}}$$

For $x_i > 0$, (for values of ρ such as 0) then $\frac{p_z+p_y}{p_x} > \frac{a_{3i}\, b_{3i}}{a_3}$ and $\frac{p_z+p_y}{p_x} > \frac{a_{3i}\,(1-b_{3i})}{a_3}$, i = 1,2.

If $b_{3i} = b_{i3} = 0{,}5$, then $z_i^3 = z_3^i$, i = 1,2.

The higher ρ (the higher the elasticity of substitution σ between the two composites for individuals 1 and 2), the lower the connections with 3 relative to the private good, i.e., the lower $\frac{z_3^i}{x_i}$ iff $\frac{p_z + p_y}{p_x} > \frac{a_{3i}\, b_{3i}}{a_3} + \frac{a_{i3}\,(1-b_{i3})}{a_i}$; and the lower $\frac{z_i^3}{x_i}$ iff $\frac{p_z + p_y}{p_x} > \frac{a_{i3}\, b_{i3}}{a_i} + \frac{a_{3i}\,(1-b_{3i})}{a_3}$.

2) $\rho_i = \rho_k = \rho$. (We have a regular CES). Reciprocity: $a_{ik}\, b_{ik} = a_{ki}\,(1 - b_{ki})$. Then: Common elasticity of substitution requires:

$$z_i^k = \left\{ \frac{p_x}{p_z + p_y} \left[\frac{a_{ik}\, b_{ik}}{a_i} x_i^{(1-r)} + \frac{a_{ki}\,(1-b_{ki})}{a_k} x_k^{(1-r)} \right] \right\}^{\frac{1}{1-r}}$$

Reciprocity implies that, regardless of income:

(53) $$z_i^k = \left\{ \frac{p_x}{p_z + p_y}\, a_{ik}\, b_{ik} \left[\frac{1}{a_i} x_i^{(1-r)} + \frac{1}{a_k} x_k^{(1-r)} \right] \right\}^{\frac{1}{1-r}} = z_k^i \left(\frac{a_{ik}\, b_{ik}}{a_{ki}\, b_{ki}} \right)^{\frac{1}{1-r}}$$

Assume further identical relative preferences for calls such that $\frac{a_{ik}\, b_{ik}}{a_i} = \frac{a_{ik}\, b_{ik}}{a_k} = \theta$, constant in the economy. Then:

(54) $$z_i^k = \left\{ \frac{p_x}{p_z + p_y}\, q \left[x_i^{(1-r)} + x_k^{(1-r)} \right] \right\}^{\frac{1}{1-r}} = z_k^i = x_i \left\{ \frac{p_x}{p_z + p_y}\, q \left[1 + \frac{x_k^{(1-r)}}{x_i^{(1-r)}} \right] \right\}^{\frac{1}{1-r}}$$

For each consumer i – because $z_i^k = y_i^k$ -, $p_z + t_i^k = p_y - t_k^i = p_x\, \theta\, \frac{z_i^{k(r-1)}}{x_i^{(r-1)}}$. $p_x x_i + (p_z + t_i^j)\, 2\, z_i^j + (p_z + t_i^{j'})\, 2\, z_i^{j'} = I^i$. Then the three equations:

(55) $$p_x x_i + 2 p_x \theta \left(\frac{p_x}{p_z + p_y}\, q \right)^{\frac{r}{1-r}} x_i^{(1-r)} \left\{ \left[x_i^{(1-r)} + x_j^{(1-r)} \right]^{\frac{r}{1-r}} + \left[x_i^{(1-r)} + x_{j'}^{(1-r)} \right]^{\frac{r}{1-r}} \right\} = I^i$$

or

(56) $$p_x x_i \left[1 + 2\, \theta \left(\frac{p_x}{p_z + p_y}\, q \right)^{\frac{r}{1-r}} \left\{ \left[1 + \left(\frac{x_j}{x_i} \right)^{(1-r)} \right]^{\frac{r}{1-r}} + \left[1 + \left(\frac{x_{j'}}{x_i} \right)^{(1-r)} \right]^{\frac{r}{1-r}} \right\} \right] = I^i$$

allow us to retrieve the x_i's – the demands.

If income distribution is homogeneous, $x_i = x_k$ and

(57) $$x_i = \frac{I_i}{p_x}\,[1 + q^{\frac{1}{1-r}} \left(\frac{p_x}{p_z + p_y}\right)^{\frac{r}{1-r}} 2^{\frac{2-r}{1-r}}]^{-1}$$

and

(58) $$z_i^k = \frac{I_i}{p_x}\,[\left(2\,\frac{p_x}{p_z + p_y}\,q\right)^{-\frac{1}{1-r}} + 2\left(\frac{p_x}{p_z + p_y}\right)^{-1}]^{-1}$$

(59) $$v_i = A\,[a_i\, x_i^{\,r} + (1 - a_i)\, z_i^{k\,r}]^{\frac{m}{r}} =$$

$$= A\left(\frac{I_i}{p_x}\right)^{m} \{a_i\,[1 + q^{\frac{1}{1-r}} \left(\frac{p_x}{p_z + p_y}\right)^{\frac{r}{1-r}} 2^{\frac{2-r}{1-r}}]^{-r} +$$

$$+ (1 - a_i)\,[\left(2\,\frac{p_x}{p_z + p_y}\,q\right)^{-\frac{1}{1-r}} + 2\left(\frac{p_x}{p_z + p_y}\right)^{-1}]^{-r}\}^{\frac{m}{r}}$$

Then, z_i^k – as $\frac{z_i^k}{x_i}$ - increases with ρ (and σ) iff $2\,\frac{p_x}{p_z + p_y}\,q > 1$ or $2\,q = 2\,\frac{a_{ik}\,b_{ik}}{a_i} > \frac{p_z + p_y}{p_x}$ - if the relative preference for the jointly consumed good is high. $\frac{\partial v_i}{\partial I^i} = \mu\, v_i\, \frac{1}{I^i} > 0$; as $\frac{\partial^2 v_i}{\partial I^{i2}} = (\mu - 1)\,\mu\, v_i\, \frac{1}{I^{i2}}$, the whole economy "overly" rejoices - $\frac{\partial^2 v_i}{\partial I^{i2}} > 0$ - with an increase in everyone's endowment provided the utility function exhibits non-decreasing returns to scale. Also, the price of z is shared equally by any two partners:

(60) $$p_z + t_i^j = p_y - t_j^i = \frac{p_z + p_y}{2}$$

Departing from (57) and summing both sides, multiplied by p, over the n individuals in the economy, equalizing to the total resource existence of x, we could solve for the general equilibrium relative full price level as:

(61) $$\frac{p_z + p_y}{p_x} = 2^{(2-r)}\,\theta \left(\frac{\sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}\right)^{(1-r)}$$

(61) implies that the equilibrium relative price of z will decrease with the resource relative availability, $\frac{\sum_{l=1}^{n} w_z^l}{\sum_{l=1}^{n} w_x^l}$ (n = 3, the total number of individuals in the economy); and it will increase with the relative preference for the jointly consumed good, θ.

(61) could then be replaced in (57) to (60), using also the income definition, but there is not much insight to gain with that exercise.

Admit that income can differ across individuals but $\rho = 0$, i.e., of Cobb-Douglas format. Then, from (55), we conclude that individual demands are linear in income [14]:

$$p_x x_i + 4 p_x \theta \ x_i = I^i$$

This implies, on the one hand, the independence of the individual demand for the private good of income levels other than that of i itself; on the other – see (66) below -, and (also due to preference symmetry) the independence of the equilibrium relative full price of z of the income distribution in the economy.

(62) $$x_i = \frac{I^i}{p_x} (1 + 4\theta)^{-1}$$

(63) $$z_i^k = \frac{p_x}{p_z + p_y} q \ (x_i + x_k) = z_k^i = \frac{I^i + I^k}{p_z + p_y} q \ (1 + 4\theta)^{-1}$$

Replacing in the utility function, we obtain i's indirect utility function, v_i:

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[14] Gorman polar forms – to which the Stone-Geary (and Cobb-Douglas), generating a linear expenditure system, subscribes - are known to generate public goods effects, or aggregate demands independent of individual income distributions (see Deaton and Muellbauer (1980), p. 144.) – because the form (quasi-homothetic utility function) implies linear individual Engel curves (exact aggregation also requires these to exhibit constant slopes across individuals – see Deaton and Muellbauer (1980), p. 150 -, satisfied then if individuals share common preferences). Quasi-linear functional forms – see Bergstrom and Cornes (1983), Lam (1988), Batina and Ihori (2005), p. 89 – are commonly used alternatives in public goods demand modelling for allowing (because the ratio of individual's marginal utilities of the public to the private good are linear and with constant slope across individuals in the latter) aggregation across individuals.

(64) $v_i = A\,(1 + 4\,\theta)^{-m_i}\,[\left(\frac{I^i}{p_x}\right)^{a_i}\left(\frac{I^i + I^j}{p_z + p_y}\,q\right)^{a_{ij}}\left(\frac{I^i + I^{j'}}{p_z + p_y}\,q\right)^{a_{ij'}}]^{m_i}$

From (63), $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$ – there will be no assortative "matching" – nor positive, nor negative. $\frac{\partial v_i}{\partial I^j} = \mu_i\, a_{ij}\, v_i\, \frac{1}{(I^i + I^j)} > 0$; as $\frac{\partial^2 v_i}{\partial I^i \partial I^j} = \mu_i\, a_{ij}\, v_i\, \frac{m_i a_i\,(I^i + I^j)(I^i + I^{j'}) + m_i a_{ij'}\, I^i\,(I^i + I^j) - (1 - m_i a_{ij}) I^i\,(I^i + I^{j'})}{(I^i + I^j)^2\,(I^i + I^{j'})\,I^i}$, (the equivalent to) positive assortative mating – subject explored in the next section - is expected - $\frac{\partial^2 v_i}{\partial I^i \partial I^j} > 0$ - with CRS or IRS ($\mu_i \geq 1$) at the utility level.

Also:

(65) $$\frac{p_z + t_i^j}{p_x} = \frac{p_y - t_j^i}{p_x} = \theta\,\frac{x_i}{z_i^k} = \frac{I_i}{I_i + I_k}\,\frac{p_z + p_y}{p_x}$$

I pays a fraction of the price of the good(s) shared with j equal to the weight of his income relative to the pooled income of the two partners.

And given the Cobb-Douglas format of the utility, consuming something of all the goods is always worthwhile.

Internalizing equilibrium price formation in the Cobb-Douglas case:

(66) $$\frac{p_z + p_y}{p_x} = 4\,\theta\,\frac{\sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}$$

The equilibrium relative price of z will decrease – here, being proportional to its inverse - with the resource relative availability, $\frac{\sum_{l=1}^{n} w_z^l}{\sum_{l=1}^{n} w_x^l}$; and it will increase with the relative preference for the jointly consumed good, θ. We can now replace them in the demands and indirect utility:

(67) $$x_i = \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \, 4q \sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l} \, (1 + 4\,\theta)^{-1}$$

(68) $$z_i^k = z_k^i = \frac{(w_x^i + w_x^k) \sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 4q \sum_{l=1}^{n} w_x^l}{4 \sum_{l=1}^{n} w_x^l} \, (1 + 4\,\theta)^{-1}$$

(69) $$v_i^k = A\,(1 + 4\,\theta)^{-m_i}$$

$$\left\{ \left(\frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \, 4q \sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l} \right)^{a_i} \left[\frac{(w_x^i + w_x^k) \sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 4q \sum_{l=1}^{n} w_x^l}{4 \sum_{l=1}^{n} w_x^l} \right]^{(1-a_i)} \right\}^{m_i}$$

(70) $$\frac{p_z + t_i^j}{p_x} = \frac{p_y - t_j^i}{p_x} = \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \, 4q \sum_{l=1}^{n} w_x^l}{(w_x^i + w_x^k) \sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 4q \sum_{l=1}^{n} w_x^l} \; \frac{p_z + p_y}{p_x} =$$

$$= \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \, 4q \sum_{l=1}^{n} w_x^l}{(w_x^i + w_x^k) \sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 4q \sum_{l=1}^{n} w_x^l} \; 4\,\theta \; \frac{\sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}$$

With fixed coefficient technologies – ρ tends to -∞ -, $x_i = z_i^k = z_k^i = \frac{I^1 + I^2 + I^3}{3\,p_x + 6\,(p_z + p_y)}$ and $v_i = A \left[\frac{I^1 + I^2 + I^3}{3\,p_x + 6\,(p_z + p_y)} \right]^m$. With perfect substitutability - ρ tends to 1 -, consumption pairs could be expected.

ii) $\lambda_{ik} = 0$, k = j, j': the sub-function is of the Cobb-Douglas type:

(71) $$\frac{U_{z_k}^i}{U_x^i} = \frac{a_{ik}\, b_{ik}\, [z_i^{k\, b_{ik}}\, y_i^{k\,(1-b_{ik})}]^{(r_i-1)}\, z_i^{k\,(b_{ik}-1)}}{a_i\, x_i^{(r_i-1)}} = \frac{a_{ik}\, b_{ik}\, y_i^{k[(1-b_{ik})(r_i-1)]}\, z_i^{k\,(r_i b_{ik}-1)}}{a_i\, x_i^{(r_i-1)}}$$

and

(72) $$\frac{U_{y_k}^i}{U_x^i} = \frac{a_{ik}\,(1-b_{ik})\,[z_i^{k\, b_{ik}}\, y_i^{k(1-b_{ik})}]^{(r_i-1)}\, y_i^{k-b_{ik}}}{a_i\, x_i^{(r_i-1)}} = \frac{a_{ik}\,(1-b_{ik})\, z_i^{k[b_{ik}(r_i-1)]}\, y_i^{k[r_i(1-b_{ik})-1]}}{a_i\, x_i^{(r_i-1)}}$$

Then:

$$\frac{a_{ik}\, b_{ik}\, z_k^{i\,[(1-b_{ik})(r_i-1)]}\; z_i^{k\,(r_i b_{ik}-1)}}{a_i\, x_i^{(r_i-1)}} + \frac{a_{ki}\,(1-b_{ki})\, z_k^{i\,[b_{ki}(r_k-1)]}\; z_i^{k\,[r_k(1-b_{ki})-1]}}{a_k\, x_k^{(r_k-1)}} = \frac{p_z+p_y}{p_x}$$, i = 1,2,3; k = j, j'

iii) $\lambda_{ik} = 1$, k = j, j': the sub-function is linear in the arguments.

(73) $$\frac{U^i_{z_k}}{U^i_x} = \frac{a_{ik}\, b_{ik}\, [b_{ik}\, z_i^k + (1-b_{ik})\, y_i^k]^{(r_i-1)}}{a_i\, x_i^{(r_i-1)}} = g_{ik}$$

and

(74) $$\frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}\,(1-b_{ik})\,[b_{ik}\, z_i^k + (1-b_{ik})\, y_i^k]^{(r_i-1)}}{a_i\, x_i^{(r_i-1)}} = \frac{1-b_{ik}}{b_{ik}}\, g_{ik}$$

For interior solutions to be possible:

$$g_{ik} + \frac{1-b_{ki}}{b_{ki}}\, g_{ki} = \frac{p_z+p_y}{p_x} \quad \text{and} \quad g_{ki} + \frac{b_{ik}}{1-b_{ik}}\, g_{ik} = \frac{p_z+p_y}{p_x}$$

If $b_{ik} = 0.5$, $z_i^k > 0$ and $y_i^k = z_k^i = 0$ iff $b_{ki} < 0.5$; $z_i^k = 0$ and $y_i^k = z_k^i > 0$ iff $b_{ki} > 0.5$. If $z_i^k > 0$ and $y_i^k = z_k^i = 0$:

(75) $$\frac{a_{ik}\, b_{ik}^{\,r_i}\; z_i^{k\,(r_i-1)}}{a_i\, x_i^{(r_i-1)}} + \frac{a_{ki}\,(1-b_{ki})^{r_k}\; z_i^{k\,(r_k-1)}}{a_k\, x_k^{(r_k-1)}} = \frac{p_z+p_y}{p_x}$$

$$\frac{a_{ki}\, b_{ki}\,(1-b_{ki})^{(r_k-1)}\; z_i^{k\,(r_k-1)}}{a_k\, x_k^{(r_k-1)}} + \frac{a_{ik}\,(1-b_{ik})\, b_{ik}^{\,(r_i-1)}\; z_i^{k\,(r_i-1)}}{a_i\, x_i^{(r_i-1)}} < \frac{p_z+p_y}{p_x}$$

If b_{ik} = 0,5 for all i,k, we fall under (3) and there will be multiple values of z_i^k and y_i^k but a unique total $(z_i^k + y_i^k) = (z_k^i + y_k^i)$ satisfying equilibrium, including the corners represented by (75) in equality.

Admit a constant $\theta = \frac{a_{ik}\, b_{ik}^{\,r_i}}{a_i} = \frac{a_{ki}\,(1-b_{ki})^{r_k}}{a_k}$ for "active" links and $\rho_i = \rho$ for all i. Connections with all individuals require:

(76) $$p_x\, x_i + p_x\,\theta \left(\frac{p_x}{p_z+p_y}\, q\right)^{\frac{r}{1-r}} x_i \left\{ \left[1+\left(\frac{x_j}{x_i}\right)^{(1-r)}\right]^{\frac{r}{1-r}} + \left[1+\left(\frac{x_{j'}}{x_i}\right)^{(1-r)}\right]^{\frac{r}{1-r}} \right\} = I^i$$

Then, we reached a similar expression to (55). Demands will be similar.

iv) $\lambda_{ik} = -\infty$, k = j, j' and the sub-function is of the fixed coefficient, Leontief, type – $[b_{ik} z_i^{k\, l_{ik}} + (1-b_{ik}) y_i^{j\, l_{ik}}]^{\frac{1}{l_{ik}}}$ tends to $Min(z_i^k, y_i^k)$. Then, at efficient consumption levels, both items equalize and:

$$(77) \qquad \frac{U^i_{z_k}}{U^i_x} = \frac{a_{ik}\ Min(z_i^k, y_i^k)^{(r_i-1)}}{a_i\ x_i^{(r_i-1)}} = \frac{a_{ik}\ z_i^{k(r_i-1)}}{a_i\ x_i^{(r_i-1)}}$$

and

$$(78) \qquad \frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}\ Min(z_i^k, y_i^k)^{(r_i-1)}}{a_i\ x_i^{(r_i-1)}} = \frac{a_{ik}\ y_i^{k(r_i-1)}}{a_i\ x_i^{(r_i-1)}}$$

$z_i^k = y_i^k$: there is perfect complementarity between calls made or received by i from each k.

For interior solutions:

$$(79) \qquad \frac{a_{ik}\ z_i^{k(r_i-1)}}{a_i\ x_i^{(r_i-1)}} + \frac{a_{ki}\ z_i^{k(r_k-1)}}{a_k\ x_k^{(r_k-1)}} = \frac{p_z + p_y}{p_x}, \quad i = 1,2,3;\ k = j, j'$$

with half of the conditions (compatible and) redundant, and

$$(80) \qquad p_x x_i + 2\, p_x \left[\frac{a_{ij}\ z_i^{j\, r_i}}{a_i\ x_i^{(r_i-1)}} + \frac{a_{ij'}\ z_i^{j'\, r_i}}{a_i\ x_i^{(r_i-1)}} \right] = I^i, \quad i = 1,2,3$$

For special cases, we arrive at solutions with similar properties as before.

. Other interesting formulations would allow for a different degree of substitution between the two composites, say:

$$(81) \qquad U^i(x_i, z_i^j, y_i^j, z_i^{j'}, y_i^{j'}) = A\, (a_i\ x_i^{r_i} + (1 - a_i)$$
$$\{a_{ij}\, [b_{ij} z_i^{j\, l_{ij}} + (1-b_{ij}) y_i^{j\, l_{ij}}]^{\frac{q_i}{l_{ij}}} + a_{ij'}\, [b_{ij'} z_i^{j'\, l_{ij'}} + (1-b_{ij'}) y_i^{j'\, l_{ij'}}]^{\frac{q_i}{l_{ij'}}} \}^{\frac{r_i}{q_i}})^{\frac{m_i}{r_i}}$$
$$0 < a_i, a_{ij}, a_{ij'}, b_{ij}, b_{ij'} < 1,\ \ a_{ij} + a_{ij'} = 1,\ \ \rho_i, \theta_i, \lambda_{ij}, \lambda_{ij'} \le 1$$

FOC require:

(82) $$\frac{U^i_{z_k}}{U^i_x} = (1 - a_i)\ \frac{a_{ik}\, b_{ik}\, [b_{ik}\, z_i^{k\, l_{ik}} + (1-b_{ik})\, y_i^{k\, l_{ik}}]^{\frac{q_i}{l_{ik}}-1}\ z_i^{k\,(l_{ik}-1)}}{a_i\ x_i^{(r_i-1)}}$$

$$\{a_{ij}\, [b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, y_i^{j\, l_{ij}}]^{\frac{q_i}{l_{ij}}} + a_{ij'}\, [b_{ij'}\, z_i^{j'\, l_{ij'}} + (1-b_{ij'})\, y_i^{j'\, l_{ij'}}]^{\frac{q_i}{l_{ij'}}}\}^{\frac{r_i}{q_i}-1} = \frac{p_z + t_i^j}{p_x}$$

and

(83) $$\frac{U^i_{y_k}}{U^i_x} = (1 - a_i)\ \frac{a_{ik}\, (1-b_{ik})\, [b_{ik}\, z_i^{k\, l_{ik}} + (1-b_{ik})\, y_i^{k\, l_{ik}}]^{\frac{q_i}{l_{ik}}-1}\ y_i^{k\,(l_{ik}-1)}}{a_i\ x_i^{(r_i-1)}}$$

$$\{a_{ij}\, [b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, y_i^{j\, l_{ij}}]^{\frac{q_i}{l_{ij}}} + a_{ij'}\, [b_{ij'}\, z_i^{j'\, l_{ij'}} + (1-b_{ij'})\, y_i^{j'\, l_{ij'}}]^{\frac{q_i}{l_{ij'}}}\}^{\frac{r_i}{q_i}-1} = \frac{p_y - t_j^i}{p_x}$$

Monogamous family formation can then be adequately modeled with reference to the threshold value of $\theta_i = 1$ or larger – representing taste for unicity…

In the limiting case where θ_i tends to $+\infty$, $\{a_{ij}\, [b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, y_i^{j\, l_{ij}}]^{\frac{q_i}{l_{ij}}} + (1 - a_{ij})\, [b_{ij'}\, z_i^{j'\, l_{ij'}} + (1-b_{ij'})\, y_i^{j'\, l_{ij'}}]^{\frac{q_i}{l_{ij'}}}\}^{\frac{1}{q_i}}$ tends to $\mathrm{Max}\{[b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, y_i^{j\, l_{ij}}]^{\frac{1}{l_{ij}}},\ [b_{ij'}\, z_i^{j'\, l_{ij'}} + (1-b_{ij'})\, y_i^{j'\, l_{ij'}}]^{\frac{1}{l_{ij'}}}\}$ – note that $\mathrm{Min}(x, y, z) = \mathrm{Max}(x^{-1}, y^{-1}, z^{-1})^{-1}$ as well as $\mathrm{Min}(x^{-1}, y^{-1}, z^{-1})^{-1} = \mathrm{Max}(x, y, z)$ and use the fact that the CES tends to Leontief - and only pair-wise connections are formed. (Provided that SOC can still apply). Let us then consider such limiting case.

With three individual types, only 1 pair will be formed, let us say i and j. Then:

(84) $$(1 - a_i)\ \frac{b_{ij}\, [b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, z_j^{i\, l_{ij}}]^{\frac{1-l_{ij}}{l_{ij}}}\ z_i^{j\,(l_{ij}-1)}}{a_i\ x_i^{(r_i-1)}}$$

$$\{[b_{ij}\, z_i^{j\, l_{ij}} + (1-b_{ij})\, z_j^{i\, l_{ij}}]^{\frac{1}{l_{ij}}}\}^{(r_i-1)} +$$

$$(1 - a_j)\ \frac{(1-b_{ji})\, [b_{ji}\, z_j^{i\, l_{ji}} + (1-b_{ji})\, z_i^{j\, l_{ji}}]^{\frac{1-l_{ji}}{l_{ji}}}\ z_i^{j\,(l_{ji}-1)}}{a_j\ x_j^{(r_j-1)}}$$

$$\{[b_{ji}\, z_j^{i\, l_{ji}} + (1-b_{ji})\, z_i^{j\, l_{ji}}]^{\frac{1}{l_{ji}}}\}^{(r_j-1)} = \frac{p_z + p_y}{p_x}$$

or

(85) $$(1 - a_i)\ \frac{b_{ij}\,[b_{ij}\, z_i^{j\,l_{ij}} + (1-b_{ij})\, z_j^{i\,l_{ij}}]^{\frac{r_i - l_{ij}}{l_{ij}}}\ z_i^{j(l_{ij}-1)}}{a_i\, x_i^{(r_i-1)}} +$$

$$(1 - a_j)\ \frac{(1-b_{ji})\,[b_{ji}\, z_j^{i\,l_{ji}} + (1-b_{ji})\, z_i^{j\,l_{ji}}]^{\frac{r_j - l_{ji}}{l_{ji}}}\ z_i^{j(l_{ji}-1)}}{a_j\, x_j^{(r_j-1)}} = \frac{p_z + p_y}{p_x}$$

j’ either may consume only x, and $x_{j'} = \frac{I^{j'}}{p_x}$ - that occurring if $\rho_{j'}$ is large (certainly larger than 0). Or, he will pay his connections to only one of the other k’s – either to i or to j, for whom the marginal utility of consumption of joint goods with j’ is 0 - in full so that:

(86) $$(1 - a_{j'})\ \frac{b_{j'k}\,[b_{j'k}\, z_{j'}^{k\,l_{j'k}} + (1-b_{j'k})\, z_k^{j'\,l_{j'k}}]^{\frac{r_{j'} - l_{j'k}}{l_{j'k}}}\ z_{j'}^{k(l_{j'k}-1)}}{a_{j'}\, x_{j'}^{(r_{j'}-1)}} = \frac{p_z + p_y}{p_x}$$

and

(87) $$(1 - a_{j'})\ \frac{(1-b_{j'k})\,[b_{j'k}\, z_{j'}^{k\,l_{j'k}} + (1-b_{j'k})\, z_k^{j'\,l_{j'k}}]^{\frac{r_{j'} - l_{j'k}}{l_{j'k}}}\ z_k^{j'(l_{j'k}-1)}}{a_{j'}\, x_{j'}^{(r_{j'}-1)}} = \frac{p_z + p_y}{p_x}$$

Then $b_{j'k}\, z_{j'}^{k(l_{j'k}-1)} = (1-b_{j'k})\, z_k^{j'(l_{j'k}-1)}$ or $b_{j'k}^{\frac{l_{j'k}}{l_{j'k}-1}}\, z_{j'}^{k\,l_{j'k}} = (1-b_{j'k})^{\frac{l_{j'k}}{l_{j'k}-1}}\, z_k^{j'\,l_{j'k}}$ and $[b_{j'k}\, z_{j'}^{k\,l_{j'k}} + (1-b_{j'k}) z_k^{j'\,l_{j'k}}]^{\frac{r_{j'} - l_{j'k}}{l_{j'k}}} = [b_{j'k} + (1-b_{j'k})^{\frac{1}{1-l_{j'k}}}\, b_{j'k}^{\frac{l_{j'k}}{l_{j'k}-1}}]^{\frac{r_{j'} - l_{j'k}}{l_{j'k}}}\ z_{j'}^{k(r_{j'} - l_{j'k})}$. The expression becomes

(88) $$(1 - a_{j'})\ [b_{j'k} + (1-b_{j'k})^{\frac{1}{1-l_{j'k}}}\, b_{j'k}^{\frac{l_{j'k}}{l_{j'k}-1}}]^{\frac{r_{j'} - l_{j'k}}{l_{j'k}}}\ \frac{b_{j'k}\ z_{j'}^{k(r_{j'}-1)}}{a_{j'}\ x_{j'}^{(r_{j'}-1)}} = \frac{p_z + p_y}{p_x}$$

His budget constraint becomes:

(89) $$p_x x_{j'} + 2\,(p_z + p_y)\,[1 + \left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{l_{j'k}-1}}]\ z_{j'}^k = I^{j'} =$$

$$= p_x x_{j'}\,\{1 + 2\,[1 + \left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{l_{j'k}-1}}]\ \left(\frac{p_z + p_y}{p_x}\right)^{\frac{r_{j'}}{r_{j'}-1}}$$

$$\left[\frac{(1-a_{j'})\, b_{j'k}}{a_{j'}}\right]^{\frac{1}{1-r_{j'}}} \; [b_{j'k} + (1-b_{j'k})^{\frac{1}{1-l_{j'k}}}\, b_{j'k}^{\frac{l_{j'k}}{l_{j'k}-1}}]^{\frac{r_{j'}-l_{j'k}}{l_{j'k}(1-r_{j'})}} \; \}$$

An equilibrium may then arise in which any of the three individuals pays its connections in full to one and only one individual, "free-riding" on the connections with other(s) – eventually, with an individual not paying.

In sum, with taste for unicity, a mating equilibrium mechanism must additionally arise…

Consider $\lambda_{ik} = \rho_i$. Then, for the pair i j, we fall back into

(90) $$(1 - a_i)\, \frac{b_{ij}\, z_i^{j\,(r_i-1)}}{a_i\, x_i^{(r_i-1)}} + (1 - a_j)\, \frac{(1-b_{ji})\, z_i^{j\,(r_j-1)}}{a_j\, x_j^{(r_j-1)}} = \frac{p_z + p_y}{p_x}$$

(91) $$(1 - a_j)\, \frac{b_{ji}\, z_j^{i\,(r_j-1)}}{a_j\, x_j^{(r_j-1)}} + (1 - a_i)\, \frac{(1-b_{ij})\, z_j^{i\,(r_i-1)}}{a_i\, x_i^{(r_i-1)}} = \frac{p_z + p_y}{p_x}$$

Budget constraints require for the pair i,j:

(92) $$p_x x_i + p_x (1 - a_i) \left[\frac{b_{ij}\, z_i^{j\, r_i}}{a_i\, x_i^{(r_i-1)}} + \frac{(1-b_{ij})\, z_j^{i\, r_i}}{a_i\, x_i^{(r_i-1)}} \right] = I^i$$

Let reciprocity of some sort require $b_{ij} = (1 - b_{ji})$. The traits of the general solution of (49) but now for two agents only are recovered.

For single payers:

(93) $$I^{j'} = p_x x_{j'},$$

$$\{1 + 2\, [1 + \left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{r_{j'}-1}}] \left(\frac{p_z + p_y}{p_x}\right)^{\frac{r_{j'}}{r_{j'}-1}} \left(\frac{(1-a_{j'})\, b_{j'k}}{a_{j'}}\right)^{\frac{1}{1-r_{j'}}} \}$$

If we allow for agent multiplicity, interior pairs can be formed only. Monogamy would be the rule against polygamy with perfect taste for unicity. Now, mating assorting can be studied not through interior consumption – z_i^j and y_i^j, more adequately qualifying "matching" -, but from corner solutions patterns – inspecting indirect utility functions properties.

In the symmetric preferences, Cobb-Douglas case ($\rho_i = 0$) for a (mated) individual i:

$$p_x x_i + 2\, p_x \theta\ x_i = I^i$$

where $\theta = \frac{(1-a_i)b_{ij}}{a_i}$. Marshallian demands, x_i and $z_i^k = z_k^i$, and indirect utility, v_i^k, of an individual i connected to individual k are given by:

(94) $$x_i = \frac{I^i}{p_x}\ (1 + 2\,\theta)^{-1}$$

(95) $$z_i^k = \frac{p_x}{p_z + p_y} q\ (x_i + x_k) = z_k^i = \frac{I^i + I^k}{p_z + p_y} q\ \ (1 + 2\,\theta)^{-1}$$

(96) $$v_i^k\ =\ A\ (1 + 2\,\theta)^{-m_i}\ [\left(\frac{I^i}{p_x}\right)^{a_i} \left(\frac{I^i + I^k}{p_z + p_y} q\right)^{(1-a_i)}]^{m_i}$$

Internalizing equilibrium price formation – now allowing for any given number of individuals in the economy, n, where each of them mates one and only one individual:

(97) $$\frac{p_z + p_y}{p_x} = 2\,\theta\ \frac{\sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}$$

(98) $$x_i = \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i\ 2q \sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}\ (1 + 2\,\theta)^{-1}$$

(99) $$z_i^k = z_k^i = \frac{(w_x^i + w_x^k)\sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\ 2q \sum_{l=1}^{n} w_x^l}{2\sum_{l=1}^{n} w_x^l}\ (1 + 2\,\theta)^{-1}$$

(100) $$v_i^k\ =\ A\ (1 + 2\,\theta)^{-m_i}$$

$$\{\left(\frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i\ 2q \sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}\right)^{a_i} \left[\frac{(w_x^i + w_x^k)\sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\ 2q \sum_{l=1}^{n} w_x^l}{2\sum_{l=1}^{n} w_x^l}\right]^{(1-a_i)}\}^{m_i}$$

(101) $$\frac{p_z + t_i^j}{p_x}\ =\ \frac{p_y - t_j^i}{p_x}\ =\ \frac{I_i}{I_i + I_k}\ \frac{p_z + p_y}{p_x}\ =$$

$$= \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \; 2q \sum_{l=1}^{n} w_x^l}{(w_x^i + w_x^k)\sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 2q \sum_{l=1}^{n} w_x^l} \; \frac{p_z + p_y}{p_x} =$$

$$= \frac{w_x^i \sum_{l=1}^{n} w_z^l + w_z^i \; 2q \sum_{l=1}^{n} w_x^l}{(w_x^i + w_x^k)\sum_{l=1}^{n} w_z^l + (w_z^i + w_z^k)\, 2q \sum_{l=1}^{n} w_x^l} \; 2\,\theta \; \frac{\sum_{l=1}^{n} w_x^l}{\sum_{l=1}^{n} w_z^l}$$

Given the special form of the utility function – the linearity of demands for the private good, with fixed (for all i) marginal increment, in $\frac{I^i}{p_x}$ (and independence of mate's income – even if linearity with fixed marginal increment also in the latter would imply the same result) -, the relative full price level is independent of resource distribution. Also due to the uniformity of the direct utility functions, it is also independent of the particular mating arrangement that should come to develop in the economy.

Nevertheless, out of similar special cases, mating dynamics are expected to feedback to it.

V. Assortative Mating and Transferability.
V.1. Introduction.

In this section, we are going to suggest some of the expected mating arrangements in an economy where individual i (i = 1,2,…, n) possesses utility potential $v_i^k(I^i, I^k)$, where $I^{i(k)}$ is i(k)'s income, if paired with k≠ i, and the equilibrium devices involved in its determination. Obviously, $v_i^k(I^i, I^k)$ may represent an indirect utility function of individual i arising from a direct utility function exhibiting taste-for-unicity and an optimization involving shared-consumption – say, such as (96).

We will further assume that $v_i^k(I^i, I^k) = v_i(I^i, I^k)$, all k and i, that the same general indirect utility function form applies for all potential mates, only differing and increasing in their income level – i.e., $\frac{\partial v_i(I^i, I^k)}{\partial I^k} > 0$ for all i, k, and all the individuals I - with the first sub-index i left in the indirect utility function just to indicate the individual to which it belongs to. This is a simplifying assumption [15]: we might as well just require that any potential mate k is preference ordered – ranked – similarly by any i in the economy.

Everybody wants to mate with the highest income. He can just mate one individual… as also the second lowest income: mating types will constitute a relatively scarce resource, the usual setting under which pricing systems naturally develop. But for pricing to occur, one must be able to pay in some other resource – i.e., to trade. Given the context - $v_i(I^i, I^k)$ -, a plausible "numeraire" would then be income I^i[16]. Another, often encountered in the family economics literature, is utility – utility units – itself: utility is then invoked to be *transferable* between the couple.

If neither utility nor endowments (income…) are transferable – individuals "must" obtain utility according to $v_i(I^i, I^k)$, because $\frac{\partial v_i}{\partial I^k} > 0$ for all i, - more generally, because the ranking of potential mates in the economy is uniform -, we expect positive assortative mating in the economy: higher income (more highly preferred as mate) individuals will cluster together starting at the highest level.

In other cases, different assignments may be generated. Some contexts have been thoroughly studied in the literature, namely, transferable utilities – see Legros and Newman (2002) for recent references. However, not all cases; and when efficiency was analyzed,

[15] Form (96) obeys it due to the uniformity of direct preferences in the economy of the special case…

[16] We might as well consider one of the two endowments… We are assuming that any of them can.

connection with the implicit supporting price system was missing. We therefore proceed to both.

V.2. Transferable Utilities.

. One can find in Becker (1973) a proof that, in the presence of *transferable utilities*, positive (negative) assortative mating is optimal in the sense that it maximizes the sum of individuals' utilities, positively dependent on the income of each of the individuals forming a pair, iff $\frac{\partial^2 v_i}{\partial I^i \partial I^k}$ > (<) 0. The condition was later generalized to the requirement of super (sub) modularity – see for instance, Legros and Newman (2002) for a definition. In this sub-section, we provide an intuition (an alternative proof) for the result, when $\frac{\partial v_i}{\partial I^k} > 0$ for all i, after characterizing a first-order condition principle for efficient matching and relate it to the supporting (general equilibrium) pricing system. We further digress on the spontaneous mating arrangement arising when matching pairs are formed with individuals of distinct groups.

. The marginal benefit obtained by individual i, with income I^i, by mating with individual k of income I^k, call it d_i^k, is the utility gain he obtains by mating with k instead of with the individual k-1 when potential mates are ordered by ascending order of income. I.e.:

$$d_i^k = v_i(I^i, I^k) - v_i(I^i, I^{k-1}) \tag{102}$$

In a decentralized economy, mating changes are expected to occur till equality of the marginal benefit of the match – the *price* (in utility units) that individuals would pay for the last match improvement - across the economy, i.e., for all the i's that mated; in the optimal assignment scheme:

$$d_i^{k*} = v_i(I^i, I^{k*}) - v_i(I^i, I^{k*-1}) = p_{MF} \ , \quad i = 1,2,\ldots, n \tag{103}$$

Such rule would stem from first-order conditions for efficiency – characterized more generally in V.5 -, i.e., maximization of $\sum_{i=1}^{n} v_i(I^i, I^{k*})$, which, *at given individual income levels* and in the presence utility transferability would appear as the natural maximand: the couple formed by i has joint utility maximized for (n/2-1) given levels of sum of couple utilities we assign to other couples.

. Let then the n individuals that are mated be ordered ascendingly according to their own income level, i (k) = 1,2…,n. Then, i pays a "net" dowry [17] to k*:

$$D_i^{k*} = p_{MF}\,(r_{k*} - r_i) \approx p_{MF}\,(k* - i) \tag{104}$$

where r_i (r_k) [18] represents the rank order of individual i (k) by individual k (i)'s preferences – and of all individuals above k (i). I.e., i obtains "net-of-transfers" utility:

$$v_i^{k*} = v_i(I^i, I^{k*}) - D_i^{k*} = v_i(I^i, I^{k*}) - p_{MF}\,(k* - i) \tag{105}$$

in the optimal match in which he is paired with k*, the one chosen to operate utility transfers with. The equalization of the marginal benefit of mating with k to the ranking points price arises naturally from FOC of the discrete choice problem facing i of determining the k that maximizes $v_i^k = v_i(I^i, I^k) - p_{MF}\,(k - i)$ – once i, i cannot change…. $v_i^{k*} + v_{k*}^{\,i} = v_i(I^i, I^{k*}) + v_k(I^k, I^{i*})$, all i,k*, and therefore transfers are confined to each pair.

p_{MF} is the price of the income ranking points in the economy for matching purposes. Those points are attributed according to a classification that ranges from 1 to n [19], (i.e., even if there is income replication, in which case the rank of equally endowed individuals could be

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[17] See Botticini and Siow (2003) for a recent overview of other rationales for dowries and bequests.

[18] They can just slightly differ from i (k) – at most, i - r_i = 1, k - r_k = 1 -, because one cannot mate with oneself…

[19] This preference ordering – quantifying quality – of the match with each individual, k, must be uniformly accepted and agreed upon in the economy – be independent of i - for the price system (competition or market for ranking points – discrete quantities, but nevertheless aggregatable quantities) to work. If not, and i_j is the preference ordering assessment of individual i by individual j in a scale of 1 (least preferred) to n-1 (most preferred) – so that i is endowed or rated with $\sum_{j=1, j\neq i}^{n} i_j$ points, uniquely appreciated by everybody -, one would speculate that an equilibrium condition could require $[v_i(i, k) - v_i(i, k\text{-}1)] / [\sum_{j=1, j\neq k}^{n} k_j - \sum_{j=1, j\neq k-1}^{n} (k-1)_j] = p$ to be constant in the optimal assignment, where k is i's pair – $v_i(i,k)$ i's utility when paired with k -, (k-1) his next preference, and p the price of all ranking points in the market – $n \sum_{i=1}^{n-1} i = (n\text{-}1)n^2 / 2$ - with $D_i^k = p\,(\sum_{j=1, j\neq k}^{n} k_j - \sum_{j=1, j\neq i}^{n} i_j)$.

the mid-rank of the individuals in the category) where n is the number of individuals *that were paired*, discrete [20] and consecutive if all incomes differ. Such pricing scheme occurs, or is due, because unicity is required at the utility level – matching with j has the opportunity cost of not being available to match with somebody else.

In equilibrium, for individuals that were not mated by the matching market (that stayed outside the group of the n mated ones – i.e., such n is, or are, endogenous), it must be the case that for unmatched j's either:

$$(106) \qquad d_j^{1*} = v_j(I^j, I^{1*}) - v_j(I^j, 0) < p_{MF} \ , \quad j = n + 1, n + 2,\ldots$$

where I^{1*} is the lowest income of the *paired* individuals. While the reverse is occurring – as in any market -, there is excess demand for matching and p_{MF} will be increasing while additional matches are being arranged, process that becomes complete only when equality holds - because of discreteness, till $d_i^{k+1*} < p_{MF} \leq d_i^{k*}$ - for all the (some…) n mated partners.

Or the closest mated income to the (an) excluded j, say j+1*, is mated with someone – k* - that would not change it for j. That is:

$$(107) \qquad d_k^{j-1*} = v_k(I^{k*}, I^j) - v_k(I^{k*}, I^{j-1*}) < p_{MF} \ , \quad j = n + 1, n + 2,\ldots$$

(106) would apply when lower incomes are not mated – arising with positive assortative mating; (107) when middle incomes are not mated, expected with negative assortative mating.

. For the resulting arrangement to be optimal for individual i – for him to achieve the maximum and not the minimum utility with marginal benefit to price equalization -, one requires the marginal benefit to be decreasing in the match, i.e., $d_i^{k*} - d_i^{k*-1} = v_i(I^i, I^{k*}) - v_i(I^i, I^{k*-1}) - [v_i(I^i, I^{k-1*}) - v_i(I^i, I^{k*-2})] < 0$ – where k* -2 is the next best match to (before income)

[20] The price will be that of a discrete ranking of potential partners, not of their income: what is as stake is a discrete location over a set of ordered alternatives. Of course, the income magnitude affects the equilibriumprice but through its effect on utility levels.

k* - 1. This is satisfied if $\frac{\partial^2 v_i}{\partial I^{k^2}} < 0$ [21] and existing income levels in the economy are equally spaced.

. Now, for d_i^{k*} to be constant in the economy, I^i and the income of the pair, I^{k*}, must change or relate according to (or close…) - differentiating (103):

$$(108) \qquad \frac{\partial d_i^k}{\partial I^i} dI^i + \frac{\partial d_i^k}{\partial I^{k*}} dI^{k*} + \frac{\partial d_i^k}{\partial I^{k*}} dI^{k*-1} = [\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^i}] dI^i + \frac{\partial v_i(I^i, I^{k*})}{\partial I^k} dI^{k*} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^k} dI^{k*-1} = 0$$

Assume that income levels are equally or uniformly spaced in the economy so that $dI^{k*} = dI^{k*-1}$. Then:

$$(109) \quad [\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^i}] dI^i = -[\frac{\partial v_i(I^i, I^{k*})}{\partial I^{k*}} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^{k*-1}}] dI^{k*}$$

Approximately, $\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^i} \approx (I^{k*} - I^{k*-1}) \frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k}$ and $[\frac{\partial v_i(I^i, I^{k*})}{\partial I^{k*}} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^{k*-1}}] \approx -(I^{k*} - I^{k*-1}) \frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^{k^2}}$. Then we expect the assignment in the economy to exhibit:

$$(110) \qquad \frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} dI^i = -\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^{k^2}} dI^{k*}$$

If $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^{k^2}} < 0$ (required by SOC for maximum benefit), then $\frac{dI^{k*}}{dI^i} > 0$ and we register positive assortative mating – as income rises, so does that of the partner – iff $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} > 0$. $\frac{dI^{k*}}{dI^i} < 0$ and we register negative assortative mating – as income rises, that of the partner tends to decrease – iff $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} < 0$.

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[21] As in conventional continuous optimization, non-convexities – e.g., increasing returns to scale – may generate equilibrium failure, as well as validity of interior FOC of the efficient allocation solution.

Similar conclusions would be obtained if we reasoned with the marginal loss from accepting k* instead of the next upper income, $l_i^{k*} = v_i(I^i, I^{k*+1}) - v_i(I^i, I^{k*})$ = constant, i = 1,2,…, n. Provided $\frac{\partial^2 v_i}{\partial I^{k^2}} < 0$ and income is evenly spaced, $l_i^{k*} < d_i^{k*}$.

. If $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^{k^2}} > 0$, marginal benefit equalization leads to minimum individual (and, thus, aggregate) utility; such minimization would be consistent with assignments such that $\frac{dI^{k*}}{dI^i} < 0$, i.e., negative (positive) assortative mating, iff $\frac{\partial^2 v_i}{\partial I^i \partial I^k} > (<) 0$. But, when SOC fail, the marginal equalization principle – and the law of one price – fails: demands for match ranking points are no longer negatively sloped. Then, one would expect that if $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} >$ 0, a match with simultaneously high income of partners generates a higher utility surplus, transferable within the couple, and there would be positive assortative mating; with $\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} < 0$, a match with dissimilar income levels would; i.e., we always (still) expect - because utility is transferable - the equilibrium assignment to be the optimal *aggregate* one. But the failure of the market match price equalization would confer bargaining power within some range to individuals within each pair – and lead to multiple possible arrangements of effective transfers occurring within the couple, eventually colliding with the optimality conditions generating the indirect utility functions...

A numerical illustration of the marginal benefit (and loss) principle is presented in the Appendix.

. Admit that mating can only occur between an individual of group M (males, 1,2,…, n_A) and another of group F (females, n_A+1, n_A+2, …, n). One could think that different prices could be formed for rankings of each group, say p_M for ranking points of males – equalized to the marginal benefit that individuals of group F are deriving from mating with those of group M - and p_F for those of females - the marginal benefit that males are deriving from mating with females [22]. An individual of group M (i = 1,2,…, Min(n_M, n_F), ordered ascendingly by income on group M, where Min(n_M, n_F) are individuals that end-up effectively mated) would pay to an individual of group F (k = 1,2,…, Min(n_M, n_F), ordered ascendingly by income on group F) a net transfer $D_i^{k*} = p_F\, k* - p_M\, i$, i = 1,2,…, Min($n_M$,

[22] As equalization of marginal benefit for each group equalizes, cross-derivative correspondence with the sign of sorting is still be valid.

n_F); k = 1,2,..., Min(n_M, n_F); consistently, an individual of group F (k = 1,2,…, Min(n_M, n_F)) would pay to an individual of group M (i = 1,2,…, Min(n_M, n_F)) a net transfer $D_k^{i*} = p_M$ i* – p_F k, k = 1,2,..., Min(n_M, n_F); i = 1,2,…, Min(n_M, n_F); the individual of each group would equalize his marginal benefit to the price of the ranking points of the other group. Yet, equilibrium would not yet be defined, once it requires additionally an overall appraisal of the two groups relative income availability. Moreover, interpersonal comparison with the own group rankings end up by being made indirectly, which is not accounted for by that pricing system.

One would therefore speculate that the previous – uniform pricing - rule still applies, with marginal benefit and ranking order of individuals – unique and uniquely priced - being calculated as if one could also mate with people of the *own* group; the equilibrium price of ranking points now adjusts till k* belongs to the opposite group. Or that, under group-specific rankings, p_M ($p_M \sum_{\substack{i=1 \\ i\in F}}^{Min(n_M,n_F)} k*$) and p_F ($p_F \sum_{\substack{i=1 \\ i\in M}}^{Min(n_M,n_F)} k*$) will approximate: the marginal benefit of a mate in the economy – the price of ranking points for matching purposes - would attempt to equalize.

Under unbalanced groups, the last rule may, again not be sufficient. If there is:

- positive assortative mating: prices should guarantee that $d_j^{1*} = v_j(I^j, I^{1*}) - v_j(I^j, 0) < p_F$ if $n_A > n\text{-}n_A$ and only n-n_A M's are mated; to $d_{j'}^{1*} = v_{j'}(I^{j'}, I^{1*}) – v_{j'}(I^{j'}, 0) < p_M$ if $n_A < n\text{-}n_A$ and only n_A F's are mated – with 1* the lowest income mated of the other group - for individuals j (of M), j' (of F) not mated (that preferred not to match in the optimal assignment) of each group. Given the positive sorting, low income levels are expected to be excluded, and the highest excluded income qualifies the relevant marginal unmated individual, j or j'. And due to the evolution of marginal benefit, the price approximation rule may be sufficient.

- negative assortative mating: prices will go up till – guarantee that - $d_i^{k*} = v_j(I^{i*}, I^j) - v_j(I^{i*}, I^{k*}) < p_M$ if $n_A > n\text{-}n_A$ and only n-n_A M's are mated; to $d_i^{k*} = v_j(I^{i*}, I^{j'}) – v_{j'}(I^{i*}, I^{k*}) < p_F$ if $n_A < n\text{-}n_A$ and only n_A F's are mated – with i* the individual mated with next lowest income relative to the excluded (not mated) individuals j (of M), j' (of F) of each group. Given the negative assorting, middle income levels are expected to be excluded, and the lowest excluded income qualifies the relevant marginal unmated individual, j or j', j or j'.

With positive assortative mating, the effective transfer between the pairs in a couple tends to 0. Yet, the ranking points price system must be at least latent – insuring (provided SOC hold) equalization of the marginal benefit across the economy and not other (non-optimal in the presence of utility transferability) mating rule. With negative assortative mating, non-negligible transfers effectively occur between pairs.

V.3. Transferable Income.

If utility is not transferable across individuals but income is, one could advance that the marginal benefit equated across individuals would be measured in income terms, i.e., d'^{k}_{i} such that [23]:

(111) $v_i(I^i - D'^{k^*-1}_i - d'^{k}_i,\ I^{k^*} + D'^{k^*-1}_i + d'^{k}_i) \ = \ v_i(I^i - D'^{k^*-1}_i,\ I^{k^*-1} + D'^{k^*-1}_i)$

that is:

(112) $v_i[I^i - (k^* - i)\, p_{MF},\ I^{k^*} + (k^* - i)\, p_{MF}] \ = \ v_i[I^i - (k^* - 1 - i)\, p_{MF},\ I^{k^*-1} + (k^* - 1 - i)\, p_{MF}]$

Individual i chooses k maximizing $v_i[I^i - (k - i)\, p_{MF},\ I^k + (k - i)\, p_{MF}]$, which would generate FOC implying that the difference between the left and right hand-sides of (112) – the marginal net-of-cost benefit - approaches zero.

p_{MF} is now a price measured in income units and $D'^{k^*}_i$ deducted to the individual i's own resources. It reflects the fact that a couple's budget constraints or resources can be *pooled*, and it incorporates a measure of the strength of the individual in the household allocation decision.

Using Taylor's expansion to the first order we can (grossly…) approximate:

(113) $$d'^{k^*}_i \approx \frac{v_i(I^i, I^{k^*}) - v_i(I^i, I^{k^*-1})}{\dfrac{\partial v_i(I^i, I^{k^*})}{\partial I^i} - \dfrac{\partial v_i(I^i, I^{k^*})}{\partial I^k}} \approx$$

[23] These are also the expected market features if both utility and income are transferable, provided that $v_i(I^i, I^k)$ is quasi-concave in the two arguments: i chooses k* by making the derivative of $v_i(I^i, I^k)$ with respect to k* – the difference between the left and right-hand side terms of each of the expressions – equal to zero.

$$\approx \frac{v_i(I^i, I^{k*})}{\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*})}{\partial I^k}} - \frac{v_i(I^i, I^{k*-1})}{\frac{\partial v_i(I^i, I^{k*-1})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*-1})}{\partial I^k}} = p_{MF}$$

Then, for adequate conclusions on mating one would advance that if the function [24]

(114) $$v'_i(I^i, I^k) = \frac{v_i(I^i, I^{k*})}{\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} - \frac{\partial v_i(I^i, I^{k*})}{\partial I^k}}$$

- that evaluates i's utility in terms of income units, positively related to I^k iff $\frac{\partial v_i(I^i, I^{k*})}{\partial I^k} > v'_i(I^i, I^k) \left[\frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^i \partial I^k} - \frac{\partial^2 v_i(I^i, I^{k*})}{\partial I^{k2}}\right]$ (provided that $\frac{\partial v_i(I^i, I^{k*})}{\partial I^i} > \frac{\partial v_i(I^i, I^{k*})}{\partial I^k}$) - is concave in I^k, $\frac{dI^{k*}}{dI^i} > (<) 0$ and we register positive (negative) assortative mating iff $\frac{\partial^2 v'_i(I^i, I^{k*})}{\partial I^i \partial I^k} > (<) 0$.

V.4. Absence of Transferability.

If neither utility nor income are transferable, we may speculate that willingness to form a pair will still be ruled by the previous mechanism – a matching market. Yet, the equilibrium is going to press the actual transfer between individuals of each couple to zero - not to equalization of marginal benefit, but of its product by the couple ratings differential to zero, i.e.:

(115) $$D_i^{k*} = [v_i(I^i, I^{k*}) - v_i(I^i, I^{k*-1})]\,(k^* - i) = 0 \ , \quad i = 1,2,\ldots, n$$

Positive assortative mating is then always expected – the absolute value of (k* - i) being minimized:

(116) $$k^* \approx i \ , \quad i = 1,2,\ldots, n$$ [25]

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[24] $v'_i(I^i, I^k)$ can be seen as inversely related to "boldness" – see Aumann and Kurz (1977) -, the semi-elasticity of the utility with respect to the argument; here, the denominator is deducted from the compensating effect through the partner's income.

as forwarded in the beginning of the section.

Here, n would include all individuals. If there are two groups, then rankings (here exogenous and fixed…) go from 1 to n for the largest group, from the difference in elements between the two groups plus 1 to n for the smallest.

Without transferability of any sort, such equilibrium is efficient as well.

V.5. The Efficient Allocation.

Some final appraisal on mating efficiency can be forwarded. Firstly, none of the conditions qualifies *social efficiency*: this requires a social welfare function *and* also some redistribution possibilities over utility, its arguments or through match dictation… With transferable utility, a Benthamite – maximizing sum of individuals' utilities [26] – optimization criterion does not guarantee a social optimum for all possible welfare functions either: the transfer dictated by the latter, not by the Benthamite one, would also have to effectively take place *afterwards*…

Also, never do we expect to approach a pure Benthamite result: the transfers occur only between members of a couple. On the one hand, the maximization rule of the sum of utilities invoked before applies only to the transferable utility case, and on the other, refers to the sum of "indirect" utilities...

An efficient allocation with monogamous matching and transferable utilities – through mating but not other - can be linked to a problem of type (8), for monogamous utility functions, with (8) replaced by $\underset{x_i, z_i^j, x_j, z_j^l, k, l, j}{Max} \; U^i(x_i, z_i^k, y_i^k) + U^k(x_k, z_k^i, y_k^i)$ and (8a) by $U^j(x_j, z_j^l, y_j^l) + U^l(x_l, z_l^j, y_l^j) \geq \overline{U}^j + \overline{U}^l = \overline{U}'^{jl}$, j ≠ i,k,l j,l = 1, 2, …, n (and j with l only);

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[25] If $v_i(i,k)$ is i's utility when paired with k and i_j is the preference ordering assessment of individual i by individual j – in a scale of 1 (least preferred) to n –1 (most preferred) -, one can adventure a simple algorithm that under non-transferable utilities would join i and k such that $\sum_{\substack{j=1 \\ j\neq i}}^{n} i_j \approx \sum_{\substack{j=1 \\ j\neq k}}^{n} k_j$, i,k = 1,2…,n- that is, minimizing the average absolute distance between the rankings in each duo (provided all i's are considered acceptable to k and vice-versa – i.e., with unacceptability to j of partner i, the algorithm should be so constrained, and possibly allow i_j to be 0 for such cases, and j choose i_j in the scale of 1 to the number of acceptable choices to him/her out of the total individuals minus 1 – of the maximum individuals in each group, n, if one cannot match with the same group, for which i_j would start at the difference plus 1.) See Gale and Shapley (1962) and Roth (1984) on optimal assignment.

[26] Which, in any case, it is not our general implicit criterion – only for matching purposes…

or in a more complex formulation, with (8) replaced by $\underset{x_i, z_i^j, x_j, z_j^l, k, l, j, \overline{U}^k, \overline{U}^j, \overline{U}^l}{Max} U^i(x_i, z_i^k, y_i^k) + \overline{U}^k$ and added of $\overline{U}^j + \overline{U}^l \geq \overline{U'}^{jl}$, jl = 1, 2, …, n/2-1. Or yet, (8) is replaced by $\underset{x_i, z_i^j, (w_i - w_k), x_j, z_j^l, (w_j - w_l), k, l, j}{Max} U^i(x_i, z_i^k, y_i^k) + (w_i - w_k)$ and (8a) by $U^j(x_j, z_j^l, y_j^l) + (w_j - w_l) \geq \overline{U}^j$ $j \neq i$, j = 1, 2, …, n: transfers are adjustable to provide optimal partnership well-being.

Without transferability, (8) is just replaced by $\underset{x_i, z_i^j, x_j, z_j^l, k, l, j}{Max} U^i(x_i, z_i^k, y_i^k)$.

With transferable endowments to a mate – but not other nor utility -, given that the shared good must be consumed at the same level for both partners but not the other, we hypothesise that (8) becomes $\underset{x_i, z_i^j, (w_i - w_k), x_j, z_j^l, (w_j - w_l), k, l, j}{Max} U^i(x_i + w_i - w_k, z_i^k, y_i^k)$ and (8a) $U^j(x_j + w_j - w_l, z_j^l, y_j^l) \geq \overline{U}^j$ $j \neq i$, j = 1, 2, …, n: income transferability between partners allows any allocations $x_i + x_j = x_i^* + x_j^*$ where the latter are the solution found for two partners i and j – then, transfers are adjustable to provide optimal partnership well-being. (Of course, for appropriate $\overline{U}^j$'s, the problem applying to the no transferability case, $\underset{x_i, z_i^j, x_j, z_j^l, k, l, j}{Max} U^i(x_i, z_i^k, y_i^k)$, generates the same solution as that of the current paragraph.)

Transferability of both endowments and utility between individuals in a pair would imply replacing (8) by $\underset{x_i, z_i^j, x_j, z_j^l, k, l, j}{Max} U^i(x_i, z_i^k, y_i^k) + U^k(x_k, z_k^i, y_k^i)$ and (8a) by $U^j(x_j, z_j^l, y_j^l) + U^l(x_l, z_l^j, y_l^j) \geq \overline{U'}^{jl}$, $j \neq i,k,l$ j,l = 1, 2, …, n (and j with l only): no definition of individual utility levels would be supplied…

As noted in section II, the Samuelson condition is expected to hold in any of the efficient allocations.

V.6. Cobb-Douglas Preferences: An Example.

. We can apply the previous rules to our utility function [27]. Using (95), $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$ – there will be no assortative “matching” – nor positive, nor negative; but the qualification relies here on the interpretation of the cross effect only on the level of z_i^k (per couple). To conclude about couple formation, one must rely on the indirect utility function properties:

From (96), $\frac{\partial v_i}{\partial I^k} = \mu_i (1 - a_i) v_i \frac{1}{(I^i + I^k)} > 0$; $\frac{\partial^2 v_i}{\partial I^{k2}} = \mu_i (1 - a_i) [\mu_i (1 - a_i) - 1] v_i \frac{1}{(I^i + I^k)^2}$. As $\frac{\partial^2 v_i}{\partial I^i \partial I^k} = \mu_i (1 - a_i) v_i \frac{m_i a_i I^k + (m_i - 1) I^i}{(I^i + I^k)^2 I^i} > 0$ iff $I^k > \frac{1 - m_i}{m_i a_i} I^i$,

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[27] See Becker (1973), p. 826 and 841, and Lam (1988).

positive assortative "mating" is expected if the direct utility function exhibits constant or increasing returns to scale – and utility is transferable across individuals.

An individual of any type will prefer to mate an individual with higher income – a higher v_i^k. If $\mu_i \geq 1$ (IRS or CRS), there will be correspondence; then linkages will sort themselves by decreasing income levels. With DRS, if in the economy, for any i,k, $I^i > \frac{m_i\, a_i}{1-m_i} I^k$, negative assorting can occur – with strongly decreasing returns to scale and a low relative preference for the individual private good; if the reverse happens, we still observe positive assorting in couple formation.

In sum, with non-decreasing returns to scale, "doubly-good" marriages will be popular - but these not necessarily longer or with more children (not involving higher z_i^k's) than just a couple's pooled income implies – because $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$…

. If utility is not transferable but income is, the mating qualification would rely on the cross effects over the function:

$$(117) \qquad v'_i(I^i, I^k) = \frac{v_i(I^i, I^k)}{\frac{\partial v_i(I^i, I^k)}{\partial I^i} - \frac{\partial v_i(I^i, I^k)}{\partial I^k}} = \mu_i^{-1}\, a_i^{-1}\, I^i$$

$\frac{\partial v'_i}{\partial I^i} = \mu_i^{-1}\, a_i^{-1} > 0$ ($\frac{\partial v'_i}{\partial I^i} = 0$) and $\frac{\partial^2 v'_i}{\partial I^i \partial I^k} = 0$: with non-transferable utility and transferable income, no assortative mating is expected.

V.7. Final Discussion.

. Congestion of linkages – say, a fixed number of linkages – would also generate a ranking market. Say r links are supported by each individual and indirect utilities are of the form $v_i(I^i, I^{k_1}, I^{k_2}, \ldots, I^{k_r})$ and utility is transferable; it is possible that, with $I^{k_i *}$ ordered ascendingly, that the equilibrium will imply that for all individuals (and one relevant group) $v_i(I^i, I^{k_1 *}, I^{k_2 *}, \ldots, I^{k_r *}) - v_i(I^i, I^{k_1 * -1}, I^{k_2 *}, \ldots, I^{k_{r-1} *}, I^{k_r}) = p = \text{constant}$ – i solves $\underset{k_1, k_2, \ldots k_{r-1}, k_r}{Max}$ $v_i(I^i, I^{k_1}, I^{k_2}, \ldots, I^{k_r}) + p\, r\, i - p\, k_1 - p\, k_2 - \ldots - p\, k_r$ -, where $I^{k_1 * -1}$ is the income of the highest income lower to I^{k_1}.

. Illustrating special arrangements, some of social others of engineering interest:

Case A. Group Formation. $1 \leftrightarrow 2 \qquad 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0$; $z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

Links between 1 (2) and 3 are too expensive. Such case may arise either due to 3's utility function valuing less communication (z's and y's) than the others; or by either 1 and 2's (or all…) utility functions embedding strong substitutability between links with different individuals (between z_i^j and $z_i^{j'}$; z_i^j and $y_i^{j'}$; and between y_i^j and $y_i^{j'}$; y_i^j and $z_i^{j'}$), but not with the same (i.e., not between z_i^j and y_i^j; nor $z_i^{j'}$ and $y_i^{j'}$).

Case B. Transit Sequence. $1 \leftrightarrow 2 \leftrightarrow 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0$.

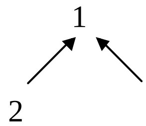
If utility is related to distance – and 1 and 3 are more distant than 2 is to either 1 or 3 – a transit sequence appears.

Case C. One-Way Transit Sequence. $1 \rightarrow 2 \rightarrow 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0$; $z_2^1 = y_2^1 = 0$; $z_3^2 = y_3^2 = 0$

This case may also suggest a multiple layer hierarchy.

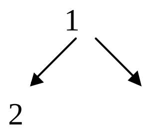
Case D. Hierarchic Sequence.



$z_1^2 = y_1^2 = 0$; $z_1^3 = y_1^3 = 0$; $z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

Attention of 1 seems more important than that of all other individuals. Notice that it may mean that equilibrium specific-transfers obtained from 1 are relatively higher in equilibrium.

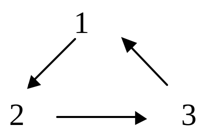
Case E. Emission Sequence.



$z_2^1 = y_2^1 = 0$; $z_3^1 = y_3^1 = 0$; $z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

1 may be an advertising point. Or, in a hierarchic chain, it has a leading role with respect to the purchase of (decisions over) z.

Case F. One-Way Circular Sequence.



$z_2^1 = y_2^1 = 0; z_2^3 = y_2^3 = 0; z_1^3 = y_1^3 = 0$

VI. Public Good vs. Shared Good.

In this section, we inspect the case where the externality is extended to more than one consumer, even if to a fixed number: if the number is not fixed, we would fall under a typical club good case. There will be an efficient allocation but the market may no longer insure its attainment...

Assume then that each z is in fact consumed by the whole economy. $z_i^j = y_j^i = y_j$. Then each z_i^j – as y_j - is replicated among the n consumers. Let us then admit it is unique or uniform. i's utility takes the form

(118) $$U^i(x_i, z_i, y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$$

We will denote it by $U^i(x_i, z_i, y_{-i})$. I obtains utility from the private good, x_i, from its own purchases of the public good, z_i, and from the purchases other consumers make, y_j, so that:

(119) $$z_j = y_j \ , \quad j = 1, 2, \dots, n$$

Of course, each z_i is then a conventional public good – we have n different public goods in the economy. A special case where a common (unique) public good is formed arises for $U^i(x_i, z_i, y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n) = U^i(x_i, z_i + y_1 + y_2 + \dots + y_{i-1} + y_{i+1} + \dots + y_n)$ with (119) holding.

Assume (118) – with (119). An efficient allocation will be obtained from the problem:

(120) $$\underset{x_i, z_i, y_i, x_j, z_j, y_j}{Max} \quad U^i(x_i, z_i, y_{-i})$$

s.t.: (121) $$U^j(x_j, z_j, y_{-j}) \geq \overline{U}^j \ , \quad j \neq i, j = 1, 2, \dots, n$$

(122) $$z_i = y_i \ , \quad i = 1, 2, \dots, n$$

(123) $$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} W_x^i$$

(124) $$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} W_z^i$$

In lagrangean form, embedding (122):

(125) $$\underset{x_i, z_i^j, x_j, z_j^l, l_j, m_x, m_z}{Max} \quad U^i(x_i, z_i, z_{-i}) + \sum_{\substack{j \neq i \\ j=1}}^{n} l_j \; [\bar{U}^j - U^j(x_j, z_j, z_{-j})] +$$
$$+ \; m_x \left(\sum_{i=1}^{n} W_x^i - \sum_{i=1}^{n} x_i\right) + m_z \left(\sum_{i=1}^{n} W_z^i - \sum_{i=1}^{n} z_i\right)$$

Interior FOC require:

(126) $$U_x^i - m_x = 0 \text{ (1 equation)}$$

(127) $$- l_j \; U_x^j - m_x = 0 \;, \quad j \neq i, j = 1, 2, \ldots, n \text{ (n-1 eqs.)}$$

(128) $$U_{z_k}^i - \sum_{\substack{j \neq i \\ j=1}}^{n} l_j \; U_{z_k}^j - m_z = 0 \;, \quad k = 1, 2, \ldots, n \text{ (n equation)}$$

(126) and (127) imply (16) that still holds

(129) $$l_j = - \frac{U_x^i}{U_x^j}, \quad j \neq i, j = 1, 2, \ldots, n$$

Replacing (129) in (128), and equating the two (and (126)) we obtain the familiar Samuelson condition(s):

(130) $$\sum_{j=1}^{n} \frac{U_{z_i}^j}{U_x^j} \left(= \frac{m_z}{U_x^i}\right) = \frac{m_z}{m_x} \;, \quad i = 1, 2, \ldots, n \qquad \text{(n equations)}$$

. Let us consider a price-cum-transfer system analogous to that of the call to decentralize that efficient solution. Each consumer i pays p_z for z_i and p_y per unit of y_j, i.e., by z_j, $j \neq i$; he pays t_i^j, $j \neq i$, to each of the other n-1 individuals for accepting his choice of z_i and receives t_j^i from each for per unit he accepts of their choice of z_j. A typical budget constraint is then:

(131) $$p_x x_i + \left(p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j\right) z_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)\, z_j = p_x W_x^i + p_z' \, W_z^i$$

The lagrangean will take the form:

(132) $$\underset{x_i, z_i, m}{Max} \quad U^i(x_i, z_i, z_{-i}) +$$

$$+\ m\ [p_x W_x^i + p_z' W_z^i - p_x x_i - (p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j)\, z_i - \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)\, z_j\ ]$$

and FOC for i = 1,2,…, n:

(133) $$U_x^i - m\ p_x = 0$$

(134) $$U_{z_i}^i - m\ (p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j) = 0$$

(135) $$U_{z_j}^i - m\ (p_y - t_j^i) = 0\ ,\quad j \neq i,\ j = 1, 2, \ldots, n$$

with the budget constraint. Then:

(136) $$\frac{U_{z_i}^i}{U_x^i} = \frac{p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j}{p_x}\ ,\quad i = 1, 2, \ldots, n \text{ (1 eq. for each i)}$$

and

(137) $$\frac{U_{z_j}^i}{U_x^i} = \frac{p_y - t_j^i}{p_x}\ ,\quad j \neq i,\ j = 1, 2, \ldots, n \text{ (n – 1 eqs. for each i)}$$

Equilibrium requires additionally:

(138) $$p_z' = p_z + (n - 1)\, p_y$$

(139) $$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} W_x^i$$

(140) $$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} W_z^i$$

A full price system can be derived: (136) and (137) and individual budget constraints generate n x (n + 1) equations that add to (138)-(140): n x (n + 1) + 3 equations with (the sum of budget constraints making) one of the last three redundant. We must generate 2 n individual consumptions, and a vector price $(\frac{p_z}{p_x}, \frac{p_y}{p_x}, \frac{p_z'}{p_x}, \frac{t_1^2}{p_x}, \ldots, \frac{t_1^n}{p_x}, \ldots, \frac{t_n^2}{p_x}, \ldots, \frac{t_n^{n-1}}{p_x})$ – with n x (n – 1) + 3 elements, i.e., n (n + 1) + 3 unknowns. Again if we fix, $\frac{p_z}{p_x}$ or $\frac{p_y}{p_x}$, a determined solution is obtained.

But if under one-to-one communication, replication of individuals of each type may insure competitive link-specific transfer price formation – we know who to charge what (even if we fix one price) *given* the actual transfer -, now, such possibility may no longer exist – and the natural spontaneity of the equilibrium breaks down…

I.e., competitive decentralization requires – apart from absence of transaction costs – a smaller number of individuals types than the total number of individuals in the economy – and responsibility for each part of, or the common purchase to be assigned to someone – some type - in particular. With some agent heterogeneity, the final cost shares will be in line with marginal utilities. But – as is well-known - perfect information and type discrimination must then be insured.

If i cannot veto – he does not directly obey (134) and, therefore, (136) - but authorities guarantee the (adequate) price $(p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j)$ for the unit of z_i *and* collect as a lump-sum $Z_i = \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)\, z_j$ from i, the efficient allocation is also insured ((136) becomes redundant) – but then not entirely through the market price system.

VII. Shared Inputs and Network Nodes Transfer Prices.

. Network nodes are passing points. Then, we can admit that there will be reception and emission of a given amount that passes through i. Then let z_i^j denote reception from j and y_i^j emission to j; we have a multiproduct cost-function of each node:

$$(141) \quad C^i(x_i, z_i^1, z_i^2, \ldots, z_i^{i-1}, z_i^{i+1}, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^{i-1}, y_i^{i+1}, \ldots, y_i^n)$$

$$i = 1, 2, \ldots, n$$

There may be, for nodes that are only passing points, the additional restriction:

$$(142) \qquad \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j$$

We assume there are no such points. When $\sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j < \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j$, i is a net emitter, having connections with a group of outside users that on aggregate send more than they receive; and vice-versa.

We want to determine the properties of an allocation which minimizes aggregate cost over the n nodes while guaranteeing a total distribution of $\sum_{i=1}^{n} W_z^i$ a fixed level of homogeneous output, $\sum_{i=1}^{n} W_x^i$. Or that maximizes $\sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j$ subject to minimum costs for j i and $\sum_{i=1}^{n} W_x^i$ restrictions.

Let z_i^j denote quantity of demand of transportation from node i to node j and y_i^j transportation from node j to node i. p_z is the price of a unit distance transportation cost, linkage formation would possibly require:

$$p_x \sum_{i=1}^{n} x_i + p_z \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + p_y \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j - \sum_{i=1}^{n} C^i(x_i, z_i^1, z_i^2, \ldots, z_i^{i-1}, z_i^{i+1}, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^{i-1}, y_i^{i+1}, \ldots, y_i^n)$$

$$i = 1, 2, \ldots, n$$

s.t. $$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j$$

Khun-Tucker conditions would generate active transportation.

. Multiproduct technologies can also benefit from the previous framework. Hypothetically, $q_i = F^i(x_i, z_i^j, y_i^j)$ could represent the production function of section or plant i – with q_i sold at price p_i - , which uses x_i exclusively and shares input z_i^j, for which i is responsible, and $y_i^j = z_j^i$, allotted to j's responsibility, with plant j. Then, joint profit maximization would generate similar conditions to the efficiency requirements encountered before; transfer prices (unit costs) among manufacturing divisions in the spirit of section III or VI would result in optimal allocations of decentralized management – of unilateral profit maximization by each of the plants.

With two plants only, $p_1 \frac{\partial F^1}{\partial z_1^2} + p_2 \frac{\partial F^2}{\partial y_2^1} = W_z$ and $p_1 \frac{\partial F^1}{\partial y_1^2} + p_2 \frac{\partial F^2}{\partial z_2^1} = W_z$. Each two terms represent – as in standard externalities - the internal net prices allocated to the divisions for the pertaining joint purchase.

Suppose that to produce the same good z, sold at price W_z, several, say, n, divisions are required – a sort of Leontief technology -, each with production requirements $z = F^i(x_i, L_i)$. Each section i implies a – standardly inferred from the minimization of $W_x x_i + W_L L_i$ s.t. $z = F^i(x_i, L_i)$ – cost function $C^i(z, W_x, W_L)$. Optimality requires a split of the marginal revenue according to $W_z = \sum_{i=1}^{n} \frac{\partial C^i}{\partial z}$ - evaluated at the z* that insures such equality. Or according to the Lagrange multipliers of the solution of

$$\underset{z, x_i, L_i, l_i}{Max} \quad W_z z - W_x \sum_{i=1}^{n} x_i - W_L \sum_{i=1}^{n} L_i - \sum_{i=1}^{n} l_i [z - F^i(x_i, L_i)]$$

i.e., insuring $W_z = \sum_{i=1}^{n} l_i$.

VIII. Summary and Conclusions.

General equilibrium of a pure exchange economy was proven to be able to generate efficient allocations in economies where share goods are present; under special arrangements, uniqueness is also guaranteed. Efficient allocations require the Samuelson (public good) rule with respect to the ratio of utilities – whether or not sharing takes the form of an externality. Optimal pricing involves common reception and emission prices – adding up to a uniquely determined quantity - along with link-specific transfers from consumers who value a specific "call" more than its charged price. End-specific roles – for adequate general price allocation - must also be pre-ordained – achieved with a (much) milder version of (than) the Arrow's dictator.

With multiple sharing by more than two individuals – because either the good is shared by more than one individual or because there are similar links between different pairs -, some indeterminacy may arise with respect to the distribution of the general aggregate unit cost. Of course, heterogeneity requires more complex identification.

CES utility functions generate interesting environments. With transferable utility, positive assortative mating is likely to arise with linear homogeneity or higher – and negative with strong DRS and/or low relative preferences for joint-consumption. Cobb-Douglas technologies, generating linear Engel curves, suggest no quantity assorting of household good demand.

Utility functions implying monogamy allowed us to study mating arrangements more profoundly. Definition of the marginal benefit of a match - and price of ranking points - was forwarded, and mechanics of an adequate (dowry) price system for an endogenous matching market explained; with transferable utility, the requirement of equalization of marginal benefit of a match across individuals provides the direction of assortative mating. If utility is not transferable but income – qualifying assorting – is, then it is the income value of the marginal benefit that is expected to equalize in the economy; this suggests the importance of the function given by the ratio of utility over the difference between the marginal utility relative to own income minus the marginal with respect to the partner's in determining the outcome of decentralized assorting.

Fruitful extensions are expected in family economic modelling and estimation, both in the static as in the intertemporal domain, with household decisions also covering labor market participation and supply, allowing for joint family investment – and taxation -, encompassing both single and multi-element unit as special cases, possibly assuming single and married, male and female (with or without children…), parameter preference differentiation.

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Appendix.

Tables A1 to A5 contain the results of assignment simulations with two six-agent economies. Paired arrangements allow for 15 different scenarios (A to O). Scenario A represents perfect positive assortative assignment or mating; scenario O, negative assortative assignment.

Economy I, where individuals, I1 to I6, have income 3, 6, 9, 12, 15 and 18 and gender (or other) is irrelevant to allow for mating.

In Economy II, there are three individual types, with income 3, 6, and 9, i.e., I1 to I6 have income 3, 3, 6, 6, 9, 9. One can either interpret the context as one in which mating can only be accomplished between individuals of different gender – between an odd (say, male, I1, I3 and I5) and an even (female, I2, I4, I6) characters which have the same income distribution; then only scenarios (in Tables below) A, C, G, H, M and O are relevant, with H and M being quantitatively indistinguishable. Or to an economy where pairs can be formed between any two individuals but there is some type replication: we just stage a less sparse income distribution; then quantitatively distinguishable scenarios are still only A, C, G, H, O. Nevertheless, marginal benefits and losses allowed for combinations – pairs of individuals - of the same type (but of the existing six characters in the economy) [28].

The indirect utility form used was:

(A1) $$v_i(I^i, I^k) = [I^{i\,a_i} (I^i + I^k)^{(1-a_i)}]^{m_i} = I^{i\,m_i a_i} (I^i + I^k)^{m_i(1-a_i)}$$

with a_i fixed at 0.3. Several values of μ_i were considered: 0.25, 0.75, 1, 1.25 and 2.25. $\frac{\partial^2 v_i}{\partial I^i \partial I^k} = \mu_i (1 - a_i) v_i \frac{m_i\, a_i\, I^k + (m_i - 1) I^i}{(I^i + I^k)^2 I^i} > 0$ for μ_i = 1, 1.25 and 2.25, and for some income levels in the economies when $\mu_i = 0.75$ (when $I^k > 1.111\, I^i$). With $\mu_i = 0.25$, $\frac{\partial^2 v_i}{\partial I^i \partial I^k} < 0$ always in the economies because for their income ranges $I^k < 10\, I^i$. Notice also that for the last case ($\mu_i = 2.25$), as $\mu_i (1 - a_i) - 1 = 0.575 > 0$, $\frac{\partial^2 v_i}{\partial I^{k2}} > 0$ (In any case, with

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[28] One could have – possibly more accurately - calculated the marginal benefit as the difference of utility obtained from joining k relative to that obtained by linking to k-2 divided by 2 - with the marginal benefit from linking to V1 as the difference obtained from joining him relative to staying single divided by 1.5. Still, adjustments would also be due for individuals that mate with contiguous classes…

$\mu_i > 2$, marginal utilities of the direct utility function fail to be decreasing, even if not quasi-concavity).

Apart from the utilities derived by each of the six individuals (columns V1 to V6), we report the average marginal loss (MeanL), the average deviation from the mean of the six individual values, marginal losses, (AVDEVL), the average marginal benefit, its average deviation (MeanB and AVDEVB) and the average marginal loss plus benefit divided by 2, along with the corresponding average deviation (MEAN and AVDEVM). The last calculations were repeated using a different procedure (in the former, the marginal benefit of individuals mating with individual I1 is calculated as the difference relative to a single status – i.e., $I^{k-1} = 0$) to evaluate the individual marginal losses (Mean B1 and AVDEVB1), with corresponding average deviations (MEAN1 and AVDEVM1).

In fact, for $\mu_i < 2$, the equalization between marginal benefits across six individuals in the economy – the minimum average deviation, AVDEV – seems to occur for a scenario close to the one generating the maximum sum of utilities. Differences from such coincidence can be attributed to the fact that in the reported calculations of marginal benefits and losses, and to the small number of individuals in the economies – and unlike the ranking-pricing scheme would suggest – we did not allow mating with one-self('s income). For $\mu_i = 2.25$, equalization between marginal benefits suggests the allocation generating the *minimum* sum of utilities.

Nevertheless, it is always true that the maximum sum of utilities is achieved with scenario A – with positive assortative mating – under $\frac{\partial^2 v_i}{\partial I^i \partial I^k} > 0$ and with O for $\frac{\partial^2 v_i}{\partial I^i \partial I^k} < 0$.

Table A1

Assign	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanL	AVDEVL	MeanB	AVDEVB	MEAN	AVDEVM	MeanB1	AVDEVB1	MEAN1	AVDEVM1
μ_i =	0.25																			
A	(I1, I2)	(I3, I4)	(I5, I6)	1.595	1.680	2.009	2.053	2.259	2.290	11.886	0.063	0.047	0.113	0.056	0.088	0.047	0.048	0.035	0.055	0.022
B	(I1, I2)	(I3, I5)	(I4, I6)	1.595	1.680	2.056	2.185	2.137	2.252	11.906	0.061	0.039	0.095	0.068	0.078	0.054	0.030	0.020	0.045	0.012
C	(I1, I2)	(I3, I6)	(I4, I5)	1.595	1.680	2.099	2.145	2.181	2.211	11.912	0.066	0.039	0.103	0.063	0.085	0.049	0.037	0.025	0.052	0.017
D	(I1, I3)	(I2, I4)	(I5, I6)	1.677	1.897	1.821	1.998	2.259	2.290	11.943	0.041	0.027	0.068	0.015	0.055	0.017	0.053	0.023	0.047	0.016
E	(I1, I3)	(I2, I5)	(I4, I6)	1.677	1.949	1.821	2.185	2.087	2.252	11.972	0.045	0.018	0.060	0.017	0.053	0.017	0.045	0.018	0.045	0.013
F	(I1, I3)	(I2, I6)	(I4, I5)	1.677	1.995	1.821	2.145	2.181	2.166	11.986	0.050	0.022	0.067	0.021	0.059	0.016	0.053	0.023	0.051	0.016
G	(I1, I4)	(I2, I3)	(I5, I6)	1.744	1.837	1.894	1.935	2.259	2.290	11.960	0.049	0.033	0.081	0.025	0.065	0.025	0.069	0.034	0.059	0.029
H	(I1, I4)	(I2, I5)	(I3, I6)	1.744	1.949	2.099	1.935	2.087	2.211	12.026	0.043	0.015	0.056	0.010	0.049	0.012	0.044	0.015	0.043	0.011
I	(I1, I4)	(I2, I6)	(I3, I5)	1.744	1.995	2.056	1.935	2.137	2.166	12.034	0.042	0.014	0.075	0.029	0.058	0.019	0.062	0.035	0.052	0.020
J	(I1, I5)	(I2, I3)	(I4, I6)	1.801	1.837	1.894	2.185	2.032	2.252	12.001	0.053	0.024	0.072	0.029	0.062	0.026	0.061	0.036	0.057	0.029
K	(I1, I5)	(I2, I4)	(I3, I6)	1.801	1.897	2.099	1.998	2.032	2.211	12.038	0.042	0.014	0.055	0.007	0.049	0.011	0.044	0.015	0.043	0.012
L	(I1, I5)	(I2, I6)	(I3, I4)	1.801	1.995	2.009	2.053	2.032	2.166	12.055	0.048	0.017	0.064	0.017	0.056	0.015	0.054	0.022	0.051	0.018
M	(I1, I6)	(I2, I3)	(I4, I5)	1.850	1.837	1.894	2.145	2.181	2.116	12.024	0.057	0.027	0.079	0.031	0.068	0.022	0.069	0.038	0.063	0.026
N	(I1, I6)	(I2, I4)	(I3, I5)	1.850	1.897	2.056	1.998	2.137	2.116	12.054	0.041	0.014	0.054	0.005	0.047	0.008	0.045	0.015	0.043	0.012
O	(I1, I6)	(I2, I5)	(I3, I4)	1.850	1.949	2.009	2.053	2.087	2.116	12.064	0.048	0.016	0.064	0.017	0.056	0.014	0.054	0.021	0.051	0.018
A	(I1, I2)	(I3, I4)	(I5, I6)	1.486	1.486	1.767	1.767	1.955	1.955	10.416	0.060	0.040	0.106	0.043	0.083	0.038	0.049	0.033	0.055	0.016
B	(I1, I2)	(I3, I5)	(I4, I6)	1.486	1.486	1.837	1.837	1.894	1.894	10.434	0.047	0.047	0.080	0.060	0.064	0.051	0.024	0.032	0.035	0.013
C	(I1, I2)	(I3, I6)	(I4, I5)	1.486	1.486	1.837	1.837	1.894	1.894	10.434	0.047	0.047	0.080	0.060	0.064	0.051	0.024	0.032	0.035	0.013
D	(I1, I3)	(I2, I4)	(I5, I6)	1.595	1.595	1.680	1.680	1.955	1.955	10.461	0.028	0.038	0.058	0.039	0.043	0.009	0.039	0.039	0.033	0.013
E	(I1, I3)	(I2, I5)	(I4, I6)	1.595	1.677	1.680	1.837	1.821	1.894	10.505	0.022	0.030	0.051	0.051	0.037	0.014	0.032	0.043	0.027	0.018
F	(I1, I3)	(I2, I6)	(I4, I5)	1.595	1.677	1.680	1.837	1.894	1.821	10.505	0.022	0.030	0.049	0.049	0.036	0.014	0.030	0.040	0.026	0.017
G	(I1, I4)	(I2, I3)	(I5, I6)	1.595	1.595	1.680	1.680	1.955	1.955	10.461	0.028	0.038	0.058	0.039	0.043	0.009	0.039	0.039	0.033	0.013
H	(I1, I4)	(I2, I5)	(I3, I6)	1.595	1.677	1.837	1.680	1.821	1.894	10.505	0.026	0.034	0.045	0.045	0.035	0.012	0.026	0.034	0.026	0.017
I	(I1, I4)	(I2, I6)	(I3, I5)	1.595	1.677	1.837	1.680	1.894	1.821	10.505	0.026	0.034	0.057	0.057	0.042	0.018	0.038	0.051	0.032	0.021
J	(I1, I5)	(I2, I3)	(I4, I6)	1.677	1.595	1.680	1.837	1.821	1.894	10.505	0.025	0.033	0.047	0.047	0.036	0.014	0.032	0.043	0.028	0.019
K	(I1, I5)	(I2, I4)	(I3, I6)	1.677	1.595	1.837	1.680	1.821	1.894	10.505	0.028	0.038	0.041	0.041	0.034	0.011	0.026	0.034	0.027	0.018
L	(I1, I5)	(I2, I6)	(I3, I4)	1.677	1.677	1.767	1.767	1.821	1.821	10.532	0.036	0.036	0.058	0.038	0.047	0.021	0.043	0.043	0.039	0.027
M	(I1, I6)	(I2, I3)	(I4, I5)	1.677	1.595	1.680	1.837	1.894	1.821	10.505	0.025	0.033	0.045	0.045	0.035	0.013	0.030	0.040	0.027	0.018
N	(I1, I6)	(I2, I4)	(I3, I5)	1.677	1.595	1.837	1.680	1.894	1.821	10.505	0.028	0.038	0.039	0.039	0.033	0.011	0.024	0.032	0.026	0.017
O	(I1, I6)	(I2, I5)	(I3, I4)	1.677	1.677	1.767	1.767	1.821	1.821	10.532	0.036	0.036	0.058	0.038	0.047	0.021	0.043	0.043	0.039	0.027

Table A2

Assign	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanL	AVDEVL	MeanB	AVDEVB	MEAN	AVDEVM	MeanB1	AVDEVB1	MEAN1	AVDEVM1
μ_i =	0.75																			
A	(I1, I2)	(I3, I4)	(I5, I6)	4.058	4.743	8.107	8.649	11.530	12.013	49.101	0.655	0.459	1.069	0.346	0.862	0.285	0.621	0.425	0.638	0.238
B	(I1, I2)	(I3, I5)	(I4, I6)	4.058	4.743	8.696	10.430	9.755	11.427	49.110	0.647	0.275	0.852	0.328	0.750	0.302	0.404	0.269	0.526	0.147
C	(I1, I2)	(I3, I6)	(I4, I5)	4.058	4.743	9.251	9.869	10.377	10.812	49.110	0.742	0.376	0.956	0.362	0.849	0.263	0.508	0.338	0.625	0.214
D	(I1, I3)	(I2, I4)	(I5, I6)	4.720	6.825	6.043	7.977	11.530	12.013	49.108	0.431	0.287	0.767	0.156	0.599	0.109	0.625	0.222	0.528	0.129
E	(I1, I3)	(I2, I5)	(I4, I6)	4.720	7.400	6.043	10.430	9.095	11.427	49.115	0.520	0.173	0.661	0.077	0.591	0.115	0.520	0.173	0.520	0.128
F	(I1, I3)	(I2, I6)	(I4, I5)	4.720	7.938	6.043	9.869	10.377	10.164	49.111	0.617	0.234	0.763	0.181	0.690	0.169	0.622	0.235	0.619	0.198
G	(I1, I4)	(I2, I3)	(I5, I6)	5.306	6.202	6.794	7.249	11.530	12.013	49.095	0.533	0.355	0.890	0.278	0.711	0.235	0.756	0.367	0.644	0.261
H	(I1, I4)	(I2, I5)	(I3, I6)	5.306	7.400	9.251	7.249	9.095	10.812	49.113	0.512	0.171	0.646	0.073	0.579	0.114	0.512	0.171	0.512	0.128
I	(I1, I4)	(I2, I6)	(I3, I5)	5.306	7.938	8.696	7.249	9.755	10.164	49.107	0.514	0.171	0.863	0.346	0.688	0.205	0.729	0.391	0.622	0.224
J	(I1, I5)	(I2, I3	(I4, I6)	5.839	6.202	6.794	10.430	8.388	11.427	49.081	0.620	0.261	0.781	0.226	0.701	0.236	0.653	0.301	0.637	0.266
K	(I1, I5)	(I2, I4)	(I3, I6)	5.839	6.825	9.251	7.977	8.388	10.812	49.091	0.510	0.176	0.642	0.072	0.576	0.121	0.515	0.172	0.512	0.131
L	(I1, I5)	(I2, I6)	(I3, I4)	5.839	7.938	8.107	8.649	8.388	10.164	49.085	0.609	0.249	0.752	0.192	0.680	0.197	0.624	0.267	0.617	0.238
M	(I1, I6)	(I2, I3)	(I4, I5)	6.332	6.202	6.794	9.869	10.377	9.475	49.050	0.723	0.340	0.880	0.306	0.802	0.215	0.757	0.388	0.740	0.297
N	(I1, I6)	(I2, I4)	(I3, I5)	6.332	6.825	8.696	7.977	9.755	9.475	49.060	0.519	0.173	0.638	0.070	0.578	0.113	0.515	0.180	0.517	0.148
O	(I1, I6)	(I2, I5)	(I3, I4)	6.332	7.400	8.107	8.649	9.095	9.475	49.059	0.616	0.240	0.749	0.188	0.683	0.188	0.627	0.271	0.621	0.239
A	(I1, I2)	(I3, I4)	(I5, I6)	3.280	3.280	5.516	5.516	7.477	7.477	32.547	0.488	0.325	0.819	0.121	0.653	0.208	0.485	0.324	0.487	0.162
B	(I1, I2)	(I3, I5)	(I4, I6)	3.280	3.280	6.202	6.202	6.794	6.794	32.553	0.373	0.373	0.573	0.382	0.473	0.278	0.239	0.319	0.306	0.102
C	(I1, I2)	(I3, I6)	(I4, I5)	3.280	3.280	6.202	6.202	6.794	6.794	32.553	0.373	0.373	0.573	0.382	0.473	0.278	0.239	0.319	0.306	0.102

D	(I1, I3)	(I2, I4)	(I5, I6)	4.058	4.058	4.743	4.743	7.477	7.477	32.556	0.239	0.319	0.509	0.339	0.374	0.036	0.357	0.357	0.298	0.099
E	(I1, I3)	(I2, I5)	(I4, I6)	4.058	4.720	4.743	6.202	6.043	6.794	32.561	0.239	0.319	0.392	0.392	0.315	0.105	0.240	0.320	0.239	0.160
F	(I1, I3)	(I2, I6)	(I4, I5)	4.058	4.720	4.743	6.202	6.794	6.043	32.561	0.239	0.319	0.396	0.396	0.317	0.106	0.244	0.325	0.241	0.161
G	(I1, I4)	(I2, I3)	(I5, I6)	4.058	4.058	4.743	4.743	7.477	7.477	32.556	0.239	0.319	0.509	0.339	0.374	0.036	0.357	0.357	0.298	0.099
H	(I1, I4)	(I2, I5)	(I3, I6)	4.058	4.720	6.202	4.743	6.043	6.794	32.561	0.235	0.314	0.387	0.387	0.311	0.104	0.235	0.314	0.235	0.157
I	(I1, I4)	(I2, I6)	(I3, I5)	4.058	4.720	6.202	4.743	6.794	6.043	32.561	0.235	0.314	0.520	0.520	0.378	0.143	0.368	0.491	0.302	0.201
J	(I1, I5)	(I2, I3)	(I4, I6)	4.720	4.058	4.743	6.202	6.043	6.794	32.561	0.243	0.324	0.381	0.381	0.312	0.104	0.240	0.320	0.241	0.161
K	(I1, I5)	(I2, I4)	(I3, I6)	4.720	4.058	6.202	4.743	6.043	6.794	32.561	0.239	0.319	0.377	0.377	0.308	0.103	0.235	0.314	0.237	0.158
L	(I1, I5)	(I2, I6)	(I3, I4)	4.720	4.720	5.516	5.516	6.043	6.043	32.559	0.354	0.354	0.509	0.339	0.431	0.199	0.368	0.368	0.361	0.251
M	(I1, I6)	(I2, I3)	(I4, I5)	4.720	4.058	4.743	6.202	6.794	6.043	32.561	0.243	0.324	0.385	0.385	0.314	0.105	0.244	0.325	0.243	0.162
N	(I1, I6)	(I2, I4)	(I3, I5)	4.720	4.058	6.202	4.743	6.794	6.043	32.561	0.239	0.319	0.381	0.381	0.310	0.103	0.239	0.319	0.239	0.160
O	(I1, I6)	(I2, I5)	(I3, I4)	4.720	4.720	5.516	5.516	6.043	6.043	32.559	0.354	0.354	0.509	0.339	0.431	0.199	0.368	0.368	0.361	0.251

Table A3

Assign	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanL	AVDEVL	MeanB	AVDEVB	MEAN	AVDEVM	MeanB1	AVDEVB1	MEAN1	AVDEVM1
μ_i =	1																			
A	(I1, I2)	(I3, I4)	(I5, I6)	6.473	7.969	16.286	17.754	26.049	27.513	102.045	1.647	1.182	2.644	0.791	2.145	0.565	1.737	1.158	1.692	0.591
B	(I1, I2)	(I3, I5)	(I4, I6)	6.473	7.969	17.882	22.790	20.844	25.738	101.695	1.662	0.669	2.058	0.472	1.860	0.479	1.151	0.767	1.407	0.427
C	(I1, I2)	(I3, I6)	(I4, I5)	6.473	7.969	19.419	21.169	22.635	23.908	101.573	1.956	0.976	2.346	0.732	2.151	0.557	1.439	0.959	1.697	0.635
D	(I1, I3)	(I2, I4)	(I5, I6)	7.917	12.946	11.008	15.938	26.049	27.513	101.371	1.082	0.721	2.017	0.466	1.549	0.285	1.682	0.684	1.382	0.315
E	(I1, I3)	(I2, I5)	(I4, I6)	7.917	14.421	11.008	22.790	18.984	25.738	100.857	1.375	0.470	1.720	0.207	1.547	0.332	1.385	0.462	1.380	0.340
F	(I1, I3)	(I2, I6)	(I4, I5)	7.917	15.834	11.008	21.169	22.635	22.016	100.579	1.688	0.701	2.006	0.470	1.847	0.532	1.671	0.719	1.679	0.670
G	(I1, I4)	(I2, I3)	(I5, I6)	9.256	11.395	12.869	14.029	26.049	27.513	101.110	1.356	0.936	2.307	0.742	1.832	0.534	1.969	0.967	1.663	0.615
H	(I1, I4)	(I2, I5)	(I3, I6	9.256	14.421	19.419	14.029	18.984	23.908	100.016	1.379	0.499	1.702	0.252	1.540	0.370	1.364	0.463	1.371	0.364
I	(I1, I4)	(I2, I6)	(I3, I5)	9.256	15.834	17.882	14.029	20.844	22.016	99.860	1.398	0.512	2.270	0.915	1.834	0.557	1.932	1.038	1.665	0.678
J	(I1, I5)	(I2, I3)	(I4, I6)	10.515	11.395	12.869	22.790	17.042	25.738	100.348	1.647	0.731	2.006	0.485	1.827	0.546	1.666	0.706	1.657	0.653
K	(I1, I5)	(I2, I4)	(I3, I6)	10.51	12.94	19.41	15.93	17.04	23.90	99.768	1.377	0.519	1.699	0.249	1.538	0.368	1.358	0.486	1.368	0.378

				5	6	9	8	2	8											
L	(I1, I5)	(I2, I6)	(I3, I4)	10.515	15.834	16.286	17.754	17.042	22.016	99.448	1.674	0.742	1.986	0.496	1.830	0.575	1.645	0.754	1.659	0.691
M	(I1, I6)	(I2, I3)	(I4, I5)	11.714	11.395	12.869	21.169	22.635	20.051	99.833	1.995	0.947	2.290	0.753	2.142	0.559	1.949	0.981	1.972	0.787
N	(I1, I6)	(I2, I4)	(I3, I5)	11.714	12.946	17.882	15.938	20.844	20.051	99.375	1.431	0.477	1.694	0.246	1.562	0.338	1.352	0.502	1.392	0.401
O	(I1, I6)	(I2, I5)	(I3, I4)	11.714	14.421	16.286	17.754	18.984	20.051	99.210	1.708	0.705	1.983	0.501	1.846	0.549	1.641	0.750	1.675	0.666
A	(I1, I2)	(I3, I4)	(I5, I6)	4.874	4.874	9.747	9.747	14.621	14.621	58.482	1.082	0.722	1.801	0.048	1.442	0.377	1.177	0.784	1.129	0.389
B	(I1, I2)	(I3, I5)	(I4, I6)	4.874	4.874	11.395	11.395	12.869	12.869	58.274	0.825	0.825	1.209	0.806	1.017	0.480	0.585	0.780	0.705	0.235
C	(I1, I2)	(I3, I6)	(I4, I5)	4.874	4.874	11.395	11.395	12.869	12.869	58.274	0.825	0.825	1.209	0.806	1.017	0.480	0.585	0.780	0.705	0.235
D	(I1, I3)	(I2, I4)	(I5, I6)	6.473	6.473	7.969	7.969	14.621	14.621	58.126	0.537	0.716	1.179	0.786	0.858	0.065	0.851	0.851	0.694	0.231
E	(I1, I3)	(I2, I5)	(I4, I6)	6.473	7.917	7.969	11.395	11.008	12.869	57.631	0.602	0.803	0.835	0.835	0.719	0.240	0.507	0.676	0.555	0.370
F	(I1, I3)	(I2, I6)	(I4, I5)	6.473	7.917	7.969	11.395	12.869	11.008	57.631	0.602	0.803	0.869	0.869	0.736	0.245	0.541	0.722	0.572	0.381
G	(I1, I4)	(I2, I3)	(I5, I6)	6.473	6.473	7.969	7.969	14.621	14.621	58.126	0.537	0.716	1.179	0.786	0.858	0.065	0.851	0.851	0.694	0.231
H	(I1, I4)	(I2, I5)	(I3, I6)	6.473	7.917	11.395	7.969	11.008	12.869	57.631	0.551	0.734	0.879	0.879	0.715	0.238	0.551	0.734	0.551	0.367
I	(I1, I4)	(I2, I6)	(I3, I5)	6.473	7.917	11.395	7.969	12.869	11.008	57.631	0.551	0.734	1.209	1.209	0.880	0.346	0.881	1.175	0.716	0.477
J	(I1, I5)	(I2, I3)	(I4, I6)	7.917	6.473	7.969	11.395	11.008	12.869	57.631	0.588	0.784	0.842	0.842	0.715	0.238	0.507	0.676	0.548	0.365
K	(I1, I5)	(I2, I4)	(I3, I6)	7.917	6.473	11.395	7.969	11.008	12.869	57.631	0.537	0.716	0.885	0.885	0.711	0.237	0.551	0.734	0.544	0.363
L	(I1, I5)	(I2, I6)	(I3, I4)	7.917	7.917	9.747	9.747	11.008	11.008	57.344	0.859	0.859	1.168	0.779	1.014	0.466	0.833	0.833	0.846	0.606
M	(I1, I6)	(I2, I3)	(I4, I5)	7.917	6.473	7.969	11.395	12.869	11.008	57.631	0.588	0.784	0.876	0.876	0.732	0.244	0.541	0.722	0.565	0.376
N	(I1, I6)	(I2, I4)	(I3, I5)	7.917	6.473	11.395	7.969	12.869	11.008	57.631	0.537	0.716	0.919	0.919	0.728	0.245	0.585	0.780	0.561	0.374
O	(I1, I6)	(I2, I5)	(I3, I4)	7.917	7.917	9.747	9.747	11.008	11.008	57.344	0.859	0.859	1.168	0.779	1.014	0.466	0.833	0.833	0.846	0.606

Table A4																				
Assign	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanL	AVDEVL	MeanB	AVDEVB	MEAN	AVDEVM	MeanB1	AVDEVB1	MEAN1	AVDEVM1
μ_i =	1.25																			
A	(I1, I2)	(I3, I4)	(I5, I6)	10.325	13.390	32.718	36.445	58.848	63.013	[illegible]14.73	3.920	2.935	6.306	1.760	5.113	1.138	4.577	3.051	4.248	1.658
B	(I1, I2)	(I3, I5)	(I4, I6)	10.325	13.390	36.773	49.794	44.537	57.971	212.788	4.062	1.746	4.805	0.659	4.434	0.884	3.076	2.051	3.569	1.156
C	(I1, I2)	(I3, I6)	(I4, I5)	10.325	13.390	40.765	45.408	49.372	52.865	212.124	4.911	2.465	5.557	1.409	5.234	1.300	3.828	2.552	4.370	1.954
D	(I1, I3)	(I2, I4)	(I5, I6)	13.280	24.557	20.050	31.846	58.848	63.013	211.594	2.555	1.703	5.042	1.478	3.799	0.794	4.298	2.106	3.427	0.896
E	(I1, I3)	(I2, I5)	(I4, I6)	13.280	28.103	20.050	49.794	39.626	57.971	208.823	3.437	1.337	4.242	0.661	3.840	0.968	3.499	1.347	3.468	1.047
F	(I1, I3)	(I2, I6)	(I4, I5)	13.280	31.586	20.050	45.408	49.372	47.688	207.385	4.371	1.975	4.993	1.412	4.682	1.667	4.249	2.103	4.310	2.039
G	(I1, I4)	(I2, I3)	(I5, I6)	16.144	20.936	24.374	27.150	58.848	63.013	210.464	3.242	2.311	5.678	1.889	4.460	1.190	4.875	2.480	4.059	1.493
H	(I1, I4)	(I2, I5)	(I3, I6)	16.144	28.103	40.765	27.150	39.626	52.865	204.652	3.498	1.406	4.232	0.765	3.865	1.086	3.430	1.332	3.464	1.073
I	(I1, I4)	(I2, I6)	(I3, I5)	16.144	31.586	36.773	27.150	44.537	47.688	203.877	3.582	1.457	5.622	2.259	4.602	1.545	4.819	2.704	4.201	1.895
J	(I1, I5)	(I2, I3)	(I4, I6)	18.936	20.936	24.374	49.794	34.626	57.971	206.635	4.124	2.005	4.876	1.043	4.500	1.348	4.025	1.753	4.074	1.589
K	(I1, I5)	(I2, I4)	(I3, I6)	18.936	24.557	40.765	31.846	34.626	52.865	203.594	3.497	1.420	4.231	0.762	3.864	1.083	3.380	1.322	3.439	1.019
L	(I1, I5)	(I2, I6)	(I3, I4)	18.936	31.586	32.718	36.445	34.626	47.688	201.998	4.322	2.059	4.930	1.306	4.626	1.582	4.079	1.988	4.201	1.866
M	(I1, I6)	(I2, I3)	(I4, I5)	21.670	20.936	24.374	45.408	49.372	42.430	204.189	5.181	2.512	5.626	1.753	5.403	1.378	4.734	2.381	4.957	1.973
N	(I1, I6)	(I2, I4)	(I3, I5)	21.670	24.557	36.773	31.846	44.537	42.430	201.812	3.705	1.288	4.229	0.758	3.967	0.994	3.336	1.313	3.521	1.015
O	(I1, I6)	(I2, I5)	(I3, I4)	21.670	28.103	32.718	36.445	39.626	42.430	200.991	4.445	1.933	4.929	1.303	4.687	1.498	4.037	1.944	4.241	1.738
A	(I1, I2)	(I3, I4)	(I5, I6)	7.241	7.241	17.222	17.222	28.589	28.589	[illegible]06.10	2.266	1.510	3.780	0.325	3.023	0.610	2.683	1.789	2.474	0.866
B	(I1, I2)	(I3, I5)	(I4, I6)	7.241	7.241	20.936	20.936	24.374	24.374	105.101	1.731	1.731	2.437	1.625	2.084	0.770	1.339	1.786	1.535	0.512

C	(I1, I2)	(I3, I6)	(I4, I5)	7.241	7.241	20.936	20.936	24.374	24.374	105.101	1.731	1.731	2.437	1.625	2.084	0.770	1.339	1.786	1.535	0.512
D	(I1, I3)	(I2, I4)	(I5, I6)	10.325	10.325	13.390	13.390	28.589	28.589	104.608	1.131	1.508	2.586	1.724	1.859	0.232	1.919	1.919	1.525	0.524
E	(I1, I3)	(I2, I5)	(I4, I6)	10.325	13.280	13.390	20.936	20.050	24.374	102.354	1.423	1.898	1.673	1.673	1.548	0.542	1.007	1.342	1.215	0.810
F	(I1, I3)	(I2, I6)	(I4, I5)	10.325	13.280	13.390	20.936	24.374	20.050	102.354	1.423	1.898	1.799	1.799	1.611	0.560	1.133	1.510	1.278	0.852
G	(I1, I4)	(I2, I3)	(I5, I6)	10.325	10.325	13.390	13.390	28.589	28.589	104.608	1.131	1.508	2.586	1.724	1.859	0.232	1.919	1.919	1.525	0.524
H	(I1, I4)	(I2, I5)	(I3, I6)	10.325	13.280	20.936	13.390	20.050	24.374	102.354	1.213	1.617	1.880	1.880	1.546	0.561	1.213	1.617	1.213	0.809
I	(I1, I4)	(I2, I6)	(I3, I5)	10.325	13.280	20.936	13.390	24.374	20.050	102.354	1.213	1.617	2.645	2.645	1.929	0.793	1.978	2.638	1.596	1.103
J	(I1, I5)	(I2, I3)	(I4, I6)	13.280	10.325	13.390	20.936	20.050	24.374	102.354	1.341	1.789	1.750	1.750	1.546	0.539	1.007	1.342	1.174	0.783
K	(I1, I5)	(I2, I4)	(I3, I6)	13.280	10.325	20.936	13.390	20.050	24.374	102.354	1.131	1.508	1.957	1.957	1.544	0.559	1.213	1.617	1.172	0.781
L	(I1, I5)	(I2, I6)	(I3, I4)	13.280	13.280	17.222	17.222	20.050	20.050	101.106	1.958	1.958	2.514	1.676	2.236	1.025	1.770	1.770	1.864	1.372
M	(I1, I6)	(I2, I3)	(I4, I5)	13.280	10.325	13.390	20.936	24.374	20.050	102.354	1.341	1.789	1.877	1.877	1.609	0.559	1.133	1.510	1.237	0.825
N	(I1, I6)	(I2, I4)	(I3, I5)	13.280	10.325	20.936	13.390	24.374	20.050	102.354	1.131	1.508	2.083	2.083	1.607	0.579	1.339	1.786	1.235	0.824
O	(I1, I6)	(I2, I5)	(I3, I4)	13.280	13.280	17.222	17.222	20.050	20.050	101.106	1.958	1.958	2.514	1.676	2.236	1.025	1.770	1.770	1.864	1.372

Table A5

Assign	Pair 1 Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanL	AVDEVL	MeanB	AVDEVB	MEAN	AVDEVM	MeanB1	AVDEVB1	MEAN1	AVDEVM1
μ_i =	2.25																		
A	(I1, I2(I3, I4	(I5, I6	66.833	106.705	532.855	647.057	1532.926	1733.678	4620.053	101.516	88.747	186.846	105.236	144.181	57.986	169.288	122.795	135.402	66.766
B	(I1, I2(I3, I5	(I4, I6	66.833	106.705	657.577	1134.784	928.313	1492.021	4386.232	122.512	68.905	134.633	57.942	128.572	53.735	117.074	78.050	119.793	62.514
C	(I1, I2(I3, I6	(I4, I5	66.833	106.705	791.611	961.270	1117.531	1263.883	4307.832	164.535	107.814	159.469	79.674	162.002	93.744	141.910	97.232	153.223	102.523
D	(I1, I3(I2, I4	(I5, I6	105.140	317.912	220.711	507.577	1532.926	1733.678	4417.942	60.678	45.919	163.637	109.926	112.157	41.714	150.234	118.861	105.456	48.415

E	(I1, I3	(I2, I5	(I4, I6	105.140	405.273	220.711	1134.784	752.241	1492.021	4110.170	108.302	67.041	128.315	59.621	118.309	55.798	114.913	73.023	111.607	60.265
F	(I1, I3	(I2, I6	(I4, I5	105.140	500.132	220.711	961.270	1117.531	1049.885	3954.669	156.689	110.947	152.690	81.497	154.690	96.222	139.288	94.899	147.988	102.923
G	(I1, I4	(I2, I3	(I5, I6	149.417	238.559	313.659	380.883	1532.926	1733.678	4349.121	79.157	62.590	173.167	103.573	126.162	37.148	154.355	116.114	116.756	44.776
H	(I1, I4	(I2, I5	(I3, I6	149.417	405.273	791.611	380.883	752.241	1263.883	3743.307	112.577	64.391	125.782	44.280	119.180	50.808	106.971	63.091	109.774	56.878
I	(I1, I4	(I2, I6	(I3, I5	149.417	500.132	657.577	380.883	928.313	1049.885	3666.206	118.941	62.727	161.854	77.852	140.398	68.998	143.043	96.664	130.992	78.403
J	(I1, I5	(I2, I3	(I4, I6	199.118	238.559	313.659	1134.784	590.087	1492.021	3968.228	126.180	81.490	137.241	45.740	131.710	50.187	112.693	65.143	119.436	50.699
K	(I1, I5	(I2, I4	(I3, I6	199.118	317.912	791.611	507.577	590.087	1263.883	3670.187	111.975	64.616	125.178	40.434	118.577	51.049	100.630	57.612	106.303	47.183
L	(I1, I5	(I2, I6	(I3, I4	199.118	500.132	532.855	647.057	590.087	1049.885	3519.134	144.967	85.154	141.609	46.929	143.288	62.313	117.061	68.874	131.014	70.776
M	(I1, I6	(I2, I3	(I4, I5	253.835	238.559	313.659	961.270	1117.531	850.755	3735.608	181.098	96.809	161.057	67.884	171.077	74.733	130.492	81.270	155.795	78.286
N	(I1, I6	(I2, I4	(I3, I5	253.835	317.912	657.577	507.577	928.313	850.755	3515.968	124.871	54.127	124.158	38.082	124.514	46.104	93.593	48.903	109.232	39.138
O	(I1, I6	(I2, I5	(I3, I4	253.835	405.273	532.855	647.057	752.241	850.755	3442.016	151.499	78.306	141.050	47.197	146.275	58.027	110.485	63.125	130.992	58.314
A	(I1, I2	(I3, I4	(I5, I6	35.289	35.289	167.866	167.866	417.992	417.992	1242.295	34.079	24.410	62.980	27.569	48.529	14.023	55.165	36.777	44.622	19.233
B	(I1, I2	(I3, I5	(I4, I6	35.289	35.289	238.559	238.559	313.659	313.659	1175.014	27.903	27.903	35.088	31.155	31.496	13.166	27.273	36.365	27.588	17.074
C	(I1, I2	(I3, I6	(I4, I5	35.289	35.289	238.559	238.559	313.659	313.659	1175.014	27.903	27.903	35.088	31.155	31.496	13.166	27.273	36.365	27.588	17.074
D	(I1, I3	(I2, I4	(I5, I6	66.833	66.833	106.705	106.705	417.992	417.992	1183.059	16.578	22.104	48.429	37.914	32.503	13.109	40.035	42.865	28.306	16.665
E	(I1, I3	(I2, I5	(I4, I6	66.833	105.140	106.705	238.559	220.711	313.659	1051.606	32.880	43.840	20.035	20.035	26.458	15.242	11.642	15.522	22.261	18.040
F	(I1, I3	(I2, I6	(I4, I5	66.833	105.140	106.705	238.559	313.659	220.711	1051.606	32.880	43.840	25.433	25.433	29.157	15.506	17.039	22.719	24.960	19.703
G	(I1, I4	(I2, I3	(I5, I6	66.833	66.833	106.705	106.705	417.992	417.992	1183.059	16.578	22.104	48.429	37.914	32.503	13.109	40.035	42.865	28.306	16.665
H	(I1, I4	(I2, I5	(I3, I6	66.833	105.140	238.559	106.705	220.711	313.659	1051.606	21.876	29.168	30.269	30.269	26.073	13.601	21.876	29.168	21.876	16.399
I	(I1, I4	(I2, I6	(I3, I5	66.833	105.14	238.55	106.705	313.659	220.711	1051.60	21.876	29.168	45.861	45.861	33.868	19.090	37.467	49.956	29.671	23.287

				0	9				6										
J	(I1, I5(I2, I3	(I4, I6	105.140	66.833	106.705	238.559	220.711	313.659	1051.606	27.582	36.777	25.044	25.044	26.313	14.672	11.642	15.522	19.612	14.508
K	(I1, I5(I2, I4	(I3, I6	105.140	66.833	238.559	106.705	220.711	313.659	1051.606	16.578	22.104	35.278	35.278	25.928	13.159	21.876	29.168	19.227	12.867
L	(I1, I5(I2, I6	(I3, I4	105.140	105.140	167.866	167.866	220.711	220.711	987.434	39.056	39.056	40.174	27.405	39.615	20.025	26.772	26.772	32.914	26.529
M	(I1, I6(I2, I3	(I4, I5	105.140	66.833	106.705	238.559	313.659	220.711	1051.606	27.582	36.777	30.442	30.442	29.012	14.084	17.039	22.719	22.311	17.054
N	(I1, I6(I2, I4	(I3, I5	105.140	66.833	238.559	106.705	313.659	220.711	1051.606	16.578	22.104	40.676	40.676	28.627	12.700	27.273	36.365	21.926	15.541
O	(I1, I6(I2, I5	(I3, I4	105.140	105.140	167.866	167.866	220.711	220.711	987.434	39.056	39.056	40.174	27.405	39.615	20.025	26.772	26.772	32.914	26.529

Table B1 stages a gender-differentiated Economy I with an unbalanced distribution of income between the two groups allowed to match – on average, odd individuals have lower income than even ones. Calculations relative to the marginal benefit (calculated allowing the pairs to remain single and income of pair to be 0) are reported for total individuals (MeanB and AVDEVB. Also the minimum, MinB, and the difference MeanB-MinB; these were hypothetical proxies for prices and distance to be minimized, but turned out to be less relevant than the two other measures), for odd individuals (MeanBOd and AVDEVBOD; MinBOD and MeanBOd-MinBOd), and even ones (MeanBEven and AVDEVBEv; MinE and MeanBEven-MinE).

One can appreciate that equalization of the marginal benefit, computed not allowing mating within each group, and to loose different prices for the two groups does not lead to the same choice always – i.e., the minimum of column AVDEVBOD differs from that of column AVDEVBEv in some cases. If one computes the marginal benefit as if one could mate own group, then the adequate values are those of Tables A1 to A5 but relevant only for pairs A, C, G, H, M, O – where the rule of equalization of the marginal benefit would also seem to validate the optimal assignment, as noted.

The first shaded column of Table B1 registers the mean of the difference between marginal benefit of odd and even individuals in each pair – the expected differential in prices. In general, (except for $\mu_i = 2.25$, which would not be expected), the minimum absolute value of such magnitudes coincides with the optimum assignment. Interestingly, the minimum absolute deviation of the marginal benefit over individuals of the lower income (AVDEVBOD) also does.

The second shaded column reports the average between AVDEVBEv and AVDEVBOd: the minimum would equalize marginal benefit to (different…) prices in the two groups. For $\mu_i = 1.25$, the minimization of such criterion does not point to the optimal assignment; therefore, a third shaded column reports an average of the previous shaded columns.

The last two columns report the difference in the minimum marginal benefit of Odd and Even individuals (another proxy for the first shaded column values; it performs very poorly), and the last one of the average of the difference between Mean and Minimum of Odd and Even (a proxy for the role of the mean of AVDEVBEv and AVDEVBOd.)

Table B1

	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	V6	Sum	MeanB	AVDEV	MinB	Mean-	MeanBO	AVDEVB	MinO	Mean-	MeaBEv	AVDBEv	MinE	Mean-	Mean(O Even)	MeanA	AVER	MinO-	Mean(Min)
	0.25																										
	(I1, I2)	(I3, I4)	(I5, I6)	1.595	1.680	2.009	2.053	2.259	2.290	11.886	0.131	0.049	0.078	0.053	0.157	0.081	0.078	0.079	0.104	0.017	0.079	0.025	0.053	0.049	0.051	-0.001	0.052
	(I1, I2)	(I3, I6)	(I4, I5)	1.595	1.680	2.099	2.145	2.181	2.211	11.912	0.128	0.050	0.090	0.037	0.154	0.083	0.090	0.064	0.101	0.009	0.092	0.009	0.054	0.046	0.050	-0.002	0.036
	(I1, I4)	(I2, I3)	(I5, I6)	1.744	1.837	1.894	1.935	2.259	2.290	11.960	0.117	0.040	0.074	0.042	0.130	0.034	0.078	0.052	0.103	0.036	0.074	0.029	0.026	0.035	0.031	0.004	0.041
	(I1, I4)	(I2, I5)	(I3, I6)	1.744	1.949	2.099	1.935	2.087	2.211	12.026	0.107	0.020	0.074	0.032	0.120	0.020	0.090	0.029	0.094	0.013	0.074	0.019	0.026	0.016	0.021	0.016	0.024
	(I1, I6)	(I2, I3)	(I4, I5)	1.850	1.837	1.894	2.145	2.181	2.116	12.024	0.111	0.032	0.056	0.055	0.121	0.028	0.094	0.027	0.102	0.037	0.056	0.046	0.019	0.032	0.025	0.038	0.036
	(I1, I6)	(I2, I5)	(I3, I4)	1.850	1.949	2.009	2.053	2.087	2.116	12.064	0.104	0.016	0.056	0.048	0.113	0.005	0.106	0.008	0.095	0.026	0.056	0.039	0.018	0.015	0.017	0.049	0.023
	0.75																										
	(I1, I2)	(I3, I4)	(I5, I6)	4.058	4.743	8.107	8.649	11.530	12.013	49.101	1.293	0.205	0.909	0.383	1.415	0.243	1.153	0.262	1.170	0.174	0.909	0.261	0.244	0.208	0.226	0.244	0.261
	(I1, I2)	(I3, I6)	(I4, I5)	4.058	4.743	9.251	9.869	10.377	10.812	49.110	1.278	0.188	0.909	0.369	1.402	0.251	1.143	0.258	1.155	0.164	0.909	0.246	0.246	0.208	0.227	0.234	0.252
	(I1, I4)	(I2, I3)	(I5, I6)	5.306	6.202	6.794	7.249	11.530	12.013	49.095	1.243	0.192	0.801	0.442	1.333	0.177	1.153	0.180	1.154	0.235	0.801	0.352	0.179	0.206	0.193	0.352	0.266
	(I1, I4)	(I2, I5)	(I3, I6)	5.306	7.400	9.251	7.249	9.095	10.812	49.113	1.200	0.152	0.801	0.399	1.288	0.123	1.143	0.145	1.112	0.207	0.801	0.311	0.176	0.165	0.171	0.342	0.228
	(I1, I6)	(I2, I3)	(I4, I5)	6.332	6.202	6.794	9.869	10.377	9.475	49.050	1.220	0.226	0.737	0.484	1.302	0.197	1.025	0.277	1.138	0.268	0.737	0.402	0.164	0.233	0.198	0.289	0.339
	(I1, I6)	(I2, I5)	(I3, I4)	6.332	7.400	8.107	8.649	9.095	9.475	49.059	1.191	0.207	0.737	0.454	1.270	0.163	1.025	0.245	1.112	0.250	0.737	0.375	0.158	0.207	0.183	0.289	0.310
	1																										
	(I1, I2)	(I3, I4)	(I5, I6)	6.473	7.969	16.286	17.754	26.049	27.513	102.045	3.267	0.433	1.969	1.298	3.435	0.025	3.414	0.021	3.100	0.754	1.969	1.131	0.335	0.390	0.362	1.444	0.576
	(I1, I2)	(I3, I6)	(I4, I5)	6.473	7.969	19.419	21.169	22.635	23.908	101.573	3.250	0.466	1.969	1.280	3.419	0.191	3.133	0.286	3.080	0.741	1.969	1.111	0.339	0.466	0.402	1.163	0.699
	(I1, I4)	(I2, I3)	(I5, I6)	9.256	11.395	12.869	14.029	26.049	27.513	101.110	3.187	0.521	2.029	1.159	3.355	0.382	2.783	0.572	3.020	0.661	2.029	0.991	0.335	0.521	0.428	0.754	0.782
	(I1, I4)	(I2, I5)	(I3, I6)	9.256	14.421	19.419	14.029	18.984	23.908	100.016	3.135	0.523	2.029	1.106	3.300	0.456	2.783	0.517	2.971	0.628	2.029	0.942	0.329	0.542	0.436	0.754	0.729
	(I1, I6)	(I2, I3)	(I4, I5)	11.714	11.395	12.869	21.169	22.635	20.051	99.833	3.145	0.594	2.051	1.094	3.326	0.579	2.458	0.868	2.964	0.609	2.051	0.913	0.362	0.594	0.478	0.407	0.890
	(I1, I6)	(I2, I5)	(I3, I4)	11.714	14.421	16.286	17.754	18.984	20.051	99.210	3.110	0.599	2.051	1.059	3.286	0.552	2.458	0.828	2.934	0.589	2.051	0.883	0.352	0.571	0.461	0.407	0.856
	1.25																										
	(I1, I2)	(I3, I4)	(I5, I6)	10.325	13.390	32.718	36.445	58.848	63.013	214.738	7.940	1.835	3.999	3.941	8.066	1.126	6.377	1.689	7.814	2.543	3.999	3.815	0.252	1.835	1.043	2.377	2.752
	(I1, I2)	(I3, I6)	(I4, I5)	10.325	13.390	40.765	45.408	49.372	52.865	212.124	7.928	1.827	3.999	3.929	8.057	1.126	6.377	1.680	7.799	2.534	3.999	3.800	0.257	1.830	1.043	2.377	2.740
	(I1, I4)	(I2, I3)	(I5, I6)	16.144	20.936	24.374	27.150	58.848	63.013	210.464	7.765	1.705	4.816	2.949	8.027	1.472	5.819	2.208	7.503	1.792	4.816	2.687	0.524	1.632	1.078	1.003	2.448
	(I1, I4)	(I2, I5)	(I3, I6)	16.144	28.103	40.765	27.150	39.626	52.865	204.652	7.732	1.798	4.816	2.916	7.991	1.448	5.819	2.172	7.473	1.975	4.816	2.657	0.518	1.712	1.115	1.003	2.414
	(I1, I6)	(I2, I3)	(I4, I5)	21.670	20.936	24.374	45.408	49.372	42.430	204.189	7.654	1.511	5.354	2.300	8.019	1.662	5.527	2.493	7.288	1.289	5.354	1.934	0.731	1.476	1.104	0.173	2.213
	(I1, I6)	(I2, I5)	(I3, I4)	21.670	28.103	32.718	36.445	39.626	42.430	200.991	7.632	1.616	5.354	2.278	7.992	1.644	5.527	2.466	7.272	1.348	5.354	1.918	0.720	1.496	1.108	0.173	2.192
	2.25																										

[illegible]	(I1, I2)	(I3, I4)	(I5, I6)	66.833	106.705	532.855	647.057	1532.93	1733.68	4620.05	245.985	137.803	50.362	195.623	229.860	123.690	54.988	174.872	262.110	141.166	50.362	211.749	-32.251	132.428	50.089	4.626	193.310
[illegible]	(I1, I2)	(I3, I6)	(I4, I5)	66.833	106.705	791.611	961.270	1117.53	1263.88	4307.83	242.789	126.743	50.362	192.428	226.344	114.238	54.988	171.356	259.234	139.248	50.362	208.872	-32.890	126.743	46.927	4.626	190.114
[illegible]	(I1, I4)	(I2, I3)	(I5, I6)	149.417	238.559	313.659	380.883	1532.93	1733.68	4349.12	230.977	141.079	82.585	148.392	223.781	127.743	82.585	141.196	238.172	154.415	112.868	125.304	-14.392	141.079	63.343	-30.284	133.250
[illegible]	(I1, I4)	(I2, I5)	(I3, I6)	149.417	405.273	791.611	380.883	752.241	1263.88	3743.31	223.916	103.193	82.585	141.331	216.928	89.562	82.585	134.343	230.904	121.483	112.868	118.035	-13.976	105.523	45.774	-30.284	126.189
[illegible]	(I1, I6)	(I2, I3)	(I4, I5)	253.835	238.559	313.659	961.270	1117.53	850.755	3735.61	212.088	85.109	104.418	107.670	214.357	100.622	104.418	109.939	209.819	69.596	131.854	77.965	4.538	85.109	44.823	-27.436	93.952
[illegible]	(I1, I6)	(I2, I5)	(I3, I4)	253.835	405.273	532.855	647.057	752.241	850.755	3442.02	208.223	56.715	104.418	103.805	211.019	71.067	104.418	106.601	205.426	40.499	166.714	38.712	5.593	55.783	30.688	-62.296	72.657

The same calculations were repeated for unbalanced groups – i.e., I6 was discarded from the economy. The same criteria proved useful in identifying the equilibrium. Additionally, the marginal benefit from mating (joining with I2) of the singleton is also reported (column MBOut): it was expected to be lower than the (average) marginal benefit of matched odd individuals (than MeanBOd) – the price of even consorts - in the (shaded) equilibrium. However, this rule should apply to positive assortative mating situations – when the cross derivative of the indirect utility function is positive - reason why it did not work for the first case.

With negative assortative mating, high income individuals are left unmated. We then expect – the cross-derivative of the indirect utility function is negative - that it is the individual that is mated with the closest income to the lowest excluded income that is better-off than (has higher marginal benefit than he would have by) mating with the excluded one.

Table B2

si	Pair 1	Pair 2	Pair 3	V1	V2	V3	V4	V5	SUM	MBOut	MeanE	AVDE	MinB	Mean-	eanBO	VDEVB	MinO	Mean-	MeaBEv	AVDBEv	MinE	Mean-	Mean(Even)	MeanA	AVER	MinO-	Mean(Min)
i	0.25																										
.	(I1, I2)	(I3, I4)	(I5)	1.595	1.680	2.009	2.053	1.968	9.305	0.119	0.157	0.061	0.115	0.042	0.197	0.082	0.115	0.082	0.116	0.001	0.115	0.001	0.081	0.042	0.061	0.000	0.042
:	(I1, I2)	(I3)	(I4, I5)	1.595	1.680	1.732	2.145	2.181	9.333	0.162	0.145	0.067	0.092	0.053	0.186	0.093	0.094	0.093	0.104	0.011	0.092	0.011	0.083	0.052	0.067	0.002	0.052
G	(I1, I4)	(I2, I3)	(I5)	1.744	1.837	1.894	1.935	1.968	9.379	0.119	0.136	0.031	0.074	0.061	0.156	0.006	0.149	0.006	0.116	0.041	0.074	0.041	0.040	0.024	0.032	0.075	0.024
[	(I1, I4)	(I2, I5)	(I3)	1.744	1.949	1.732	1.935	2.087	9.448	0.162	0.114	0.021	0.074	0.039	0.134	0.015	0.119	0.015	0.093	0.019	0.074	0.019	0.041	0.017	0.029	0.045	0.017
I	(I1)	(I2, I3)	(I4, I5)	1.316	1.837	1.894	2.145	2.181	9.374	0.279	0.126	0.033	0.092	0.034	0.128	0.034	0.094	0.034	0.125	0.032	0.092	0.032	0.003	0.033	0.018	0.002	0.033
)	(I1)	(I2, I5)	(I3, I4)	1.316	1.949	2.009	2.053	2.087	9.414	0.279	0.116	0.003	0.111	0.004	0.117	0.002	0.115	0.002	0.114	0.003	0.111	0.003	0.003	0.003	0.003	0.003	0.003
	0.75																										
.	(I1, I2)	(I3, I4)	(I5)	4.058	4.743	8.107	8.649	7.622	33.180	1.473	1.350	0.239	0.909	0.441	1.546	0.233	1.313	0.233	1.155	0.246	0.909	0.246	0.391	0.239	0.315	0.403	0.239
:	(I1, I2)	(I3)	(I4, I5)	4.058	4.743	5.196	9.869	10.377	34.244	1.598	1.298	0.240	0.909	0.388	1.531	0.248	1.283	0.248	1.065	0.155	0.909	0.155	0.466	0.202	0.334	0.373	0.202
G	(I1, I4)	(I2, I3)	(I5)	5.306	6.202	6.794	7.249	7.622	33.174	1.473	1.277	0.252	0.801	0.475	1.423	0.175	1.248	0.175	1.130	0.329	0.801	0.329	0.293	0.252	0.273	0.447	0.252
[	(I1, I4)	(I2, I5)	(I3)	5.306	7.400	5.196	7.249	9.095	34.246	1.598	1.180	0.189	0.801	0.379	1.360	0.112	1.248	0.112	1.000	0.198	0.801	0.198	0.361	0.155	0.258	0.447	0.155
I	(I1)	(I2, I3)	(I4, I5)	2.280	6.202	6.794	9.869	10.377	35.522	1.779	1.390	0.139	1.220	0.170	1.440	0.158	1.283	0.158	1.339	0.120	1.220	0.120	0.101	0.139	0.120	0.063	0.139
)	(I1)	(I2, I5)	(I3, I4)	2.280	7.400	8.107	8.649	9.095	35.531	1.779	1.346	0.091	1.198	0.148	1.393	0.080	1.313	0.080	1.299	0.101	1.198	0.101	0.093	0.091	0.092	0.114	0.091
	1																										
.	(I1, I2)	(I3, I4)	(I5)	6.473	7.969	16.286	17.754	15.000	63.483	3.984	3.146	0.589	1.969	1.177	3.445	0.028	3.418	0.028	2.847	0.878	1.969	0.878	0.598	0.453	0.525	1.448	0.453
:	(I1, I2)	(I3)	(I4, I5)	6.473	7.969	9.000	21.169	22.635	67.247	3.869	3.127	0.579	1.969	1.158	3.562	0.089	3.473	0.089	2.692	0.723	1.969	0.723	0.870	0.406	0.638	1.504	0.406
G	(I1, I4)	(I2, I3)	(I5)	9.256	11.395	12.869	14.029	15.000	62.548	3.984	3.026	0.621	2.029	0.998	3.326	0.543	2.783	0.543	2.727	0.698	2.029	0.698	0.598	0.621	0.610	0.754	0.621
[	(I1, I4)	(I2, I5)	(I3)	9.256	14.421	9.000	14.029	18.984	65.689	3.869	2.955	0.550	2.029	0.927	3.383	0.601	2.783	0.601	2.527	0.499	2.029	0.499	0.856	0.550	0.703	0.754	0.550
I	(I1)	(I2, I3)	(I4, I5)	3.000	11.395	12.869	21.169	22.635	71.068	3.473	3.590	0.170	3.415	0.175	3.760	0.109	3.651	0.109	3.420	0.005	3.415	0.005	0.340	0.057	0.198	0.236	0.057
)	(I1)	(I2, I5)	(I3, I4)	3.000	14.421	16.286	17.754	18.984	70.446	3.473	3.538	0.316	3.026	0.512	3.701	0.283	3.418	0.283	3.376	0.350	3.026	0.350	0.325	0.316	0.321	0.391	0.316
	1.25																										
.	(I1, I2)	(I3, I4)	(I5)	10.325	13.390	32.718	36.445	29.520	122.397	10.106	7.004	1.816	3.999	3.004	7.360	0.984	6.377	0.984	6.647	2.648	3.999	2.648	0.713	1.816	1.265	2.377	1.816
:	(I1, I2)	(I3)	(I4, I5)	10.325	13.390	15.588	45.408	49.372	134.083	8.785	7.271	2.083	3.999	3.272	8.061	1.685	6.377	1.685	6.481	2.482	3.999	2.482	1.580	2.083	1.832	2.377	2.083
G	(I1, I4)	(I2, I3)	(I5)	16.144	20.936	24.374	27.150	29.520	118.123	10.106	6.741	1.424	4.816	1.926	7.302	1.483	5.819	1.483	6.181	1.365	4.816	1.365	1.121	1.424	1.273	1.003	1.424
[	(I1, I4)	(I2, I5)	(I3)	16.144	28.103	15.588	27.150	39.626	126.611	8.785	6.977	1.660	4.816	2.161	7.962	2.144	5.819	2.144	5.991	1.176	4.816	1.176	1.971	1.660	1.815	1.003	1.660
I	(I1)	(I2, I3)	(I4, I5)	3.948	20.936	24.374	45.408	49.372	144.038	6.377	8.760	0.607	7.546	1.214	9.266	0.480	8.785	0.480	8.255	0.709	7.546	0.709	1.011	0.595	0.803	1.239	0.595
)	(I1)	(I2, I5)	(I3, I4)	3.948	28.103	32.718	36.445	39.626	140.839	6.377	8.728	0.972	7.167	1.561	9.225	0.881	8.344	0.881	8.231	1.064	7.167	1.064	0.994	0.972	0.983	1.177	0.972
	2.25																										
.	(I1, I2)	(I3, I4)	(I5)	66.833	106.705	532.855	647.057	442.798	1796.25	309.443	147.680	95.005	50.362	97.318	137.092	82.104	54.988	82.104	158.268	107.906	50.362	107.906	-21.176	95.005	36.915	4.626	95.005
:	(I1, I2)	(I3)	(I4, I5)	66.833	106.705	140.296	961.270	1117.53	2392.63	173.363	196.213	143.538	50.362	145.851	210.139	155.151	54.988	155.151	182.287	131.925	50.362	131.925	27.852	143.538	85.695	4.626	143.538

;	(I1, I4)	(I2, I3	(I5)	149.417	238.559	313.659	380.883	442.798	1525.31	309.443	125.167	27.441	82.585	42.583	127.974	45.389	82.585	45.389	122.361	9.493	112.868	9.493	5.612	27.441	16.527	-30.284	27.441
[	(I1, I4)	(I2, I5	(I3)	149.417	405.273	140.296	380.883	752.241	1828.11	173.363	167.902	70.770	82.585	85.318	196.014	113.429	82.585	113.429	139.791	26.923	112.868	26.923	56.223	70.176	63.199	-30.284	70.176
I	(I1)	(I2, I3	(I4, I5	11.845	238.559	313.659	961.270	1117.53	2642.86	54.988	246.180	93.571	131.854	114.326	269.326	95.964	173.363	95.964	223.033	91.179	131.854	91.179	46.293	93.571	69.932	41.508	93.571
)	(I1)	(I2, I5	(I3, I4	11.845	405.273	532.855	647.057	752.241	2349.27	54.988	240.382	47.427	166.714	73.668	264.320	45.124	219.196	45.124	216.444	49.730	166.714	49.730	47.876	47.427	47.651	52.482	47.427