When and How to Dismantle Nuclear Weapons*

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Abstract This paper shows that allowing the option of destroying the auctioned item by the seller improves the optimal auction when identity-dependent externalities exist between the seller and bidders. The sufficient and necessary conditions for item destruction to be optimal both in terms of nonparticipating threat and allocation outcome are also provided in this paper. A modified second-price sealed-bid auction with appropriately set nonnegative entry fees and reserve price is established as the optimal auction. The optimal auction induces full participation of bidders, and a feature of the optimal auction is that each losing bidder's payment includes a component (positive or negative) equal to the externality on him at the outcome of the auction. These components eliminate the impact of externalities on strategic bidding behavior. It thus follows intuitively that a second-price auction with these additional payments is optimal. The above findings hold when players have private information on externalities they create for others, or when externality to every bidder is proportional to total payments of other bidders or all bidders.

Keywords: Auctions design; Endogenous participation; Externality.

JEL classifications: D44, D82.

1 Introduction

Auctions design with externalities among bidders has been studied by a number of papers. Jehiel, Moldovanu and Staccheti (1996), Varma (2002), and Brocas (2003, 2005) among others consider negative identity-specific externalities imposed on losers by the winning bidder, while Maasland and Onderstal (2002, 2005) and Goeree, Maasland, Onderstal and Turner (2005) study the cases where positive externalities among bidders are proportional to the total payments of other bidders or all bidders. Jehiel, Moldovanu and Staccheti (1999) study auctions design when bidders have private information of externalities on themselves in a multi-dimensional setting. While the literature largely focuses on externalities among bidders, clearly in many cases there exist externalities between the seller and bidders other than those among bidders. One recent example is the North Korea's nuclear weapon case, where the seller (North Korea) puts great externalities on the bidders (China, Japan, Russia, South Korea, US) if it keeps its nuclear arsenal.

We study in this paper the optimal auction while allowing externalities among all players, including the seller and bidders. [1] One major contribution of this paper lies in that the analysis brings in the option for the seller to destroy the item (i.e., dismantle its nuclear arsenal) at a cost. In previous auctions design literature, destroying the auctioned item has not been formulated as a possible outcome or as a nonparticipation threat. The significance of bringing in this option is the following. First, we are particularly interested in addressing when and how to dismantle the nuclear weapons, i.e. under what conditions is it optimal for the seller to destroy the object and what actions should be taken by the seller to maximize his revenue if he destroys the object. Second, allowing this new option enlarges the freedom of auctions design with externalities. Specifically, destroying the

[1] Potipiti (2005) also considers a setting of selling retaliation in the WTO where externalities exist between the seller and buyers. However, both the setting and the focus of Potipiti (2005) are quite different from that of this paper.

item can be an optimal allocation outcome for the seller or be used by the seller as an optimal nonparticipation threat, since it eliminates the externalities imposed on bidders. In particular, eliminating these externalities has two potential effects. First, seller's threat to a bidder who refuses to participate can be made more severe. This happens when a bidder enjoys positive externalities whoever else gets the object. In this case, the most severe nonparticipation threat is to destroy the object. Second, the seller may extract higher rent when he destroys the object if the object is unsold. This occurs when the sum of the seller's valuation, the destroying cost of the seller and the total externalities to the bidders is negative, if the seller keeps the item. In this situation, the seller can be better off by destroying the object and collecting a payment from each bidder which equals the externality to him.

We start with and derive our main findings in a baseline setting where the identity-specific externalities among all players are common-knowledge. [2] We also allow heterogeneity in the externalities among the players. In addition, our analysis does not require the externalities to be uniformly positive or negative.

The optimal auction is fully characterized in terms of the nonparticipation threats, the winning probabilities, the probability of destroying the item, and especially the payments of bidders. A modified second-price sealed-bid auction with appropriately set nonnegative entry fees and reserve price is established as the optimal auction. Specifically, the features of the optimal auction are as follows. (i) All bidders participate. The optimal nonparticipation threats take the following form: If only i does not participate, the item is then assigned to the one (including the seller) generating bidder i the smallest externality provided this externality is nonpositive. Otherwise the seller destroys the object.

[2] We adopt this setting because the multi-dimensional setting in Jehiel, Moldovanu and Staccheti (1999) is too complicated for a thorough characterization of the optimum, and the Jehiel, Moldovanu and Staccheti (1996) setting is fundamentally one-dimensional.

(ii) Every participant pays a nonnegative entry fee, which equals the absolute value of the smallest possible externality to him. This smallest externality must be nonpositive due to the option of destroying the object by the seller. (iii) The highest bidder wins if his bid is higher than the reserve price, and he pays the second highest bid or the reserve price whichever is higher. Each bidder pays an additional payment (positive or negative) that equals the externality on him at the outcome of the auction. This additional payment is a unique feature which is first discovered in the literature to my best knowledge. (iv) If no bidder bids higher than the reserve price, the seller may keep the item or destroy it, depending on the sum of the seller's valuation, the destroying cost of the seller and the total externalities to all bidders when the seller keeps the item. The seller keeps the item if this sum is positive, and destroys it if this sum is negative. Moreover, the optimal reserve price is set differently depending on whether the above mentioned sum is positive or negative.

As pointed out in the previous paragraph, besides the entry fee, each bidder pays an additional payment that equals the externality on him at the outcome of the auction. When the payoffs of bidders are adjusted by these additional payments, we face a situation that mimics a standard auctions design problem with zero externalities on bidders. Intuitively, this is why a modified second-price auction with these additional payments is optimal, if the entry fee and reserve price are set appropriately.

We develop a formal procedure to establish that there is no loss of generality to consider only the mechanisms which induce full participation of bidders for the optimal auction. The significance of this result can be seen from the following arguments. First, there exist occasions where full participation is not the unique optimum. Note that in the Myerson (1981) setting with zero externalities, the seller still gets the optimal expected revenue if the bidders whose valuations are lower than the optimal reserve price do not participate. Second, in the case of positive externalities, bidders with lower valuations may prefer not to participate due to the free-riding incentive. What is the intuition behind the optimality of full participation? As has been pointed out in the previous paragraph, the additional payments in the optimal auction lead to a situation that mimics a standard auctions design problem with zero externalities on bidders. Note that the sum of the additional payment in the optimal auction and the entry fee for each bidder is nonnegative, as the entry fee is the absolute value of the lowest externality to the bidder. Thus, the seller gains (weakly) from the participation of every type of bidders.[3] This explains why the seller wants every type of bidders to participate, although some types have no chance of winning.

An interesting question is whether the above insights also apply in a multi-dimensional setting where bidders have private information on the externalities they create for others. Jehiel, Moldovanu and Staccheti (1996) have shown that when bidders have private information on the externalities they create for others, the optimal auction design problem can be transformed into a one-dimensional problem. Therefore intuitively, the findings obtained in the common-knowledge setting are still valid if we replace the common-knowledge externalities in our setting by the expectations of the private-information externalities.

The key findings from our baseline setting also apply to the settings of Maasland and Onderstal (2002) and Goeree, Maasland, Onderstal and Turner (2005), i.e., when externality to any bidder is proportional to the other bidders' total payments or those of all bidders. Maasland and Onderstal (2002) assume that externality to any bidder is proportional to the other bidders' total payments. They show that a lowest-price all-pay auction with proper entry fee and reserve price is optimal in this setting. Based on the insights from our baseline setting, we put forward an alternative modified second-price

[3] Besides the additional payment, each bidder (winner or loser) makes another nonnegative payment as in a standard second-price auction without externalities.

auction which is revenue-equivalent to the lowest-price all-pay auction in Maasland and Onderstal (2002). Similar result is obtained for the Goeree, Maasland, Onderstal and Turner (2005) setting where positive externalities among bidders are proportional to the total payments of all bidders. Being consistent with the findings from our baseline setting, in the alternative auctions constructed every bidder's payment consists of a component which equals the externalities on them at the outcome of the auction.

This paper is organized as follows. Section 2 derives the optimal mechanism in a baseline setting with common-knowledge externalities. The implementation of the optimal auction is established. The findings are shown to hold when bidders have private information on the externalities they create for others. Section 3 shows that the main findings from the baseline setting apply to the case of financial externalities. Section 4 concludes the paper. The technical proofs are given in the appendix.

2 The Optimal Auction

In this section we fully establish the optimal auction mechanism when there are externalities among players, including the seller and bidders. Externalities lead to an auctions design problem in which the bidders have mechanism-dependent reservation utilities. We will establish concretely through a formal procedure that the full participation is the optimal participation in terms of the seller's expected revenue. For this purpose, we explicitly deal with the optimal endogenous participation.

Following Stegeman (1996), we define participating in the auction as submitting a bid. Since an auction mechanism implementing endogenous participation essentially cannot require the bidders who do not participate to submit bids, we consider the mechanisms based on only the signals submitted by the participating bidders. Following Jehiel, Moldovanu and Staccheti (1996), we assume that the bidders who do not participate have no chance to win the object and their payments to the seller are zero. This assumption is consistent with the **no passive reassignment** (NPR) assumption adopted by Stegeman (1996).

2.1 The Setting

There is one seller who wants to sell one indivisible object to N potential bidders through an auction. We use $\mathcal{N} = \{1, 2, ..., N\}$ to denote the set of all potential bidders, where $\mathcal{N}$ is assumed to be common knowledge. The seller's value for the object is v_0, which is public information. Hereafter, we let the seller to be player 0, and bidder i to be player i. The ith bidder's private value of the object is v_i, which is his/her private information. These values v_i, $i \in \mathcal{N}$ are independently distributed on interval $[\underline{v}_i, \overline{v}_i]$ following respectively cumulative distribution function $F_i(\cdot)$ with density function $f_i(\cdot)$. We assume the regularity condition that the virtual valuation functions $J_i(v) = v - (1 - F_i(v))/f_i(v)$ are increasing on interval $[\underline{v}_i, \overline{v}_i]$. The density $f_i(\cdot)$ is assumed to be common knowledge. Every bidder observes his private value before his participation decision. The seller and the bidders are assumed to be risk neutral.

Player i suffers or enjoys an externality $e_{i,j}$ when player j keeps the item, $i, j = 0, 1, ..., N$. By definition, $e_{i,i} = 0$, $i = 0, 1, ..., N$. These externalities are public information. The auctioned item can be destroyed by the seller at a cost of $c_0 \geq 0$. If the item is destroyed, no player suffers or enjoys any externality. As a result, bidder i's payoff is $v_i - x_i$ if he wins and pays x_i; his payoff is $e_{i,j} - x_i$ if he pays x_i while another player j (seller or bidder) wins.

The timing of the auction is as follows.

Time 0: The externalities $e_{i,j}$, the seller's value v_0 and destroying cost c_0 are revealed by Nature as public information. Every bidder i observes his/her private value v_i, $i \in \mathcal{N}$.

Time 1: The seller announces the rule of the auction. The possibility of destroying the item by the seller is allowed. We assume that the seller has the power of committing to the proposed rule.

Time 2: The bidders simultaneously and confidentially make their participation decisions and announce their bids if they decide to participate.

Time 3: The payoffs of the seller and all bidders are determined according to the announced rule at time 1.

In this paper, we look for the optimal mechanism among the threshold-participation mechanisms. Here, threshold-participation refers to that the bidders only participate if their valuations are equal to or higher than their corresponding thresholds. Based on the "semirevelation" principle established by Lemma 1 in Stegeman (1996) that allows no participation, we only need to look at the direct truthful semirevelation mechanism, which requires bidders to reveal their true types if and only if they participate. Denote thresholds vector $(v_c^{(1)}, ..., v_c^{(N)})$ by $\mathbf{v_c}$, where $v_c^{(i)}$ is the threshold for bidder i and it takes values in $[\underline{v}_i, \overline{v}_i]$. In a direct truthful semirevelation mechanism implementing threshold-participation $\mathbf{v_c}$, bidder i announces his/her valuation if and only if his/her valuation is equal to or greater than threshold $v_c^{(i)}$. The seller determines how to allocate the object and how much each bidder pays, using a set of outcome functions which accommodate all participation possibilities. Following Stegeman (1996), we introduce a null message $\emptyset$ to denote the signal of a nonparticipant. Let $\mathbf{m} = (m_1, m_2, ..., m_N)$, where m_i is the signal of bidder i and it takes values in $\mathcal{M}_i = [\underline{v}_i, \overline{v}_i] \cup \{\emptyset\}, \ \forall i \in \mathcal{N}$. Define $\mathcal{M} = \prod_{i=1}^N \mathcal{M}_i$. These outcome functions announced by the seller consist of the probability $p_0(\mathbf{m})$ for the seller to keep the item, the payment functions $x_i(\mathbf{m})$ and winning probability functions $p_i(\mathbf{m})$ of bidder $i, \forall i \in \mathcal{N}$. Note that $1 - \sum_{i=0}^N p_i(\mathbf{m})$ is the probability of destroying the item by the seller. This set of allocation functions are denoted by $(\mathbf{p}, \mathbf{x})$.

We say $(\mathbf{p}, \mathbf{x})$ is a direct truthful semirevelation mechanism implementing threshold participation $\mathbf{v_c}$ if and only if the following conditions hold.

(a) The bidders with private values lower than their participation thresholds do not participate, i.e., if they participate, they get expected utility which is equal to or lower than their expected utility from nonparticipation. Thus these types of bidders submit the null signal;

(b) The bidders with private values equal to or higher than their participation thresholds do participate and reveal their true valuations.

(c) $p_i(\mathbf{m}) \geq 0,\ \forall i \in \mathcal{N}$, with $\sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$.

(d) $p_i(\mathbf{m}) = 0$ and $x_i(\mathbf{m}) = 0$ if $m_i = \emptyset,\ \forall i \in \mathcal{N}$.[4]

Define $m_i(x) = x$ if $x \in [v_c^{(i)}, \overline{v}_i]$ and $m_i(x) = \emptyset$ if $x \in [\underline{v}_i, v_c^{(i)})$. As pointed out before, the probability of destroying the object is $1 - \sum_{i=0}^{N} p_i(\mathbf{m})$ and the destroying cost is c_0. In addition, when the object is destroyed, the seller does not enjoy v_0. We use $\mathbf{m}(\mathbf{v})$ to denote $(m_1(v_1), ..., m_N(v_N))$. For any direct truthful semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ implementing threshold participation $\mathbf{v_c}$, the seller's expected revenue is given by:

$$\begin{aligned}
&R(\mathbf{p}, \mathbf{x}, \mathbf{v_c}) \\
&= E_{\mathbf{v}}\Big\{(v_0 + e_{0,0})p_0(\mathbf{m}(\mathbf{v})) + \sum_{i=1}^{N} e_{0,i}\ p_i(\mathbf{m}(\mathbf{v})) - c_0\Big(1 - \sum_{i=0}^{N} p_i(\mathbf{m}(\mathbf{v}))\Big) + \sum_{i=1}^{N} x_i(\mathbf{m}(\mathbf{v}))\Big\} \\
&= E_{\mathbf{v}}\Big\{(v_0 + c_0 + e_{0,0})p_0(\mathbf{m}(\mathbf{v})) + \sum_{i=1}^{N}(e_{0,i} + c_0)\ p_i(\mathbf{m}(\mathbf{v})) + \sum_{i=1}^{N} x_i(\mathbf{m}(\mathbf{v}))\Big\} - c_0, \qquad (1)
\end{aligned}$$

where $\mathbf{v} = (v_1, v_2, ..., v_N)$. Denote the support of $\mathbf{v}$ by $\mathcal{V} = \prod_{i=1}^{N}[\underline{v}_i,\ \overline{v}_i]$.

For bidder i with private value v_i, if he submits signal $m_i \in \mathcal{M}_i$, his interim expected payoff is given by:

$$\begin{aligned}
&U_i(v_i, m_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \\
&= E_{\mathbf{v}_{-i}}\Big(v_i\ p_i(m_i, \mathbf{m_{-i}}(\mathbf{v_{-i}})) + \sum_{j\geq 0} e_{i,j}\ p_j(m_i, \mathbf{m_{-i}}(\mathbf{v_{-i}})) - x_i(m_i, \mathbf{m_{-i}}(\mathbf{v_{-i}}))\Big), \qquad (2)
\end{aligned}$$

where $\mathbf{v}_{-i} = (v_1, , ..., v_{i-1}, v_{i+1}, ..., v_N)$, and $\mathbf{m_{-i}}(\mathbf{v_{-i}}) = (m_1(v_1), ..., m_{i-1}(v_{i-1}), m_{i+1}(v_{i+1}),$ $..., m_N(v_N))$. Denote the support of $\mathbf{v}_{-i}$ by $\mathcal{V}_{-i} = \prod_{j=1, j\neq i}^{N}[\underline{v}_j,\ \overline{v}_j]$.

[4]This is from the **no passive reassignment** (NPR) assumption.

The seller's optimization problem is to find the optimal participation thresholds $\mathbf{v_c^*}$ and the optimal direct truthful semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ implementing $\mathbf{v_c^*}$, i.e.,

$$\max_{(\mathbf{p},\mathbf{x},\mathbf{v_c})} \; R(\mathbf{p}, \mathbf{x}, \mathbf{v_c}) \tag{3}$$

Subject to:

$$(i)\; U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \geq U_i(v_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c});\; \forall i \in \mathcal{N},\; \forall v_i \in [v_c^{(i)}, \overline{v}_i], \tag{4}$$

$$(ii)\; U_i(v_i, v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \leq U_i(v_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c});\; \forall i \in \mathcal{N},\; \forall v_i < v_c^{(i)},\; \forall v_i' \in [v_c^{(i)},\; \overline{v}_i], \tag{5}$$

$$(iii)\; U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \geq U_i(v_i, v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c});\; \forall i \in \mathcal{N},\; \forall v_i \in [v_c^{(i)},\; \overline{v}_i],\; v_i' \in [v_c^{(i)},\; \overline{v}_i], \tag{6}$$

$$(iv)\; p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0 \; if \; m_i = \emptyset,\; p_i(\mathbf{m}) \geq 0,\; \forall i \in \mathcal{N} \;\; \sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\; \forall \mathbf{m} \in \mathcal{M}. \tag{7}$$

Restrictions (4)-(7) come from conditions (a)-(d). Since we consider threshold-participation at $v_c^{(i)}$ for bidder i, there is no loss of generality to restrict the message space for bidder i as $[v_c^{(i)},\; \overline{v}_i] \cup \{\emptyset\}$

2.2 The Optimal Participation and the Optimal Auction

We next derive the optimal participation thresholds $\mathbf{v_c^*}$ and the optimal direct truthful semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ implementing $\mathbf{v_c^*}$.

For any direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$, we define

$$Q_i(v_i; \mathbf{p}, \mathbf{v_c}) = E_{\mathbf{v}_{-i}} p_i(\mathbf{m}(\mathbf{v})). \tag{8}$$

If $(\mathbf{p}, \mathbf{x})$ is a direct truthful semirevelation mechanism implementing threshold participation $\mathbf{v_c}$, then $Q_i(v_i; \mathbf{p}, \mathbf{v_c})$ is the conditional expected probability that bidder i wins the object if his private value is v_i. Note that $Q_i(v_i; \mathbf{p}, \mathbf{v_c}) = 0$ if $v_i < v_c^{(i)}$.

The following Lemma gives the necessary and sufficient conditions for a direct semirevelation mechanism to be a direct truthful semirevelation mechanism implementing threshold participation at given $\mathbf{v_c}$.

Lemma 1: *Direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *is a direct truthful semirevelation mechanism implementing threshold participation* $\mathbf{v_c}$*, if and only if* $\forall i \in \mathcal{N}$ *the following conditions and (5), (7) hold:*

$$Q_i(v_i; \mathbf{p}, \mathbf{v_c}) \geq Q_i(s_i; \mathbf{p}, \mathbf{v_c}), \ \forall v_c^{(i)} \leq s_i \leq v_i \leq \overline{v}_i, \tag{9}$$

$$U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) = U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{v_c^{(i)}}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i, \ \forall v_i \in [v_c^{(i)}, \overline{v}_i], \tag{10}$$

$$U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \geq U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c}). \tag{11}$$

Proof: see appendix.

Based on Lemma 1, we can replace (4) and (6) by (9), (10) and (11) in the seller's optimization problem. Applying Lemma 1, the expected revenue of the seller from an auction mechanism $(\mathbf{p}, \mathbf{x}, \mathbf{v_c})$ satisfying conditions (4)-(7) is given in the following Lemma.

Lemma 2: *For a direct truthful semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *implementing threshold participation* $\mathbf{v_c}$*, the seller's expected revenue can be written as*

$$R(\mathbf{p}, \mathbf{x}, \mathbf{v_c}) = E_{\mathbf{v}}\Big\{ \sum_{i=0}^{N} p_i(\mathbf{m}(\mathbf{v})) \tilde{J}_i(v_i) \Big\} - c_0 - \sum_{i=1}^{N} A_i. \tag{12}$$

where $A_i = U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) F_i(v_c^{(i)}) + (1 - F_i(v_c^{(i)})) U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c})$ *and* $\tilde{J}_i(v_i) = J_i(v_i) + c_0 + \sum_{j \geq 0} e_{j,i}$*. We call* $\tilde{J}_i(v)$ *the augmented virtual value function.*

Proof: see appendix. Note that $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j \geq 0} e_{j,0}$.

Here and hereafter, if $x < \underline{t}_i - \frac{1}{f_i(\underline{t}_i)}$, $\tilde{J}_i^{-1}(x)$ is defined as $\underline{t}_i$; if $x > \overline{t}_i$, $\tilde{J}_i^{-1}(x)$ is defined as $\overline{t}_i$. Based on Lemma 2, we are then able to characterize the optimal participation and the optimal auction.

Proposition 1: *(i) All types of bidders participate. The optimal nonparticipation threats take the following form: If only bidder i does not show up, the item is assigned to the one (including the seller) generating him the smallest externality given that this smallest externality is nonpositive, otherwise the seller destroys the item. (ii) Every participating bidder pays a nonnegative entry fee, which equals the absolute value of the smallest possible* *externality to him. This smallest externality must be nonpositive due to the option of destroying the object by the seller. (iii) If all bidders participate, the object is assigned to the player (including the seller) with the highest "augmented virtual value", provided this value is nonnegative. If this value is negative, the object is destroyed by the seller. The winning bidder i pays* $\tilde{J}_i^{-1}(\max\{0, \max_{j=0,j\neq i}^N \tilde{J}_j(v_j)\})$ *besides the entry fee. On top of the entry fee, each losing bidder pays a payment (positive or negative) equal to the externality on him at the outcome of the auction.*

Proof of Proposition 1: From (12), a truthful direct semirevelation mechanism must be optimal if it satisfies the following 2 conditions. First, it minimizes $F_i(v_c^{(i)})U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c})$ $+(1-F_i(v_c^{(i)}))U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c})$, $\forall i \in \mathcal{N}$. Second, it also maximizes $\sum_{i=0}^N p_i(\mathbf{m}(\mathbf{v}))\tilde{J}_i(v_i)$, $\forall \mathbf{v} \in \mathcal{V}$. We next put forward a mechanism $(\mathbf{p}^*, \mathbf{x}^*, \mathbf{v_c^*})$ satisfying the above criterion and thus maximizes the seller's expected revenue.

First, set $\mathbf{v_c^*} = (v_c^{*(i)})$ where $v_c^{*(i)} = \underline{v}_i$, $i \in \mathcal{N}$. In this case, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$ can be pushed to take the lowest possible value $\min_{j\geq 0} e_{i,j}$. This can be achieved by the following specification. Set $p_{j_0}(m_i, \mathbf{v}_{-i}) = 1$ where $m_i = \emptyset$, $j_0 = argmin_{j\geq 0, j\neq i} e_{i,j}$ if $e_{i,j_0} \leq 0$. Otherwise set $p_j(m_i, \mathbf{v}_{-i}) = 0$ for $j \geq 0, j \neq i$. Note that if $\min_{j\geq 0, j\neq i} e_{i,j} > 0$, the above specification implies that the item is then destroyed by the seller if bidder i does not participate. When the item is destroyed, externalities cease to exist. As $e_{i,i} = 0$, $\forall i \in \mathcal{N}$, we have that $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*}) = \min_{j\geq 0} e_{i,j}$ always holds for the above specifications. Note that $\min_{j\geq 0} e_{i,j} \leq 0$ is the strongest threat possible for bidder i's nonparticipation.

Second, $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$ can be driven down to be exactly equal to $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$, which in turn equals $\min_{j\geq 0} e_{i,j}$. Note that $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$ can not be lower than $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$ from the participation constraint. The full participation payment $x_i^*(\mathbf{v})$ pushing $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$ to $\min_{j\geq 0} e_{i,j}$ is defined as below for any given set of full participation winning probabilities $p_i(\mathbf{v})$, $\forall 0 \leq i \leq N$. From Lemma 1, we have $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*})$

$= U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c^*}) - \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c^*}) ds_i$. Thus from (2) $x_i^*(\mathbf{v})$ must satisfy

$$\min_{j\geq 0} e_{i,j} = E_{\mathbf{v}_{-i}}\left(v_i\ p_i(\mathbf{v}) + \sum_{j\geq 0} e_{i,j}\ p_j(\mathbf{v}) - x_i^*(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} p_i(s_i, \mathbf{v}_{-i}) ds_i\right), \forall i \in \mathcal{N}. \quad (13)$$

Naturally, we define

$$x_i^*(\mathbf{v}) = v_i\ p_i(\mathbf{v}) + \sum_{j\geq 0} e_{i,j}\ p_j(\mathbf{v}) - \min_{j\geq 0} e_{i,j} - \int_{\underline{v}_i}^{v_i} p_i(s_i, \mathbf{v}_{-i}) ds_i,\ \forall i \in \mathcal{N}. \quad (14)$$

Clearly, the above defined $\mathbf{v_c^*}$, the nonparticipation threats and full-participation payments $\mathbf{x}^*(\mathbf{v}) = (x_i^*(\mathbf{v}))$ minimize $F_i(v_c^{(i)}) U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + (1 - F_i(v_c^{(i)})) U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c})$, $\forall i \in \mathcal{N}$, for any given set of full participation winning probabilities $p_i(\mathbf{v}),\ \forall 0 \leq i \leq N$.

Third, we define the full-participation winning probabilities $\mathbf{p}^* = (p_i^*(\cdot))$ which maximize $\sum_{i=0}^N p_i(\mathbf{v}) \tilde{J}_i(v_i)$. Clearly, the winning probability of player $i,\ i = 0, 1, ..., N$ should be defined as follows.

$$p_i^*(\mathbf{v}) = \begin{cases} 1, & if\ \tilde{J}_i(v_i) > \max_{j=0, j\neq i}^N \tilde{J}_j(v_j)\ and\ \tilde{J}_i(v_i) \geq 0, \\ 0, & otherwise. \end{cases} \quad (15)$$

As usual, tie is broken randomly. Clearly, the above defined $\mathbf{v_c^*}$ and $\mathbf{p}^*(\mathbf{v})$ maximize $\sum_{i=0}^N p_i(\mathbf{m}(\mathbf{v})) \tilde{J}_i(v_i)$, $\forall \mathbf{v} \in \mathcal{V}$.

The full-participation payments $\mathbf{x}^*(\mathbf{v})$ are then set as follows according to (14). For bidder $i,\ i \in \mathcal{N}$,

$$x_i^*(\mathbf{v}) = \begin{cases} \tilde{J}_i^{-1}(\max\{0, \max_{j=0, j\neq i}^N \tilde{J}_j(v_j)\}) + E_i, & if\ i\ wins, \\ e_{i,j} + E_i, & if\ \ j(\geq 0)\ wins, \\ E_i, & if\ the\ object\ is\ destroyed, \end{cases} \quad (16)$$

where $E_i = -\min_{j\geq 0} e_{i,j}$.

The full-participation winning probabilities and payments functions $\mathbf{x}^*$ and $\mathbf{p}^*$ together with the optimal threats lead to a Nash equilibrium in which every bidder participates and reveals truthfully their types, because the conditions in lemma 1 are satisfied. Since

$\mathbf{v}_\mathbf{c}^*$ and $\mathbf{p}^*$ maximize the seller's expected revenue in (12), we have that $\mathbf{v}_\mathbf{c}^* = (\underline{v}_i)$ is the set of optimal participation thresholds, and the full-participation winning probabilities and payments functions $\mathbf{x}^*$ and $\mathbf{p}^*$ together with the optimal threats constitute the optimal auction rule. In the same spirit of Jehiel, Moldovanu and Staccheti (1996), there is no need to consider the joint deviation from the Nash equilibrium.[5] Thus all the other winning probabilities and payments functions which are not relevant to the equilibrium path can be specified in any way. □

If $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0$, the object is never destroyed by the seller. If instead $\tilde{J}_0(v_0) = v_0 + c_0 + \sum_{j\geq 0} e_{j,0} < 0$, the object is destroyed by the seller in probability $\prod_{i=1}^N Prob(p_i^*(\mathbf{v}) = 0)$, i.e., $\prod_{i=1}^N F_i(J_i^{-1}(-c_0 - \sum_{j\geq 0} e_{j,i}))$.

2.3 Implementation of the Optimal Auction in Symmetric Setting

We now consider the implementation of the optimal mechanism derived above in a symmetric setting. In this symmetric setting, $F_i(\cdot) = F(\cdot),\ f_i(\cdot) = f(\cdot)$ on support $[\underline{v}, \overline{v}]$, $\forall i \in \mathcal{N}$. In addition, $e_{i,0} = e_{j,0},\ e_{0,i} = e_{0,j},\ e_{i,j} = e, \forall i, j \in \mathcal{N}$. As usual, we assume the regularity condition that the virtual valuation $J(v) = v - \frac{1-F(v)}{f(v)}$ is an increasing function.

Based on (15) and (16), we have the winning probability of player i, $\forall i \in \mathcal{N}$, which is defined as

$$p_i^*(\mathbf{v}) = \begin{cases} 1 & \text{if} \quad v_i \geq z_i(\mathbf{v}_{-i}), \\ 0 & \text{if} \quad v_i < z_i(\mathbf{v}_{-i}), \end{cases} \tag{17}$$

[5] Please refer to note 11 in Jehiel, Moldovanu and Staccheti (1996).

and

$$x_i^*(\mathbf{v}) = \begin{cases} z_i(\mathbf{v}_{-i}) - \min_{j\geq 0} e_{i,j}, & if\ i\ wins, \\ e_{i,j} - \min_{j\geq 0} e_{i,j}, & if\ \ j(\geq 0)\ wins, \\ -\min_{j\geq 0} e_{i,j}, & if\ the\ object\ is\ destroyed, \end{cases} \tag{18}$$

where $z_i(\mathbf{v}_{-i}) = \max\{\max_{j\neq i, j\in\mathcal{N}} v_j, \tilde{J}^{-1}(\max\{0, v_0 + c_0 + \sum_{j\geq 0} e_{j,0}\})\}$. In addition,

$$p_0^*(\mathbf{v}) = \begin{cases} 1, & if\ \tilde{J}(\max_{j=1}^N v_j) < v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0, \\ 0, & otherwise. \end{cases} \tag{19}$$

Eq. (18) means that every bidder i pays an entry fee of $-\min_{j\geq 0} e_{i,j}$. Moreover, if bidder i wins, he pays an additional payment of $z_i(\mathbf{v}_{-i})$. If he loses, he pays an additional payment that equals the externality he enjoys or suffers. From (19), if $v_0 + c_0 + \sum_{j\geq 0} e_{j,0} \geq 0$, the seller keeps it when the object is not sold out, while if $v_0 + c_0 + \sum_{j\geq 0} e_{j,0} < 0$, it is optimal for the seller to destroy the unsold object. In case that the sum of the seller's valuation, the destroying cost of the seller and the total externalities to the bidders is negative when the seller keeps the item, intuitively the seller is better off by destroying the object and collecting a payment from each bidder which equals the externality to him.

Based on the above results, we have the following proposition that describes the implementation of the optimal auction.

Proposition 2: *In a symmetric setting with externalities among the seller and bidders, a modified second-price sealed-bid auction with the following features is the optimal auction. (a) All types of bidders participate. The optimal nonparticipation threats take the same form as in Proposition 1(i). Every participant pays a nonnegative entry fee that is defined as in Proposition 1(ii). (b) If all bidders participate, the highest bidder wins if his bid is higher than the reserve price, and he pays the second highest bid or the reserve price, whichever is higher. The reserve price is set at* $\tilde{J}^{-1}(\max\{0, v_0 + c_0 + \sum_{j\geq 0} e_{j,0}\})$*. Each losing bidder pays an additional payment (positive or negative) equal to the externality on* *him at the outcome of the auction. (c) If no bidder bids higher than the reserve price, the seller may keep or destroy the item. The necessary and sufficient condition for the seller to destroy the auctioned item (dismantling nuclear weapon) is that the sum of the seller's valuation, the destroying cost of the seller and the total externalities to the bidders if the seller keeps the item is negative.*

Each bidder's payment is adjusted by the amount of externality on him, while he/she suffers or enjoys this externality at the same time. This leads to an environment in which bidders bid as if there is no externality on them. Based on similar arguments for the standard second-price auction, bidding his/her true valuation when participating is a weakly dominant strategy for every bidder in the auction specified in Proposition 2. This is intuitively why a modified second-price auction with these additional payments is optimal, provided that the reserve price and entry fee are properly set. According to Proposition 2, the entry fee is set at the highest possible level which can be supported by the optimal nonparticipation threats, and the optimal reserve price is set in a way to reflect the bidders' contribution adjusted by the externalities involved.

When $e_{i,0}$ is negative enough such that $\min_{j\geq 0} e_{i,j} = e_{i,0}, \forall i$, for the optimal auction we then have $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}, \underline{\mathbf{v}}_{\mathbf{i}}) = U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}, \underline{\mathbf{v}}_{\mathbf{i}}) = e_{i,0}, \forall i$. From (12), the optimal expected revenue for the seller is

$$R^*(\mathbf{p_0}, \mathbf{x_0}) = -c_0 - \sum_{j\geq 0} e_{j,0} + \int_{\mathcal{V}} \Big\{ p_0^0(\mathbf{m}(\mathbf{v}))(v_0 + c_0 + \sum_{j\geq 0} e_{j,0}) \\ + \sum_{i=1}^{N} p_i^0(\mathbf{m}(\mathbf{v}))(v_i + c_0 + \sum_{j\geq 0} e_{j,i} - \frac{1 - F_i(v_i)}{f_i(v_i)}) \Big\} \mathbf{f}(\mathbf{v}) d\mathbf{v},$$

where $(\mathbf{p_0}, \mathbf{x_0})$ is the optimal auction rule when the externalities are $e_{i,0},\ i \in \mathcal{N}$. Let $R'^*(\mathbf{p_0}, \mathbf{x_0})$ denote the value of the right-hand-side of $R^*(\mathbf{p_0}, \mathbf{x_0})$ when $e_{i,0}$ decreases to $e'_{i,0}\ i \in \mathcal{N}$. Clearly $R'^*(\mathbf{p_0}, \mathbf{x_0}) \geq R^*(\mathbf{p_0}, \mathbf{x_0})$ as $p_0(\mathbf{m}(\mathbf{v})) \in [0,1]$. Suppose when $e_{i,0}$ decreases to $e'_{i,0}\ i \in \mathcal{N}$, the corresponding optimal auction rule changes to $(\mathbf{p}', \mathbf{x}')$. Denote the optimal expected revenue by $R^*(\mathbf{p}', \mathbf{x}')$ when the externalities are $e'_{i,0},\ i \in \mathcal{N}$. We must have $R^*(\mathbf{p}', \mathbf{x}') \geq R'^*(\mathbf{p_0}, \mathbf{x_0})$. Therefore, $R^*(\mathbf{p}', \mathbf{x}') \geq R^*(\mathbf{p_0}, \mathbf{x_0})$, i.e., the seller's optimal expected revenue increases as $e_{i,0}$ decreases. This help s to explain why North Korea tries to convince the relevant countries that it owns very powerful nuclear weapon.

2.4 When Bidders Have Private Information on the Externalities They Create for Others

An interesting question is to what extent the results obtained in a common-knowledge externalities setting apply in a multi-dimensional setting where bidders have private information on the externalities they create for others. Specifically, we consider the setting where $e_{j,i}, \forall 0 \leq j \leq N$, are bidder i's private information. For simplicity, we assume that all v_k and $e_{j,i}$, $\forall 0 \leq k, i, j \leq N$ are mutually independent.

Jehiel, Moldovanu and Staccheti (1996) look at a 2-dimensional setting where winning bidder impose the same externality on other bidders. They show that the winning probability of any bidder must not depend on his/her externality signal because of the rationality condition. Therefore the private information on externality is a redundant signal, and thus the auction design problem is a one-dimensional program in nature. Although Jehiel, Moldovanu and Staccheti (1996) show these results while assuming losing bidders suffer the same externality, it is clear that all these results still hold when players experience heterogenous externalities, as long as players' private information is the externalities they create for other players. Specifically, the optimal auction problem can be transformed into a one-dimensional program by first integrating over the externalities dimensions. The one-dimensional program obtained is fundamentally the same as the problem assuming common-knowledge externalities. The only difference is that in the obtained one-dimensional problem, the common-knowledge externalities in the baseline setting are replaced by the expected externalities. Note that these expected externalities are also common-knowledge as the distributions of the externalities are common-knowledge.

Based on the above arguments, clearly the results obtained in the common-knowledge setting still hold if we replace the common-knowledge externalities by the expectations of the private-information externalities.

3 The Case of Financial Externalities

In the last section, we studied the optimal auction design when the externalities among the players are exogenous. As a special feature, the optimal auction requires that every bidder's payment consists of a component which equals the externalities to him at the outcome of the auction. This section investigates whether these findings apply to the case of financial externalities.

Let us first consider a setting of financial externalities formulated in Maasland and Onderstal (2002). In their setting, bidders' valuations follow the same distribution on $[\underline{v}, \overline{v}]$. Each bidder enjoys a positive externality which equals a proportion (denoted by $\phi < \frac{1}{N-1}$) of the total payments of the other bidders. Here, N is the number of bidders as before. These are all the externalities considered in this setting, there is no externality between the seller and bidders. Denote the payment of bidder i by x_i. Bidder i's payoff is then $v_i - x_i + \phi \sum_{j=1, j \neq i}^{N} x_j$ if he wins and pays x_i; his payoff is $(\phi \sum_{j=1, j \neq i}^{N} x_j) - x_i$ if he pays x_i while another player j (seller or bidder) wins.[6]

Under this specification, Maasland and Onderstal (2002) show that a lowest-price all-pay auction with proper entry fee and reserve price is optimal in a symmetric independent private value setting. All types of bidders participate. However, there exists a bidding threshold $\hat{v}$ ($> \underline{v}$) which is also the threshold winning type. Only the bidders whose valuations are no less than $\hat{v}$ bid. The bidders whose valuations are lower than $\hat{v}$ pay the entry fee but abstain from bidding. The bidding function is increasing, and the bidder

[6]Destroying the object is never optimal in this setting.

with lowest valuation has zero expected payoff.

It seems that this lowest-price all-pay auction has little similarity with the optimal auction discovered in section 2.3. Thus an interesting question would be whether the findings for the exogenous externality setting still hold when the externalities are specified as in Maasland and Onderstal (2002). The answer is positive, as will be shown below.

Maasland and Onderstal (2002) establish a revenue-equivalence theorem which says that the payoff of the lowest type and the winning probabilities fully determine the seller's expected revenue. Based on this revenue-equivalence theorem, we put forward an alternative modified second-price auction discribed below, which is revenue-equivalent to the above lowest-price all-pay auction.

(a.1) There is no entry fee, the reserve price is $\hat{v}$ $(> \underline{v})$;

(a.2) If at least one bidder does not participate, the seller keeps the item to create zero externality for all bidders;

(a.3) If all participate, the highest bidder wins if his bid is no less than the reserve price $\hat{v}$, and his payment consists of two components. First, he pays b_1, which is the second highest bid or the reserve price $\hat{v}$ $(> \underline{v})$, whichever is higher. Second, he pays $\phi(N-1)b_2$, where $b_2 = \frac{\phi b_1}{1-\phi(N-2)-\phi^2(N-1)} > 0$. Every losing bidder pays b_2. If the highest bid is less than $\hat{v}$, the seller keeps the item, and no one pays.

It can be directly verified that under rule (a.3), all bidders (winner or loser) make a payment component which is equal to the financial externalities on them. This component eliminates the impact of the financial externalities on the bidding behavior. For the winner, the first component b_1 of his payment does not depend on the payments of other bidders; while the second component of $\phi(N-1)b_2$ equals exactly the financial externalities due to the payments of other bidders. For any losing bidder, the financial externality on him is $\phi(b_1 + \phi(N-1)b_2 + (N-2)b_2)$ which equals b_2 by the construction of b_2. In other words, each loser pays the financial externality on him like the optimal action established in section 2.3. Based on the above discussion, we see that the payoff structure under rules (a.1)-(a.3) mimics that of a standard second-price auction without externalities. Thus it is not a surprise that we have the following result.

Lemma 3: *For the auction defined by* $(a.1)$, $(a.2)$ *and* $(a.3)$*, all bidders participate and bid their ture values. Moreover, all losing bidders get zero payoff.*

Proof: see appendix.

Base on Lemma 3, the lowest type has no chance to win in the auction defined by $(a.1)$, $(a.2)$ and $(a.3)$, thus his expected payoff is zero. Moreover, the winning probability function is the same as that of the lowest-price all-pay auction. Therefore, based on the revenue-equivalence theorem in Maasland and Onderstal (2002), these two auctions are revenue-equivalent. This result is formally stated in the following proposition.

Proposition 3: *The modified second price auction defined by* $(a.1)$, $(a.2)$ *and* $(a.3)$ *is revenue equivalent to the optimal lowest-price all-pay auction established by Maasland and Onderstal (2002).*

In Goeree, Maasland, Onderstal and Turner (2005), each bidder instead enjoys a positive externality which equals a proportion (denoted by ϕ, assume $\phi < \frac{1}{N}$) of the total payments of all bidders. Denote the payment of bidder i by x_i. Bidder i's payoff is then $v_i - x_i + \phi \sum_{j=1}^{N} x_j$ if he wins and pays x_i; his payoff is $(\phi \sum_{j=1}^{N} x_j) - x_i$ if he pays x_i while another player j (seller or bidder) wins. Similar to Maasland and Onderstal (2002), they show that a two-stage lowest-price all-pay auction with proper entry fee and reserve price is optimal. In the first stage, bidders make the decision whether to pay the entry fee and participate. All types of bidders participate, however, there exists a bidding threshold $\hat{v}'$ $(> \underline{v})$ which is also the threshold of winning type. Only the bidders whose valuations are no less than $\hat{v}'$ bid. The bidding function is increasing, and the bidder with lowest valuation has zero expected payoff.

In this setting, we can establish similar results as Proposition 3 by two slight modifications to (a.1) and (a.3). First, $\hat{v}$ in (a.1) is replaced by the bidding threshold $\hat{v}'$ in Goeree, Maasland, Onderstal and Turner (2005). Second, in (a.3) every bidder instead pays an additional payment of $b_2 = \frac{\phi b_1}{1-\phi N}$.

Corollary 1: *If we replace the $\hat{v}$ in (a.1) by the bidding threshold as in Goeree, Maasland, Onderstal and Turner (2005) and let $b_2 = \frac{\phi b_1}{1-\phi N}$ in (a.3), then the modified second price auction defined by* (a.1), (a.2) *and* (a.3) *is revenue equivalent to the optimal two-stage lowest-price all-pay auction established by Goeree, Maasland, Onderstal and Turner (2005).*

Consistent with the findings in sections 2.2 and 2.3, for these alternative auctions, every bidder's payment consists of a component which equals the externalities to them at the outcome of the auction. Moreover, the spirit of (a.1) and (a.2) also catch the essence of constructing the optimal nonparticipation threats and optimal entry fees in section 2.2 and 2.3. All these results suggest that the findings from our baseline setting remain valid when the financial externalities are considered.

4 Conclusion

This paper provides a complete characterization of the optimal auction maximizing the seller's expected revenue when the externalities among all players (seller and bidders) are allowed. The externalities are not restricted to be uniformly negative or positive. We show that introducing the possibility for the seller to destroy the auctioned item at a cost enlarges the freedom of optimal auction design with externalities. Specifically, destroying the item can be both an optimal threat and an optimal outcome. As optimal threat for bidder i's nonparticipation, the item is assigned to the one (including the seller) generating bidder i the smallest externality provided this smallest externality is nonpositive. Otherwise the seller uses destroying the item as an optimal threat. Due to this optimal threat, the optimal entry fees are always nonnegative. The necessary and sufficient condition for the seller to destroy the auctioned item (dismantle nuclear weapon) as an optimal outcome is that the sum of the seller's valuation, the dismantling cost of the seller and the total externalities on the bidders if the seller keeps the item is negative. Furthermore, the optimal reserve price is set differently depending on whether the above sum is positive. Moreover, when bidders suffer highly negative externalities if the seller holds the item, we show that the seller's expected revenue increases as these externalities become more negative. This provides an alternative explanation why North Korea tries to convince relevant countries that its nuclear weapons are powerful.

Jehiel, Moldovanu and Staccheti (1996) point out that the seller is better off by not selling at all if the sum of externalities generated by a sale is larger than all valuations. Our analysis further reveals that the seller is better off by dismantling while extracting payments from all buyers, if the above mentioned sum of the seller's valuation, the dismantling cost and the total externalities to the bidders when the seller keeps the item is negative. This discloses that the crucial force driving the dismantling result is the externalities on the bidders imposed by the seller instead of those caused by a sale.

A unique feature of the optimal auction established is that every bidder's payment consists of a component which equals the externalities on them at the outcome of the auction. This component eliminates the impact of the externalities on strategic bidding behavior. Thus, introducing these additional payments leads to a situation that mimics a standard auctions design problem with zero externalities on bidders. This is why a modified second-price auction with these additional payments is optimal, provided the entry fees and reserve price are appropriately set. In addition, since the sum of this additional payment and the entry fee is always nonnegative, intuitively there is no loss of generality to consider only the full-participation mechanisms for the optimal auction with externalities.[7] This result has been shown in this paper through a formal procedure.

Although the above feature of the optimal auction and other findings are established in a baseline setting where the externalities are common-knowledge, these are still valid in an environment where players have private information on the externalities they create for others. Moreover, we show that these insights hold even when the externality to every bidder depends on other bidders' total payments or those of all bidders. Based on these insights, revenue-equivalent modified second-price auctions are constructed for those optimal lowest-price all-pay auctions discovered by Maasland and Onderstal (2002) and Goeree, Maasland, Onderstal and Turner (2005).

[7]Please refer to footnote 3.

Appendix

Proof of Lemma 1: From (2), we have

$$U_i(v_i, v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) = U_i(v_i', v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + (v_i - v_i')Q_i(v_i'; \mathbf{p}, \mathbf{v_c}),\ \forall v_i,\ v_i' \in [v_c^{(i)},\ \overline{v}_i],\ \forall i \in \mathcal{N}. \quad \text{(A.1)}$$

From (6) and (A.1), we have

$$U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \geq U_i(v_i', v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + (v_i - v_i')Q_i(v_i'; \mathbf{p}, \mathbf{v_c}),\ \forall v_i,\ v_i' \in [v_c^{(i)},\ \overline{v}_i],\ \forall i \in \mathcal{N}. \quad \text{(A.2)}$$

Note (6) is equivalent to (A.2).

Using (A.2) twice, we have for $\forall\ v_i,\ v_i' \in [v_c^{(i)},\ \overline{v}_i],\ \forall\ i \in \mathcal{N}$,

$$(v_i - v_i')Q_i(v_i'; \mathbf{p}, \mathbf{v_c}) \leq U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) - U_i(v_i', v_i'; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \leq (v_i - v_i')Q_i(v_i; \mathbf{p}, \mathbf{v_c}). \quad \text{(A.3)}$$

(A.3) implies (9). From (A.3), we have for $\forall\ s_i,\ s_i + \delta \in [v_c^{(i)},\ \overline{v}_i],\ \forall\ i \in \mathcal{N}$,

$$Q_i(s_i; \mathbf{p}, \mathbf{v_c})\delta \leq U_i(s_i + \delta, s_i + \delta; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) - U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \leq Q_i(s_i + \delta; \mathbf{p}, \mathbf{v_c})\delta. \quad \text{(A.4)}$$

Since $Q_i(s_i; \mathbf{p}, \mathbf{v_c})$ increases with s_i, (A.4) implies

$$\frac{U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c})}{ds_i} = Q_i(s_i; \mathbf{p}, \mathbf{v_c}),\ \forall i \in \mathcal{N},\ \forall\ s_i \in [v_c^{(i)}, \overline{v}_i], \quad \text{(A.5)}$$

where $Q_i(s_i; \mathbf{p}, \mathbf{v_c})$ is Riemann integrable, so

$$\int_{v_c^{(i)}}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c})ds_i = U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) - U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c}). \quad \text{(A.6)}$$

(A.6) implies (10), and (11) is directly from (4). Thus (9)-(11) are derived from (4) and (6). Now we have to show (4) and (6) from (8)-(11).

$\forall v_c^{(i)} \leq s_i \leq v_i \leq \overline{v}_i,\ \forall i \in \mathcal{N}$, (9) and (10) imply

$$\begin{aligned} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) &= U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{s_i}^{v_i} Q_i(r_i; \mathbf{p}, \mathbf{v_c})dr_i \\ &\geq U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{s_i}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c})dr_i \\ &= U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + (v_i - s_i)Q_i(s_i; \mathbf{p}, \mathbf{v_c}). \end{aligned}$$

Similarly, $\forall v_c^{(i)} \le v_i \le s_i \le \overline{v}_i,\ \forall i \in \mathcal{N}$, (9) and (10) imply

$$\begin{aligned} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) &= U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{v_i}^{s_i} Q_i(r_i; \mathbf{p}, \mathbf{v_c}) dr_i \\ &\ge U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{s_i}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c}) dr_i \\ &= U_i(s_i, s_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + (v_i - s_i) Q_i(s_i; \mathbf{p}, \mathbf{v_c}). \end{aligned}$$

Thus we have (A.2), i.e., (6) is shown. Equation (4) is directly derived from (8), (10) and (11). □

Proof of Lemma 2: From (2),

$$\begin{aligned} &\int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, m_i(v_i); \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i \\ &= \int_{\underline{v}_i}^{\overline{v}_i} \left(\int_{\mathcal{V}_{-i}} (v_i\ p_i(\mathbf{m}(\mathbf{v})) + \sum_{j\ge 0} e_{i,j}\ p_j(\mathbf{m}(\mathbf{v})) - x_i(\mathbf{m}(\mathbf{v}))) \mathbf{f_{-i}}(\mathbf{v_{-i}}) d\mathbf{v}_{-i} \right) f_i(v_i) dv_i \\ &= \int_{\mathcal{V}} (v_i\ p_i(\mathbf{m}(\mathbf{v})) + \sum_{j\ge 0} e_{i,j}\ p_j(\mathbf{m}(\mathbf{v})) - x_i(\mathbf{m}(\mathbf{v}))) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \end{aligned} \tag{A.7}$$

where $\mathbf{f_{-i}}(\mathbf{v_{-i}}) = \prod_{j=1, j\ne i}^{N} f_j(v_j)$ is the density of $\mathbf{v}_{-i}$, and $\mathbf{f}(\mathbf{v}) = \prod_{i=1}^{N} f_i(v_i)$ is the density of $\mathbf{v}$.

From (A.7), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, m_i(v_i); \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i \\ &= \int_{\mathcal{V}} (p_0(\mathbf{m}(\mathbf{v})) \sum_{j\ge 1} e_{j,0} + \sum_{i=1}^{N} [(v_i + \sum_{j\ge 1} e_{j,i})\ p_i(\mathbf{m}(\mathbf{v})) - x_i(\mathbf{m}(\mathbf{v}))]) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \end{aligned} \tag{A.8}$$

Note that $e_{i,i} = 0,\ \forall i \ge 0$. From (1) and (A.8),

$$\begin{aligned} &R(\mathbf{p}, \mathbf{x}, \mathbf{v_c}) = -c_0 - \sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, m_i(v_i); \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i \\ &+ \int_{\mathcal{V}} \Big(p_0(\mathbf{m}(\mathbf{v}))(v_0 + c_0 + \sum_{j\ge 0} e_{j,0}) + \sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{v}))[(v_i + c_0 + \sum_{j\ge 0} e_{j,i})] \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \end{aligned} \tag{A.9}$$

From (10), we have

$$\int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, m_i(v_i); \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i$$

$$
\begin{aligned}
&= U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c})F(v_c^{(i)}) + \int_{v_c^{(i)}}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i \\
&= U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c})F_i(v_c^{(i)}) + \int_{v_c^{(i)}}^{\overline{v}_i} [U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \int_{v_c^{(i)}}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i] f_i(v_i) dv_i \\
&= A_i + \int_{v_c^{(i)}}^{\overline{v}_i} [\int_{v_c^{(i)}}^{v_i} Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i] f_i(v_i) dv_i = A_i + \int_{v_c^{(i)}}^{\overline{v}_i} [\int_{s_i}^{\overline{v}_i} f_i(v_i) dv_i] Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i \\
&= A_i + \int_{v_c^{(i)}}^{\overline{v}_i} [1 - F_i(s_i)] Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i = A_i + \int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i, \qquad \text{(A.10)}
\end{aligned}
$$

where $A_i = U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c})F_i(v_c^{(i)}) + (1 - F_i(v_c^{(i)}))U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c})$.

From (8), we have

$$
\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] Q_i(s_i; \mathbf{p}, \mathbf{v_c}) ds_i \\
&= \int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] \left\{ \int_{\mathcal{V}_{-i}} p_i(m_i(s_i), \mathbf{m_{-i}}(\mathbf{v_{-i}})) \mathbf{f_{-i}}(\mathbf{v_{-i}}) d\mathbf{v}_{-i} \right\} ds_i \\
&= \int_{\mathcal{V}} p_i(m_i(s_i), \mathbf{m_{-i}}(\mathbf{v_{-i}})) \frac{1 - F_i(s_i)}{f_i(s_i)} \mathbf{f_{-i}}(\mathbf{v_{-i}}) f_i(s_i) d\mathbf{v}_{-i} ds_i. \qquad \text{(A.11)}
\end{aligned}
$$

From (A.10) and (A.11), we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) f_i(v_i) dv_i \\
&= \sum_{i=1}^{N} F_i(v_c^{(i)}) U_i(v_c^{(i)}, \emptyset; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) + \sum_{i=1}^{N} (1 - F_i(v_c^{(i)})) U_i(v_c^{(i)}, v_c^{(i)}; \mathbf{p}, \mathbf{x}, \mathbf{v_c}) \\
&\quad + \int_{\mathcal{V}} \Big(\sum_{i=1}^{N} p_i(\mathbf{m}(\mathbf{v})) \frac{1 - F_i(v_i)}{f_i(v_i)} \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \qquad \text{(A.12)}
\end{aligned}
$$

From (A.9) and (A.12), we have the desired result. □

Proof of Lemma 3: Let us consider any bidder $i \in \mathcal{N}$. Denote the highest bid among all others' bids by $b^{(1)}$.

First of all, note that a losing bidder always gets zero payoff by construction. Thus every bidder participates.

Second, we show that it is a weakly dominant strategy for every bidder to bid their true valuations. Suppose $v_i < b^{(1)}$. If his bid b_i is no greater than $b^{(1)}$, this does not change his winning status, he still gets zero payoff as a losing bidder. He has no incentive to bid higher than $b^{(1)}$, since he gets negative payoff if $b_i \geq \hat{v}$. If $b_i < \hat{v}$, his payoff is still zero.

Suppose $v_i > b^{(1)}$. He has no incentive to bid higher than v_i if this does not change his winning status. If this changes his winning status, it must be the case that $v_i < \hat{v}$. However, if this is the case, he decreases his payoff by overbidding. He also has no incentive to underbid since this may decrease his payoff by losing the auction.

In summary, bidder i has no incentive to deviate from bidding v_i. $\square$

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