Voting with Preferences over Margins of Victory

Humberto Llavador[*]

Universitat Pompeu Fabra

January 2006

First version

Abstract

This paper analyzes a two-alternative voting model with the distinctive feature that voters have preferences over the support that each alternative receives, and not only over the identity of the winner. The main result of the paper is the existence of a unique equilibrium outcome with a very intuitive characterization: in equilibrium voters who prefer a higher support for one of the alternatives vote for such alternative. Its computation is equally simple: the equilibrium outcome is the unique fixed point of the *connected* survival function associated to the distribution of the electorate. This characterization works for electorates with a finite number of citizens as well as with a continuum of agents, and for scenarios with and without abstention. Finally, *strategic* voting (voting for the least preferred alternative) is common for a fraction of the electorate who favor electorally "balanced" results.

[*]humberto.llavador@upf.edu
Departament d'Economia i Empresa, Universitat Pompeu Fabra
R. Trias Fargas, 25-27, 08005 Barcelona (Spain)
tel.: (+34) 93 542 2381   fax: (+34) 93 542 1746, .

1 Introduction

It is well accepted that in most voting situations individuals show interest not only in the identity of the winner but also in the support received by each alternative. However, the literature on voting behavior restricts individual preferences by making individual utility depend only on the identity and policy of the winner. The purpose of this paper is to study, in a very general scenario, voting equilibria when agents hold preferences over vote allocations.

Our setting throughout the paper involves two alternatives and a set of individuals who can vote for one alternative or abstain. Observe that the restriction is on the number of alternatives, and not on the dimensionality of the policy space.[1] We take the view that individuals have preferences over vote allocations. Preferences are single-peaked, but no other condition is required. In particular, preferences may present discontinuities (see figure 1), and individuals may differ not only in their ideal point, but also in the form of their utility function. We analyze both an electorate with a finite number of agents and the case of a continuum of agents.

In order to keep the framework as general as possible, we take individuals' preferences over vote allocations as primitives of the model. Note, however, that one could easily derive them from preferences over policies by specifying an institutional context like a divided government (Alesina and Rosenthal, 1995), proportional representation (Ortuño-Ortín, 1997; De Sinopoli and Iannantuoni, 2003), or any other generic policymaking function that relates policies with the electoral support received by each alternative (Llavador, forthcoming).[2]

The main result of the paper proves the existence of a unique voting equilibrium outcome and provide, graphically and analytically, a very intuitive characterization: the equilibrium outcome is the unique fixed point of the *connected* survival function associated to the

---

[1]Because preferences will be defined over vote allocations and alternatives may represent bundles of policies, there is no restriction on the dimensionality of the policy space.

[2]In fact, the present voting equilibrium could be easily incorporated into any of those models. For instance, Llavador (forthcoming) offers an illustration of a political competition model with *sophisticated* voters who behave at equilibrium consistently with our findings, although with a more *ad hoc* description of voters' behavior.

distribution of the electorate (see figures 2 and 3).

The paper is organized as follows. Section 2 presents the general model. Sections 3 and 4 study and characterize equilibria for a finite electorate and for a continuum of agents, respectively. Section 5 concludes.

2 The Model

Consider a group of n individuals and two alternatives A and B. Each voter can vote for one of the alternatives or abstain, $s_i \in S = \{A, B, O\}$. A profile of actions $s = (s_1, \ldots, s_n)$ determines an electoral outcome defined as the fraction of votes for each alternative. If at least one individual votes, the electoral outcome is fully described by the fraction of votes that alternative A receives. Letting $\hat{E}_k = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$ represent possible outcomes when k agents vote, the set of all possible electoral outcomes can be constructed as $E_n = \bigcup_{k=1}^{n} \hat{E}_k$.

Define the *electoral outcome function* $\tilde{e} : S \times \cdots \times S \to E_n$ such that[3]

$$\tilde{e}(s) = \frac{|\{s_i \in s : s_i = A\}|}{|\{s_i \in s : s_i = A\}| + |\{s_i \in s : s_i = B\}|}. \tag{1}$$

The distinguishing feature of the current approach is that individuals have preferences over electoral outcomes. Let individual i's preferences be represented by the utility function $u_i : [0, 1] \to \Re$. (Observe that voters may differ in the functional form of their utilities.) Assume that preferences are single-peaked, denote by $e_i = \arg\max_{e \in E_n} u_i(e)$ voter i's preferred electoral outcome among those feasible, and assume it is unique.[4] As figure 1 shows, these are a very mild conditions. In particular, observe that preferences do not need to be continuous, concave or symmetric.

[Figure 1 about here.]

---

[3]A precise definition should assign a value to the case when nobody vote. However, since turnout will always be positive, the analysis does not depend on the assigned value.

[4]Observe that a sufficient, and not necessary, condition is that voters' ideal policies are feasible.

Let $f_n : [0,1] \to [0,1]$ represent the distribution function of voters' preferred electoral outcomes. And let $F_n(x) = \sum_{z \leq x} f_n(z)$ be the corresponding CDF.

We use Nash Equilibrium in pure strategies as our concept of equilibrium. Therefore, a voting Nash equilibrium will be a profile of strategies $s^* \in S^n$ such that no citizen has incentive to change the current electoral outcome $e^* = \tilde{e}(s^*)$ by choosing a different action from her equilibrium action s_i^*.

Definition 1 *A **voting Nash equilibrium** is a profile of strategies $s^* \in S^n$ such that for all $i = 1, \ldots, n$*

$$u_i(\tilde{e}(s^*)) \geq u_i(\tilde{e}(s_i, s_{-i}^*)) \qquad \text{for all } s_i \in S.$$

3 Equilibrium results for a finite number of voters

First we show that for a given number of voters n and a distribution of preferred policies f_n, there exists a unique electoral equilibrium outcome e^*. Furthermore, the electoral outcome e^* acts as a dividing type (not related to the *median* type), such that all voters with $e_i < e^*$ vote for B, while all voters with $e_i > e^*$ vote for A. (Voters of type e^* may split in their support or abstain.)

Before stating the theorem, we present the following lemma.

Lemma 1 *Let f_n and F_n be the PDF and CDF of a population of n individuals. Construct the correspondence $\phi_n : [0,1] \twoheadrightarrow [0,1]$ as $\phi_n(x) = [1 - F_n(x), 1 - F_n(x) + f_n(x)]$. Then there exists a unique $x_n^* \in [0,1]$ such that $x_n^* \in \phi_n(x_n^*)$.*

Proof:

Observe that $\phi_n : [0,1] \twoheadrightarrow [0,1]$. Hence, if ϕ_n is closed, it follows from Kakutani's fixed point theorem that ϕ_n has a fixed point.

Take $x^k \to \bar{x}$, $y^k \in \phi_n(x^k)$ and $y^k \to \bar{y}$. By construction of the correspondence ϕ, we can always find a ball $B_\epsilon(\bar{x})$ around $\bar{x}$ such that $\forall x \in B_\epsilon(\bar{x})$, $\phi_n(x) \subseteq \phi_n(\bar{x})$. Therefore, for a sufficiently large N and or all $k > N$, $|x^k - \bar{x}| < \epsilon$ and hence $\phi_n(\bar{x}) \supseteq \phi_n(x_n) \ni y^k$. Since

$y^k \rightarrow \bar{y}$ and $y^k \in \phi_n(\bar{x})$ for all sufficiently large k, then $\bar{y} \in \phi_n(\bar{x})$ and hence ϕ_n is closed. Therefore there exists x^* such that $x^* \in \phi_n(x^*)$.

Uniqueness follows from the fact that the survival function $1 - F_n$ is a non-increasing function and satisfies that for all $x' < x$

$$1 - F_n(x) + f_n(x) \leq 1 - F_n(x').$$

Take $x < x^*$. Recall that $x^* \in \phi(x^*)$ and hence $1 - F_n(x^*) \leq x^* \leq 1 - F_n(x^*) + f_n(x^*)$. Then

$$1 - F_n(x) \geq 1 - F_n(x^*) + f_n(x^*) \geq x^* > x. \quad \text{Hence } x \notin \phi_n(x) \text{ for all } x > x^*.$$

Similarly, take $x > x^*$. Then,

$$1 - F_n(x) + f_n(x) \leq 1 - F_n(x^*) \leq x^* < x. \quad \text{Hence } x \notin \phi_n(x) \text{ for all } x < x^*..$$

We conclude then that ϕ_n has a unique fixed point. $\quad\square$

Theorem 1 *Consider a n-citizen electoral game with single-peaked preferences over electoral outcomes. Then:*

1. *There exists an electoral Nash equilibrium $s^* = (s_1^*, \ldots, s_n^*)$.*

2. *The electoral equilibrium outcome is unique. Namely, $e^* = \tilde{e}(s^*)$ for all electoral Nash equilibria.*

3. *If e^* is an equilibrium electoral outcome, then $s_i^* = A$ for all i with $e_i > e^*$ and $s_i^* = B$ for all i with $e_i < e^*$.*

[Figure 2 about here.]

Proof:

1. First we show that an equilibrium exists. Let x_n^* be the unique fixed point of ϕ_n, as

4

defined in lemma 1. We will consider several cases and construct an equilibrium for each one.

a. $x_n^* = 1 - F_n(x_n^*)$. [Figure 2(a).]

Consider the strategy $s_i^* = A$ for $e_i > x_n^*$ and $s_i^* = B$ for $e_i \leq x_n^*$. The electoral outcome is then $\tilde{e}(s^*) = 1 - F_n(x_n^*) = x_n^*$. Individuals with $e_i > x_n^*$ ($e_i < x_n^*$) want a larger (smaller) support for A and are already voting for A (B), hence they have no incentive to change their vote. Individuals with $e_i = x_n^*$ obtain their preferred electoral outcome and do not want to change it. Everybody is playing a best response and hence s^* is an electoral Nash equilibrium.

b. $x_n^* = 1 - F_n(x^*) + f_n(x_n^*)$. [Figure 2(b).]

Consider the strategy $s_i^* = A$ for $e_i \geq x_n^*$ and $s_i^* = B$ for $e_i < x_n^*$. The electoral outcome is now $\tilde{e}(s^*) = 1 - F(x_n^*) + f_n(x_n^*) = x_n^*$. As in the previous case, everybody is playing a best response and hence s^* is an electoral Nash equilibrium.

c. $1 - F_n(x^*) < x_n^* < 1 - F_n(x_n^*) + f_n(x_n^*)$. [Figure 2(c).]

Let $s_i^* = A$ for $e_i > x_n^*$ and $s_i^* = B$ for $e_i < x_n^*$. Let the $f_n(x_n^*)$ voters with $e_i = x_n^*$ allocate their votes or abstain so that $e(s^*) = 1 - x_n^*$, which we know it is feasible since $x_n^* \in E_n$. Those who want a larger support for A (B) are voting for A (B). Any strategy is optimal for those who are obtaining their preferred electoral outcome. It follows that s^* is an electoral Nash equilibrium.

3. Second, we show that if s^* is an electoral Nash equilibrium, then $s_i^* = A$ for all $e_i > \tilde{e}(s^*)$, and $s_i^* \neq B$ for all $e_i < \tilde{e}(s^*)$.

Let $e^* = \tilde{e}(s^*)$. Suppose that $s_i^* = A$ for some voter with $e_i < e^*$. Then $e_i \leq \tilde{e}(B, s_{-i}^*) < e^*$ and hence voter i would be better off changing her vote to B, contradicting the fact that s^* is a Nash equilibrium. A similar argument shows that there cannot exist a voter with $e_i > e^*$ not voting for A.

2. Finally, we prove uniqueness of the electoral outcome. Let e^* be an equilibrium outcome.

It follows from the previous point that $1 - F_n(e^*) \leq e^* \leq 1 - F_n(e^*) + f_n(e^*)$. But then, by Lemma 1, there exists a unique such e^*. $\qquad\square$

This theorem has the following implications. First, the equilibrium outcome is graphically very appealing (see figure 2) and is very easy to calculate as the unique fixed point of the correspondence ϕ_n (constructed by "connecting" the discontinuities of the survival function $1 - F_n$). Second, only citizens obtaining their preferred outcome may abstain. Abstention is then a *strategic* decision: citizens abstain in order not to change the electoral outcome. Third, we can easily identify (even graphically) the behavior of almost all individuals. (The only unidentified behavior is perhaps that of those individuals obtaining their most preferred electoral outcome.) Finally, there will exist typically a group of voters who favor an electorally balanced result and vote *strategically* for their least preferred alternative as their favorite alternative receives a too large support from the rest of the electorate.

4 Continuum of voters

The previous analysis cannot be trivially extended to a voting game with a continuum of agents. In such games the action chosen by an individual agent does not affect the electoral outcome and hence any profile of strategies is an equilibrium. We therefore utilize a different approach. We have opted here for a limit argument, weakly approaching the distribution of the society by a continuum of agents with a sequence of *finite* societies and then analyzing the behavior of the sequence of electoral equilibrium outcomes. (Recall that there exist a unique electoral equilibrium outcome associated to a society with a finite number of individuals. See Theorem 1.) The following theorem shows that no matter how we approach the continuum CDF, the sequence of electoral equilibria always converges to the same value e^*, defined as the unique fixed point of the survival function: $e^* = 1 - F(e^*)$ (see Figure 3).

[Figure 3 about here.]

Theorem 2 *Let F be a continuous distribution function. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions that weakly converges to F, where F_n represents the CDF of a population with n agents. Letting e_n^* be the electoral equilibrium outcome associated to the distribution F_n, $n = 1, 2, \ldots$, the sequence $\{e_n^*\}_{n=1}^{\infty}$ converges to e^* as n goes to ∞, where e^* is defined as the unique solution to $e = 1 - F(e)$.*

Proof:

We want to show that

$$\forall \epsilon > 0 \ \exists N : \forall n > N \quad |e_n^* - e^*| < \epsilon$$

Take $\epsilon > 0$. Since F_n weakly converges to F we know that

$$\exists N \text{ such that } \forall n > N \quad |F_n(e^*) - F(e^*)| < \epsilon \tag{2}$$

Take $n > N$. Suppose first that $F_n(e^*) = F(e^*) = 1 - e^*$. It follows from the characterization of e_n^* that $e_n^* = e^*$ for all n, and hence $|e_n^* - e^*| = 0 < \epsilon$.

Suppose next that $F_n(e^*) < F(e^*) = 1 - e^*$. From weak convergence (see 2)

$$F_n(e^*) > F(e^*) - \epsilon = 1 - e^* - \epsilon.$$

Since F_n is a non-decreasing function, $F_n(e^* + \epsilon) \geq F_n(e^*) > 1 - (e^* + \epsilon)$. Therefore, we have that

$$F_n(e^*) < 1 - e^* \text{ and } F_n(e^* + \epsilon) > 1 - (e^* + \epsilon).$$

It follows that $e_n^* \in (e^*, e^* + \epsilon)$ and hence $|e_n^* - e^*| < \epsilon$.

Finally, suppose that $F_n(e^*) > F(e^*) = 1 - e^*$. From (2)

$$F_n(e^*) < F(e^*) + \epsilon = 1 - e^* + \epsilon.$$

Since F_n is a non-decreasing function, $F_n(e^* - \epsilon) \leq F_n(e^*) < 1 - (e^* - \epsilon)$. Therefore, we have that

$$F_n(e^*) > 1 - e^* \text{ and } F_n(e^* - \epsilon) < 1 - (e^* - \epsilon).$$

And it follows that $e_n^* \in (e^* - \epsilon, e^*)$ and hence $|e_n^* - e^*| < \epsilon$.

Therefore, we have proved that for all $n > N$ we have $|e_n^* - e^*| < \epsilon$ and hence that e_n^* converges to e^* as n goes to ∞. $\square$

5   Final remarks

This paper analyzes a two-alternative voting model with the distinctive feature that voters have preferences over margins of victory. The main result of the paper is a very intuitive characterization of the unique equilibrium outcome: at equilibrium voters who prefer a higher support for one of the alternatives vote for such alternative. Its computation is equally simple: the equilibrium outcome is the unique fixed point of the connected survival function associated to the distribution of the electorate. This characterization works for electorates with a finite number of citizens as well as with a continuum of agents. It is also worth noticing that, although the analysis is presented for the more general case where abstention is always an option, the results would remain unchanged had we made voting mandatory, except for altering the set of feasible outcomes.

The cleanness of the results makes them easy to incorporate in existing and future political economy analysis where the distribution of the vote is relevant in determining political outcomes. The analysis also applies to one candidate elections where voters may vote in favor or against the *official* candidate. In these contexts, our model provides a formal argument of why it is not uncommon for a fair number of members to vote against the candidate, even when it is known that she will be elected.

Finally, suppose no voter stresses the margin of victory, preferring outcomes where their favorite alternative gets the largest support. That is, the electorate's preferred outcomes

concentrate on 0 and 1. Then, sincere voting obtains, replicating the Nash equilibrium in weakly undominated strategies of traditional voting models.

References

Alesina, Alberto and Howard Rosenthal, *Partisan Politics, Divided Government, and the Economy*, Cambridge: Cambridge University Press, 1995.

De Sinopoli, Francesco and Giovanna Iannantuoni, "A Spatial Voting Model Where Proportional Rule Leads to Two-Party Equilibria," 2003. CEIS Tor Vergata - Research Paper Series No. 31.

Llavador, Humberto, "Electoral Platforms, Implemented Policies and Abstention." Forthcoming in *Social Choice and Welfare*.
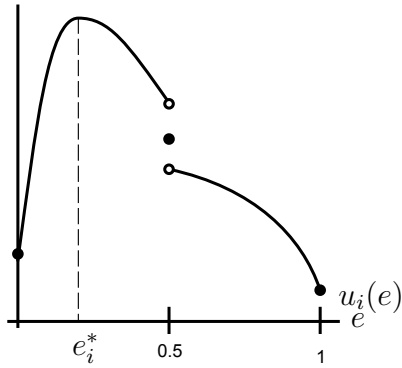
Ortuño-Ortín, Ignacio, "A Spatial Model of Political Competition and Proportional Representation," *Social Choice and Welfare*, 1997, *14*, 427–438.
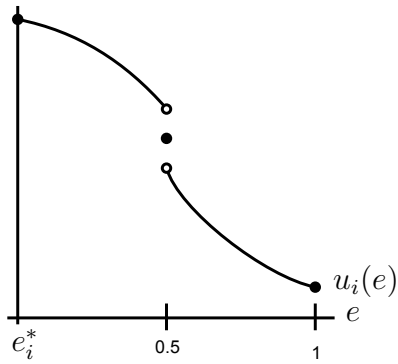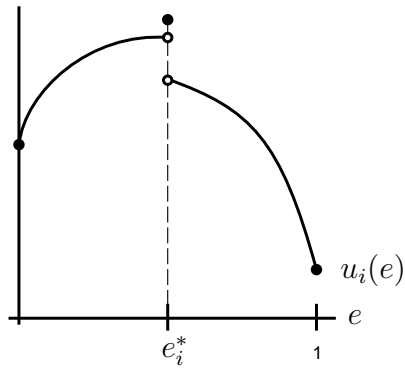
(a) Voter i's favorite outcome is B getting 100% of the vote.
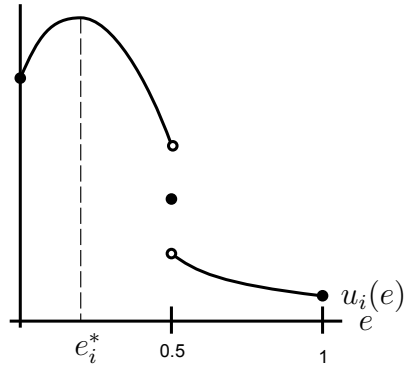
(b) Voter i prefers any outcome with B winning, but she'd rather have a little diversification of the vote.

(c) Voter i wants B to win, but she prefers A winning by a narrow margin than B getting a too large share of the vote.

(d) Voter i has a preference for close results. Anything different from a tie is "much" worse. Nevertheless, she also shows a preference for B.

Figure 1: **Examples of preferences over electoral outcomes** for an individual who shows a preference for alternative B. The examples show that continuity, concavity, or symmetry conditions are not required for the analysis. Electoral outcomes are represented by e: the fraction of the vote for alternative A.

(a) $x_n^* = 1 - F_n(x_n^*)$

(b) $x_n^* = 1 - F_n(x^*) + f_n(x_n^*)$

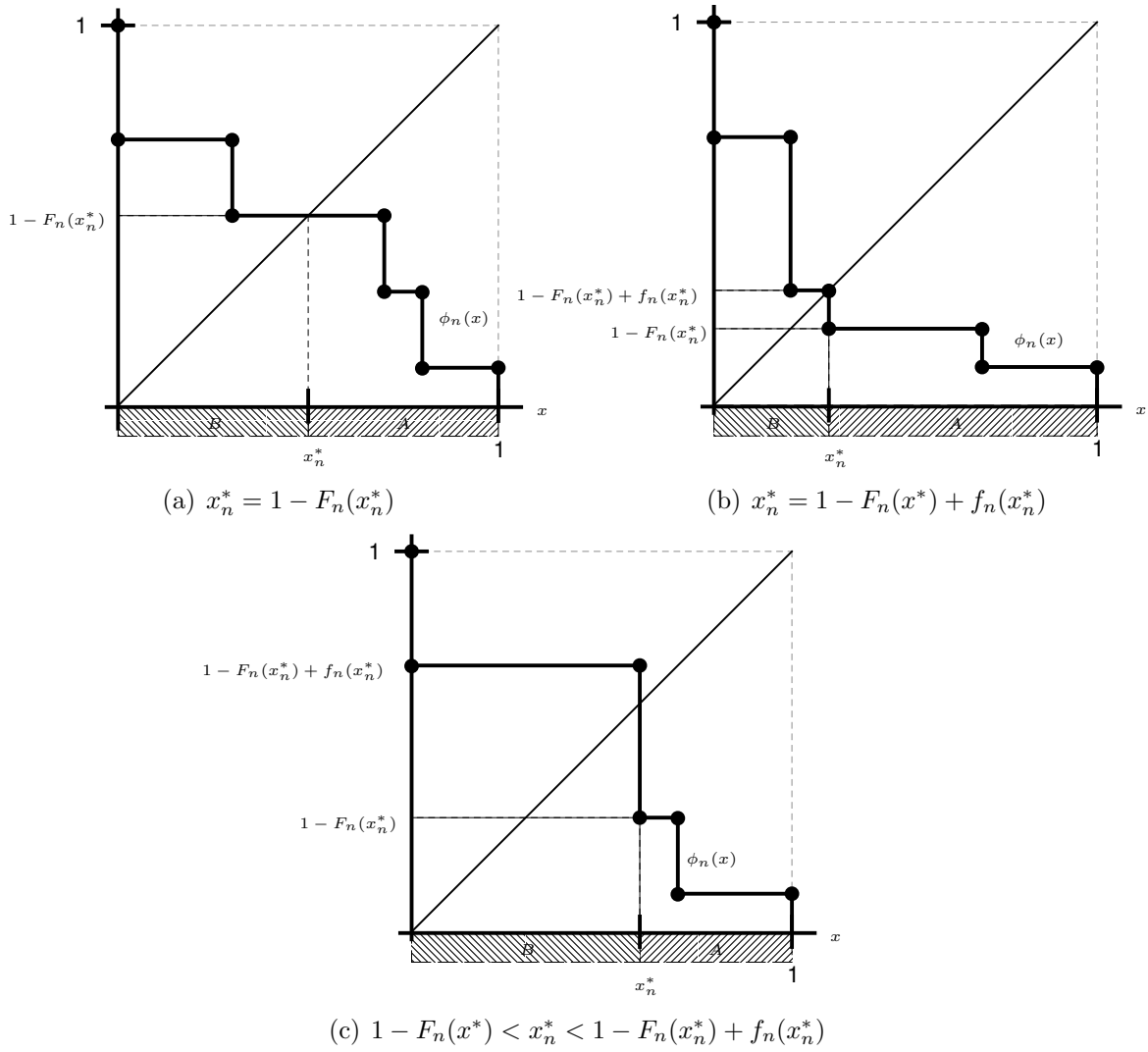(c) $1 - F_n(x^*) < x_n^* < 1 - F_n(x_n^*) + f_n(x_n^*)$

Figure 2: **Electoral equilibria for three different distributions of a n-voter electorate**. The equilibrium electoral outcome, $x_n^* \in [0, 1]$ represents the fraction of the vote for alternative A. Voters with a lower(higher) ideal electoral outcome vote for $B(A)$. The correspondence ϕ is obtained by connecting the survival function: $\phi_n(x) = [1 - F_n(x), 1 - F_n(x) + f_n(x)]$.
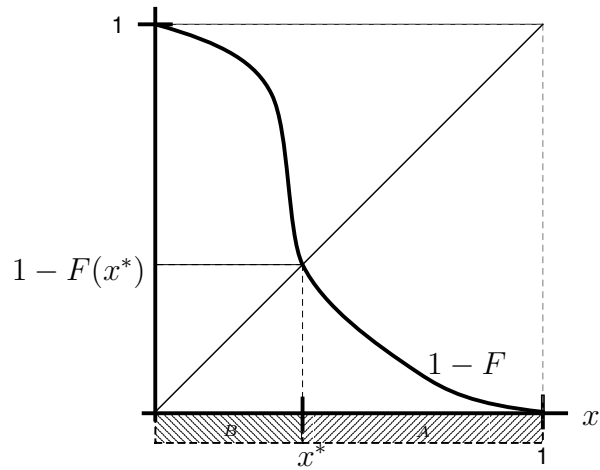
11

Figure 3: **Equilibrium outcome for an electorate with a continuum of voters**. The equilibrium electoral outcome x^* represents the fraction of the vote for alternative A. At equilibrium, the fraction of voters with a higher ideal electoral outcome $(1 - F(x^*))$ equals the fraction of voters voting for A (i.e. x^*).