The worst absolute surplus loss in the problem of commons: random priority vs. average cost*

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Summary. A good is produced with increasing marginal cost. A group of agents want at most one unit of that good. The two classic methods that solve this problem are average cost and random priority. In the first method users request a unit ex ante and every agent who gets a unit pay average cost of the number of produced units. Under random priority users are ordered without bias and the mechanism successively offers the units at price equal to marginal cost. We compare these mechanisms by the worst absolute surplus loss and find that random priority unambiguously performs better than average cost for any cost function and any number of agents. Fixing the cost function, we show that the ratio of worst absolute surplus losses will be bounded by positive constants for any number of agents, hence the above advantage of random priority is not very large.

Keywords and Phrases: Cost-sharing, Increasing marginal cost, Surplus loss, Price of anarchy.

JEL Classification Numbers: C70, D61, D70.

1 Introduction

Indivisible goods are produced with increasing marginal cost. A fixed number of risk neutral buyers want at most one unit of that good. Every agent decides independently to buy or not buy one unit of the good based on his utility and the price he faces. A mechanism is a random variable that assigns at most one unit and some cost to the agents. The total charge collected by mechanism should cover the production cost.

One interesting application of this problem arises in the context of scheduling (Lawler et al.[8], Cres and Moulin [1], Moulin [10]). Every user has a job that takes one unit of time. The planner schedules one job at a time. Each agent has the option to leave the queue at time 0 or wait until his job has been processed. The disutility (cost) of the agent is the waiting time until served. Hence those

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agents who expect a waiting time higher than their utility will balk at time 0. The management of queues in networks, for instance internet, is the canonical example of this problem (Shenker [15]).

We compare the two classic decentralized mechanisms for this allocation problem: average cost (*ac*) and random priority (*rp*) (Cres and Moulin [1][2]). Both mechanisms are the most accepted for being easy to implement and by their incentive properties.

Average cost is more familiar and simpler to implement than random priority. It is the mechanism in which all agents ex ante pay the same price. Formally it is the mechanism in which all agents simultaneously decide to buy or not buy a unit. Those agents who buy will be ordered without bias and pay true marginal cost. Those who did not buy pays nothing. In the queuing interpretation *ac* is the so called unorganized queue (Cres and Moulin [2]). Agents decide to enter the queue and server picks at random one of the agents remaining in the queue.

Under random priority users are randomly ordered without bias. The mechanism starts offering to the agents following this ordering a unit of good at cost equal to true marginal cost. Every agent decides to buy or not buy the offered unit. Those who did not get a unit of good pay nothing.[1] In the queuing interpretation *rp* is the so called organized queue (Cres and Moulin [2]). Server picks a random order without bias of the agents and they decide to enter the queue after learning their number in the queue.

If both mechanism are available, which one should we choose? Cres and Moulin [2] compared the welfare performance of these two mechanisms when the number of agents is large. They showed that neither mechanism outperforms the other. The relative performance of the two mechanisms depends much on the configuration of the agent utilities. In a large economy they concluded that *rp* manages better the crowded commons. In this case random priority will collect more surplus and overproduce less than average cost. The more crowded the economy (as a replicating process), the more *rp* outperforms *ac*, up to the point where *ac* will overproduce infinitely more than *rp*. In this limit case, *ac* will not collect any surplus relative to the efficient production, whereas *rp* will collect a positive share of the efficient surplus. These results give us powerful arguments to choose *rp* against *ac* when the commons are crowded and there are many agents. But what are we going to choose when they are not crowded? Several difficulties arise in this case.

We use a simple benchmark to compare the two mechanisms, namely the worst case scenario. For a fixed number of agents and a given cost function, this should be the utility profile that wastes the largest amount of surplus relative to the efficient surplus.

The index used in the recent literature of the price of anarchy (Koutsoupias and Papadimitriou [4] , Moulin [11]) is the worst relative gain, that is the infimum of the ratios of the relative and efficient surplus. If we use these measure

[1] A big downplay of *rp* is that lotteries may no be available, hence we may not be able to implement it. On the other hand, *ac* does not have the problem of implementation with or without lotteries. In this paper, we focus on the problem with lotteries.

rp outperforms ac. In fact, the worst relative gain of ac is 0 whereas of rp is $\frac{1}{n}$ where n is the number of agents.

On the other hand, with a fixed number of agents, we can also define the worst absolute surplus loss (*wal*) of a mechanism with respect to the efficient surplus, that is the supremum of the differences of the efficient surplus and the surplus of the mechanism in discussion, where the supremum runs over all utility profiles. This is always positive and bounded for rp and ac without any assumption on the agent configuration utilities (see lemma 1). Like the relative gain, this will give a complete order of the mechanisms.

The interpretation of the two indexes relative loss and absolute loss are interestingly different. While the first measure is normalized to treat low utility society similar to high utility society, the second does not. The worst absolute loss takes into account that a big loss in a society should not be considered equal to irrelevant small losses. To illustrate this, consider the mechanism that allocates at most one unit to the agents. It selects randomly an agent and offer him a unit at price equal to the marginal cost of the first unit c_1. No offer is made to the other agents. This mechanism has an expected relative surplus gain of $\frac{1}{n}$[2]. Therefore it outperforms ac and is equally ranked to rp in the relative gain sense. On the other hand, it has an infinite worst absolute surplus loss[3] and hence it is outperformed by rp and ac in the *wal* sense. This mechanism alert us to the more general fact that whenever a mechanism does not guarantee a unit of good to those agents with utility large enough, the worst absolute loss will be infinite and hence inferior to most mechanism in the *wal* sense, whereas it may be well ranked in the relative gain sense.

Outline of the results

The main result of the paper shows that the worst absolute surplus loss of random priority $wal(n, c, rp)$ will be smaller than the worst absolute surplus loss of average cost $wal(n, c, ac)$ for any number of agents n and any marginal cost function c. In other words, rp always outperforms ac in the *wal* sense (Theorem 1).

In the second result we estimate how large is the outperformance given by Theorem 1. We compute upper bounds for the ratio $\frac{wal(n,c,ac)}{wal(n,c,rp)}$ when the number of agents n goes to infinity. We show that for any cost function of order m,[4] the sequences $\{wal(n, c, rp)\}_n$ and $\{wal(n, c, ac)\}_n$ will also have order m (Theorem 2). Hence even though random priority outperforms average cost in this worst case scenario, this is not as strong as in the crowded economy with many agents of Cres and Moulin [2].

Related literature

[2]The agents that get a unit at the efficient surplus will get a unit at price c_1 with probability $\frac{1}{n}$, hence it collects at least $\frac{1}{n}$ of the efficient surplus. The gain is not more that $\frac{1}{n}$ because in the utility profile with exactly one agent with utility bigger than c_1 it collects exactly $\frac{1}{n}$ of the efficient surplus.

[3]Consider the utility profile with exactly two agents with utility λc_1 and the rest with utility zero. As λ goes to infinity, the efficient mechanism serves the two agents whereas this mechanism serves at most one of them, hence the loss will be unbounded.

[4]This is that the cost function is bounded by a polynomial cost function of degree m.

This work is related to the large and growing literature in computer sciences of the worst case scenario. Particularly, it is with the recent literature on the price of anarchy, introduced to measure the effects of selfish routing in a congested network. For instance Koutsoupias and Papadimitriou [4], Roughgarden and Tardos [13] and Roughgarden [14] offer the first results in this topic.

This paper is also related to the applications of the price of anarchy to the more general model of cost-sharing, where the agents share the one to one technology with increasing marginal cost. Every user request independently an amount of output and the mechanism produces together this amount and allocates the cost between agents. The combination of the price of anarchy along with related models can be found at Johari and Tsilikis [5], Johari, Mannor and Tsilikis [7], Moulin [11].

In particular, the main result of this paper is similar to the findings of Moulin [11]. He compares four classic mechanism by the worst relative surplus gain in the more general context of cost sharing with a divisible good. He finds that the relative surplus gain of the serial mechanism is $O(\frac{1}{log(n)})$ whereas the relative surplus gain of the other three mechanism: average cost pricing, incremental cost pricing and marginal cost sharing is $O(\frac{1}{n})$, where n is the number of agents and the marginal cost function is convex or concave with bounded elasticity.

2 Random Priority and Average Cost

2.1 The Model

The problem consists of the cost function $C : \mathbb{N} \to \mathbb{R}$ homogeneous in the units of the good produced with increasing marginal cost and a finite set of potential buyers $N \subset \mathbb{N}$. The marginal cost of the i-th unit is denoted by c_i, $0 < c_1 < c_2 < \cdots < c_n < \ldots$, $C(i) = c_1 + \cdots + c_i$. The derivative of the marginal cost, that is the cost increment of the $i-$th unit with respect to the $(i-1)-$th unit is denoted by δ_i, hence $c_i = \delta_1 + \ldots \delta_i$. A vector of utility profiles is denoted by $u = (u_1, \ldots, u_n) \in \mathbb{R}^N_+$. Given such utility profile, the local demand $p(c)$ is the number of agent whose utility is equal to c. The demand function is the number of agents whose utility is bigger than or equal to c, that is, $d(c) = \sum_{x \geq c} p(x)$. The demand for the $q - th$ unit is denoted by $d_q = d(c_q)$ and the number of agents with utility in $[c_q, c_{q+1})$ is denoted by p_q.

A mechanism (method) is a random variable ξ such that every utility profile $u \in \mathbb{R}^N_+$ is mapped to an allocation where every agent get at most one unit of good and the price for being served $y \in \mathbb{R}^N_+$. If the agents in S get a unit of good then the production cost is covered by the price charged to those agents: $y_S = \sum_{i \in S} y_i = C(|S|)$.

The *efficient* allocation (eff) produces q^{eff} units and serve the agents giving priority to higher utility agents, where q^{eff} is chosen such that $d_q \geq q$ for all $q \leq q^{eff}$, and $d_q < q$ for all $q \geq q^{eff} + 1$.

2.2 The mechanisms

Average cost (ac) is the mechanism where every agent decides to buy or not buy at time 0. Those agents who buy will be ordered without bias and assigned a unit of good at true marginal cost: the agent ranked t gets a good at price c_t. Those who do not buy pay nothing. Since agents are risk neutral we think this mechanism as the Nash equilibrium of the game (not necessarily unique) where every agent decides independently to buy or not buy one unit. If q^{ac} agents buy, these agents will pay $\frac{C(q^{ac})}{q^{ac}}$. Those who do not buy pay nothing. Without loss of generality (see below) we can compute the equilibrium of ac by the intersection $p^{ac} = ac(q^{ac})$ of the demand function and the ac function. It charges p^{ac} to the q^{ac} agents with highest utilities.

Another mechanism is *random priority* (rp). This method draws with uniform probability an order of the agents and offer them the goods at price equal to marginal cost. Hence agent i will get offered a unit at price c_{k+1} where k is the number or agents ranked before i who bought a unit.

Throughout the paper we think the agents are not altruistic. Whenever an agent is indifferent between buying or not buying, he will buy. This implies there is more overproduction with rp, hence rp collects less surplus. This assumption is without loss of generality.

rp has unique and unambiguous equilibrium outcomes. However, ac does not: multiple equilibriums are possible. For instance, if agent 1 has utility $\frac{c_1+c_2+c_3}{3} - \epsilon$ and the remaining agents have utility $u = \frac{c_1+c_2}{2} + \epsilon$, $\epsilon < \frac{2c_3-c_1-c_2}{6}$. Lets assume $\frac{c_1+c_2+c_3}{3} \leq c_2$. Then the rp equilibrium serves exactly one agent and it requires every agent to get a good with probability $\frac{1}{n}$. Whereas any profile where exactly two agents buy a good is an equilibrium for ac. Notice the equilibriums of ac are also welfare different. This multiplicity of equilibriums does not affect the computation of the worst case scenario of ac, we simply assume without loss of generality that the agents with higher utility get a good.

The surplus $\sigma^{\xi}(u)$ of the method ξ in the utility profile u is the difference between aggregate utility and cost paid by those agents who get a good. We also denote $\sigma^{\xi}(p)$ by the surplus of ξ in the utility profile whose local demand is p. The efficient surplus σ^{eff} is easily computed by ordering the agents from high to low utility. It is given by $\sigma^{eff}(u) = \sum(u_i - c_i)_+$ whenever $u_1 \geq \cdots \geq u_n$ and $(x)_+ = max(0, x)$.

If q^{ac} agents buy a unit of good with ac, then $\sigma^{ac}(u) = \sum_{i=1}^{q^{ac}}(u_i - c_i)$ whenever $u_1 \geq \cdots \geq u_n$. Remember we are choosing the equilibrium of ac whose agents have highest utility (i.e. the equilibrium that collects more surplus).

On the other hand, the surplus of rp is the expected surplus of the mechanism where every agent is served in the expected order. Formally speaking, let $prio^{\theta}$ the mechanism where the agents are offered the goods following the fixed order θ. Let $\theta(k)$ the agent ranked k by the order θ. Let $prio_1^{\theta} = \theta(j)$ where j is smallest integer such that $u_{\theta(j)} \geq c_1$. Similarly, $prio_k^{\theta} = \theta(m)$ means the agents ranked strictly between $prio_{k-1}^{\theta}$ and $\theta(m)$ have utility smaller that c_k and $u_{\theta(m)} \geq c_k$.

The priority surplus is simply $\sigma^\theta(u) = \sum(u_{prio_i^\theta} - c_i)$. The surplus of rp is the average of the priority surpluses over all orders θ: $\sigma^{rp}(u) = \frac{1}{n!}\sum_\theta \sigma^\theta(u)$.

A method ξ outperforms ξ' if $\sigma^\xi(u) \geq \sigma^{\xi'}(u)$ for every $u \in \mathbb{R}_+^N$. Neither ac or rp is outperformed by the other. Indeed, consider a profile where all agents have utility $\frac{c_1+c_2}{2} + \epsilon$, where $\epsilon < \min\{\frac{c_2-c_1}{2}, \frac{2c_3-c_1-c_2}{6}\}$. Then rp serves exactly one agent. It collects a fully efficient surplus of $\sigma^{rp} = \frac{c_2-c_1}{2} + \epsilon$. On the other hand ac is fully inefficient, it serves two agents and collects a surplus of $\sigma^{ac} = \epsilon$. When ϵ goes to zero, ac does not collect any surplus[5] whereas rp collects a positive surplus.

On the other hand, consider the next example proposed by Cres and Moulin [2] in the context of queuing. It involves two types of agents. There are $n-1$ agents of type 1 with utility $c_1 + \epsilon$ and one agent of type 2 with utility $c_2 + \epsilon$, $\epsilon < \frac{c_2-c_1}{2}$. Under rp, type 2 agent buys at price c_1 with probability $\frac{1}{n}$ and at price c_2 otherwise. Hence the expected surplus with rp is $\frac{1}{n}(c_2-c_1+\epsilon)+\frac{n-1}{n}(2\epsilon)$. On the other hand, the equilibrium of ac is fully efficient, it involves only the agent of type 2, hence the surplus is $c_2 - c_1 + \epsilon$. When ϵ goes to zero the surplus of rp goes to $\frac{c_2-c_1}{n}$, whereas the surplus of ac goes to $c_2 - c_1$.

3 The worst absolute loss

Definition 1 *Let n be the number of agents and c the marginal cost function. The worst absolute loss (wal) of the method ξ is*

$$wal(n, c, \xi) = \max_{u \in \mathbb{R}_+^N} \sigma^{eff}(u) - \sigma^\xi(u)$$

We say that a mechanism ξ satisfies consumer sovereignty if for every agent i there is a utility $\bar{u}_i$ such that $\xi(u_i, u_{N\setminus i})$ allocates a unit of good to agent i with probability 1 for any utilities of the remaining agents $u_{N\setminus i}$. Consumer sovereignty was defined by Moulin [9] and plays a key role in group strategy proof mechanisms in the similar problem with decreasing marginal cost (Moulin and Shenker [12], Immorlica et al. [3]).

The worst absolute loss is positive and bounded for any mechanism that satisfies consumer sovereignty. The mechanisms that does not satisfy this property will have infinite absolute loss. Hence the order is interestingly different than the best relative gain (see section 5).

Lemma 1 *Any method ξ that satisfies consumer sovereignty satisfies $wal(n, c, \xi) < \infty$ for any number of agents n and any marginal cost c.*

Proof. Let m such that any agent with utility $u_i > m$ get a unit of good with ξ. Let $M = \max\{c_n, m\}$. Notice that any agent with utility bigger than M is guaranteed a unit of good with eff and ξ.

[5] Here we observe the famous tragedy of the commons.

Let $u \in \mathbb{R}_+^N$ a utility profile. Let $S \subset N$ such that $u_s > M$ for all $s \in S$ and $u_t \leq M$ for all $t \notin S$. Then every agent in S has a guaranteed unit of good with eff and ξ. Thus for E and T the coalition of agents that get service with eff and ξ respectively: $\sigma^{eff} = u_S + u_{E\backslash S} - C(\mid E \mid)$ and $\sigma^{\xi} = u_S + u_{T\backslash S} - C(\mid T \mid)$ where $u_i \leq M$ for all $i \in (E \cup T)\backslash S$.

$$\sigma^{eff} - \sigma^{\xi} = u_{E\backslash S} - C(\mid E \mid) - (u_{T\backslash S} - C(\mid T \mid)) \tag{1}$$

Since $u_{E\backslash S} - C(\mid E \mid) \leq nM$ and $u_{T\backslash S} - C(\mid T \mid) \leq nM$, then equation (1) is bounded above by nM. ■

In particular, notice that rp and ac satisfy consumer sovereignty hence both methods have finite worst absolute surplus loss. For the former method notice that any agent with utility bigger than or equal to c_n has a guaranteed object in any priority method, hence with rp. On the other hand, any agent with utility bigger than or equal to $\frac{C(n)}{n}$ has a guaranteed object with ac.

$wal(n, c, ac)$ is simpler to calculate than $wal(n, c, rp)$. The biggest surplus loss will be given in the famous tragedy of the commons.

Lemma 2 *The utility profile where all agents have utility $\bar{u} = \frac{C(n)}{n}$ gives the worst absolute loss of ac. At this profile ac collects zero surplus. Hence*

$$wal(n, c, ac) = \max_{1 \leq s \leq n} s\frac{C(n)}{n} - C(s) \tag{2}$$

Proof. Consider a utility profile u, $u_1 \geq \cdots \geq u_n$, and assume the agents of T form an equilibrium of ac. Without loss of generality we can assume $T = \{1, \ldots t\}$, that is it contains the $t = \mid T \mid$ agents with highest utility. Then $u_i \geq \frac{c_1 + \cdots + c_t}{t}$ for all $i \in T$.

Since the production of ac is at least the production of eff then the efficient production will be contained in T. Hence the loss will be:

$$\begin{aligned}
\sigma^{eff}(u) - \sigma^{ac}(u) &= \max_{S \subseteq T} U(S) - C(|S|) - (U(T) - C(|T|)) \\
&= \max_{0 \leq s \leq t} (c_1 + \cdots + c_t) - (u_{s+1} + \cdots + u_t) - C(s) \qquad (3) \\
&\leq \max_{0 \leq s \leq t} s\frac{c_1 + \cdots + c_t}{t} - C(s) \qquad (4)
\end{aligned}$$

Where the last inequality follows because $u_i \geq \frac{c_1 + \cdots + c_t}{t}$ and thus every term in (4) is not smaller than every term in (3).

Furthermore, notice (4) represents the loss when all agents have utility $\frac{c_1 + \cdots + c_t}{t}$ (the equilibrium of ac at this profile contains all agents, hence $\sigma^{ac} = 0$).

Finally, equation 2 follows because the efficient surplus is monotone in the utility profiles. Hence the efficient surplus when all agents have utility $\bar{u} = \frac{c_1 + \ldots c_n}{n}$ is not smaller than the efficient surplus when all agents have utility $\bar{u} = \frac{c_1 + \ldots c_t}{t}$ where $t < n$.

■

3.1 Main Result

The calculation of $wal(n, c, rp)$ is not as explicit as $wal(n, c, ac)$. This comes from the radically different surpluses of the the $n!$ priority methods in the computation of the surplus of rp. In general, it is not possible to give a simple formula for this surplus, however, Cres and Moulin [1] offer a computer algorithm to do it.

In the proof of theorem 1 we reduce the number of utility profiles needed to compute $wal(n, c, rp)$. The maximum loss will be given at a utility profile where all agents have utility in $\{c_1, \ldots, c_n\}$. In fact, it is not difficult to check that $wal(n, c, rp)$ is achieved at a utility profile where there are k agents with utility c_n and $n - k$ agents with utility in $\{c_1, \ldots, c_k\}$ for some $0 < k < n$.

For instance, for two agents $wal(2, c, rp)$ is achieved at the local demand $p = (1, 1)$. Hence $wal(2, c, rp) = \frac{\delta_2}{2}$. Contrary to other cases (see below) $wal(2, c, rp) = wal(2, c, ac)$ for any marginal cost function c.

For three agents $wal(3, c, rp)$ is achieved at the local demand $p = (2, 0, 1)$ or $p = (0, 1, 2)$. Then $wal(3, c, rp) = \max\{\frac{2}{3}\delta_2, \frac{2}{3}\delta_3\}$.

With four agents, $wal(4, c, rp)$ is achieved at one of the local demands: $(3, 0, 0, 1)$, $(2, 0, 0, 2)$, $(1, 1, 0, 2)$, $(0, 2, 0, 2)$, $(1, 0, 0, 3)$, $(0, 1, 0, 3)$ or $(0, 0, 1, 3)$. Then $wal(4, c, rp)$ will be maximized at one of the corresponding surpluses: $\frac{3}{4}\delta_2$, $\frac{1}{2}\delta_2 + \frac{1}{2}\delta_3$, $\frac{1}{4}\delta_2 + \frac{5}{6}\delta_3 + \frac{1}{12}\delta_4$, $\delta_3 + \frac{1}{6}\delta_4$, $\frac{1}{4}\delta_2 + \frac{1}{4}\delta_3 + \frac{1}{4}\delta_4$, $\frac{1}{2}\delta_3 + \frac{1}{2}\delta_4$, or $\frac{3}{4}\delta_4$.

In general we can reduce the computation of $wal(n, c, rp)$ to at most $2^{n-1} - 1$ utility profiles. Hence as the number of agents increases, this computation becomes hard. However, we can always say that the maximum surplus loss of rp will be smaller them the maximum surplus loss of ac.

Theorem 1 *For any marginal cost function c and the number of agents n bigger than or equal to three:*

$$wal(n, c, rp) < wal(n, c, ac)$$

This result differs from previous literature in three ways. The first is that we do it for any number of agents. Related papers usually works the case of many agents or a continuous of agents. The second difference is that it holds for any increasing marginal cost function, we do not require any particular shape of the cost function.

Finally the most important difference is that we consider the absolute surplus loss. The relative surplus loss (gain) case is the most common work in the literature (Moulin [11]).

4 Asymptotic behavior of $\frac{wal(n,c,ac)}{wal(n,c,rp)}$

In this section we estimate how much rp outperform ac in the wal sense. We use $\frac{wal(n,c,ac)}{wal(n,c,rp)}$ for a fixed number of agents as a measure of the total overall loss

n	$wal(n,c,ac)$	$wal(n,c,rp)$	$\frac{wal(n,c,ac)}{wal(n,c,rp)}$
3	1	0.6666	1.5
4	3	1.1666	2.5714
5	3	1.7	1.7666
6	6	2.2833	2.6277
7	6	3.0190	1.9873
8	10	3.7797	2.6456

Table 1: Worst absolute surplus loss comparison of linear marginal cost.

of ac with respect to rp[6]. Contrary to the crowded commons case where rp infinitely outperforms ac, in this case rp will weakly outperform ac at most by a finite constant.

We say that the sequence $\{c_n\}_n$ has strong order m if the sequence $\{\frac{c_n}{n^m}\}_n$ converges to a positive constant. We say that the sequence $\{w_n\}_n$ has order m if the sequence $\{\frac{w_n}{n^m}\}_n$ is bounded above and below by strictly positive constants.

Theorem 2 *If the marginal cost function c has strong order m, $1 < m < \infty$ then $wal(n,c,rp)$ and $wal(n,c,ac)$ have order $m+1$ (as a function of n). Therefore, $sup_n \frac{wal(n,c,ac)}{wal(n,c,rp)} < \infty$.*

The property of finite strong order of the marginal cost function is a little more general than the property of bounded elasticity of marginal cost introduced by Moulin [11]. This property requires that for marginal cost c $\inf\{\frac{zc'(z)}{c(z)-c(0)}\} = p > 0$ and $\sup\{\frac{zc'(z)}{c(z)-c(0)}\} = p < \infty$ for concave and convex marginal marginal cost function respectively. These properties imply that the marginal cost can be written as $c(z) = z^p\phi(z)$ where $\phi(z)$ is non decreasing (c concave) or non increasing (c convex). The property of finite order simply requires that the marginal cost can be written as: $c(z) = z^p\phi(z)$ where $\phi(z)$ converges to a positive constant.

4.1 Examples

Tables 1 and 2 show calculations of the worst absolute surplus loss for linear and quadratic marginal cost ($\delta_i = 1$ and $\delta_i = i-1$ respectively). By theorem 1, $\frac{wal(n,c,ac)}{wal(n,c,rp)} \geq 1$ for any number of agents and any cost function. In both cases we can see that this ratio tends to grow in the number of agents.[7] By theorem

[6] Notice, by the examples discussed before, $\sup_u \frac{\sigma^{eff}(u)-\sigma^{ac}(u)}{\sigma^{eff}(u)-\sigma^{rp}(u)} = \infty$ and $\inf_u \frac{\sigma^{eff}(u)-\sigma^{ac}(u)}{\sigma^{eff}(u)-\sigma^{rp}(u)} = 0$, hence these measures are not informative.

[7] This ratio is not nicely increasing in the number of agents. This problem seems to be related with the discontinuities that generate having a finite number of agents instead of a continuos number.

n	$wal(n,c,ac)$	$wal(n,c,rp)$	$\frac{wal(n,c,ac)}{wal(n,c,rp)}$
3	1.666	1.333	1.25
4	4	2.5	1.600
5	8	4.8	1.6666
6	13.5	8	1.6875
7	22	12	1.8333
8	32.5	17.4285	1.8647

Table 2: Worst absolute surplus loss comparison of quadratic marginal cost.

2 we know these numbers are bounded above. Proposition 3 give us bounds for these examples.

We finish this section with examples of particular marginal cost functions. We are particularly interested in the case of many users, hence the calculations are done when the number of agents is arbitrarily large. The first is linear marginal cost in which the ratio $\frac{wal(n,c,ac)}{wal(n,c,rp)}$ is bounded above by 2.78. The second is quadratic marginal cost in which the ratio is bounded above by 2.43. For exponential marginal cost the maximum surplus losses do not differ in the limit.

Proposition 3 *i. For linear marginal cost* $l_n = n$, $\lim_{n\to\infty} \frac{wal(n,l,ac)}{wal(n,l,rp)}$ *is bounded above by* $\frac{1+\sqrt{5}}{28-12\sqrt{5}} \approx 2.7725$

ii. For quadratic marginal cost $q_n = \frac{(n-1)n}{2}$, $\lim_{n\to\infty} \frac{wal(n,q,ac)}{wal(n,q,rp)}$ *is bounded above by* 2.43

iii. For exponential marginal cost $e_n = x^n$ $x > 1$, $\lim_{n\to\infty} \frac{wal(n,e,ac)}{wal(n,e,rp)}$ *is bounded above by* $\frac{x}{x-1}$.[8] *For quadratic exponential marginal cost,* $e^2_n = x^{n^2}$ $x > 1$, $\lim_{n\to\infty} \frac{wal(n,e^2,ac)}{wal(n,e^2,rp)} = 1$.

In the last proposition of the paper we go back to the case of a fixed finite number of agents. We show that if marginal cost is too convex, then $wal(n,c,ac)$ and $wal(n,c,rp)$ will not differ. Therefore, intuitively, the more convex the marginal cost function, the less rp will outperform ac.[9] Hence among polynomial marginal cost functions, linear marginal cost seems to be the case where rp outperform most ac in the wal sense.

Proposition 4 *If the number of agents n is fixed then* $\lim_{k\to\infty} \frac{wal(n,c^k,ac)}{wal(n,c^k,rp)} = 1$ *for marginal cost function* $c^k_n = n^k$.

[8] I conjecture that $\lim_{n\to\infty} \frac{wal(n,e,ac)}{wal(n,e,rp)} = 1$. This conjecture is in spirit similar to proposition 4: As we increase the power of a polynomial cost fuction there will be no difference between rp and ac.

[9] This observation is similar to one on Moulin [11]. He cannot guarantee a surplus gain for the serial mechanism in this case, e.g. the exponential case mentioned before.

5 Proofs

Theorem 1.

Proof.

We fix the number of agents n and the marginal cost function c, so we write $wal(n, c, \xi)$ simply as $wal(\xi)$.

We consider an auxiliary mechanism called *random assignment* (ra). It draws with uniform probability an order θ of the agents and computes the surplus σ^{ra^θ} of the mechanism ra^θ as follows: Agent i ranked $\theta(i)$ is offered a unit of good at price $c_{\theta(i)}$. Hence agent i buys a unit of good with ra^θ if $u_i \geq c_{\theta(i)}$. Since the marginal cost is increasing, production cost is not bigger than collected money, hence it is feasible to allocate the goods for any θ. Notice this mechanism is strategy proof but is not budget balanced, the budget surplus will not be redistributed.

To prove the theorem, we will prove that $wal(rp) < wal(ra) = wal(ac)$.

In steps 1a and 1b we prove that $wal(rp)$ and $wal(ra)$ are achieved at a utility profile such that every agent i has utility equal to a marginal cost c_h, for some $1 \leq h \leq n$. We denote the set of such utility profiles by their local demand:

$$P^N = \{p \in \mathbb{N}^N \mid \sum_{i \in N} p_i = n\}$$

Step 1a. $wal(rp) = \max_{p \in P^N} \sigma^{eff}(p) - \sigma^{rp}(p)$.

Let $u \in \mathbb{R}^N$ a utility profile that maximizes $wal(rp)$, and assume $c_j < u_i < c_{j+1}$ for some agent i. Without loss of generality, assume there is no other agent with utility strictly between (c_j, u_i). We analyze the next two cases.

Case 1. Agent i gets a unit of good with $eff(u)$.

First notice that for any order of N, agent i gets a unit of good with rp. Otherwise, consider the utility profile $\bar{u}$ where only the utility of agents i increases by $\epsilon = \frac{c_{j+1} - u_i}{2}$ with respect to u. Then $\sigma^{rp}(\bar{u})$ will increase by less that ϵ with respect to $\sigma^{rp}(u)$ because in the order that he does not get the unit of good with u, he will neither get a unit with $\bar{u}$. On the other hand, $\sigma^{eff}(\bar{u})$ will increase by exactly ϵ because agent i still gets a unit with eff. Therefore $\sigma^{eff}(\bar{u}) - \sigma^{rp}(\bar{u}) > \sigma^{eff}(u) - \sigma^{rp}(u)$.

Hence agent i certainly get a unit with rp. Now, consider the utility profile $\tilde{u}$ where only the utility of agent i is reduced to c_j with respect to u. Then $\sigma^{rp}(\tilde{u})$ will be reduced by $u_i - c_j$ because agent i will still certainly get a unit with rp, and so will $\sigma^{eff}(\tilde{u})$. Hence $\sigma^{eff}(\tilde{u}) - \sigma^{rp}(\tilde{u}) = \sigma^{eff}(u) - \sigma^{rp}(u)$.

Case 2. Agent i does not get a unit of good with $eff(u)$.

Consider the utility profile $\hat{u}$ where only the utility of agent i is reduced by $\epsilon = \frac{u_i - c_j}{2}$ with respect to u. Then $\sigma^{rp}(\hat{u})$ strictly decreases with respect to $\sigma^{rp}(u)$ because it does in every order that he gets a good. On the other hand, i still does not get a unit with $eff(\hat{u})$, so $\sigma^{eff}(\hat{u}) = \sigma^{eff}(u)$. Hence $\sigma^{eff}(\hat{u}) - \sigma^{rp}(\hat{u}) > \sigma^{eff}(u) - \sigma^{rp}(u)$.

Step 1b. $wal(ra) = \max_{p \in P^N} \sigma^{eff}(p) - \sigma^{ra}(p)$.

The proof is very similar to step 1a. Consider a utility profile $u \in \mathbb{R}^N$ and let $c_j < u_i < c_{j+1}$. Then agent i has a guaranteed unit with ra only if $j \geq n$. If this is the case, then consider the utility profile $\tilde{u}$ where only the utility of agent i is reduced to c_n with respect to u. Then $\sigma^{ra}(\tilde{u})$ will be reduced by $u_i - c_n$ because agent i will still certainly get a unit with ra, and so will $\sigma^{eff}(\tilde{u})$. Hence $\sigma^{eff}(\tilde{u}) - \sigma^{ra}(\tilde{u}) = \sigma^{eff}(u) - \sigma^{ra}(u)$.

If $c_j < u_i < c_{j+1}$ where $j < n$ then we can increase the loss when agent i does not get a unit with $eff(u)$. Indeed, consider the utility profile where only the utility of agent i is reduced by $\epsilon = \frac{u_i - c_j}{2}$. This profile keeps the same efficient surplus while reducing the the surplus of ra.

If agent i gets a unit with $eff(u)$ then we can increase the loss by increasing only the utility of agent i by $\epsilon = \frac{c_{j+1} - u_i}{2}$. In this case the efficient surplus increases by ϵ. On the other hand, σ^{ra} will increase by less than ϵ because agent i does not get a unit of good when he is at position j.

Step 2.

$$wal(ac) = \max_{1 \leq s \leq n-1} \frac{n-s}{n} \sum_{t=1}^{s} (t-1)\delta_t + \frac{s}{n} \sum_{t=s+1}^{n} (n-t+1)\delta_t. \tag{5}$$

We rewrite $wal(ac)$ in lemma 2 as a function of $\delta_1, \ldots, \delta_n$.

$$\begin{aligned}
wal(ac) &= \max_{u} \sigma^{eff} - \sigma^{ac} \\
&= \max_{1 \leq s \leq n-1} s \frac{c_1 + \cdots + c_n}{n} - (c_1 + \cdots + c_s) \\
&= \max_{1 \leq s \leq n-1} \frac{s}{n}(c_{s+1} + \cdots + c_n) - \frac{n-s}{n}(c_1 + \cdots + c_s) \\
&= \max_{1 \leq s \leq n-1} \frac{s}{n}[(n-s)(\delta_1 + \cdots + \delta_s) + (n-s)\delta_{s+1} + \cdots + \delta_n] \\
&\quad - \frac{n-s}{n}[s\delta_1 + (s-1)\delta_2 + \cdots + \delta_s] \\
&= \max_{1 \leq s \leq n-1} \frac{n-s}{n}[\delta_2 + \cdots + (s-1)\delta_s] + \frac{s}{n}[(n-s)\delta_{s+1} + \cdots + \delta_n]
\end{aligned}$$

Step 3. $wal(rp) \leq wal(ra)$

Since marginal cost is increasing, the surplus generated by ra will not be bigger than the surplus generated by rp in every utility profile and every order of the agents. Indeed, take such order θ of the agents. Notice that those agents who get a unit with ra^θ will also get a unit with $prio^\theta$. If an agent get a unit with ra^θ, he will pay less with $prio^\theta$ than with ra^θ, hence this agent contribute more to σ^θ than to σ^{ra^θ}. The remaining agents who get a unit with $prio^\theta$ contribute a nonnegative amount to σ^θ that do not contribute to σ^{ra^θ}. Thus $\sigma^\theta \geq \sigma^{ra^\theta}$.

Hence the average of the surplus generated by every order will keep the same relation, that is $\sigma^{rp}(u) \geq \sigma^{ra}(u)$ for every $u \in \mathbb{R}^N_+$. Therefore the worst absolute loss of rp is bounded above by the worst absolute loss of ra.

Step 4. $wal(ra) = wal(ac)$.

Let $p \in P^N$. To compute σ_{ra}, we notice that with probability $\frac{p_i}{n}$ agent of type c_i will be in the first position, and will contribute $\delta_2 + \cdots + \delta_i$ to the surplus σ_{ra}. Thus the expected surplus of the first position is given by:

$$\frac{1}{n}[p_2\delta_2 + p_3(\delta_2 + \delta_3) + \cdots + p_n(\delta_2 + .. + \delta_n)].$$

Similarly, the expected surplus of the position k is:

$$\frac{1}{n}[p_{k+1}\delta_{k+1} + p_{k+2}(\delta_{k+1} + \delta_{k+2}) + \cdots + p_n(\delta_{k+1} + .. + \delta_n)].$$

Adding the n expected surplus we obtain:

$$\begin{aligned}
\sigma^{ra}(p) &= \frac{1}{n}[(\sum_{i=2}^{n} p_i)\delta_2 + 2(\sum_{i=3}^{n} p_i)\delta_3 + \cdots + (n-1)(\sum_{i=n}^{n} p_i)\delta_n] \\
&= \frac{1}{n}[d_2\delta_2 + 2d_3\delta_3 + \cdots + (n-1)d_n\delta_n]. \qquad (6)
\end{aligned}$$

On the other hand, the efficient production i^* is determined by $d_i \geq i$ for all $i \leq i^*$ and $d_j < j$ for all $j > i^*$. Hence the efficient surplus σ_{eff} is given by:

$$\begin{aligned}
\sigma^{eff}(p) &= p_n c_n + ... + p_{i^*+1}c_{i^*+1} + (i^* - d_{i^*+1})c_{i^*} - (c_1 + \cdots + c_{i^*}) \\
&= (1)\delta_2 + \cdots + (i^*-1)\delta_{i^*} + (p_{i^*+1} + \cdots + p_n)\delta_{i^*+1} + \cdots + p_n\delta_n \\
&= (1)\delta_2 + \cdots + (i^*-1)\delta_{i^*} + d_{i^*+1}\delta_{i^*+1} + \cdots + d_n\delta_n \qquad (7)
\end{aligned}$$

Subtracting (6) to (7) we get:

$$\begin{aligned}
\sigma^{eff}(p) - \sigma^{ra}(p) = \frac{1}{n}[&(n-d_2)\delta_2 + \cdots + (i^*-1)(n-d_{i^*})\delta_{i^*} + \\
&+ d_{i^*+1}(n-i^*)\delta_{i^*+1} + \cdots + d_n(1)\delta_n] \qquad (8)
\end{aligned}$$

Consider the local demand p^{i^*} where $n - i^*$ agents have utility c_1 and i^* agents have utility c_n. Then

$$\sigma^{eff}(p) - \sigma^{ra}(p) = \frac{1}{n}[(n-i^*)(\delta_2 + \cdots + (i^*-1)\delta_{i^*}) + \\ +i^*((n-i^*)\delta_{i^*+1} + \cdots + (1)\delta_n)] \quad (9)$$

Then equation (9) is an strict upper bound of equation (8) for any p such that $p^{i^*} \neq p$. Indeed, notice the coefficients of δ_k are increasing in k for $k \leq i^*$ and decreasing in k for $k > i^*$. Also, $n - d_{i^*} \leq n - i^*$ and $d_{i^*+1} \leq i^*$. Therefore every coefficient in (8) is less or equal than its corresponding coefficient in (9). Hence by maximizing over each p^{i^*} we conclude that $wal(ra)$ is given by:

$$\max_p \sigma^{eff} - \sigma^{ra} = \max_{1\leq s\leq n-1} \frac{n-s}{n}[1\delta_2 + \cdots + (s-1)\delta_s] + \frac{s}{n}[(n-s)\delta_{s+1} + \cdots + (1)\delta_n]$$

Step 5. $wal(rp) < wal(ac)$.

Combining steps 3 and 4, we have that $wal(rp) \leq wal(ra) = wal(ac)$.

To prove the strict inequality, by step 3 and the comparison of equations (9) and (8), for all $p \in P^N$ with efficient production i^* and $p \neq p^{i^*}$:

$$\sigma^{eff}(p) - \sigma^{rp}(p) \leq \sigma^{eff}(p) - \sigma^{ra}(p) < \sigma^{eff}(p^{i^*}) - \sigma^{ra}(p^{i^*})$$

Therefore, we just have to prove that $\sigma^{eff}(p^{i^*}) - \sigma^{rp}(p^{i^*}) < \sigma^{eff}(p^{i^*}) - \sigma^{ra}(p^{i^*})$. Indeed, consider the demand p^{i^*} and an order of the agents θ such that an agent with utility c_1 is at position 2 and an agent with utility c_n is at position 3 (we can do this because $n \geq 3$). Then at this order the agent with utility c_n is guaranteed a unit of good at price c_2 or less with $prio^\theta$ while he will get a unit at price c_3 with ra^θ. Therefore $\sigma^{ra^\theta}(p^{i^*}) < \sigma^\theta(p^{i^*})$ and $\sigma^{ra^\phi}(p^{i^*}) \leq \sigma^\phi(p^{i^*})$ for any other order $\phi \neq \theta$. Hence $\sigma^{ra}(p^{i^*}) < \sigma^{rp}(p^{i^*})$. ■

Theorem 2.

Proof.

The lower bound of $wal(n, c, rp)$ will be given by restricting the domain of demands to the set of local demands with only two types of agents. We denote by $p^{k,a}$ the demand where k agents have utility c_n and $n-k$ agents have utility c_a, $1 \leq a \leq k$.

Step 1.

$$\sigma^{eff}(p^{k,a}) - \sigma^{rp}(p^{k,a}) = \sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(c_{k+1} + \cdots + c_{k+a-s} - (a-s)c_a) \quad (10)$$

Consider an order θ of the agents in $p^{k,a}$ where exactly $a-s$ agents with utility a get a unit with rp^θ. Then the production at this profile is $k+a-s$ and the surplus is given by

$$\sigma^{\theta}(p^{k,a}) \,=\, kc_n + (a-s)c_a - (c_1 + ... + c_{k+a-s}) \tag{11}$$

On the other hand, the probability to choose such order θ is equal to $\frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}$. Indeed, notice such order θ should put exactly s agents with utility n and $a-s$ agents with utility a in the first a positions. We can choose such configuration of agents in $\binom{n-k}{a-s}\binom{k}{s}$ ways. Given the two groups of agents, the first with a agents and the second with $n-a$ agents, we do not care for the order of agents between groups. That is, a permutation of agents in the same group still gives s agents with utility n and $a-s$ agents with utility a in the first a positions. We can permute this two groups in $a!(n-a)!$. Hence there are $\binom{n-k}{a-s}\binom{k}{s}a!(n-a)!$ orders where the first a positions are filled by s agents with utility n and $a-s$ agents with utility a. Hence the probability of choosing such order θ is given by:

$$\frac{\binom{n-k}{a-s}\binom{k}{s}a!(n-a)!}{n!} = \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}.$$

Finally, notice that the efficient production with $p^{k,a}$ is k units, hence $\sigma^{eff}(p^{k,a}) = kc_n - (c_1 + \cdots + c_k)$. Step 1 follows from last two equations.

Step 2. $wal(n, c, rp)$ has order at least $m+1$.

Consider a local demand $p^{k,k}$ where $k = [\lambda n]$, where $\lambda \in \mathbb{Q} \cap (\frac{1}{2}, 1)$. As we increase the number of agents n, we are reproducing the economy and the utilities are scaled taking marginal cost as a reference. The claim is that $wal(n, c, p^{k,k})$ has order $m+1$.

By step 1, it suffices to prove that the next equation has order $m+1$:

$$\sum_{s=2k-n}^{k} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} (c_{k+1} + \cdots + c_{k+k-s} - (k-s)c_k) \tag{12}$$

Since c_n has order m, then we can represent $c_n = n^m h(n)$ where $h : \mathbb{N} \to \mathbb{R}_+$ is such that $\lim_{n\to\infty} h(n) = L > 0$. Thus for any $\delta > 0$, there is N large such that:

$$[L+\delta][(k+1)^m + \cdots + (k+k-s)^m - (k-s)k^m] \geq c_{k+1} + \cdots + c_{k+k-s} - (k-s)c_k$$

for all s, $2k-n < s < k$ and for all $n > N$.

Hence the order of (12) is bigger than or equal to the order of the equation:

$$\sum_{s} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} [(k+1)^m + \cdots + (k+k-s)^m - (k-s)k^m] \tag{13}$$

Let $\epsilon > 0$ be small. We write $s = \gamma n$. Since $2k - n \leq s < k$ then $2\lambda - 1 \leq \gamma < \lambda$. We abuse of notation by writing the set $\{\gamma \mid 2\lambda - 1 = \gamma < \lambda - \epsilon,\ \gamma n \in \mathbb{N}\}$ simply as $\{2\lambda - 1 = \gamma < \lambda - \epsilon\}$.

$$\frac{1}{k^m(\epsilon n)} \sum_s \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}}[(k+1)^m + \cdots + (k+k-s)^m - (k-s)k^m] \geq$$
$$\sum_{2\lambda-1=\gamma<\lambda-\epsilon} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}}[\frac{(1+\frac{1}{k})^m + \cdots + (1+\frac{k-s}{k})^m}{k-s} - 1] +$$
$$\sum_{\lambda>\gamma>\lambda-\epsilon} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} \frac{(1+\frac{1}{k})^m + \cdots + (1+\frac{k-s}{k})^m - (k-s)k^m}{\epsilon n} \tag{14}$$

Notice for every γ such that $2\lambda - 1 = \gamma < \lambda - \epsilon$, $\lim_{n\to\infty}(1 + \frac{k-s}{k})^m = (1 + \frac{\lambda-\gamma}{\lambda})^m > 1$, then:

$$\lim_{n\to\infty} \frac{(1+\frac{1}{k})^m + \cdots + (1+\frac{k-s}{k})^m}{k-s} = (\int_0^{\frac{\lambda-\gamma}{\lambda}} (1+x)^m dx)(\frac{\lambda}{\lambda-\gamma}) \tag{15}$$
$$= (\frac{(1+\frac{\lambda-\gamma}{\lambda})^{m+1}}{m+1} - \frac{1}{m+1})(\frac{\lambda}{\lambda-\gamma})$$
$$> (\frac{(1+(m+1)\frac{\lambda-\gamma}{\lambda}}{m+1} - \frac{1}{m+1})(\frac{\lambda}{\lambda-\gamma}) = 1 \tag{16}$$

Where equation (15) holds because $\frac{(1+\frac{1}{k})^m+\cdots+(1+\frac{k-s}{k})^m}{(k-s)} \frac{\lambda-\gamma}{\lambda}$ is a superior (upper) sum of the interval $[0, \frac{\lambda-\gamma}{\lambda}]$ with partition of size $k - s \approx (\lambda - \gamma)n$ and function $f(x) = (1+x)^m$. As n increases, the partition becomes finer and hence such sum converges to such integral.

Then equation (14) is bigger than equation (17) because the partition is a superior sum (thus it is bigger than the integral) and the second term in right hand side of equation (14) is positive.

$$\sum_{2\lambda-1=\gamma<\lambda-\epsilon} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} \nu(\gamma) \tag{17}$$

where $\nu(\gamma) = (\frac{(1+\frac{\lambda-\gamma}{\lambda})^{m+1}}{m+1} - \frac{1}{m+1})(\frac{\lambda}{\lambda-\gamma}) - 1 > 0$ for all $\gamma < \lambda$.

Since $\sum_{2\lambda-1=\gamma\leq\lambda} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} = 1$ for all n because this is the sum of probabilities that sum up to 1. Then $\lim_{n\to\infty} \sum_{2\lambda-1=\gamma<\lambda} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} = 1$ because

$\lim_{n\to\infty} \frac{1}{\binom{n}{k}} = 0$. Thus $\lim_{n\to\infty} \sum_{2\lambda-1=\gamma<\lambda} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} \nu(\gamma) > 0$ because $\nu(\gamma) > 0$ for all $\gamma \in [2\lambda-1, \lambda)$.

Therefore we can choose small $\epsilon > 0$ such that

$$\lim_{n\to\infty} \sum_{2\lambda-1=\gamma<\lambda-\epsilon} \frac{\binom{n-k}{k-s}\binom{k}{s}}{\binom{n}{k}} \nu(\gamma) > 0.$$

Hence the sequence has order at least $m+1$.

Step 3. $wal(n,c,ac)$ has order at most $m+1$.

By lemma 2,

$$wal(n,c,ac) = \max_{1\leq s\leq n-1} \frac{s}{n} C(n) - C(s). \tag{18}$$

For every n, let s_n^* the number that maximizes $wal(n,c,ac)$. Since $1 \leq s_n^* \leq n-1$, then the sequence $\{\frac{s_n^*}{n}\}_n$ has order at most 1, therefore $\frac{s_n^*}{n} C(n) - C(s_n^*)$ has order at most $m+1$, and so does $\{wal(n,c,ac)\}_n$.

We complete the proof of theorem by noticing that $wal(n,c,ac) \geq wal(n,c,rp)$ for any n and any c, then the order of $\{wal(n,c,ac)\}_n$ is bigger than or equal to the order of $\{wal(n,c,rp)\}_n$. Therefore along with steps 2 and 3 both orders are equal to $m+1$. ■

Proposition 3

Proof.

Part (i)

Step 1. $wal(n,l,ac) \leq \frac{n^2}{8}$.

$$\begin{aligned} wal(n,l,ac) &= \max_{1\leq s\leq n-1} \frac{n-s}{n} \sum_{t=1}^{s} (t-1) + \frac{s}{n} \sum_{t=s+1}^{n} (n-t+1) \\ &= \max_{1\leq s\leq n-1} \frac{1}{2}(s)(n-s). \end{aligned} \tag{19}$$

This equation is parabolic with vertex in $s = \frac{n}{2}$. Hence

$$wal(n,l,ac) = \frac{1}{2}([\frac{n}{2}])([\frac{n}{2}]+1) < \frac{n^2}{8}.$$

Step 2.

$$\sigma^{eff}(p^{k,a}) - \sigma^{rp}(p^{k,a}) = \frac{(k-a+1)(n-k)a}{n} + \frac{a(a-1)(n-k)(n-k-1)}{2n(n-1)} \tag{20}$$

By step 1 on proof of theorem 2,

$$\begin{aligned}
\sigma^{eff} - \sigma^{rp}(p^{k,a}) &= \sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(c_{k+1} + \cdots + c_{k+a-s} - (a-s)c_a) \\
&= \sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}((a-s)k + \frac{(a-s)(a-s+1)}{2} - (a-s)a) \\
&= (k - a + \frac{1}{2})\sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(a-s) + \frac{1}{2}\sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(a-s)^2
\end{aligned}$$

On the other hand, notice that:

$$\begin{aligned}
\sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(a-s) &= \sum_s \frac{\binom{n-k-1}{a-s-1}\binom{k}{s}}{\binom{n}{a}}(n-k) \\
&= \sum_s \frac{\binom{n-k-1}{a-s-1}\binom{k}{s}}{\binom{n-1}{a-1}}(n-k)\frac{\binom{n-1}{a-1}}{\binom{n}{a}} \\
&= (n-k)\frac{\binom{n-1}{a-1}}{\binom{n}{a}} \qquad (21) \\
&= \frac{(n-k)a}{n} \qquad (22)
\end{aligned}$$

where (21) holds because the previous sum of combinatorial coefficients represents same probabilities but with $n-1$ agents and $a-1$ positions, hence they sum up to 1. We use the same trick in the next equations.

$$\begin{aligned}
\sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(a-s)^2 &= \sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}((a-s) + (a-s)(a-s-1)) \\
&= \frac{(n-k)a}{n} + \sum_s \frac{\binom{n-k}{a-s}\binom{k}{s}}{\binom{n}{a}}(a-s)(a-s-1) \\
&= \frac{(n-k)a}{n} + (\sum_s \frac{\binom{n-k-2}{a-s-2}\binom{k}{s}}{\binom{n-2}{a-2}})\frac{\binom{n-2}{a-2}(n-k)(n-k-1)}{\binom{n}{a}} \\
&= \frac{(n-k)a}{n} + \frac{\binom{n-2}{a-2}(n-k)(n-k-1)}{\binom{n}{a}} \\
&= \frac{(n-k)a}{n} + \frac{(a)(a-1)(n-k)(n-k-1)}{n(n-1)} \qquad (23)
\end{aligned}$$

Finally, we substitute equations (22) and (23) in the expected loss equation to prove equation (20).

Consider $a(n) = \frac{\sqrt{5}-1}{\sqrt{5}+1}n$ and $k(n) = \frac{\sqrt{5}-1}{2}n$. Clearly, $1 < a(n) < k(n) < n$. To simplify notation we write $a^* = a(n)$ and $k^* = k(n)$. Then:

$$\lim_{n\to\infty} \frac{\frac{n^2}{8}}{\frac{a^*(a^*-1)(n-k^*)(n-k^*-1)}{2n(n-1)} - \frac{(a^*-k^*-1)(n-k^*)a^*}{n}} = \frac{1+\sqrt{5}}{28-12\sqrt{5}}. \tag{24}$$

Finally, notice that replacing a^* and k^* in (24) by its respective integer parts, will not modify such limit. Therefore the limit

$$\lim_{n\to\infty} \frac{wal(n,l,ac)}{wal(n,l,rp)} \tag{25}$$

is bounded by $\frac{1+\sqrt{5}}{28-12\sqrt{5}}$.

Part (ii)

By space reasons, this part is available upon request to the author.

Part (iii)

With exponential cost function, from (5) we have:

$$wal(n,e,ac) = \max_{1\le s\le n-1} \frac{n-s}{n}\sum_{t=1}^{s}(t-1)\delta_t + \frac{s}{n}\sum_{t=s+1}^{n}(n-t+1)\delta_t$$

Let $\delta_t = (x-1)x^{t-1}$. Then

$$\sum_{t=1}^{s}(t-1)\delta_t = \frac{-x^{s+1}+x^{s+1}s-x^s s+x}{(x-1)}$$

and

$$\sum_{t=s+1}^{n}(n-t+1)\delta_t = \frac{x^{n+1}-x^{s+1}n+x^s n-x^{s+1}+x^{s+1}s-x^s s}{(x-1)}.$$

Hence

$$\begin{aligned} wal(n,e,ac) &= \max_{1\le s\le n-1} \frac{-x^{s+1}n+nx-sx+sx^{n+1}}{n\,(x-1)} \\ &= \max_{1\le s\le n-1} \frac{x(s(x^n-1)-n(x^s-1))}{n\,(x-1)} \end{aligned} \tag{26}$$

Since $x > 1$, when n is large $s(x^n - 1) - n(x^s - 1)$ is maximized at $s = n - 1$. Hence

$$wal(n, e, ac) = \frac{-x^n n + x + x^{n+1} n - x^{n+1}}{n\,(x-1)^2}.$$

Finally, notice $\sigma^{eff}(p^{n-1,n-1}) - \sigma^{rp}(p^{n-1,n-1}) = \frac{n-1}{n}\delta_n = \frac{(n-1)(x-1)x^{n-1}}{n}$. Therefore $\lim_{n\to\infty} \frac{wal(n,e,ac)}{wal(n,e,rp)} \leq \lim_{n\to\infty} \frac{-x^n n + x + x^{n+1} n - x^{n+1}}{n(x-1)^2 \frac{n-1}{n} x^{n-1}} = \frac{x}{x-1}$.

On the other hand, for quadratic exponential function, we know $\delta_{n+1} = x^{(n+1)^2} - x^{n^2}$, therefore $\lim_{n\to\infty} \frac{\delta_{n+1}}{n^2 \delta_n} = \infty$.

The remaining part is an argument similar to proof of proposition 4. For a fixed large number of agents n, we are maximizing over a set of $n-1$ linear equations in δ_i, $i \in \{1 \dots n\}$ The coefficient of each δ_i is smaller than n. The above limit tells us δ_n will be bigger than any linear combination of $\{\delta_1, \dots, \delta_{n-1}\}$ with coefficients smaller than n (any of such linear combinations is smaller than $n^2\delta_{n-1}$.)

The equation in (5) that has the biggest coefficient in δ_n is when $s = n - 1$. Thus,

$$\lim_{n\to\infty} \frac{wal(n, e^2, ac)}{\frac{n-1}{n}\delta_n} = 1 \tag{27}$$

On the other hand, notice $wal(n, e^2, rp) \geq \sigma^{eff}(p^{n-1,n-1}) - \sigma^{rp}(p^{n-1,n-1}) = \frac{n-1}{n}\delta_n$. Therefore $\frac{wal(n,e^2,rp)}{\frac{n-1}{n}\delta_n} \geq 1$ for all n. This equation along with theorem 1 and equation (27) implies: $\lim_{n\to\infty} \frac{wal(n,e^2,ac)}{wal(n,e^2,rp)} = 1$.

■

Proposition 4

Proof.

First notice that by equation (5) the calculation of $wal(n, c^k, ac)$ involves the maximization over n linear equations on $\delta_1, \dots, \delta_n$. The coefficient of δ_i on each equation is independent of the cost function.

Let $\delta_1^k \dots \delta_n^k$ such coefficients associated to the marginal cost c^k. Then:

$$\lim_{k\to\infty} \frac{\delta_{i+1}^k}{\delta_i^k} = \lim_{k\to\infty} \frac{(i+1)^k - i^k}{i^k - (i-1)^k} = \lim_{k\to\infty} \frac{(1+\frac{1}{i})^k - 1}{1 - (1-\frac{1}{i})^k} = \infty \ \forall\ i \tag{28}$$

This implies that as k goes to infinite, δ_n^k will be infinitely bigger with respect to any linear combination of the remaining coefficients $\delta_1^k, \dots, \delta_{n-1}^k$. Hence for arbitrarily large k, $wal(n, c^k, ac)$ will be achieved on the equation that has the biggest coefficient on δ_n^k. From equation (5) we can check that such equation is given when $s = n - 1$. That is, for every $1 \leq s < n - 1$

$$\lim_{k\to\infty} \frac{\frac{n-s}{n}\sum_{t=1}^{s}(t-1)\delta_t^k + \frac{s}{n}\sum_{t=s+1}^{n}(n-t+1)\delta_t^k}{\frac{n-1}{n}\delta_n^k} =$$

$$= \lim_{k\to\infty} \frac{(n-s)\sum_{t=1}^{s}(t-1)\frac{\delta_t^k}{\delta_n^k} + s\sum_{t=s+1}^{n}(n-t+1)\frac{\delta_t^k}{\delta_n^k}}{(n-1)} = \frac{s}{n-1} < 1$$

Hence,

$$\lim_{k\to\infty} \frac{wal(n, c^k, ac)}{\frac{n-1}{n}\delta_n^k} = 1 \tag{29}$$

On the other hand, notice $wal(n, c^k, rp) \geq \sigma^{eff}(p^{n-1,n-1}) - \sigma^{rp}(p^{n-1,n-1}) = \frac{n-1}{n}\delta_n^k$. Therefore

$$\frac{wal(n, c^k, rp)}{\frac{n-1}{n}\delta_n^k} \geq 1 \ \forall \ k \tag{30}$$

The proposition follows immediately by equation (29), (30) and theorem 1: $wal(n, c^k, ac) > wal(n, c^k, rp)$ for all k.

■

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