# Optimal Strategic Communication: Can a Less Informed Expert be More Informative?*

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**Abstract**

This paper investigates an extended version of Crawford–Sobel's (1982) communication game in which the principal can control the quality of the expert's information. We prove that the optimal quality of information is always bounded away from the full information and characterize the optimal information structure that maximizes players' ex-ante payoffs. Based on this, we show that our mechanism provides a superior ex-ante payoff for the principal, compared to both Crawford–Sobel's most informative equilibrium and optimal delegation. We then study multi-stage communication. This modification results in further ex-ante Pareto improvement since it allows for the step-by-step refinement of the expert's information, preserving truth-telling communication at every stage. Finally, we construct a mechanism in which approximately full information is revealed for a large sub-interval of the state space.

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## 1 Introduction

Situations in which principals do not have enough information and have to consult experts before implementing a policy can be found almost everywhere. Auctioneers consult experts about an object's value before setting auction rules, managers consult financial and marketing analysts before making corporate decisions, and politicians consult advisors on special subjects.

Despite the apparently different nature of these situations, several common features characterize virtually every process of communication. The first is a conflict of interest between the involved parties. As a result, the expert may want to withhold true information or provide it only partially, since releasing all information could be harmful for her. One can expect that the larger the conflict of interest, the less useful will be the information

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provided by the expert. The second feature is the imperfect primary information of the expert. Even the most knowledgeable expert may not be completely informed. Moreover, quite often the principal has the power to control the flow of information available to the expert. For example, governments (principals) usually collect reports from oil companies (experts) to estimate the amount of oil in oilfields before making a decision about a lease sale. Depending on the company's report, a government decides whether to sell an oilfield separately (if it consists of several parts) or as a whole piece. It may conduct an auction or approach just one company. In the former case, the government sets auction rules: its type, a reserve price, etc. This behavior constitutes an incentive for companies to distort their information in a favorable way, since the government's policies are influenced by their estimates. For its part, the government can restrict the company's quality of information by specifying the number and locations of test drills. Finally, the communication process between parties can be conducted on a multi-stage basis – the government can request a new report after each test drill. Thus, the principal affects the precision of the expert's information before communication at every stage, whereas the expert can update her report afterwards.[1]

The first analysis of strategic communication is due to Crawford and Sobel [6] in their seminal paper. They introduce a model of a perfectly informed expert and an uninformed principal whose payoffs depend on a random state of nature. After a private observation of the true state, the expert sends a costless message to the principal. On the basis of the message received, the principal implements an action, determining the parties' payoffs. Crawford and Sobel show that full information revelation is never possible unless players' interests perfectly match. In addition, when a conflict of interest arises, the quality of the disclosed information falls, eventually resulting in an equilibrium with no useful information conveyed.

Crawford and Sobel's characterization of the equilibria is predicated upon two assumptions. First, the expert is *perfectly* informed about the realization of the state of nature. Second, the communication process consists of one stage only. This paper studies the effects of relaxing both of these assumptions. In many cases, even the most professional experts may have only insufficient or noisy information. Furthermore, a principal can directly restrict their access to information. In addition, instead of a one-stage communication process, the principal can gradually improve information precision of the expert in every stage and request a new report, conditional on the expert's updated information.

We study the simplest model which incorporates both discussed features: communication through multiple stages and imperfect information of the sender, the quality of which is controlled by the receiver at every stage.

Our major contribution can be summarized as follows. We demonstrate that by prop-

[1] The Mineral Management Service (MMS) of the U.S. Department of the Interior does not perform any direct data-collection activities. Instead, it issues permits to industry for collecting prelease geological and geophysical data. In general, companies wishing to collect data on the Outer Continental Shelf prior to a lease sale must obtain a permit from the MMS. The permits set forth the specific details for each data-gathering activity, including the area where the data are collected, the timing of the data-gathering activity, approved equipment and methods, and other similar detailed information relevant to each specific permit. After a permit is granted, the MMS monitors all field data collection activities to ensure compliance with the terms of the permit. It is empowered to select and obtain data that are collected by private firms. The MMS uses the obtained data for several purposes, including evaluation of tracts' market values, determination of bidding procedures, leasing, and post-lease operations.

erly restricting the quality of the expert's information, the principal can foster expert's incentives to communicate. This results in a higher quality of the information transmitted, compared to Crawford–Sobel (hereafter, CS) uniform-quadratic model. Thus, more information may be obtained from a less informed expert. We characterize the optimal information structure of the expert and show that the trade-off between the imperfect information of the expert and her incentives to reveal it results in ex-ante Pareto improvement. Moreover, our model leads to a superior outcome for the principal than optimal delegation (whenever the informative communication exists).[2]

In generalizations of our base setting, we demonstrate how to extend the analysis to a wider class of players' preferences and distribution functions. In the more general setting, the principal prefers communication with an imperfectly informed expert to that with a perfectly informed expert whenever informative communication is feasible, and to optimal delegation as long as players' interests are sufficiently close.

The main result for multi-stage communication is that the combination of multi-stage communication with imperfect information of the expert is so powerful that the principal can achieve (almost) full information revelation over a large interval of states. We introduce a mechanism through which the expert truthfully discloses all available information at each stage of the communication process. This allows the principal to implement a policy as close as needed to his ideal policy when the number of communication stages is arbitrary large. The result basically relies on the following intuition: step-by-step updating of the expert's information at every stage can be organized in such a way that the expert has a possibility to induce only those actions which are either optimal for the policymaker or essentially different from the expert's ideal policy, given expert's current information.

The paper proceeds as follows. Section 3 highlights an example, which illustrates that the optimal information structure is coarse and involves a finite number of partition elements. Section 4 presents the formal model of one-stage communication. The general analysis of the one-stage model is provided in Section 5. Section 6 extends the analysis to the multi-stage case. Section 7 introduces an extension of the multi-stage model and discusses results. Section 8 concludes the paper.

## 2 Literature Review

The fact that quality of information of the principal is not monotone in that of the expert was first recognized by Fischer and Stocken [8]. They, however, restrict the set of possible biases in players' preferences $b$, introduced by Crawford and Sobel [6], to that of the discrete form $b = \frac{1}{2c}$, where $c$ is an integer. In addition, they analyze pure-strategy equilibria only. Their main result for the "uniform-quadratic" setting[3] is that the optimal structure of the informational partition is uniform of size $c$, that is, equally spaced. This is not a general

[2] Delegation of decision rights of a policy maker to an expert has been suggested as an efficient alternative to communication (see, for instance, [7], [10]). If parties' interests (and so ideal policies) are sufficiently close, then by delegating his rights, the decision maker can benefit by letting the expert himself make decisions. Also, in most cases, the decision maker can delegate decision rights only partially by applying some rules to restrict policies that can be chosen by the expert. Optimal delegation imposes the policy restrictions that maximize expected payoff of the decision maker. Because this work highlights a different mechanism of communication through which the policy maker can acquire higher utility, it is natural to compare the principal's expected payoffs in these mechanisms.

[3] That is, for the uniform distribution of the state and quadratic preferences of players.

feature of the model for other values of $b$. In general, as shown below, there exist equilibria with non-uniform partitions, which provide a higher expected payoff to the principal. In this paper, we characterize the optimal information structure for all $b$. Moreover, we study the benefits of multi-stage communication.

Another approach to improve the receiver's quality of information is to establish communication through multiple stages. Aumann and Hart [2] consider two-person games with two-sided cheap talk in which one side is better informed than the other, and completely characterize the equilibrium payoffs. Their general analysis is restricted to the class of games with incomplete information with discrete types and a bimatrix structure of players' strategies and payoffs. Krishna and Morgan [12] show that even with only two stages there exists an equilibrium that almost always ex-ante Pareto dominates all of the equilibria identified by Crawford and Sobel. Moreover, there are informative equilibria in multi-stage communication even if the conflict of interest between players is so large that no informative equilibria exist in the CS model.[4] Battaglini [3] considers a model with multi-dimensional signals and multiple imperfectly informed experts. He demonstrates that when experts have different preferences, the number of experts is large, and the principal has a limited ability to commit, then it is possible to construct an equilibrium in which the quality of extracted information is arbitrarily close to complete information.[5]

An issue of endogenous quality of information to mechanism design is also studied by Bergemann and Pesendorfer [4]. They consider an auction in which the seller determines the precision of bidders' valuations and to whom to sell at what price. In this case, optimal information structures in the optimal auction are coarse and represented by the finite number of monotone partitions.[6]

In the light of the literature, the main contribution of our paper is that control of the quality of expert's information strictly improves communication whenever informative communication is achievable. This result holds for a wide class of players' preferences and distributions of the state. Also, we show that the principal can obtain arbitrarily precise information about a state with only one expert through multiple stages of communication.

## 3 An Example

We start with the uniform-quadratic variant of the communication model introduced by Crawford and Sobel [6]. Two players, the uniformed receiver (R) and the better informed sender (S), communicate on some state of nature, which is represented by a random variable $\theta$, uniformly distributed on the unit interval $\Theta = [0, 1]$. We can treat the sender as an expert (she) and the receiver as a principal (he). The expert sends a costless message $m$ to the principal, who then implements some action $a$, which affects payoffs of both players.

[4] These equilibria have a non-monotonic structure, that is, a sender of a high type can be associated with a lower action.

[5] The assumption of the principal's limited ability to commit can be omitted, if the game is played through an arbitrary, but finite number of periods, where a new state of nature and new experts' signals are drawn in each period.

[6] An interesting property of the optimal structure is that the partitions are asymmetric across agents even for symmetric distributions of object's values.

Players' state-relevant utility functions are quadratic:

$$U_R(a,\theta) = -(a-\theta)^2, \text{ and } U_S(a,b,\theta) = -(a-b-\theta)^2, \tag{1}$$

where a parameter $b > 0$ reflects the bias in the players' interests.

First, consider the case of the *perfectly* informed expert. Crawford and Sobel demonstrate that all the equilibria are characterized by finite monotone partitions. That is, for any $b$ there exists at most $N^{CS}(b) < \infty$ number of intervals on the state space such that the expert sends one message for each interval $(w_k, w_{k+1}]$, which is associated with a corresponding action $a_k = \frac{w_k+w_{k+1}}{2}$.[7] Also, there are exactly $N^{CS}(b)$ equilibria with $1, 2, ..., N^{CS}(b)$ intervals, where the equilibrium with $N^{CS}(b)$ intervals is Pareto superior to all other equilibria.

For example, if $b = \frac{1}{5}$, then $N^{CS}(b) = 2$, and the expert sends one message if $\theta \leq \frac{1}{10}$, and another message otherwise. This equilibrium is not very informative for values of $\theta > \frac{1}{10}$. As a result, the principal's expected payoff $U_R^{CS} \simeq -\frac{1}{16}$ insignificantly exceeds his payoff $-\frac{1}{12}$ in the case of no communication.

However, if the principal controls the expert's information structure in a such way that the expert knows only whether $\theta$ is higher or lower than $\frac{1}{2}$, then she must estimate the average utility across all states, given available information. This shifts her preferences in a way favorable for the principal. Then, the expert truthfully reveals her information that provides the principal's expected utility $U_R = -\frac{1}{48}$, which is essentially higher than that in the case of the perfectly informed expert.

Moreover, there exists an equilibrium with three messages, namely, for $\theta$ less than $\frac{1}{5}$, between $\frac{1}{5}$ and $\frac{4}{5}$, and higher than $\frac{4}{5}$, which provides expected utility $U_R \simeq -\frac{1}{52}$. A finer information structure violates the sender's incentives to communicate truthfully, which results in distortion of information and lower principal's payoffs.[8]

# 4 The Model

Consider a uniform-quadratic setup of the CS model, in which the principal takes control over the quality of the expert's information about the state.[9] We call this modification the CWIIE model (Communication With an Imperfectly Informed Expert). The key modification of our model is a preliminary stage, in which the receiver specifies the sender's **information structure** at zero cost. In particular, the receiver partitions $\Theta$ into a finite number $n$ of intervals $W_k = (w_k, w_{k+1}]$, $k \in K = \{0, 1, ..., n-1\}$, $w_0 = 0$, $w_n = 1$. Equivalently, a partition is described by a strictly increasing sequence $(w_k)_0^n$ of its boundary points. We call a partition **uniform** of size $n$ if $(w_k)_0^n = \left(\frac{k}{n}\right)_0^n$.

The timing of the game is as follows. At the first stage, the receiver specifies a partition $\Omega = \{W_k\}_0^{n-1}$, and a message set $M$. At the second stage, a realization of the state $\theta$

[7] Formally, Crawford and Sobel define equilibrium strategies in a slightly different way. They require $m(\theta)$ to be uniformly distributed on $[w_k, w_{k+1}]$, if $w \in (w_k, w_{k+1})$, and $a(m) = \frac{w_k+w_{k+1}}{2}$ for all $m \in (w_k, w_{k+1})$.

[8] Like Crawford and Sobel, we use the term "finer" informally, implying a partition with a larger number of elements.

[9] Later, we will analyze generalizations of this setting in terms of players' preferences, distributions of the state, and the number of stages of communication.

occurs, and the sender privately observes an element of the partition $W_k$, which contains $\theta$. We denote this sender as a $k$-type. Thus, the sender's imperfect information about a state is determined by the uniform distribution over $W_k$, and a measure of the sender's imprecision about the state is $P(W_k) = \Pr(\theta \in W_k) = w_{k+1} - w_k$. At the third stage, the sender transmits a costless message $m \in M$ to the receiver. In general, the sender may mix over messages with a conditional distribution $\sigma(m|k)$. After receiving the message, the receiver updates his beliefs about the state and implements an action $a$ that determines the players' payoffs.

Let $\bar{M}(\bar{a}) = \{m : a(m) = \bar{a}\}$. We say that an action $\bar{a}$ is **induced** by a $k$-type, if $\int_{\bar{M}(\bar{a})} \sigma(m|k)\, dm > 0$, and is **purely induced** if $\int_{\bar{M}(\bar{a})} \sigma(m|k)\, dm = 1$.

Notice that the described information structure assumes monotonicity of partitions. That is, $\theta \in W_k$, $\theta' \in W_j$, $j > k$ implies that $\theta < \theta'$. This feature of the model is consistent with the argument of feasibility: it is difficult for the principal to implement an information system such that the expert's information has a form of "a true state is either high or low, but not intermediate". In addition, all characterized equilibria in the CS model have the information structure of the monotone partitional form.

## 4.1 Equilibrium

Given information structure $\Omega$, a perfect Bayesian equilibrium (hereafter, equilibrium) $(\sigma(m|k), a(m), \Omega)$ consists of a signaling strategy $\sigma : \Omega \to \Delta M$, which specifies a probability distribution $\sigma(m|k)$ over the space of messages for each type $k$, the principal's action's rule $a : M \to \mathbb{R}$, and a belief function $G : M \to \Delta\Theta$, which specifies a probability distribution over $\Theta$ for each message $m$.

The action's rule $a(m)$ maximizes the receiver's utility $U_R(a|m) = E[U_R(a,\theta)|m]$ given his belief function $G(\theta|m)$, which is constructed on the basis of Bayes' rule.[10] Given the action's rule $a(m)$, the signaling strategy maximizes the sender's type-relevant utility function

$$U_S(a,b|W_k) = E_\theta[U_S(a,b,\theta)|\theta \in W_k] = -\frac{1}{P(W_k)} \int_{w_k}^{w_{k+1}} (a-\theta)^2\, d\theta.$$

That is, the sender's strategy $\sigma(m|k)$ satisfies

$$\text{if } \bar{m} \in \operatorname{supp} \sigma(.|k), \text{ then } \bar{m} \in \arg\max_{m \in M} U_S(a(m), b|W_k), \text{ and} \qquad (2)$$
$$\int_M \sigma(m|k)\, dm = 1, \text{ for all } k \in K.$$

Notice that $U_S(a,b|W_k)$ can be written as

$$U_S(a,b|W_k) = U_S(a,b,\bar{w}_k) - D(W_k), \qquad (3)$$

where $\bar{w}_k = E[\theta|\theta \in W_k] = \frac{w_k + w_{k+1}}{2}$ is a conditional mean of the state given the sender's

[10] Due to the strict concavity of the principal's utility function over actions, he never mixes between actions.

type, and $D(W_k) = \frac{1}{12}(w_{k+1} - w_k)^2$ is a conditional residual variance of $\theta$.

Similarly, the principal's type-relevant utility function $U_R(a|W_k)$ can be represented as

$$U_R(a|W_k) = U_R(a, \bar{w}_k) - D(W_k). \tag{4}$$

The action's rule $a(m)$ is the solution of the principal's problem given his belief function $G(\theta|m)$:

$$a(m) = \arg\max_{a\in\mathbb{R}} U_R(a|m) = \arg\max_{a\in\mathbb{R}} \int_0^1 U_R(a,\theta)\, dG(\theta|m).$$

The density of the belief function $g(\theta|m) = G'(\theta|m)$ is constructed on the basis of Bayes' rule

$$g(\theta|m) = \sum_{k=0}^{n-1} \frac{\sigma(m|k)}{g(m)} 1_{W_k}(\theta),$$

where $1_{W_k}(\theta)$ is the indicator function and $g(m) = \sum_{k=0}^{n-1} P(W_k)\sigma(m|k)$.

Then, we can represent $U_R(a|m)$ as

$$U_R(a|m) = \sum_{k=0}^{n-1} g_k(m)\, U_R(a|W_k), \tag{5}$$

where $g_k(m) = \frac{P(W_k)\sigma(m|k)}{g(m)}$, and $U_R(a|W_k) = U_S(a, 0|W_k)$ is the principal's type-relevant utility function.

The receiver's expected utility is

$$\begin{aligned} U_R &= \int_M U_R(a(m)|m)\, g(m)\, dm = \sum_{k=0}^{n-1} \int_{w_k}^{w_{k+1}} \int_M \sigma(m|k)(a(m) - \theta)^2\, d\theta dm \\ &= \int_M \sum_{k=0}^{n-1} P(W_k)\sigma(m|k)\, U_R(a(m)|W_k)\, dm. \end{aligned}$$

The following section provides the general analysis of the model.

# 5 One-Stage Communication

This section characterizes the optimal information structure in the one-stage version of the model. We demonstrate that if the conflict of interest between parties is such that CS communication is informative, then the principal prefers communication with the imperfectly informed expert to both CS communication and optimal delegation. These results hold in more general settings.

To highlight the main intuition behind better communication with the imperfectly informed expert, consider the above example of $b = \frac{1}{5}$. The most-informative two-element CS partition is determined by the sender of the marginal type $w_1 = \frac{1}{10}$, who is indifferent between induced actions $a_0$ and $a_1$ (See Fig. 1). For lower types $\theta < w_1$, the action $a_0$ is strictly better than $a_1$, and vice versa.

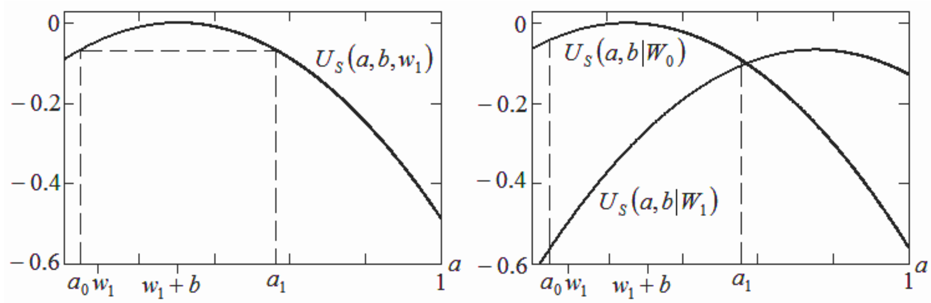


Figure 1: A CS equilibrium and a replicated equilibrium in the CWIIE model.

However, if sender's information is only whether a state is lower or higher than $w_1$, then she estimates the average utility across all states in the partition's element. The type-relevant utility function $U_S(a, b|W_k)$ is strictly concave in $a$ and symmetric with respect to the optimal policy $a_k^S = \bar{w}_k + b$. Note that $a_k^S < a^S(w_{k+1})$, where $a^S(w_{k+1}) = w_{k+1} + b$ is the optimal policy of the CS type $w_{k+1}$. That is, if the sender knows that $\theta \in W_0 = (0, w_1]$, then she strictly prefers action $a_0$ to $a_1$. Similarly, if the sender knows that $\theta \in W_1 = (w_1, 1]$, then she strictly prefers action $a_1$ to $a_0$. Through sending corresponding messages, the sender conveys all available information regardless of her type. Since specified actions are the receiver's best response to the sender's strategy, we construct an equilibrium, which replicates the CS equilibrium in terms of disclosed information. Moreover, since utility functions $U_S(a, b|W_k)$ and actions $a_0$ and $a_1$ are continuous in $w_1$, a partition $(0, w', 1)$ is still incentive-compatible for all $w'$ in some neighborhood of $w_1$. Thus, the principal can effectively modify the CS information structure without violating sender's incentives to communicate truthfully.

## 5.1 Equilibrium Characterization

In this subsection, we outline the basic characteristics of equilibrium strategies. It follows from (3) and (4) that for any sender's type, players' preferences over actions are purely determined by means $\bar{w}_k$. That is, $U_S(a, b|W_k) \geq U_S(a', b|W_k)$ if and only if $U_S(a, b, \bar{w}_k) \geq U_S(a', b, \bar{w}_k)$, and $U_R(a|W_k) \geq U_R(a'|W_k)$ if and only if $U_R(a, \bar{w}_k) \geq U_R(a', \bar{w}_k)$. Thus, type-relevant utility functions $U_S(a, b|W_k)$ and $U_R(a|W_k)$ inherit all important properties of state-relevant functions: strict concavity over actions, single-crossing, and symmetry with respect to optimal actions $a^S(\bar{w}_k) = \bar{w}_k + b$ and $a^R(\bar{w}_k) = \bar{w}_k$. This gives the no-crossing property: $a^S(\bar{w}_k) > a^R(\bar{w}_k)$, $k \in K$. Based on these observations and using the same technique as that developed in Lemma 1 in Crawford and Sobel [6], it follows that the number of induced actions in equilibrium is finite. All proofs can be found in the Appendix.

**Lemma 1** *In any equilibrium, the number of induced actions is finite. Further, the distance between any two actions is not less than $2b$.*

Formally, the number of actions is finite, since the strict concavity of $U_S(a, b|W_k)$ guarantees that the sender of each type induces at most two actions. However, the result of this lemma is stronger: it demonstrates that finiteness of the number of actions comes from the bias in the players' interests rather than from the cardinality of the type space. Thus, an increase in fineness of an information structure eventually does not bring further informational benefits, since the sender chooses among a finite set of actions. As a result, for a substantially fine partition, the sender's signaling strategy is no longer invertible, which leads to losses in information.

Thus, we may restrict the message space to a finite set $M' = \{m_i\}_0^{I-1}$, where $m_i \in \bar{M}(a_i)$, $i \in \mathcal{I} = 0, 1, ..., I-1$. Then, conditional distributions $\sigma(m|k)$ can be replaced by conditional probabilities $\sigma_{i,k}$, where $\sigma_{i,k} = \int\limits_{\bar{M}(a_i)} \sigma(m|k)\, dm$ is a conditional probability to send a message $m_i$, $i \in \mathcal{I}$, by the sender of a type $k \in K$.

From (5), the principal's best response $a_i = a(m_i)$, $i \in \mathcal{I}$, is

$$a_i = E[\theta|m_i] = E[\bar{w}_k|m_i] = \sum_{k=0}^{n-1} P(W_k|m_i)\, \bar{w}_k = \frac{1}{2} \frac{\sum\limits_{k=0}^{n-1} \sigma_{i,k}\left(w_{k+1}^2 - w_k^2\right)}{\sum\limits_{k=0}^{n-1} \sigma_{i,k}\left(w_{k+1} - w_k\right)}, \tag{6}$$

and the expected utility is

$$U_R = \sum_{i=0}^{I-1} P(m_i)\, U_R(a_i|m_i) = \sum_{i=0}^{I-1} \sum_{k=0}^{n-1} P(W_k)\, \sigma_{i,k} U_R(a_i|W_k),$$

where $P(m_i) = \sum\limits_{k=0}^{n-1} P(W_k)\, \sigma_{i,k}$ and $U_R(a|m_i) = \sum\limits_{k=0}^{n-1} \frac{P(W_k)\sigma_{i,k}}{P(m_i)} U_R(a|W_k)$.

The solution of a $k-$type sender's problem is

$$m_i \in \arg\max_{m \in M'} U_S(a(m), b|W_k), \text{ if}$$
$$U_S(a_i, b, \bar{w}_k) \geq U_S(a_j, b, \bar{w}_k) \text{ for all } j \in \mathcal{I}.$$

The family of inequalities $U_S(a_i, b, \bar{w}_k) \geq U_S(a_j, b, \bar{w}_k)$, $i, j \in \mathcal{I}$, $k \in K$, can be written as

$$\begin{aligned} &1)\ a_i + a_j \geq w_k + w_{k+1} + 2b \text{ for all } a_j > a_i, \text{ and} \\ &2)\ a_i + a_j \leq w_k + w_{k+1} + 2b \text{ for all } a_j < a_i. \end{aligned} \tag{7}$$

The following lemma characterizes the sender's equilibrium strategies.

**Lemma 2** *Any equilibrium signaling strategy* $(\sigma_{i,k})$ *satisfies the following conditions:*
*(A)* $\sigma_{i,k} > 0$ *implies* $\sigma_{j,k} = 0$ *for all* $j < i-1$ *and* $j > i+1$,
*(B)* $\sigma_{I-1,n-1} = 1$, *and* $\sigma_{i,n-1} = 0$ *for all* $i < I-1$,
*(C)* $\sigma_{i,k} > 0$ *implies* $\sigma_{j,s} = 0$ *for all* $s < k$, $j > i$, *and* $s > k$, $j < i$,
*(D)* $\sigma_{i,k} > 0$ *and* $\sigma_{i+1,k} > 0$ *imply* $\sigma_{i+1,k+1} > 0$ *for all* $k < n-1$, *and*
*(E)* $\sigma_{i,k} > 0$ *and* $\sigma_{i,k'} > 0$ *imply* $\sigma_{i,s} = 1$ *for all* $s$ *such that* $k < s < k'$.

The first condition states that mixing is possible only between two messages that induce adjacent actions. The second requires the highest-type sender to purely induce the highest action. The third condition is the "monotonicity condition," which implies that if a sender of some type induces an action, then no sender of a higher type can induce a lower action, and vice versa. Condition $(D)$ argues that if the $k-$type sender induces two actions, then a $(k+1)-$type must induce the higher action also. Finally, condition $(E)$ states that if some action is induced by types $k$ and $k'$, then this action is purely induced by all types between $k$ and $k'$.

Although Lemma 2 characterizes equilibrium strategies, it still leaves a lot of freedom in terms of players' expected payoffs. To narrow the set of payoffs, we need to formulate a model-specific revelation principle, which is described in the next section.

### 5.2 Revelation Principle

The lack of the principal's ability to commit to actions results in the failure of the standard revelation principle, which restricts the set of all equilibria outcomes to that of truth-telling direct equilibria. Two examples from contracting with imperfect commitment are due to Bester and Strausz [5] and Krishna and Morgan [11]. In both cases, the sender is characterized by a binary type, whereas three actions are induced in equilibria. No direct mechanism can replicate these equilibria in terms of induced actions and outcomes. Nevertheless, Bester and Strausz prove that for a finite set of states any incentive-efficient mechanism (i.e., that which provides equilibrium payoffs on the Pareto frontier) is payoff-equivalent to some direct mechanism. Similarly, Krishna and Morgan demonstrate that in the case of a continuum of types, any equilibrium outcome of an indirect mechanism can be replicated in a direct mechanism.

The following lemma proves that we can restrict attention to direct equilibria only; that is, the cardinality of the message space can be chosen to be equal to that of the type space, or $I = n$.

**Lemma 3** *Any equilibrium in the CWIIE model is payoff equivalent to some direct equilibrium.*

To demonstrate the result of the above lemma, we show that there is no equilibrium in the model, in which the number of induced actions exceeds the number of types. Basically, in order to have an indirect equilibrium, there must be a type which induces two actions, such that the higher action is induced by this type only. However, this contradicts property (D) of Lemma 2.[11]

Further, consider direct truth-telling, or **incentive-compatible equilibria**, in which the expert of each type $k = 0, ..., n-1$ discloses all available information by sending a type-specific message $m_k$. Given this signaling strategy, the receiver's best-response is the action's rule $a_k = E\left[\bar{w}_k | m_k\right] = \bar{w}_k$. Thus, a partition $\Omega$ is called **incentive-compatible** if there exists an incentive-compatible equilibrium.

From the sender's problem (7), the sender prefers to induce an action $a_k$ instead of

[11] In Krishna and Morgan's [11] example of an indirect equilibrium, the main incentive for a highest-type sender to induce lower actions is a higher transfer for sending lower messages, which is sufficient compensation for an undesirable policy implemented afterwards. Lack of such transfers in our setup narrows the set of equilibria.

$a_{k+1}$ (and all $a > a_{k+1}$), if

$$a_k + a_{k+1} = \bar{w}_k + \bar{w}_{k+1} = \frac{w_k + w_{k+1}}{2} + \frac{w_{k+1} + w_{k+2}}{2} \geq w_k + w_{k+1} + 2b, \tag{8}$$

which can be simplified to $w_{k+2} - w_k \geq 4b$. Similarly, the condition to induce $a_k$ instead of $a_{k-1}$ implies $w_{k+1} - w_{k-1} \geq 4b$. Therefore, a necessary and sufficient condition for a partition to be incentive-compatible is

$$w_{k+2} - w_k \geq 4b, \;\; k = 0, 1, ..., n-2. \tag{9}$$

This family of inequalities is called the **incentive-compatibility (IC) constraints.**

The following lemma proves that any pure-strategy equilibrium is payoff equivalent to some incentive-compatible equilibrium under a slightly modified partition. It is constructed from the initial one by the collapsing partition's elements that induce identical actions.

**Lemma 4** *For any pure-strategy equilibrium, there exists an incentive-compatible equilibrium, which is payoff equivalent.*

Now, we turn to mixed-strategy equilibria.

### 5.2.1 Mixed-strategy equilibria

In addition to pure-strategy equilibria, there exist multiple types of mixed-strategy equilibria even for the same information structure. The table below illustrates three examples of these equilibria. Notice that for the same partition, the last two equilibria in the table contrast in the number of induced actions: three in the former one and two in the latter one.

| $(w_k)$ | $(\sigma_{i,k})$ | $a\,(m_i)$ | $U_R \simeq$ |
|---|---|---|---|
| $(0, \frac{1}{2}, 1)$ | 3/4 0<br>1/4 1 | $0.25, 0.65$ | $-\frac{1}{43}$ |
| $(0, 0.07, 0.81, 1)$ | 0.873 0 0<br>0.127 0.962 0<br>0 0.038 1 | $0.035, 0.435, 0.845$ | $-\frac{1}{30}$ |
| $(0, 0.07, 0.81, 1)$ | 1 0.983 0<br>0 0.017 1 | $0.404, 0.876$ | $-\frac{1}{21}$ |

Nevertheless, mixing between messages is detrimental, which is demonstrated by the following "no-mixing" lemma.

**Lemma 5** *For any mixed-strategy equilibrium, there exists an incentive-compatible equilibrium, which is payoff superior.*

The superior equilibrium is constructed in two steps. First, we derive all sender's types that play mixed strategies and reassign the corresponding probabilities as follows. If some type induces two actions, we assign probability one to the lower action. Second, given the modified signaling strategy, we collapse the partition elements that induce identical actions and adjust the receiver's beliefs and actions to a new signaling strategy.

The derived lemmas constitute a model-specific "revelation principle", which relates to the result of Bester and Strausz – any optimal equilibrium payoff can be replicated in an incentive-compatible equilibrium.

### 5.3 Optimal information structure

To find the optimal incentive-compatible partition, we first determine the maximal size of the incentive-compatible partition $n(b)$ and all sequences of boundary points $(w_i)_0^{n(b)}$, which satisfy boundary conditions $w_0 = 0$, $w_{n(b)} = 1$, and the IC constraints (9). It can be shown that if $b \neq \frac{1}{2c}$ for some *even* integer $c$, then

$$n(b) = 2\langle \frac{1}{4b} \rangle + 1, \tag{10}$$

where $\langle x \rangle$ is the largest integer smaller than or equal to $x$. If $b = \frac{1}{2c}$ for some even integer $c$, then $n(b) = c$. For example, for $b = \frac{1}{5}$, $n(\frac{1}{5}) = 2\langle \frac{5}{4} \rangle + 1 = 3$. Notice that for $b = \frac{1}{4}$, the finest partition has two elements, so communication is informative, in contrast to the CS model. However, for $b > \frac{1}{4}$ no informative communication is feasible. The next result describes the structure of the optimal partition.

**Proposition 1** *For any $b$, there exists $b^*(c) \in (\frac{1}{2c}, \frac{1}{2(c-1)})$, where $c = n(b)$, such that if $b > b^*(c)$, then the optimal partition is uniform of size $c - 1$. For $b \leq b^*(c)$, the optimal partition is one of size $c$ such that: 1) if $\frac{1}{2(c+1)} < b \leq \frac{1}{2c}$, then the optimal partition is uniform, and 2) if $\frac{1}{2c} < b \leq b^*(c)$, then the IC constraints (9) are binding for all boundary points $w_k$.*

According to this proposition, the optimal partition belongs to one of two classes: either a possibly non-uniform of size $n(b)$, or the uniform of size $n(b) - 1$. For the bias $b = \frac{1}{5}$, which is used in Example 1, we have $c = n(\frac{1}{5}) = 3$ and the cutoff level $b^*(3) \simeq 0.202$. The principal's expected payoffs under the three-element partition $\left(0, \frac{1}{5}, \frac{4}{5}, 1\right)$ with the binding IC constraints and the two-element uniform partition $\left(0, \frac{1}{2}, 1\right)$ are $-\frac{1}{52}$ and $-\frac{1}{48}$, respectively. Thus, the receiver's losses due to non-uniform structure of the three-element partition are offset by benefits due to its fineness. Nevertheless, for $b = 0.22$, principal's payoffs are $-\frac{1}{27}$ and $-\frac{1}{48}$ for the three- and two-element partitions, respectively. This is because a larger bias results in the less uniform structure of the finest incentive-compatible partition $(0, 0.12, 0.88, 1)$ due to the IC constraints. In contrast, there is no a such effect for the uniform partition of a smaller size, which is still incentive-compatible.

Thus, we can establish the Pareto dominance of the CWIIE model over CS communication.

**Theorem 2** *If the sender' bias is $b \leq \frac{1}{4}$, then there exists an equilibrium in the CWIIE model, which is Pareto superior to all equilibria in the CS model.*

Compared to the CS model, an essential increase in the receiver's payoffs is driven by two factors. The first factor is the *maximal number* of partitions' elements. It grows as $\frac{1}{b}$ in the CWIIE model in contrast to $\frac{1}{b^{1/2}}$ in the CS model.[12] The second factor is the *difference*

[12] For example, for $b = \frac{1}{30}$, $N^{CS}(b) = 4$ in the CS model and $n(b) = 15$ in ours. This results in the principal's expected utilities $U_R^{CS} = -\frac{1}{93}$ and $U_R = -\frac{1}{2700}$ in the CS and CWIIE models, respectively.

*in lengths* of partition elements. Due to the specific structure of equilibria, intervals of any CS partition essentially differ in their lengths, which results in higher informational losses for large values of a state.[13] On the other hand, for any equilibrium CS partition, a uniform partition of the same size is incentive-compatible in our model, which decreases a residual variance of $\theta$.

### 5.4 Imperfect Information versus Delegation

Delegation is broadly considered as a pervasive alternative to communication. Instead of relying on expert's non-verifiable information, the policymaker can delegate his power to the expert and gain from informational efficiency (see, for example, [1], [7], [10], [13]). However, informational benefits are impaired by losses because the expert's decisions are biased. Nevertheless, in a variety of situations, an aggregate effect leads to ex-ante Pareto-improvement as compared to communication. Another useful feature of delegation is its ease in implementations: generally, there are no costs to empower the expert with a right to carry out policies. Due to these factors, more and more firms decentralize their structures (see [7]). The following example demonstrates how delegation can bring larger payoffs to the principal than those in the case of CS communication. However, these payoffs are smaller than those in the case of communication with the imperfectly informed expert.

**Example 2.** Consider the uniform-quadratic setup with the sender's bias $b = \frac{1}{5}$. In the CS communication, the most informative equilibrium provides the principal's ex-ante payoff $U_R^{CS} \simeq -\frac{1}{16}$.

If the principal delegates his rights completely, that is, without restrictions on the set of sender's feasible policies, then for any state $\theta$, the sender implements her optimal policy $a^S(\theta) = \theta + b$, which has a constant bias $b$ relative to the receiver's optimal policy $a^R(\theta) = \theta$. This brings ex-post utilities to the sender and the receiver $U_S^D(\theta) = 0$ and $U_R^D(\theta) = -b^2 = -\frac{1}{25}$, which are equal to corresponding ex-ante utilities. Therefore, the expected payoff of the principal exceeds that in CS communication. However, if the receiver partitions the state space into two equal intervals, then his expected payoff in the incentive-compatible equilibrium under this partition is $U_R = -\frac{1}{48}$, which is larger than that under delegation.

Before testing the generality of these results, note that the full or complete delegation is not necessarily optimal in the space of all delegation sets, that is, the sets of actions that can be delegated to the sender. Melumad and Shibano [13] prove that the optimal delegation set for the uniform-quadratic settings is an interval $[0, y']$, where the upper bound $y' = 1 - b$ if $b \leq \frac{1}{2}$ and $y' = \frac{1}{2}$ otherwise. Also, Dessein [7] shows that in the same model, delegation is always beneficial for the principal as opposed to CS communication whenever communication is informative.

Delegation and communication with the imperfectly informed sender utilize different factors for the payoffs' improvement. Delegation allows for the receiver to acquire benefits from the expert's informational advantage. If endowed with power, the sender conducts a policy which is close enough to the receiver's optimal policy. Communication with the

[13] A length of a $(k+1)$-th interval of a CS partition exceeds that of a $k$-th interval by $4b$, for all $k$.

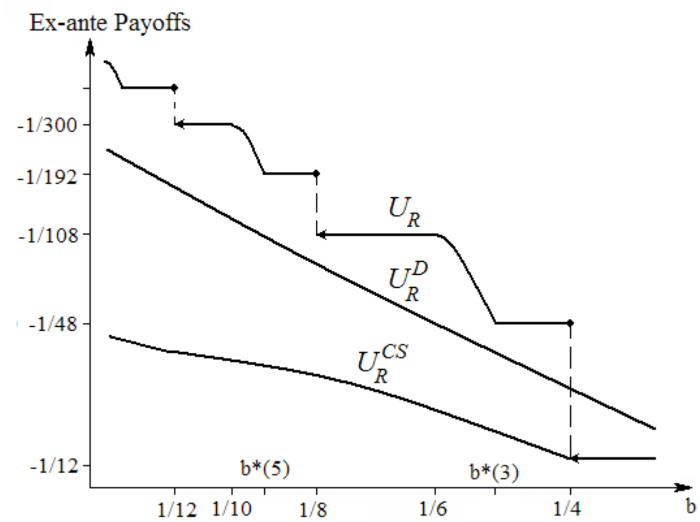


Figure 2: Payoffs in the CWIIE model, optimal delegation, and the CS model.

imperfectly informed expert smooths misalignment between players' preferences which arises from senders of "boundary" types in the CS model. This results in the inefficient structure of informational partitions, where communication is less informative for high values of the state. As a result, there is no clear intuition about what effect is more beneficial for the principal.

Now, we show that if informative communication is feasible ($b \leq \frac{1}{4}$), then communication with the imperfectly informed sender strictly dominates delegation in terms of receiver's expected payoff. This result is formalized by the following theorem.

**Theorem 3** *If the informative communication is feasible, then there exists an equilibrium in the CWIIE model which provides a higher expected payoff to the principal than optimal delegation.*

Fig. 2 demonstrates the principal's expected payoff under the optimal partition in the CWIIE model, optimal delegation, and the most informative equilibrium in the CS model.

Compared to other models, an interesting feature of the CWIIE model is the discontinuity of payoffs in the sender's bias $b$ due to the "regime switching" effect. When $b$ falls, this effect takes place at $\tilde{b}(n) = \frac{1}{2n}$ for even $n$, and the uniform partition of the *even* size $n$ becomes incentive-compatible. This leads to a switch from the uniform partition of the *odd* size $n-1$ to the uniform of size $n$. As a result, the ex-ante payoffs jump from $-\frac{1}{12(n-1)^2}$ to $-\frac{1}{12n^2}$. In terms of the analysis above, a continuous change in the sender's bias can have a discontinuous impact upon equilibrium payoffs. However, the relative change in the payoffs $\lim_{b \uparrow \tilde{b}(n)} U_R(b)$ and $\lim_{b \downarrow \tilde{b}(n)} U_R(b)$ converges to 0 as $n \to \infty$.

In contrast, there is no discontinuity at $b^*(n)$, the point of a switch from the uniform partition of the *even* size $n-1$ to the non-uniform one of the *odd* size $n$. For almost all $b$ such that the partition of *even* size $n-1$ is incentive-compatible,[14] there exists a partition

[14]Namely, for all $b \neq \frac{1}{2m}$ for some even $m$.

of the odd size $n$, which is also incentive-compatible (namely, for which the IC conditions (9) are binding). However, this partition is non-uniform for $b > \frac{1}{2n}$ and provides payoff $U_R^n(b)$, which is continuous with respect to $b$.[15] Thus, the switch between partitions at $b^*(n)$ such that $U_R^n(b^*) = -\frac{1}{12(n-1)^2}$ is not accompanied by a discontinuous change in payoffs.

### 5.5 Generalizations

This section examines the robustness of the previous results to changes in the specification of the model, namely players' utility functions and distributions of the state. First, we consider the generalized form of the players' utility functions similar to that used by Dessein [7]. The receiver's utility function $U_R(a,\theta)$ has a unique maximum for $a=\theta$ and can be written as

$$U_R(a,\theta) = U_1(\theta,\theta) + U_2(|a-\theta|), \tag{11}$$

where $U_2(.)$ is twice continuously differentiable, and $U_2'(0) \leq 0$, $U_2''(x) < 0$. If $U_2'(0) = 0$, we additionally require $U_2'''(.)$ to be continuous in the neighborhood of 0.

Similarly, the sender's utility function $U_S(a,b,\theta)$ has a maximum for $a = \theta + b$ and can be written as

$$U_S(a,b,\theta) = V_1(\theta+b,\theta) + V_2(|a-b-\theta|), \tag{12}$$

where $V_2'(x) \leq 0$ and $V_2''(x) < 0$. For future references, we will refer to (11) and (12) as **symmetric preferences**.

Given these conditions, it can be shown that for any $W_k = (w_k, w_{k+1}]$, the sender's type-relevant utility function $U_S(a,b|W_k) = E[U_S(a,b,\theta)|\theta \in W_k]$ is symmetric in $a$ with respect to $a_S = \bar{w}_k + b$. Similarly, the receiver's utility function $U_R(a|W_k)$ is concave in $a$ and symmetric with respect to $\bar{w}_k$. This implies that the receiver's best-response to the truth-telling signaling strategy $m(k) = m_k$ is $a(m_k) = \bar{w}_k$. Therefore, the IC constraints (9) also hold, and the optimal information structure is the same as determined by Proposition 1 up to values $b^*(c)$. Essentially, communication can be informative only if the bias $b \leq \frac{1}{4}$. Based on these observations, the generalization of Theorem 2 can be proved straightforwardly.

**Theorem 4** *If $\theta$ is uniformly distributed and preferences are symmetric, then for $b \leq \frac{1}{4}$, there exists an equilibrium in the CWIIE model which is Pareto superior to all equilibria in the CS model.*

As in the case of the quadratic preferences, given any CS partition, the uniform partition of the same size in the CWIIE model is incentive-compatible. Due to risk-aversion of the principal, it provides the superior expected utility.

Before we compare principal's payoff in our model with that in delegation, notice that for the class of interval delegation sets (i.e., a expert's policy must belong to a single interval) and $b \leq \frac{1}{2}$, the optimal delegation set is still of a form $[0, 1-b]$. Then, we can generalize the result of Theorem 3: communication with the imperfectly informed expert performs better than optimal delegation, when the bias $b$ is sufficiently small.

[15] The exact formula for $U_R^n(b)$ can be found in the Appendix.

**Theorem 5** *If $\theta$ is uniformly distributed and preferences are symmetric, then there is $b^*$ such that for all $b < b^*$, there exists an equilibrium in the CWIIE model, which provides a superior outcome to the receiver compared to that in optimal delegation.*

This result is weaker than Theorem 3 for the case of the quadratic preferences since it does not guarantee that the CWIIE model performs better than delegation whenever informative communication is feasible. An example demonstrates that this result cannot be strengthened because of the *risk-aversion* of the principal.[16] In communication, an induced action is unbiased on average, but there is a chance that it is far from the optimal action $\theta$ (if a state is close to a boundary of a partition element). This increases informational losses for essentially concave utility functions. Delegation, however, provides a permanent bias $b$, which is more preferable by the highly risk-averse principal. Nevertheless, when the bias is small, the optimal information structure becomes sufficiently fine to reduce the variance between optimal and induced actions, which results in better performance of the CWIIE model over delegation.

Another approach to generalize model's settings is to extend the case of the uniform distribution function of the state. In particular, we restrict attention to the class of distributions with a positive and continuously differentiable density and supported on a bounded interval. In this case, the result of Theorem 2 is completely robust.

**Theorem 6** *Suppose preferences are quadratic. For any distribution function of $\theta$ with a positive and continuously differentiable density on a bounded support, there exists an equilibrium in the CWIIE model which is superior to all informative equilibria in the CS model.*

In general, any informative CS partition is characterized by high informational losses for large values of $\theta$. Thus, it can be modified by the principal in a such way that it is incentive-compatible in the CWIIE model and "more uniform", which reduces a residual variance of $\theta$.

Similarly, the theorem below compares principal's payoffs in the CWIIE model with that in complete delegation. It demonstrates that the result of Theorem 3 for general distributions holds, if the sender's bias is small.[17]

**Theorem 7** *Suppose preferences are quadratic. For any distribution function of $\theta$ with a positive and continuously differentiable density on a bounded support, there is $\tilde{b}$ such that for all $b < \tilde{b}$, there exists an equilibrium in the CWIIE model, which provides a superior payoff to the receiver compared to that in the complete delegation.*

[16] Consider the principal's utility function $U_1(w, w) = 0$, $U_2(w) = -|w|^7$, and the bias $b = 0.126$. Then the optimal partition in the CWIIE model is the uniform three-element one. It is informative and provides expected utility $U_R \simeq -4.5 \cdot 10^{-7}$. However, optimal delegation gives $U_R^D = -3.9 \cdot 10^{-7}$, which is superior to that in the CWIIE model.

[17] The problem of optimal delegation for general distributions and quadratic preferences is solved by Alonso and Matouschek [1]. They provide necessary and sufficient conditions for delegation sets to be optimal for cases of complete delegation, centralization (the delegation set that contains only the principal's preferred actions given prior information), and interval delegation. However, when the players' preferences are sufficiently close, one can expect that principal's incentives to restrict sender's actions are small, and the optimal delegation set and players' outcomes converge to that of the case of complete delegation.

When the sender's bias converges to zero, a size of intervals in the finest incentive-compatible partition converges to $2b$ regardless of the distribution. Equivalently, the number of elements $n(b)$ in the finest incentive-compatible partition grows as $\frac{1}{2b}$, exactly as in the case of the uniform distribution. This implies that the principal's expected utility in the most informative equilibrium falls as $-\frac{1}{12\times n(b)^2}$ or $-\frac{b^2}{3}$, in contrast to $-b^2$ in delegation.

## 6 Multi-Stage Communication

In general, communication between players may not be restricted to a single stage. Moreover, the example of oil lease sales demonstrates that the principal can determine the precision of the expert's information in every round, and request a new report afterwards. In this context, our central result is that by proper updating the expert's information from stage to stage, the principal can disclose approximately full information in some interval of the state space due to expert's truth-telling communication in all stages.

Krishna and Morgan [12] describe multi-stage communication in the CS model without updating sender's information. In this case, the set of equilibrium outcomes is identical to that in the one-stage communication game. Since the expert knows all information before the communication starts, she sends the sequence of messages that induces the most preferable action. However, without information update from stage to stage, the receiver infers the same information about the state as in the one-stage case. Thus, the set of induced actions is also not affected, and any equilibrium in an multi-stage game is equivalent to that in the one-stage game. This argument can be directly reapplied to the case of an imperfectly informed expert without information updating. In contrast, if the expert's information is insignificantly updated at every stage, the outcome of the multi-stage communication differs significantly from the one-stage case.

To introduce such updating in the model, the receiver specifies a **communication schedule**: a family of sets $\{W_k^s\}_{k=0,s=1}^{n_s-1,T}$, where $n_s$ sets $\{W_k^s\}_{k=0}^{n_s-1}$ form a partition of $\Theta$ at every round of communication $s = 1, ..., T < \infty$. Once chosen, a communication schedule becomes common knowledge.

In every stage $s$, the sender observes an index $i_s$ of the partition's element $W_{i_s}^s$, which contains $\theta$, and transmits a message $m_s \in M$ to the receiver. Thus, the imprecision of the sender's information about the state is determined by a measure of the set $M_s = \bigcap_{\tau=1}^{s} W_{i_\tau}^\tau$. The sender's signaling strategy $\sigma$ is a mapping from the space of all sequences $(i_s)_1^T$ to a probability distribution over the message set $\underset{s=1,...,T}{\times} M$. After receiving a sequence of messages $(m_s)_1^T$, the receiver updates his posterior beliefs about the state and implements an action $a$. An example below illustrates how multi-stage communication results in the Pareto improvement.

**Example 3**. Take the sender's bias $b = 0.21$. In the one-stage game, the uniform two-element partition is optimal and provides receiver's ex-ante payoff $-\frac{1}{48}$.

Now, consider communication through two stages such that the receiver determines the communication schedule $W_0^1 = (0, 84]$, $W_1^1 = (0.84, 1]$, $W_0^2 = (0, 0.42]$, $W_1^2 = (0.42, 1]$ and implements actions $a(m_0, m_0) = a_{00} = 0.21$, $a(m_0, m_1) = a_{01} = 0.63$, and $a(m_1, .) = a_1 = 0.92$.

Suppose that at the first stage the sender observes $i_1 = 1$, which means that $\theta$ is uniformly distributed over the interval $W_1^1$. Given this information, the sender's utility function $U_S\left(a, b|W_1^1\right)$ is maximized at $a' = \frac{w_1+w_0}{2} + b = 1.13$. Also, her current information will not be updated in the next round since $W_1^1 \subset W_1^2$. Then, the message $m_1$ induces the action $a_1$ for any message in the second stage.[18] The message $m_0$ can induce actions $a_{00}$ and $a_{01}$, depending on the message in the next stage. Since $U_S\left(., b|W_1^1\right)$ is increasing for all $a < a'$ and $\max\left(a_{00}, a_{01}\right) < a_1 < a'$, then the sender strictly prefers to transmit the message $m_1$.

If $i_1 = 0$, then the sender infers that $\theta \in W_0^1$, and her current information will be updated in the next stage. Then, the message $m_1$ induces the action $a_1$, which provides the expected payoff $U_S\left(a_1, b|W_0^1\right) \simeq -\frac{1}{7}$. In contrast, sending message $m_0$ in this stage and truthful communication in the next stage results in the expected payoff

$$E[U_S\left(a\left(m_0, m_k\right), b|W_k^2\right)) = -\frac{1}{w_1}\int\limits_0^{w_2}(a_{00} - b - \theta)^2 d\theta - \frac{1}{w_1}\int\limits_{w_2}^{w_1}(a_{01} - b - \theta)^2 d\theta \simeq -\frac{1}{17}.$$

Thus, the sender still has no incentives to distort information.

In the second round, let $i_2 = 1$. If $i_1 = 1$, then according to the analysis above, the sender induces the action $a_1$ in the first stage by sending the message $m_1$. If $i_1 = 0$, then she infers that $\theta \in W_0^1 \cap W_1^2 = (0.42, 0.84]$, and her optimal action becomes $a'' = \frac{w_2+w_1}{2} + b = 0.84$. Given the message $m_0$ in the first round, messages $m_1$ and $m_0$ in the second round induce actions $a_{01}$ and $a_{00}$, respectively. Because $a_{00} < a_{01} < a''$, it follows that sending message $m_1$ is strictly preferable to $m_0$.

If $i_2 = 0$, then the sender deduces that $\theta \in W_0^2$. Given the message $m_0$ in the first stage, the sender can induce actions $a_{00}$ and $a_{01}$ only. Notice that $U_S\left(a_{00}, b|W_0^2\right) = U_S\left(a_{01}, b|W_0^2\right)$, since the distance between sender's optimal policy $a''' = \frac{w_3+w_2}{2} + b = 0.42$ and actions $a_{01}$ and $a_{00}$ is the same. Thus, the sender still cannot deviate from revealing her information. It can be easily seen that induced actions are the receiver's best-response to the sender's truth-telling strategy.

The expected utility of the receiver in this equilibrium is approximately $-\frac{1}{79}$, which exceeds that in the most informative equilibrium in the one-stage game. This is because conveyed information in the described equilibrium is equivalent to that in the one-stage communication under a partition $(w_k) = (0, 0.42, 0.84, 1)$ and the truth-telling signaling strategy. However, truth-telling is not the equilibrium strategy, since it violates the IC constraints (9): $w_3 - w_1 = 0.58 < 0.84 = 4b$.

## 6.1 The revealing mechanism

Now, we present our major result for the multi-stage communication. There exists a communication schedule, through which the receiver can reveal (almost) all information in the interval $[4b, 1]$.

First, restrict attention to two-element partitions $W_0^s = [0, w_s]$ and $W_1^s = (w_s; 1]$ at each stage $s = 1, ..., T$. Equivalently, such communication schedule is determined by a sequence of boundary points $(w_s)_0^{T+1}$, where we let $w_0 = 1$ and $w_{T+1} = 0$.

[18] Another interpretation of this action's rule is that communication stops as the sender conveys $m_1$.

Second, consider a decreasing communication schedule, that is, such that a sequence $(w_s)_0^{T+1}$ is decreasing. Given a state $\theta$, define $\tilde{s} = \min\{s : i_s = 1\}$ to be the first stage, in which the sender observes a higher index. If $i_s = 0$ for all $s$, then put $\tilde{s} = T+1$. For a decreasing communication schedule, $i_s = 1$ for all $s \geq \tilde{s}$, since $W_1^s \subset W_1^{s+1}$ for all $s$, and the sender's information is not updated after the stage $\tilde{s}$. Thus, the space of all sequences $(i_s)_{s=1}^T$ consists of $T+1$ non-decreasing sequences $I_0 = (0)_1^T$ and $I_k = \{(i_s)_1^T : i_s = 1$ for $s \geq k$, $i_s = 0$ for $s < k\}$, $k = 1, ..., T$.

Consider a decreasing communication schedule, depicted in Fig.3, such that

$$w_{T-1} \geq 4b,\ 0 < w_T < w_{T-1},\ w_0 = 1, \text{ and } w_{T+1} = 0. \tag{13}$$

Then, the sequence $(\bar{w}_s)_1^{T+1}$, where $\bar{w}_s = \frac{w_s + w_{s-1}}{2}$, $s = 1, ..., T+1$, is also decreasing.

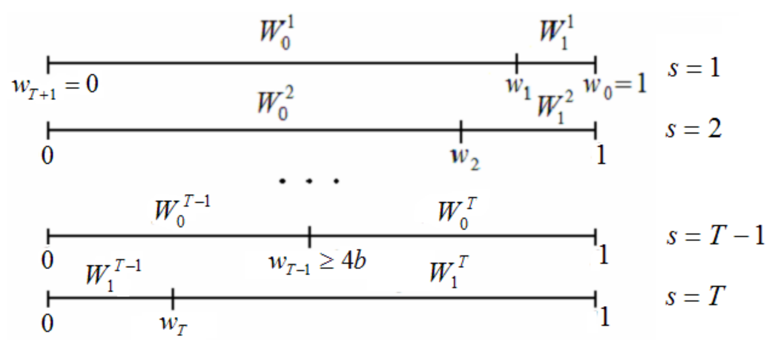


Figure 3: A decreasing communication schedule

Define $\Theta_i = (w_i; w_{i-1}]$, $i = 1, ..., T+1$. At the end of the communication process, the receiver's beliefs about $\theta$ can be expressed as $\mu(\Theta_i | (m_s)_{s=1}^T)$, which means a belief that $\theta$ is uniformly distributed on $\Theta_i$ with probability $\mu(\Theta_i|.)$.

Given this setup, the main result is characterized by the following theorem.

**Theorem 8** *For any decreasing communication schedule $(\theta_s)_{s=0}^{T+1}$ which satisfies (13), there exists an equilibrium such that:*

*1) $m_s(I_s) = i_s$, $s = 1, ..., T$,*

*2) $a\left((m_s)_{s=1}^T\right) = \bar{w}_j$, where $j = T+1$, if $m_s = 0$ for all $s$, and $j = \min\{s : m_s = 1\}$ otherwise, and*

*3) $\mu(\Theta_j | (m_s)_{s=1}^T) = 1$ and $\mu(\Theta_i | (m_s)_{s=1}^T) = 0$, $\Theta_i \neq \Theta_j$.*

Condition 1 describes the truth-telling signaling strategy of the sender. Condition 2 is the receiver's best-response, given his posterior beliefs. Condition 3 outlines principal's posterior probabilities of $\theta \in \Theta_i$ for both non-zero-probability and zero-probability messages of the sender.

**Corollary 1** *By choosing a decreasing communication schedule $(w_s)_1^T$ such that $w_{T-1} = 4b$, $0 < w_T < 4b$, and $\max\limits_{s=1,...,T-1} |w_s - w_{s-1}| \to 0$ as $T \to \infty$, the receiver discloses approximately full information in the interval $[4b, 1]$ in the above equilibrium.*

**Corollary 2** *For $b < \frac{1}{4}$, there exists a communication schedule and an equilibrium under this communication schedule in the model of multi-stage communication, which is Pareto superior to any equilibria in the model of one-stage communication.*

By slight updating information at all stages except the last two, the sender seems to get a small piece of information at every stage. This argument is misleading, however. If the sender observes $i_{s-1} = 0$ in stage $s-1$ and $i_s = 1$ in stage $s$, then she infers that $\theta \in W_0^{s-1} \cap W_1^s = (w_s, w_{s-1}]$. Hence, her information about the state becomes very precise. The main argument is that given this updated information and the receiver's beliefs, the sender's best feasible action is the one which will be implemented after revealing her information truthfully. In contrast, if $i_s = 0$, the sender's information is still essentially vague and will be improved in the future. Due to risk-aversion and sufficiently imprecise information, the sender's expected payoff from transmitting a distorted message $m_1$ (and inducing the action $a = \bar{w}_s$) is quite low, and this signaling strategy is strictly dominated by *providing truthful information at this and all future stages*. The crucial condition is that, given $i_s = 0$, the quality of sender's information must be sufficiently imperfect, which is achieved by choosing the a sufficiently coarse structure of partitions in the last two stages ($w_{T-1} \geq 4b$).

# 7 Extensions and Discussion

In this section, we discuss several issues: the principal's utility in the multi-stage model as the number of stages increases without bound; a comparison of efficiency of multi-stage communication versus one-stage communication; and the possibility to commit to actions in some stages of the communication process.

## 7.1 The limit of disclosed information

When $\max_{s=1,\ldots,T-1} |w_s - w_{s-1}| \to 0$ as $T \to \infty$, approximately full information is revealed in the interval $[4b, 1]$. Thus, the principal's expected utility in the described equilibrium is determined by boundary points $w_{T-1}$ and $w_T$ in the last two stages

$$U_R^{\lim}(w_{T-1}, w_T) = -\sum_{\tau=T-1}^{T} \int_{w_{\tau+1}}^{w_\tau} \left( \frac{w_{\tau+1} + w_\tau}{2} - \theta \right)^2 d\theta = -\frac{1}{12} w_T^3 - \frac{1}{12}(w_{T-1} - w_T)^3.$$

Given constraint (13), $U_R^{\lim}(w_{T-1}, w_T)$ is maximized at $w_{T-1} = 4b$ and $w_T = 2b$, which results in the limit of the expected utility

$$U_R^{\lim} = -\frac{4}{3} b^3.$$

The full disclosure of information in the interval $[4b, 1]$ requires infinitely many stages of communication. Given the principal's utility $U_R^T$ in the game with $T$ stages, a relative difference between $U_R^T$ and the limiting utility $U_R^{\lim}$, that is,

$$\varepsilon = \left| \frac{U_R^{\lim} - U_R^T}{U_R^{\lim}} \right|$$

can serve as the measure of imperfection of disclosed information.

Referring to the case of $b = 0.21$, the limit of the receiver's expected utility in the multi-stage equilibrium is $U_R^{\lim} \simeq -\frac{1}{81}$. However, Example 3 demonstrates that just two stages of communication provide $U_R^2 \simeq -\frac{1}{79}$, so that $\varepsilon = \frac{U_R^{\lim} - U_R^2}{U_R^{\lim}} \simeq 2.5\%$. In general, it is routine to show that the number of communication stages $T$ increases as $\varepsilon^{-1/2}$.[19] Thus, to decrease inefficiency, say, from 4% to 1%, the number of rounds of communication must be doubled.

## 7.2 One-stage versus multi-stage communication

As mentioned above, when the bias in players' preferences $b$ tends to 0, the number of intervals $n(b)$ of the optimal partition in one-stage communication grows as $\frac{1}{2b}$, so that the length of an interval $\Delta w$ decreases as $2b$. Thus, the principal's expected utility grows as $-n(b)\frac{(\Delta w)^3}{12} \sim -\frac{b^2}{3}$.

In multi-stage communication, information cannot be fully revealed only if $\theta < 4b$, which implies that the principal's expected utility increases as a third power of $b$. As a result, *the efficiency of multi-stage communication relative to one-stage rises infinitely as the conflict of interest falls.*

## 7.3 Commitment

In the previous analysis, we have considered a pure cheap-talk game, that is, unconditionally on the expert's information, the principal has full authority over policies. Here, we introduce an extension of the multi-stage model to a combination of communication in some stages with delegation in others. The main result is that such combination extends the interval, in which approximately full information can be revealed, from $[4b, 1]$ to $\left[\frac{5}{3}b, 1\right]$.

In particular, we modify the multi-stage model as follows. The principal specifies a decreasing communication schedule $(w_s)_1^{T-1}$ for stages $s = 1, ..., T-1$ and the perfect information structure in the interval $[0, w_{T-1}]$. Also, he implements an action $a\left((m_s)_1^{T-1}\right) = \frac{w_j + w_{j-1}}{2}$, where $j = \min\{s : m_s = 1\}$. If $m_s = 0$ for all $s = 1, ..., T-1$, then in the last stage $T$ he delegates authority to the expert by determining the delegation set $[0, w_{T-1}]$. That is, if $\theta \in [0, w_{T-1}]$ and the expert truthfully reveals her information in stages $s = 1, ..., T-1$ by sending message $m_s = 0$, then in the last stage the expert knows the state perfectly, but she can implement only policies in this interval. Using the same approach as in Theorem 8, one can show that there exists an equilibrium such that the expert truthfully reveals information in all stages up to stage $T-1$ if $w_{T-1} \geq \frac{5}{3}b$. That is, when $\max_{s=1,...,T-1} |w_s - w_{s-1}| \to 0$ as $T \to \infty$, then approximately full information can be revealed in the interval $\left[\frac{5}{3}b, 1\right]$, and the principal's limiting expected utility rises from $-\frac{4}{3}b^3$ to $-b^3$. This implies that there exists an informative communication even when

[19] Since conveyed information in the multi-stage equilibrium under a communication schedule $(w_s)_0^{T+1}$ is equivalent to the truthful communication in the one-stage game with the partition $(w_s)_0^{T+1}$, the most informative communication schedule is such that $w_{T-1} = 4b$, $w_T = 2b$, and $w_s = 1 - \frac{1-4b}{T-1}s$, $s = 0, ..., T-2$, which results in the receiver's expected utility $U_R^T = U_R^{\lim} - \frac{1}{12}\frac{(1-4b)^3}{(T-1)^2}$.

the bias is so large ($b \in \left(\frac{1}{4}, \frac{3}{5}\right)$) that no informative communication is achievable in the cheap-talk game.

Notice that the possibility to commit increases the principal's utility not because of expert's informational superiority, but through a different channel. If the state $\theta < w_{T-1}$, then it will be imperfectly revealed in the last stage of the cheap-talk multi-stage game. However, communication in the last stage is equivalent to the one-stage communication game with an imperfectly informed expert, which provides a higher payoff than delegation. Thus, the principal cannot benefit from the expert's informational superiority. The possibility to implement the expert's favorite policy in the last stage serves as an attractive "carrot", which enforces her incentives to communicate truthfully in the previous stages.

# 8 Conclusion

We have demonstrated that by properly restricting the quality of the expert's information, the principal can obtain more information, and get a higher payoff, than in the CS model of communication. Moreover, our model leads to a superior expected payoff than that provided by delegation. These results generally remain true for a wide class of preferences and distributions.

Communication with an imperfectly informed expert in multiple rounds, where the principal controls the quality of the expert's information in every round, can elicit almost all information for a large interval of the state space. This results in an ex-ante Pareto-improvement compared to one-stage communication. When considering the example of an oilfield lease, the government can get more precise geological data from private companies (which collect data) if it imposes proper restrictions on the number and locations of test drills, and obtains copies of the reports after each stage of the process of exploration.

Another important aspect of the presented model is the number of equilibria, significantly exceeding the number of equilibria in Crawford and Sobel. In addition to pure-strategy equilibria, there exist multiple mixed-strategy equilibria even with the same partition. Nevertheless, despite the fact that all mixed-strategy equilibria are payoff inferior to pure-strategy ones, they can still be superior to equilibria in the CS model. In the case of the multi-stage communication, the constructed locally revealing equilibrium is not unique. There exist other less informative babbling and semi-babbling equilibria such that the sender does not reveal information in some stages of the communication process.

We did not address deliberately the case when the person who determines the quality of the expert's information is the expert herself. In this case, if there exists a credible mechanism of the expert's commitment, i.e., the expert commits "not to know too much", then the result will be the same in terms of disclosed information due to the closeness of the expert's and principal's interests.

# 9 Appendix

In this section we provide proofs of the lemmas and theorems.

**Proof of Lemma 1.** Let $a$ and $a'$ be two induced actions, where $a' > a$. Consider types $k$ and $k'$ which induce corresponding actions, that is $U_S(a, b|W_k) \geq U_S(a', b|W_k)$ and $U_S(a', b|W_{k'}) \geq U_S(a, b|W_k)$. Then, it follows from (3) that $U_S(a, b, \bar{w}_k) \geq U_S(a', b, \bar{w}_k)$ and $U_S(a', b, \bar{w}_{k'}) \geq$

$U_S(a, b, \bar{w}_{k'})$.

The single-crossing property of the sender's state-relevant utility function $\frac{d^2}{dad\theta}U_S(a, b, \theta) > 0$ implies that there exists a state $\theta \in (\bar{w}_k, \bar{w}_{k'})$ such that $U_S(a', b, \theta) = U_S(a, b, \theta)$. Also, this property leads to (i) $a < a^S(\theta) < a'$, where $a^S(\theta) = \theta + b$, (ii) $a$ is not induced by any type $i$ such that $\bar{w}_i > \theta$, and (iii) $a'$ is not induced by any type $i$ such that $\bar{w}_i < \theta$. The last two properties along with the single-crossing property of $U_R(a, \theta)$ imply $a \leq a^R(\theta) = \theta \leq a'$.

In addition, the symmetry of $U_S(a, b, \theta)$ with respect to $a^S(\theta)$ implies that $a' - \theta - b = \theta + b - a$, or $a^S(\theta) = \theta + b = \frac{a+a'}{2}$. This means that both $a^S(\theta)$ and $a^R(\theta)$ belong to the interval $[a, \frac{a+a'}{2}]$. Since $a^S(\theta) - a^R(\theta) = b$, it means that $\frac{a+a'}{2} - a \geq b$, or $a' - a \geq 2b$.

To complete the proof, notice that the set of induced actions is bounded by $a^R(0)$ and $a^R(1)$. ■

**Proof of Lemma 2.** $(A)$ This property follows from a strict concavity of $U_S(a, b|W_k)$ in $a$. By contradiction, let $\sigma_{i,k} > 0$ and $\sigma_{j,k} > 0$, where $j > i + 1$, for some $k$. This implies that $U_S(a_i, b|W_k) = U_S(a_j, b|W_k) \geq U_S(a_l, b|W_k)$ for all $l \in \mathcal{I}$. Since $a_{i+1}$ can be represented as a convex combination of $a_i$ and $a_j$, $a_{i+1} = \lambda a_i + (1 - \lambda) a_j$ for some $\lambda \in (0, 1)$, this results in a contradiction $U_S(a_{i+1}, b|W_k) > \lambda U_S(a_i, b|W_k) + (1 - \lambda) U_S(a_j, b|W_k) = U_S(a_i, b|W_k)$.

$(B)$ From (6), the maximal induced action $a_{I-1} \leq \bar{w}_{n-1} < \bar{w}_{n-1} + b$. Since $U_S(a, b|W_k)$ is strictly increasing in $a$, for all $a < \bar{w}_k + b$, the result follows immediately.

$(C)$ Let $\sigma_{i,k} > 0$ and $\sigma_{j,s} > 0$ for some $s > k$ and $j < i$. $\sigma_{i,k} > 0$ implies $U_S(a_i, b|W_k) \geq U_S(a_j, b|W_k)$, and $\sigma_{j,s} > 0$ implies $U_S(a_j, b|W_s) \geq U_S(a_i, b|W_s)$. Combining these inequalities results in $U_S(a_i, b|W_s) - U_S(a_j, b|W_s) \leq 0 \leq U_S(a_i, b|W_k) - U_S(a_j, b|W_k)$, which contradicts the single-crossing property $U_S(a_i, b|W_s) - U_S(a_j, b|W_s) > U_S(a_i, b|W_k) - U_S(a_j, b|W_k)$.

$(D)$ By contradiction, let $\sigma_{i,k} > 0$, $\sigma_{i+1,k} > 0$, and $\sigma_{i+1,k+1} = 0$ for some $k < n-1$. Condition $(C)$ for $\sigma_{i,k} > 0$ implies $\sigma_{i+1,s} = 0$ for all $s < k$. Condition $(C)$ for $\sigma_{i+1,k}$ implies $\sigma_{j,k+1} = 0$ for all $j < i + 1$. Since $\sigma_{i+1,k+1} = 0$, then $\sigma_{j',k+1} > 0$ for some $j' > i + 1$. Again, using condition $(C)$ for $\sigma_{j',k+1}$, we have $\sigma_{i+1,s} = 0$ for all $s > k + 1$. Hence, $\sigma_{i+1,s} = 0$ for all $s \neq k$. It follows from (6) that $a_{i+1} = \bar{w}_k$. Then, $\sigma_{i,k} > 0$ and $\sigma_{i+1,k} > 0$ imply $U_S(a_i, b|W_k) = U_S(a_{i+1}, b|W_k)$, which results in a contradiction $a_i < \bar{w}_k + b < a_{i+1} = \bar{w}_k$.

$(E)$ This is a corollary of property $(C)$. If $\sigma_{i,k} > 0$, then $\sigma_{j,s} = 0$ for all $j < i$ and $s > k$. Similarly, $\sigma_{i,k'} > 0$ implies $\sigma_{j,s} = 0$, $j > i$ , $s < k'$. Thus, for all $s$ such that $k < s < k'$, we have $\sigma_{j,s} = 0$, $j \neq i$, that gives the desired result. ■

**Proof of Lemma 3.** By contradiction, suppose that there exists an indirect equilibrium, in which the number of induced actions exceeds the number of types, that is, $I > n$. This implies that there exists a type $k$ such that $\sigma_{i,k} > 0$, $\sigma_{i+1,k} > 0$, and $\sigma_{i+1,k+1} = 0$. Since the highest type $n - 1$ does not mix (by property $(B)$ of Lemma **2)**, it follows that $k < n - 1$. Then, property $(D)$ of Lemma **2** is violated, which completes the proof. ■

Given an equilibrium signaling strategy $(\sigma_{i,k})$, define a correspondence $p : \mathcal{I} \Longrightarrow K$ such that $p(i) = \{k \in K : \sigma_{i,k} > 0\}$. Thus, $p(i)$ determines the subset of types, inducing action $a_i$. Similarly, define a function $j : K \rightarrow \mathcal{I}$ such that $j(k) = \min\{i \in \mathcal{I} : \sigma_{i,k} > 0\}$, and a correspondence $\mu : \mathcal{I} \Longrightarrow K$ such that $\mu(i) = j^{-1}(i) = \{k \in K : j(k) = i\}$. Function $j(k)$ determines the minimal action induced by the sender of $k$-type. Conversely, given an action $a_i$, $\mu(i)$ determines the set of types, for which this action is minimal. That is, if $k \in \mu(i)$, then $\sigma_{l,k} = 0$ for all $l < i$. The next lemma describes properties of $p(i)$, $j(k)$, and $\mu(i)$.

**Lemma 6** *$p(i)$, $j(k)$, and $\mu(i)$ satisfy the following properties.*

*a) $j(k)$ is (weakly) increasing,*

*b) $\mu(i)$ is non-empty, strictly increasing, and convex-valued,*

*c) $p(i)$ is non-empty, (weakly) increasing, and convex-valued. In a pure-strategy equilibrium, $p(i)$ is strictly increasing, and*

*d) $\mu(i) \subset p(i)$ and $\max \mu(i) = \max p(i)$.*

**Proof** a) $j(k) = i$ implies $\sigma_{i,k} > 0$. Property $(C)$ of Lemma **2** results in $\sigma_{i',k'} = 0$ for all $i' < i$ and $k' > k$, which leads to $j(k') \geq j(k)$.

b) For a given $i$, if $\sigma_{i,k} = 1$ for some $k$, then $j(k) = i$. Hence, $k \in \mu(i)$. If $\sigma_{i,k} > 0$ and $\sigma_{i+1,k} > 0$, then property $(A)$ implies $\sigma_{i',k} = 0$ for all $i' < i$, thus $j(k) = i$ and $k \in \mu(i)$. If $\sigma_{i-1,k} > 0$ and $\sigma_{i,k} > 0$, then property $(D)$ implies $\sigma_{i,k+1} > 0$, and property $(C)$ gives $\sigma_{i',k+1} = 0$ for all $i' < i$. Hence, $j(k+1) = i$ and $k+1 \in \mu(i)$. Thus, $\mu(i)$ is non-empty. To prove that $\mu(i)$ is strictly increasing, by contradiction let $i' > i$ and $k' \leq k$, for some $k' \in \mu(i')$, $k \in \mu(i)$. Then, $k' \in \mu(i')$ implies $j(k') = i'$ and $\sigma_{i',k'} > 0$. Similarly, we have $j(k) = i$ and $\sigma_{i,k} > 0$. If $k' = k$, then $i' = j(k') = j(k) = i$ and we have a contradiction. If $k' < k$, then $\sigma_{i',k'} > 0$ and $\sigma_{i,k} > 0$ contradict condition $(C)$. To show that $\mu(i)$ is convex-valued, let $k' \in \mu(i)$, and $k'' \in \mu(i)$. Thus, $\sigma_{i,k'} > 0$, $\sigma_{i,k''} > 0$. Then, property $(E)$ implies $\sigma_{i,k} = 1$ for all $k$ such that $k' < k < k''$.

c) The first part of the statement can be easily proved using the same techniques as those developed in the $\mu(i)$ context. The second part follows from the fact that $p(i) = \mu(i)$ in a pure-strategy equilibrium, hence $p(i)$ is strictly increasing.

d) For any $k \in \mu(i)$, we have $j(k) = i$. This results in $\sigma_{i,k} > 0$, so $k \in p(i)$ and $\mu(i) \subset p(i)$.

Now, for a given $i'$ consider $k'' = \max \mu(i') = \max \{k \in K : j(k) = i'\}$. Therefore, $\sigma_{i',k''} > 0$, and $\sigma_{i',k} = 0$ for all $k > k''$. If not, i.e., $\sigma_{i',k} > 0$ for some $k > k''$, then condition $(C)$ implies $\sigma_{i,k} = 0$ for all $i < i'$. This means $i' = \min(i : \sigma_{i,k} > 0) = j(k)$, which contradicts $k'' = \max \mu(i')$. Thus, $\sigma_{i',k} = 0$ for all $k > k''$, which results in $k \leq k''$ for all $k \in p(i')$. Thus, $\max p(i) \leq \max \mu(i)$. ■

Since both $\mu(i)$ and $p(i)$ are non-empty, there exists $k'(i) = \min \mu(i)$, $k''(i) = \max \mu(i)$, $k'_p(i) = \min p(i)$, and $k''_p(i) = \max p(i)$, $i \in \mathcal{I}$. That is, $k'_p(i)$ and $k''_p(i)$ are the smallest and the largest types, respectively, which induce action $a_i$. Similarly, $k'(i)$ and $k''(i)$ are the smallest and the largest types for which $a_i$ is the minimal action. Property $(d)$ of Lemma 6 implies $k''(i) = k''_p(i)$ and $k'_p(i) \leq k'(i)$.

Because $p(i)$ is convex-valued, the receiver's best-response can be written as

$$a_i = \frac{1}{2} \frac{\sum_{k \in p(i)} \sigma_{i,k} \left(w_{k+1}^2 - w_k^2\right)}{\sum_{k \in p(i)} \sigma_{i,k} \left(w_{k+1} - w_k\right)} = \frac{1}{2} \frac{\sum_{k=k'_p(i)}^{k''_p(i)} \sigma_{i,k} (w_{k+1}^2 - w_k^2)}{\sum_{k=k'_p(i)}^{k''_p(i)} \sigma_{i,k} \left(w_{k+1} - w_k\right)}, \; i \in \mathcal{I}. \tag{14}$$

**Proof of Lemma 4.** In a pure-strategy equilibrium, the receiver's best-response is

$$a_i = \frac{w_{k'_p(i)} + w_{k''_p(i)+1}}{2} = \frac{w_{k'_p(i)} + w_{k'_p(i+1)}}{2}, \; i \in \mathcal{I},$$

and the expected payoff is

$$U_R = -\sum_{i=0}^{I-1} \int_{w_{k'_p(i)}}^{w_{k'_p(i+1)}} (a_i - \theta)^2 \, d\theta = -\frac{1}{12} \sum_{i=0}^{I-1} \left( w_{k'_p(i+1)} - w_{k'_p(i)} \right)^3 .$$

Consider the partition $\{W'_i\}_{i \in \mathcal{I}}$ such that $W'_i = \bigcup_{k \in p(i)} W_k = (w_{k'_p(i)}, w_{k'_p(i+1)}]$, $i \in \mathcal{I}$, and the signaling strategy $m(i) = m_i$, $i \in \mathcal{I}$. It easily follows that the receiver's best-response is not affected by these transformations.

Since $(\sigma_{i,k})$ is an equilibrium strategy, then for any $i \in \mathcal{I}$, we have

$$a_i + a_l \geq w_{k''_p(i)} + w_{k'_p(i+1)} + 2b \geq w_{k'_p(i)} + w_{k'_p(i+1)} + 2b \text{ for all } a_l > a_i.$$

According to (7), this implies $U_S(a_i, b | W'_i) \geq U_S(a_l, b | W'_i)$ for all $a_l > a_i$. Similarly,

$$a_i + a_l \leq w_{k'_p(i)} + w_{k'_p(i)+1} + 2b \leq w_{k'_p(i)} + w_{k'_p(i+1)} + 2b \text{ for all } a_l < a_i$$

implies $U_S(a_i, b | W'_i) \geq U_S(a_l, b | W'_i)$ for all $a_l < a_i$. Therefore, the described strategies constitute an incentive-compatible equilibrium under the modified partition. Payoff equivalence between the initial and the constructed equilibria follows straightforwardly. ■

Now, consider a sequence $\bar{a}_i = \frac{1}{2} \frac{\sum_{k \in \mu(i)} (w_{k+1}^2 - w_k^2)}{\sum_{k \in \mu(i)} (w_{k+1} - w_k)}$, $i \in \mathcal{I}$. Since $\mu(i)$ is convex-valued, it follows that

$$\bar{a}_i = \frac{1}{2} \frac{\sum_{k=k'(i)}^{k''(i)} (w_{k+1}^2 - w_k^2)}{\sum_{k=k'(i)}^{k''(i)} (w_{k+1} - w_k)} = \frac{w_{k'(i)} + w_{k''(i)+1}}{2} = \frac{w_{k'(i)} + w_{k'(i+1)}}{2}. \tag{15}$$

**Lemma 7** *In any equilibrium $((\sigma_{i,k}), (a_i), \Omega)$, we have $\bar{a}_i \geq a_i$, $i \in \mathcal{I}$.*

**Proof** For any $i \in \mathcal{I}$, if $\sigma_{i,k'_p(i)} = \sigma_{i,k''_p(i)} = 1$, then $\mu(i) = p(i)$, and

$$a_i = \frac{1}{2} \frac{\sum_{k \in p(i)} \sigma_{i,k} (w_{k+1}^2 - w_k^2)}{\sum_{k \in p(i)} \sigma_{i,k} (w_{k+1} - w_k)} = \frac{1}{2} \frac{\sum_{k \in p(i)} (w_{k+1}^2 - w_k^2)}{\sum_{k \in p(i)} (w_{k+1} - w_k)} = \frac{1}{2} \frac{\sum_{k \in \mu(i)} (w_{k+1}^2 - w_k^2)}{\sum_{k \in \mu(i)} (w_{k+1} - w_k)} = \bar{a}_i.$$

If $\sigma_{i,k''_p(i)} < 1$, then property $(d)$ of Lemma 6 implies $k''(i) = k''_p(i)$. If $\sigma_{i,k'_p(i)} < 1$, then by property $(A)$ of Lemma 2, either $\sigma_{i+1,k'_p(i)} > 0$ or $\sigma_{i-1,k'_p(i)} > 0$. In the former case, condition $(C)$ of the Lemma 2 implies $\sigma_{i,k} = 0$ for all $k > k'_p(i)$. Thus, $k''_p(i) = k'_p(i)$ and $p(i)$ is a singleton. Since $\mu(i)$ is non-empty and a subset of $p(i)$, we have $\mu(i) = p(i)$ and $a_i = \bar{a}_i$. If $\sigma_{i-1,k'_p(i)} > 0$, then condition $(D)$ of Lemma 2 requires $\sigma_{i,k'_p(i)+1} > 0$. By condition $(C)$ of Lemma 2, $\sigma_{l,k'_p(i)+1} = 0$ for all $l < i$. Thus, $j\left(k'_p(i) + 1\right) = i$ and $k'_p(i) + 1 \in \mu(i)$. Since $j(k)$ is increasing and $j\left(k'_p(i)\right) = i - 1$, there is no $k < k'_p(i) + 1$ such that $j(k) = i$. Therefore, $k'(i) = k'_p(i) + 1$. From (14),

$$a_i = \frac{1}{2} \frac{\sigma_{i,k'_p(i)}(w_{k'_p(i)+1}^2 - w_{k'_p(i)}^2) + \sum_{k=k'_p(i)+1}^{k''_p(i)-1} (w_{k+1}^2 - w_k^2) + \sigma_{i,k''_p(i)}(w_{k''_p(i)+1}^2 - w_{k''_p(i)}^2)}{\sigma_{i,k'_p(i)}(w_{k'_p(i)+1} - w_{k'_p(i)}) + \sum_{k=k'_p(i)+1}^{k''_p(i)} (w_{k+1} - w_k) + \sigma_{i,k''(i)}(w_{k''(i)+1} - w_{k''(i)})}.$$

Comparing the last expression with (15), it follows that

$$\bar{a}_i = \frac{1}{2}\frac{\sum\limits_{k=k'(i)}^{k''(i)}(w_{k+1}^2 - w_k^2)}{\sum\limits_{k=k'(i)}^{k''(i)}(w_{k+1} - w_k)} = \frac{1}{2}\frac{\sum\limits_{k=k'_p(i)+1}^{k''_p(i)}(w_{k+1}^2 - w_k^2)}{\sum\limits_{k=k'_p(i)+1}^{k''_p(i)}(w_{k+1} - w_k)} = a_i(\sigma_{i,k'_p(i)} = 0, \sigma_{i,k''_p(i)} = 1).$$

Finally, we complete the proof by showing that $a_i$ is decreasing in $\sigma_{i,k'_p(i)}$ and increasing in $\sigma_{i,k''_p(i)}$. A derivative of $a_i$ with respect to $x = \sigma_{i,k'_p(i)}$ is

$$\frac{da_i}{dx} = \frac{\sum\limits_{k=k'_p(i)+1}^{k''_p(i)} \sigma_{i,k}\left(w_{k+1} - w_k\right)\left(\bar{w}_{k'_p(i)} - \bar{w}_k\right)}{\left(\sum\limits_{k=k'_p(i)}^{k''_p(i)} \sigma_{i,k}\left(w_{k+1} - w_k\right)\right)^2}.$$

Since $\bar{w}_{k'_p(i)} - \bar{w}_k < 0$ for all $k > k'_p(i)$, then $\frac{da_i}{dx} < 0$. A similar approach for $\frac{da_i}{dy}$, where $y = \sigma_{i,k''_p(i)}$, results in $\frac{da_i}{dy} > 0$, which implies $\bar{a}_i \geq a_i$. ■

**Proof of Lemma 5.** Using property $(A)$ of Lemma (2), we may represent the receiver's expected utility in an equilibrium $((a_i), (\sigma_{i,k}), \Omega)$ as

$$U_R\left((a_i), (\sigma_{i,k}), \Omega\right) = \sum_{k=0}^{n-1} P\left(W_k\right)\left(\sigma_{j(k),k} U_R\left(a_{j(k)}|W_k\right) + \sigma_{j(k)+1,k} U_R\left(a_{j(k)+1}|W_k\right)\right).$$

Modify the signaling strategy $(\sigma_{i,k})$ as follows: derive all types $\bar{K}$ that induce two actions, and put $\sigma'_{j(k),k} = 1$ for all $k \in \bar{K}$. That is, if the sender of $k$-type induced two actions in the initial equilibrium, now she purely induces a lower action.

Notice that $U_S\left(a_{j(k)}, b|W_k\right) = U_S\left(a_{j(k)+1}, b|W_k\right)$ for all $k \in \bar{K}$. The single-crossing property $\frac{d^2}{dadb} U_S\left(a, b|W_k\right) > 0$ implies

$$U_S\left(a_{j(k)+1}, b|W_k\right) - U_S\left(a_{j(k)}, b|W_k\right) > U_S\left(a_{j(k)+1}, 0|W_k\right) - U_S\left(a_{j(k)}, 0|W_k\right).$$

Since $U_S\left(a_{j(k)+1}, b|W_k\right) = U_S\left(a_{j(k)}, b|W_k\right)$ and $U_R\left(a|W_k\right) = U_S\left(a, 0|W_k\right)$, then $U_R\left(a_{j(k)}|W_k\right) > U_R\left(a_{j(k)+1}|W_k\right)$. Multiplying each term by $P\left(W_k\right)$ and summing across all $k \in K$ result in $U_R\left((a_i), \left(\sigma'_{i,k}\right), \Omega\right) > U_R\left((a_i), (\sigma_{i,k}), \Omega\right)$. Notice that $a_i$, $i \in \mathcal{I}$, is not the best-response to the signaling strategy $\left(\sigma'_{i,k}\right)$.

By construction, $\sigma'_{i,k} = 1$ if and only if $k \in \mu(i)$. Then, given the strategy $\left(\sigma'_{i,k}\right)$, the receiver's best response is $\bar{a}_i$, $i \in \mathcal{I}$, and $U_R\left(\bar{a}_i|m_i\right) \geq U_R\left(a_i|m_i\right)$, $i \in \mathcal{I}$. Multiplying each term by $P\left(m_i\right) = \sum\limits_{k=0}^{n-1} P\left(W_k\right)\sigma'_{i,k} = w_{k'(i+1)} - w_{k'(i)}$ and summing across all $i \in \mathcal{I}$ result in $U_R\left((\bar{a}_i), \left(\sigma'_{i,k}\right), \Omega\right) \geq U_R\left((a_i), \left(\sigma'_{i,k}\right), \Omega\right)$.

Now, consider the partition $\bar{\Omega} = \left\{\bar{W}_i\right\}_{i \in \mathcal{I}}$ such that $\bar{W}_i = \bigcup\limits_{k \in \mu(i)} W_k = (w_{k'(i)}, w_{k'(i+1)}]$, $i \in \mathcal{I}$, and the signaling strategy $(\bar{\sigma}_{i,s})$, $i,s \in \mathcal{I}$, such that $m(i) = m_i$, $i \in \mathcal{I}$. A collapse of partition's elements does not affect the receiver's best-response, so the optimal action's rule is $\bar{a}_i$, $i \in \mathcal{I}$. This implies $U_R\left(\left(\bar{a}_i\right),\left(\bar{\sigma}_{i,s}\right),\bar{\Omega}\right)=U_R\left(\left(\bar{a}_i\right),\left(\sigma'_{i,k}\right),\Omega\right)$, and

$$U_R\left(\left(\bar{a}_i\right),\left(\bar{\sigma}_{i,s}\right),\bar{\Omega}\right)\geq U_R\left(\left(a_i\right),\left(\sigma'_{i,k}\right),\Omega\right)>U_R\left(\left(a_i\right),\left(\sigma_{i,k}\right),\Omega\right).$$

We complete the proof by showing that $(\bar{\sigma}_{i,s})$ is incentive-compatible. That is, $w_{k'(i+2)}-w_{k'(i)}\geq 4b$ for all $i=0,...,I-2$.

Since $\sigma_{i,k''(i)}$ belongs to the initial equilibrium profile for each $i\in\mathcal{I}$, we have $U_S\left(a_i,b|W_{k''(i)}\right)\geq U_S\left(a_{i+1},b|W_{k''(i)}\right)$. This implies

$$a_i+a_{i+1}\geq w_{k''(i)}+w_{k''(i)+1}+2b=w_{k''(i)}+w_{k'(i+1)}+2b.$$

From Lemma 7, $a_i\leq\bar{a}_i$ and $a_{i+1}\leq\bar{a}_{i+1}$. Combining these inequalities results in

$$\begin{aligned}w_{k'(i)}+w_{k'(i+1)}+2b\leq w_{k''(i)}+w_{k'(i+1)}+2b\ &\leq a_i+a_{i+1}\leq\bar{a}_i+\bar{a}_{i+1}\\&=\frac{w_{k'(i)}+w_{k'(i+1)}}{2}+\frac{w_{k'(i+1)}+w_{k'(i+2)}}{2}=\frac{w_{k'(i)}}{2}+w_{k'(i+1)}+\frac{w_{k'(i+2)}}{2},\end{aligned}$$

which gives $w_{k'(i+2)}-w_{k'(i)}\geq 4b$. ■

**Lemma 8** *If the uniform partition of size n is incentive-compatible, then the incentive-compatible equilibrium under this partition is payoff superior to any incentive-compatible equilibrium under a partition of the same size.*

**Proof** The ex-ante utility of the receiver in an incentive-compatible equilibrium is

$$\begin{aligned}U_R&=-\sum_{k=0}^{n-1}\int_{w_k}^{w_{k+1}}(a_k-\theta)^2d\theta=\sum_{k=0}^{n-1}P\left(W_k\right)\left(U_S\left(\bar{w}_k,b,\bar{w}_k\right)-D\left(W_k\right)\right)=\\&=-\sum_{k=0}^{n-1}P\left(W_k\right)D\left(W_k\right)=-\sum_{k=0}^{n-1}\frac{\left(w_{k+1}-w_k\right)^3}{12}=-\sum_{k=0}^{n-1}\frac{\Delta w_k^3}{12}=\sum_{k=0}^{n-1}f\left(\Delta w_k\right),\end{aligned}\tag{16}$$

where $\Delta w_k=w_{k+1}-w_k>0$ and $f\left(x\right)=-\frac{1}{12}x^3$.

Clearly, $f\left(x\right)$ is strictly concave for $x>0$ and $\sum_{k=0}^{n-1}\Delta w_k=1$. For the uniform partition of size $n$, $\Delta w'_k=\frac{1}{n}$ for all $k$. For any other incentive-compatible partition of the same size, the Jensen's inequality implies

$$U_R=\sum_{k=0}^{n-1}f\left(\Delta w_k\right)<nf\left(\frac{1}{n}\sum_{k=0}^{n-1}\Delta w_k\right)=nf\left(\frac{1}{n}\right)=\sum_{k=0}^{n-1}f\left(\Delta w'_k\right)=U'_R.$$

■

**Lemma 9** *If a partition of size n is incentive-compatible, then the uniform partition of size $n-1$ is incentive-compatible also.*

**Proof** Since a partition $(w_k)_0^n$ is incentive-compatible, we have $w_n=1\geq w_{n-2}+4b\geq...\geq w_1+\frac{n-1}{2}4b\geq\frac{n-1}{2}4b$ for odd $n$, and $w_n=1\geq w_{n-2}+4b\geq...\geq w_0+\frac{n-1}{2}4b=\frac{n-1}{2}4b$ for even $n$. In both cases, $\frac{2}{n-1}\geq 4b$. Then, for the uniform partition $(w'_k)_0^{n-1}$, we have $w'_{j+2}-w'_j=\frac{j+2}{n-1}-\frac{j}{n-1}=\frac{2}{n-1}\geq 4b$. ■

**Lemma 10** *Among all partitions of an odd size n such that $\frac{1}{2n}<b\leq\frac{1}{2(n-1)}$, the highest ex-ante*

*payoff in the incentive-compatible equilibrium is reached under the partition with all IC constraints (9) binding.*

**Proof** We prove the lemma using the Karamata's inequality.[20] Let sequences $(x_k)_1^n$ and $(y_k)_1^n$ be non-increasing, so $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$. If all the following conditions satisfied: $x_1 \geq y_1$, $x_1 + x_2 \geq y_1 + y_2$, $x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3$, ..., $x_1 + x_2 + ... + x_{n-1} \geq y_1 + y_2 + ... + y_{n-1}$, and $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$, then we say that $(x_i)_1^n$ **majorizes** $(y_i)_1^n$. The Karamata's inequality states that if $(x_i)$ majorizes $(y_i)$, and a function $f(x)$ is continuos and concave, then $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$.

From (16), the receiver's ex-ante payoff in the incentive-compatible equilibrium is $U_R\left((w_k)_0^n\right) = \sum_{k=0}^{n-1} f(\Delta w_k)$, where $\Delta w_k = w_{k+1} - w_k > 0$, and $f(x) = -\frac{1}{12}x^3$, which is continuous and strictly concave for $x > 0$.

Consider the sequence $(y_k)_0^n$, for which the IC conditions are binding, so $y_k = 2kb$ for even $k$, and $y_k = 1 - 2b(n-k)$ for odd $k$. We need to show that if $\frac{1}{2n} < b \leq \frac{1}{2(n-1)}$, then $U_R\left((y_k)_0^n\right) \geq U_R\left((w_k)_0^n\right)$ for any partition $(w_k)_0^n$, which satisfies (9).

The IC conditions (9) can be written as

$$w_{k+2} - w_k = w_{k+2} - w_{k+1} + w_{k+1} - w_k = \Delta w_{k+1} + \Delta w_k \geq 4b, \;\; k = 0, 1..., n-2.$$

For the sequence $(y_k)_0^n$, we have $\Delta y_k = y_{k+1} - y_k = 1 - 2b(n-k-1) - 2bk = 1 - 2b(n-1)$ for even $k$. The condition $b < \frac{1}{2(n-1)}$ implies $\Delta y_k > 0$. Similarly $\Delta y_k = 4b - \Delta y_{k-1} = 4b - 1 + 2b(n-1) = 2b(n+1) - 1$ for odd $k$, and $b > \frac{1}{2n} > \frac{1}{2(n+1)}$ implies $\Delta y_k > 0$. In addition, for odd $k$, $\Delta y_k - \Delta y_{k-1} = 2b(n+1) - 1 - 1 + 2b(n-1) = 2(2bn-1) > 0$. Thus, by permuting $(\Delta y_k)_0^{n-1}$ we get a non-increasing sequence $(Y_k)_1^n = (Y_1, Y_2, ..., Y_{\frac{n-1}{2}}, Y_{\frac{n+1}{2}}, ..., Y_n)$, where $Y_k = 2b(n+1) - 1$ for $k \in S_1 = 1, 2, ..., \frac{n-1}{2}$, and $Y_k = 1 - 2b(n-1)$ for $k \in S_2 = \frac{n+1}{2}, ..., n$. Note that $S_1$ has one element less than $S_2$, since $n$ is odd. Also, the IC conditions imply that $Y_k + Y_j = 4b$, $k \in S_1$, $j \in S_2$.

Now, consider a sequence $(w_k)_0^n$, which satisfies (9). We need to show that a non-increasing permutation $(X_k)_1^n$ of $(\Delta w_k)_0^{n-1}$ majorizes $(Y_k)_1^n$.

First, for even $k$, we have $w_k \geq w_{k-2} + 4b \geq ... \geq w_0 + \frac{k}{2}4b = 2kb = y_k$. Similarly, for odd $k$, $w_k \leq y_k$. Therefore, $\Delta w_k = w_{k+1} - w_k \geq y_{k+1} - y_k = \Delta y_k$ for odd $k$, and $\Delta w_k \leq \Delta y_k$ for even $k$. Thus, a non-increasing permutation $(X_k)_1^n$ of $(\Delta w_k)_0^{n-1}$ can be represented as $(X_k)_1^n = (X_1, X_2, ..., X_{\frac{n-1}{2}}, X_{\frac{n+1}{2}}, ..., X_n)$, where $X_j \geq Y_k$ for all $j, k \in S_1$, and $X_j \leq Y_k$ for all $j, k \in S_2$. This means that $\sum_{k \in S_1'} X_k \geq \sum_{k \in S_1'} Y_k$ for any $S_1' \subset S_1$, and $\sum_{k \in S_2'} X_k \leq \sum_{k \in S_2'} Y_k$ for any $S_2' \subset S_2$.

Also, the IC conditions require that for any $k \in \tilde{S}_2 = S_2 - \{n\} = \frac{n+1}{2}, ..., n-1$, there must exist $q(k) \in S_1$ such that $X_{q(k)} + X_k \geq 4b$, which we define as follows. Denote $i_n$ to be the index of the smallest element $\Delta w_{i_n}$ of the sequence $(\Delta w_k)$, which implies $\Delta w_{i_n} = X_n$. Then, for all $X_k$, $k \in \tilde{S}_2$, if $X_k = \Delta w_i$, then $X_{q(k)} = \Delta w_{i+1}$ for $i < i_n$, and $X_{q(k)} = \Delta w_{i-1}$ for $i > i_n$. Notice that $k \neq k'$ for any $k, k' \in S_2$ implies $q(k) \neq q(k')$.

[20] See, for example, [9]

Clearly, $X_1 \geq Y_1$, $X_1 + X_2 \geq Y_1 + Y_2, ...,$ $X_1 + ... + X_{\frac{n-1}{2}} \geq Y_1 + ... + Y_{\frac{n-1}{2}}$. Also,

$$X_1 + ... + X_{\frac{n-1}{2}} + X_{\frac{n+1}{2}} = \sum_{k \in S_1 - q\left(\frac{n+1}{2}\right)} X_k + X_{q\left(\frac{n+1}{2}\right)} + X_{\frac{n+1}{2}} \geq \sum_{k \in S_1 - q\left(\frac{n+1}{2}\right)} X_k + 4b$$
$$\geq \sum_{k \in S_1 - k\left(\frac{n+1}{2}\right)} Y_k + 4b = \sum_{k \in S_1 - q\left(\frac{n+1}{2}\right)} Y_k + Y_{q\left(\frac{n+1}{2}\right)} + Y_{\frac{n+1}{2}} = Y_1 + ... + Y_{\frac{n-1}{2}} + Y_{\frac{n+1}{2}}.$$

The argument can be reapplied iteratively for all $k \in \tilde{S}_2$. Since $\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} Y_k = 1$, this completes the proof. ■

**Proof of Theorem 1.** We can rewrite $n(b)$ as follows: if $\frac{1}{2(c+1)} < b < \frac{1}{2(c-1)}$ for some odd $c$, then $n(b) = c$, otherwise, for $b = \frac{1}{2(c-1)}$, $n(b) = c - 1$. Then, by Lemma (9), the uniform partition of size $c - 1$ is incentive-compatible, and provides the ex-ante payoff (in the incentive-compatible equilibrium)

$$U_R^{c-1} = -\sum_{k=0}^{c-2} \frac{(w_{k+1} - w_k)^3}{12} = -\sum_{k=0}^{c-2} \frac{1}{12(c-1)^3} = -\frac{1}{12(c-1)^2}.$$

From Lemma 8, the incentive-compatible equilibrium under this partition is superior to all equilibria under partitions of the same size. Also, this equilibrium is superior to to all equilibria under partitions of a smaller size (for a partition of size $c' < c - 1$, the superior payoff $U_R^{c'} = -\frac{1}{12(c')^2}$ is reached in the incentive-compatible equilibrium under the uniform partition, which is smaller than $U_R^{c-1}$).

Now, consider two cases: $\frac{1}{2(c+1)} < b \leq \frac{1}{2c}$ and $\frac{1}{2c} < b \leq \frac{1}{2(c-1)}$. In the first case, the uniform partition of size $c$ is incentive-compatible, hence it is optimal and brings ex-ante utility to the receiver $U_R = -\frac{1}{12c^2}$. In the second case, Lemma (10) implies that among all partitions of size $c = n(b)$, the superior partition is that with binding IC constraints (9). It provides the receiver's ex-ante payoff

$$U_R^c = -\frac{1}{12}\left(4b^2\left(c^2 - 1\right)(4bc - 3) + 1\right). \tag{17}$$

For $b = \frac{1}{2c}$, we have $U_R^c = \frac{1}{12c^2}$, and the principal's expected payoff is equal to that under the uniform partition of size $c$. For $b = \frac{1}{2(c-1)}$, $U_R^c = \frac{1}{3(c-1)^2} = \frac{1}{12\left(\frac{c-1}{2}\right)^2}$, which is a payoff under the uniform partition of size $\frac{c-1}{2}$.

Since $n(b) = c$ for all $b \in \left(\frac{1}{2c}, \frac{1}{2(c-1)}\right)$, taking the derivative of (17) with respect to $b$ gives

$$\frac{d}{db} U_R^c(b) = -2b\left(c^2 - 1\right)(2bc - 1).$$

Thus, $\frac{d}{db} U_R^c(b) < 0$ for $b > \frac{1}{2c}$. Moreover, $U_R^c\left(\frac{1}{2c}\right) = \frac{1}{12c^2} > U_R^{c-1} = \frac{1}{12(c-1)^2} > \frac{1}{3(c-1)^2} = U_R^c\left(\frac{1}{2(c-1)}\right)$. Hence, there exists a unique $b^* \in \left(\frac{1}{2c}, \frac{1}{2(c-1)}\right)$, such that $U_R^c(b^*) = U_R^{c-1}$, which completes the proof. ■

**Proof of Theorem 2.** Formally, it is straightforward to prove that for any equilibrium partition in the CS model, the uniform partition of the same size is incentive-compatible in the CWIIE model and provides a superior ex-ante payoff to the receiver. However, Theorem 3 below proves that for $b \leq \frac{1}{4}$, there exists an equilibrium in the CWIIE model which provides a higher expected payoff to the principal than optimal delegation. Due to Dessein [7], delegation performs better than CS communication for $b \leq \frac{1}{4}$, which completes the proof. ■

**Proof of Theorem 3.** Informative communication is feasible, if $b \leq \frac{1}{4}$. Melumad and Shibano

[13] prove that for $b \le \frac{1}{2}$, the optimal delegation set is the interval $[0, 1-b]$. In this case, expert's actions are

$$a^S(\theta) = \begin{cases} \theta + b \text{ if } \theta \le 1 - 2b \\ 1 - b \text{ if } \theta > 1 - 2b \end{cases}, \tag{18}$$

which provides the ex-ante payoff to the receiver

$$U_R^D(b) = \int_0^1 U_R\left(a^S(\theta), \theta\right) d\theta = -\int_0^{1-2b} (\theta + b - \theta)^2 d\theta - \int_{1-2b}^{1} (1 - b - \theta)^2 d\theta = -b^2 + \frac{4}{3}b^3. \tag{19}$$

By Lemma (9), a uniform partition of size $n(b) - 1 = 2\langle \frac{1}{4b} \rangle$ is incentive-compatible. Then, receiver's ex-ante utility under this partition is $U_R(b) = -\frac{1}{12 \times \left(2\langle \frac{1}{4b} \rangle\right)^2} = -\frac{1}{48 \times \langle \frac{1}{4b} \rangle^2}$. Since $\langle \frac{1}{4b} \rangle \ge \frac{1}{4b} - 1$, we obtain $U_R(b) \ge -\frac{1}{48\left(\frac{1}{4b} - 1\right)^2} = -\frac{b^2}{3(1-4b)^2}$, and

$$U_R(b) - U_R^D(b) \ge -\frac{b^2}{3(1-4b)^2} + b^2 - \frac{4}{3}b^3 = \frac{2}{3}b^2 \frac{1 - 14b + 40b^2 - 32b^3}{(1-4b)^2}.$$

The function $A(b) = 1 - 14b + 40b^2 - 32b^3$ has three roots. Only one of them, namely, $b_0 = \frac{1}{8}\left(3 - \sqrt{5}\right) \simeq \frac{1}{11}$ is in the interval $\left[0, \frac{1}{4}\right]$. Since $A(0) = 1$, it follows that $U_R - U_R^D > 0$ for all $b < b_0$.

For $b \in [b_0, \frac{1}{4}]$, consider three cases. If $b \in [\frac{1}{6}, \frac{1}{4}]$, then the uniform partition of size 2 is incentive-compatible, and provides the expected payoff to the receiver $U_R = -\frac{1}{12 \times 2^2} = -\frac{1}{48}$ in the pure-strategy equilibrium. Then, $D(b) = U_R(b) - U_R^D(b) = -\frac{1}{48} + b^2 - \frac{4}{3}b^3$. $D(b)$ is increasing in $b$, so, it reaches the minimum $\frac{1}{1296}$ for $b = \frac{1}{6}$. This implies that $U_R(b) - U_R^D(b) > 0$ for all $b \in [\frac{1}{6}, \frac{1}{4}]$.

For $b \in [\frac{1}{8}, \frac{1}{6})$, the uniform three-element partition is incentive-compatible, and bring ex-ante payoff $-\frac{1}{108}$in the incentive-compatible equilibrium. Then, $D(b) = U_R(b) - U_R^D(b) = -\frac{1}{108} + b^2 - \frac{4}{3}b^3 > 0$ for all $b \in [\frac{1}{8}, \frac{1}{6})$, since $D\left(\frac{1}{8}\right) = \frac{13}{3456}$. Finally, for $b \in [b_0, \frac{1}{8})$, the uniform 4-element partition is incentive-compatible, which results in $U_R = -\frac{1}{192}$. Using the same technique as for $b \ge \frac{1}{6}$, one can show that $D(b) > D\left(\frac{1}{12}\right) = \frac{5}{5184}$, which completes the proof. ■

**Proof of Theorem 4.** For symmetric preferences, the "arbitrage condition" in the CS model and the IC conditions (9) in the CWIIE model are the same as in the case of the quadratic preferences. Hence, for any $b \le \frac{1}{4}$, the most informative equilibrium in the CS-model has a partition of size $N^{CS}(b) = \langle -\frac{1}{2} + \frac{1}{2}(1 + \frac{2}{b})^{1/2} \rangle^{(-)}$, where $\langle x \rangle^{(-)}$ is the smaller integer greater than or equal to $x$. Notice that

$$N^{CS}(b) = \langle -\frac{1}{2} + \frac{1}{2}(1 + \frac{2}{b})^{1/2} \rangle^{(-)} \le \langle \frac{1}{2} + \frac{1}{2}(1 + \frac{2}{b})^{1/2} \rangle \le \frac{1}{2} + \frac{1}{2}(1 + \frac{2}{b})^{1/2},$$

where $\langle x \rangle$ is the largest integer smaller than or equal to $x$. Then, $2N^{CS}(b)\, b \le 2b(\frac{1}{2} + \frac{1}{2}(1 + \frac{2}{b})^{1/2}) = b(1 + (1 + \frac{2}{b})^{1/2}) = v(b)$. Since $v(\frac{1}{4}) = 1$, and $v'(b) = 1 + \frac{1+b}{\sqrt{b(2+b)}} > 0$, then $2N^{CS}(b)\, b < 1$ and the uniform partition of size $n = N^{CS}(b)$ is incentive-compatible.

The receiver's expected utility in the most informative CS equilibrium is

$$U_R^{CS} = \sum_{k=0}^{n-1} \int_{w_k}^{w_{k+1}} U_1(\theta,\theta) + U_2\left(\left|\frac{w_k + w_{k+1}}{2} - \theta\right|\right) d\theta = EU_1 + \sum_{k=0}^{n-1} \int_{-\frac{w_{k+1}-w_k}{2}}^{\frac{w_{k+1}-w_k}{2}} U_2(|t|)\, dt$$

$$= EU_1 + 2\sum_{k=0}^{n-1} \int_0^{\frac{w_{k+1}-w_k}{2}} U_2(t)\, dt = EU_1 + \sum_{k=0}^{n-1} f(\Delta w_k),$$

where $EU_1 = \int_0^1 U_1(\theta,\theta)\, d\theta$, $\Delta w_k = w_{k+1} - w_k$, and $f(\Delta w_k) = 2\int_0^{\frac{\Delta w_k}{2}} U_2(t)\, dt$. Then, for $x > 0$, we have $f'(x) = \frac{d}{dx}\int_0^{\frac{x}{2}} 2U_2(t)\, dt = U_2\left(\frac{x}{2}\right)$, and $f''(x) = \frac{1}{2}U_2'\left(\frac{x}{2}\right) < 0$. The receiver's ex-ante utility in the incentive-compatible equilibrium in the CWIIE model under the uniform partition of size $n$ is

$$U_R = EU_1 + \sum_{k=0}^{n-1} \int_{w_k}^{w_{k+1}} U_2\left(\left|\frac{w_k + w_{k+1}}{2} - \theta\right|\right) d\theta = EU_1 + 2\sum_{k=0}^{n-1} \int_0^{\frac{w_{k+1}-w_k}{2}} U_2(t)\, dt \tag{20}$$

$$= EU_1 + 2n\int_0^{\frac{1}{2n}} U_2(t)\, dt = EU_1 + nf\left(\frac{1}{n}\right) = EU_1 + nf\left(\frac{1}{n}\sum_{k=0}^{n-1} \Delta w_k\right).$$

Since $f(x)$ is strictly concave and $\Delta w_{k+1} = \Delta w_k + 4b \neq \Delta w_k$ from the CS "arbitrage condition", then the Jensen's inequality implies $f\left(\frac{1}{n}\sum_{k=0}^{n-1} \Delta w_k\right) > \frac{1}{n}\sum_{k=0}^{n-1} f(\Delta w_k)$ or $U_R > U_R^{CS}$. ■

**Proof of Theorem 5.** If preferences are symmetric, we use Proposition 4 from Alonso and Matouschek [1], which implies that the optimal delegation set is the same as for quadratic preferences, hence, it is the interval $[0, 1-b]$. Similarly, the sender's policy is determined by (18). This results in the receiver's ex-ante utility

$$U_R^D - EU_1 = \int_0^1 U_2\left(\left|a^S(\theta) - \theta\right|\right) d\theta = \int_0^{1-2b} U_2(b)\, d\theta + \int_{1-2b}^1 U_2(|1-b-\theta|)\, d\theta$$

$$= U_2(b)(1-2b) + 2\int_0^b U_2(\theta)\, d\theta.$$

Now, consider the CWIIE model. If $b \neq \frac{1}{2n}$ for any integer $n$, then the partition of size $n(b) = 2\langle\frac{1}{4b}\rangle + 1$ is incentive-compatible. From Lemma 9, the uniform partition of size $c = n(b) - 1 = 2\langle\frac{1}{4b}\rangle \geq \frac{1}{2b} - 1$ is incentive-compatible also. If $b = \frac{1}{2n}$ for some integer $n$, then the uniform partition of size $n = \frac{1}{2b}$ is incentive-compatible, and so is the uniform partition of size $\frac{1}{2b} - 1$. From (20), the receiver's ex-ante utility under the uniform partition of size $c$ is

$$U_R(c) = EU_1 + 2c\int_0^{\frac{1}{2c}} U_2(\theta)\, d\theta = EU_1 + E\left[U_2(\theta)\,|\theta < \frac{1}{2c}\right].$$

Since $U_2(.)$ is decreasing, it follows that $U_R$ is increasing in $c$. Then,

$$U_R(c) \geq U_R\left(\frac{1}{2b}-1\right) = EU_1 + 2\left(\frac{1}{2b}-1\right)\int_0^{\frac{1}{2\left(\frac{1}{2b}-1\right)}} U_2(\theta)\,d\theta = EU_1 + \frac{1-2b}{b}\int_0^{\frac{b}{1-2b}} U_2(\theta)\,d\theta.$$

Thus, $\left(U_R - U_R^D\right)\frac{b}{1-2b} \geq \int_0^{\frac{b}{1-2b}} U_2(\theta)\,d\theta - U_2(b)\,b - \frac{2b}{1-2b}\int_0^b U_2(\theta)\,d\theta = \phi(b)$. Clearly, $\phi(0) = 0$. Taking a derivative of $\phi(b)$ with respect to $b$ gives

$$\phi'(b) = U_2\left(\frac{b}{1-2b}\right)\frac{1}{(1-2b)^2} - U_2'(b)\,b - U_2(b) - \frac{2}{(1-2b)^2}\int_0^b U_2(\theta)\,d\theta - \frac{2b}{1-2b}U_2(b).$$

From the last expression, $\phi'(0) = 0$. Taking the second derivative results in $\phi''(0) = -U_2'(0) \geq 0$. If $U_2'(0) < 0$, then by Taylor's formula $\phi(b) = \phi(0) + \phi'(0)\,b + \frac{1}{2}\phi''\left(\tilde{b}\right)b^2 = \frac{1}{2}\phi''\left(\tilde{b}\right)b^2$, where $\tilde{b} \in [0,b]$. Since $\phi''(0) > 0$ and $\phi''(b)$ is continuous, then there exists $b^*$ such that $\phi''(b) > 0$, and hence, $\phi(b) > 0$ for all $b \in (0,b^*)$. If $U_2'(0) = 0$, then $\phi''(0) = 0$. Taking the third derivative gives $\phi'''(0) = -2U_2''(0) > 0$. By Taylor's formula, $\phi(b) = \phi(0) + \phi'(0)\,b + \frac{1}{2}\phi''(0)\,b^2 + \frac{1}{6}\phi'''(b^*)\,b^3 = \frac{1}{6}\phi'''(b^*)\,b^3$, where $b^* \in [0,b]$. Since $\phi'''(0) > 0$ and $\phi'''(b)$ is continuous, then $\phi(b) > 0$ for all $b$ in a some neighborhood of 0. ■

**Proof of Theorem 6.** The "arbitrage condition" in the CS model is

$$w_{k+1} + b - a_k = a_{k+1} - w_{k+1} - b, \tag{21}$$

where

$$a_k = E\left[\theta|\theta \in (w_k, w_{k+1}]\right] = \frac{1}{F(w_{k+1}) - F(w_k)}\int_{w_k}^{w_{k+1}} \theta dF(\theta). \tag{22}$$

In the CWIIE model, the sender's type-relevant utility function is

$$U_S(a, b|W_k) = -\frac{1}{F(\theta_{k+1}) - F(\theta_k)}\int_{w_k}^{w_{k+1}} (a-b-\theta)^2\,dF(\theta).$$

This function is concave and symmetric with respect to $a_k^S = a_k + b$. Thus, the IC constraints $a_k^S - a_k \geq a_{k+1} - a_k^S$ can be written as

$$a_{k+1} - a_k \geq 2b,\; k = 0, ..., n-2. \tag{23}$$

The condition (21) can be expressed as $a_{k+1} - a_k = 2\left(w_{k+1} - a_k\right) + 2b > 2b$, since $w_{k+1} > a_k = E\left[\theta|\theta \in (w_k, w_{k+1}]\right]$ for $f(\theta) = F'(\theta) > 0$. Thus, any CS partition $(w_k)_0^n$ is incentive-compatible in the CWIIE model. Moreover, IC conditions (23) are satisfied for all $w_k'$ in some neighborhood of $w_k$, $k = 1, ..., n-1$, since $a_k$, $k = 0, ..., n-1$, are continuous in all $w_k$.

The receiver's ex-ante utility in the incentive-compatible equilibrium is

$$U_R = -\sum_{k=0}^{n-1}\int_{w_k}^{w_{k+1}} (a_k - \theta)^2\,dF(\theta). \tag{24}$$

Then,

$$\frac{dU_R}{dw_1} = -\left(a_0 - w\right)^2 f\left(w_1\right) - \frac{da_0}{dw_1}\int_0^{w_1}\left(a_0 - \theta\right) dF\left(\theta\right) + \left(a_1 - w_1\right)^2 f\left(w_1\right) - \frac{da_1}{dw_1}\int_0^{w_1}\left(a_1 - \theta\right) dF\left(\theta\right).$$

From (22), the second and the last terms in the expression above are equal to 0, which implies

$$\frac{dU_R}{dw_1} = f\left(w_1\right)\left(a_1 - a_0\right)\left(a_0 + a_1 - 2w_1\right) = 2f\left(w_1\right)\left(a_1 - a_0\right) b > 0.$$

Thus, the partition $(0, w_1', w_2, ..., 1)$, where $w_1'$ is sufficiently close to $w_1$, is incentive-compatible and provides strictly higher ex-ante payoff.[21] ■

**Proof of Theorem 7.** If $\Delta_k = w_{k+1} - w_k$ is the length of a partition's element $W_k$, then the receiver's optimal action (22) can be represented by the Taylor's formula around $w_k$ as

$$a_k = w_k + \frac{1}{2}\Delta_k + \frac{1}{12}\frac{f'\left(\tilde{w}_k\right)}{f\left(\tilde{w}_k\right)}\Delta_k^2, \tag{25}$$

where $\tilde{w}_k \in [w_k, w_{k+1}]$. Similarly, $a_{k-1} = w_k - \frac{1}{2}\Delta_{k-1} + \frac{1}{12}\frac{f'(\tilde{w}_{k-1})}{f(\tilde{w}_{k-1})}\Delta_{k-1}^2$, where $\tilde{w}_{k-1} \in [w_{k-1}, w_k]$. Then, the IC constraints (23) become

$$\Delta_{k-1} + \Delta_k + \frac{1}{6}\frac{f'\left(\tilde{w}_k\right)}{f\left(\tilde{w}_k\right)}\Delta_k^2 - \frac{1}{6}\frac{f'\left(\tilde{w}_{k-1}\right)}{f\left(\tilde{w}_{k-1}\right)}\Delta_{k-1}^2 \geq 4b. \tag{26}$$

Similarly, expanding the density $f(\theta)$ by the Taylor's formula around $w_k$ results in

$$f\left(\theta\right) = f\left(w_k\right) + f'\left(\hat{w}_k\right)\left(\theta - w_k\right), \tag{27}$$

where $\hat{w}_k \in [0, \theta]$. Using (25) and (27), a sum's element $U_R^k = -\int_{w_k}^{w_{k+1}}\left(a_k - \theta\right)^2 f\left(\theta\right) d\theta$ in the principal's ex-ante utility (24) can be estimated as

$$U_R^k = -\frac{1}{12} f\left(w_k\right)\Delta_k^3 + O\left(\Delta_k^4\right), \tag{28}$$

where $O\left(\Delta_k^4\right)$ has an order $\Delta_k^4$. Then, taking the length of the uniform partition's element $\Delta_k = cb$, where $c \in \left(2, 2\sqrt{3}\right)$ is chosen to satisfy $cbN = 1$ for some integer $N$, transforms (26) into

$$\left(2c - 4\right) b + b^2\left(\frac{1}{6}\frac{f'\left(\tilde{w}_k\right)}{f\left(\tilde{w}_k\right)}c^2 - \frac{1}{6}\frac{f'\left(\tilde{w}_{k-1}\right)}{f\left(\tilde{w}_{k-1}\right)}c^2\right) \geq 0,$$

which is satisfied for a sufficiently small $b$. Also, (28) can be written as

$$U_R^k = -\frac{1}{12} f\left(w_k\right) c^3 b^3 + O\left(b^4\right).$$

The principal's ex-ante utility in the case of complete delegation is

$$U_R^D = -b^2 = \sum_{k=0}^{N-1} U^k,$$

[21] This argument can be reapplied to all equilibrium boundary points $w_k$, $0 < k < n - 1$.

where

$$U^k = -b^2\left(F\left(w_{k+1}\right) - F\left(w_k\right)\right) = -b^2\left(f\left(w_k\right)\Delta_k + O\left(\Delta_k^2\right)\right) = -f\left(w_k\right)cb^3 + O\left(b^4\right).$$

This implies that for sufficiently small $b$,

$$U_R^k - U^k = f\left(w_k\right)\left(c - \frac{c^3}{12}\right)b^3 + O\left(b^4\right) = f\left(w_k\right)\left(1 - \frac{c^2}{12}\right)cb^3 + O\left(b^4\right) > 0,$$

and summing across all $k = 0, ..., N-1$ results in $U_R > U_R^D$. ■

**Proof of Theorem 8.** We prove the statement by induction. That is, we show that the sender cannot benefit by distorting information in any stage, conditional on the truth-telling at all previous stages.

1) For $s = \tilde{s}$ we have $i_s = 1$, and $i_\tau = 0$ for $\tau < s$ (the last condition is omitted for $\tilde{s} = 1$). The sender infers that $\theta \in M_s = [w_s, w_{s-1}]$, and the optimal action for her is $a_S\left(M_s\right) = \bar{w}_s + b$. Given truth-telling in previous stages, we have $m_\tau = 0$, $\tau < s$. Then, if $m_s = 1$, due to the receiver's beliefs, we have $j = s$, and the induced action is $a_s = \bar{w}_s$. On the other hand, if $m_s = 0$, then $j > s$, and the induced action is $a_j = \bar{w}_j < a_s < \bar{w}_s + b$, which implies $U_S\left(a_j, b|M_1\right) < U_S\left(a_s, b|M_1\right)$. Hence, $m_{\tilde{s}} = 1$.

Also, for all $s > \tilde{s}$, we have $i_s = 1$. Since $m_{\tilde{s}} = 1$, the induced action is $a_{\tilde{s}} = \bar{w}_{\tilde{s}}$ for any $m_s$, $s > \tilde{s}$. Thus, the sender still cannot beneficially deviate from $m_s = 1$, $s > \tilde{s}$.

2) For $s$ such that $i_s = 0$, we have $i_\tau = 0$ for $\tau < s$. Given this information, the sender infers that $\theta \in M_s = [0, w_s]$. Assuming $m_\tau = 0$ for $\tau < s$, the message $m_s = 1$ induces the action $a_s = \bar{w}_s = \frac{w_s + w_{s-1}}{2}$, for any $m_\tau$, $\tau > s$. This brings the expected utility to the sender $U_S\left(a_s, b|M_s\right) = \frac{1}{w_s}\int\limits_0^{w_s} U_S\left(a_s, b, \theta\right)d\theta = -\frac{1}{w_s}\int\limits_0^{w_s}\left(\frac{w_s + w_{s-1}}{2} - b - \theta\right)^2 d\theta = -(\frac{w_s^2}{12} + \frac{w_{s-1}^2}{4} - bw_{s-1} + b^2)$.

Now, consider the sender's expected utility from the signaling strategy $\bar{m}_\tau = i_\tau$, $\tau \geq s$. Since $i_\tau = 0$ and $m_\tau = 0$ for $\tau < s$ (assuming truth-telling in previous stages), we can denote $\bar{m}_\tau = i_\tau$, for all $\tau = 1, ..., T$.

If $s = T$, then $\theta \in M_{T+1} = [0, w_T]$. The message $m_T = 0$ induces the action $a_{T+1} = \bar{w}_{T+1} = \frac{w_T}{2}$, and the message $m_T = 1$ induces $a_T = \frac{w_T + w_{T-1}}{2} \geq \frac{w_T + 4b}{2} = \frac{w_T}{2} + 2b$. This gives $U_S\left(a_{T+1}, b|M_s\right) \geq U_S\left(a_T, b|M_s\right)$, since $a_T + a_{T+1} = \frac{w_T + w_{T-1}}{2} + \frac{w_T}{2} \geq \frac{w_T}{2} + 2b + \frac{w_T}{2} = w_T + 2b$.

For $s < T$, the sender's ex-ante utility from $(\bar{m}_\tau)_{\tau=1}^T$ is

$$U_S\left(a((\bar{m}_s)_{s=1}^T), b|M_s\right) = \frac{1}{w_s}\sum_{\tau=s}^{T}\int\limits_{w_{\tau+1}}^{w_\tau} U_S\left(\bar{w}_{\tau+1}, b, \theta\right)d\theta = -\frac{1}{w_s}\sum_{\tau=s}^{T}\frac{\left(\Delta w_\tau\right)^3}{12} - b^2,$$

where $\Delta w_\tau = w_\tau - w_{\tau+1} > 0$.

Then,

$$-\frac{1}{12w_s}\sum_{\tau=s}^{T}\left(\Delta w_\tau\right)^3 - b^2 > -\frac{1}{12w_s}\left(\sum_{\tau=s}^{T}\Delta w_\tau\right)^3 - b^2 = -\frac{w_s^3}{12w_s} - b^2 = -\frac{w_s^2}{12} - b^2.$$

In addition, $w_{s-1} \geq 4b$ implies $-\frac{w_s^2}{12} - b^2 - U_S\left(a_s, b|M_s\right) = -\frac{w_s^2}{12} - b^2 - (-1)\left(\frac{w_s^2}{12} + \frac{w_{s-1}^2}{4} - bw_{s-1} + b^2\right) = \frac{1}{4}w_{s-1}\left(w_{s-1} - 4b\right) \geq 0$, which leads to $U_S\left(a((\bar{m}_s)_{s=1}^T), b|M_s\right) > -\frac{w_s^2}{12} - b^2 \geq U_S\left(a_s, b|M_s\right)$. Hence, for $i_s = 0$, $s = 1, ..., T$ the sender is worse off by sending $m_s = 1$ instead of $m_s = 0$.

To complete the proof, it easily follows that the described signaling strategy generates beliefs that $\theta$ is uniformly distributed on $\Theta_i = (w_i; w_{i-1}]$, and $a\left((m_s)_{s=1}^T\right) = \bar{w}_j$ is a best-response of the receiver given his beliefs. ■

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