Absorbing Sets for Roommate Problems with Strict Preferences*

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Abstract

The purpose of this paper is to determine the absorbing sets, a solution that generalizes the notion of (core) stability, of the *entire* class of the roommate problems with strict preferences.

Keywords: roommate problems, stability and absorbing sets.

Preliminary draft

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1 Introduction

The stable roommate problem is a generalization of the stable marriage problem in which each person in a set ranks all the other people, including oneself, in a preference list. A matching is a partition of the set into disjoint pairs of persons and/or single ones. A matching is said to be unstable if there are two persons preferring each other to their partners in the matching, or there is one person preferring to be single to her partner in the matching . Roommate problems are *solvable* if they admit a stable matching; otherwise they are said to be *unsolvable* and yet we believe that it is interesting to "solve" them. While (core) stability for roommate problems has been investigated by Gale and Shapley [5], Roth and Sotomayor [8], Irving [6], Tan [12], Abeledo and Isaak [1] Chung [3] and Diamantoudi, Miyagawa and Xue [4], to the best of our knowledge, with the exception of the papers by Tan [11] and Abraham, Biró and Manlove [2], there are not works studying unsolvable roommate problems. In this paper we propose a solution that generalizes the notion of core-stability (stability hereafter, if there is no confusion) that solves the *entire* class of the roommate problems with strict preferences.

The solution considered is that of the *absorbing sets* which it has appeared in the game theory literature under various names as a solution for abstract systems.[1] A roommate problem may be formulated as a particular abstract system (a *matching system*) with a dominance relation defined over the set of matchings so that the notion of absorbing sets is a candidate for its solution. The idea of stability lying in this solution says that any two elements in an absorbing set do dominate each other if not directly through a path, and that no element outside an absorbing set dominates an element in it. This solution gives unitary absorbing sets which in this context correspond to stable matchings or cyclical absorbing sets in which case multiple stable matchings are provided. In this last case the stability notion of the absorbing sets allows to discard with confidence those matchings lying out of the absorbing sets. It happens that for these problems there are either unitary stable matchings or matchings placed in cyclical absorbing sets.

[1] An absorbing set is a minimal closed subset of the *admissible set*, Kalai and Schmeidler [7] and it coincides with the *elementary dynamic solution* of Shenoy [10].

Tan [12] identifies a necessary and sufficient condition for the existence of stable roommate matchings based on a new figure called *stable partition.* A stable partition divides the set of agents in two subsets: The even party agents set which contains the agents whose preferences give stable matching pairs, and the odd party agents set which contains the agents whose preferences form odd rings or they are singletons. Tan shows that an instance is not solvable if and only if its preference profile has a stable partition with an odd ring. It happens that it is the relationship between the preferences of the agents of these two subsets what explains the pattern of absorbing sets in roommate problems. The results of this paper may be summarized as follows:

Firstly, for every stable partition P we define specific matchings, called *P-stable matchings*. Then we identify a type of stable partitions called *strongly stable,* which result crucial in the determination of the absorbing sets. We prove that for any unstable matching there exists a finite sequence of successive myopic blocking pairs leading to a P-strongly stable matching. This result is analogous to that of Roth and Vande Vate [9] for marriage problems and that of Diamantoudi, Miyagawa and Xue [4] for solvable roommate problems.

Secondly, we determine the specific matchings that form the absorbing sets, as well as their number and size. Notice that few absorbing sets of small size imply a greater number of matchings being ruled out as solutions for roommate problems.

Specifically we prove that if a preference profile of a roommate problem has *at most* one strongly stable partition then there is *only one* absorbing set. It is shown how the size of this absorbing set depends only on the relationship among the preferences of the agents in the odd party.

With respect to the case of multiple absorbing sets we find that it is the relationship between the preferences of the odd party agents and the preferences of the even party agents what determines the number of absorbing sets. In particular, we show that all the absorbing sets are of the same size and that for each strongly stable partition there is one absorbing set. Furthermore we prove that for any two absorbing sets the matching pairs restricted to the even party agents differ while the matching pairs of the odd party agents coincide.

The paper is organized as follows: Section 2 contains the preliminaries of the paper. Section 3 is devoted to the study of the properties of P-stable matchings and Section 4 contains the analysis of the absorbing sets. Section 5 concludes.

2 Preliminaries

This section contains the preliminaries of this paper.

A *roommate problem* is a pair $(N, (\succcurlyeq_i)_{i\in N})$ where $N = \{1, ..., n\}$ is a finite set of agents and for each $i \in \{1, ..., n\}$, $\succcurlyeq_i$is a complete and transitive preference relation defined over N. The strict preference associated with $\succcurlyeq_i$ is denoted by $\succ_i$. We limit ourselves to a roommate problem in which preferences are strict.

A *matching* is a function $\mu : N \longrightarrow N$ such that for all $i, j \in N$, if $\mu(i) = j$, then $\mu(j) = i$. Here, $\mu(i)$ denotes the agent with whom agent i is matched. We allow $\mu(i) = i$, which means that agent i is alone.

A matching $\mu \in M$ is *blocked* by a pair of agents $\{i, j\} \subseteq N$ (possibly $i = j$) if

$$j \succ_i \mu(i) \; and \; i \succ_j \mu(j) \qquad (1)$$

That is, i and j both prefer each other to their mates at μ. We allow $i = j$, in which case (1) means that i prefers being alone to being matched with $\mu(i)$. When (1) holds, we call $\{i, j\}$ a *blocking pair* of μ.

A matching is *individually rational* if there exists no blocking pair $\{i, j\}$ with $i = j$. A matching is *stable* if there exists no blocking pair.

Given a blocking pair $\{i, j\}$ of a matching μ, another matching μ' is obtained from μ by satisfying the pair if $\mu'(i) = j$ and for all $k \in N\backslash\{i, j\}$,

$$\mu'(k) = \begin{cases} k & if \; \mu(k) \in \{i, j\} \\ \mu(k) & \text{otherwise.} \end{cases}$$

That is, once i and j are matched, their mates (if any) at μ are alone in μ' and the other agents are matched as in μ.

An *abstract system associated with a roommate problem* is a pair $(M, >)$ where M is the set of matchings and $>$ is the binary relation defined over M as follows: $\mu' > \mu$ if and only if μ' is obtained from μ by satisfying a blocking pair of μ. An abstract system can be represented by means of a directed graph (digraph) whose vertices are the elements of the set M (matchings) and the arcs represent the binary relation between them.

Let $>^T$ be the *transitive closure* of $>$. Then, $\mu' >^T \mu$ if and only if there exists a finite sequence of matchings $\mu = \mu_0, \mu_1, ..., \mu_m = \mu'$ such that, for all $i \in \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair of μ_{i-1}.

A set $S \subseteq M$ is an *absorbing set*[2] if: *i)* for all $\mu, \mu' \in S$ $(\mu \neq \mu')$: $\mu' >^T \mu$, and *ii)* for all $\mu \in S$ there is no $\mu' \notin S$ such that $\mu' > \mu$.

Each of the absorbing sets satisfy two conditions. Condition *i)* says that S is a strongly connected set for $>$. Condition *ii)* says that S is a $>$-closed set.

Note that the set of unitary absorbing sets coincides with the core of $(M, >)$.

3 The P-stable matchings

In this section starting from the notion of stable partition introduced by Tan [12] we define the P-stable matchings associated to a stable partition P and study some of their properties. As we shall see some of these matchings, the P-strongly stable matchings play a crucial role in the analysis of the stability of roommate problems.

Consider a roommate problem $(N, (\succ_i)_{i \in N})$. Let $A = (x_1, ..., x_k)$ be an ordered set of agents contained in N. A is a *ring* if $k \geq 3$ and for all $i \in \{1, ..., k\}$, $x_{i+1} \succ_{x_i} x_{i-1} \succ_{x_i} x_i$ (subscript modulo k)[3]. A is a *pair of mutually acceptable agents* if $k = 2$ and for all $i \in \{1, 2\}$, $x_{i-1} \succ_{x_i} x_i$ (subscript modulo 2). A is a *singleton* if $k = 1$.

Definition 1 *A stable partition is a partition P of N in which each set either is a ring or mutually acceptable pair of agents or a singleton, and for any sets $A = (x_1, ..., x_k)$ and $B = (y_1, ..., y_l)$ of P (without ruling out $A = B$), the following condition is verified:*

if $y_j \succ_{x_i} x_{i-1}$ *then* $y_{j-1} \succ_{y_j} x_i$, *for all* $i \in \{1, ..., k\}$, $j \in \{1, ..., l\}$ $(y_j \neq x_{i+1})$.

[2] The *admissible set* of $(M, >)$ (Kalai and Schmeidler[7]) is the set

$$A(M, >) = \{\mu \in M : \mu' >^T \mu \Leftrightarrow \mu >^T \mu'\}$$

This set is the union disjoint of the minimally $>$-closed subsets of M, which are the absorbing sets.

[3] We will omit subscript modulo k

A set $A \in P$ is called a *party* of P. An odd party (even party, respectively) is a party having odd (even respectively) cardinality.

From now on, by Lemma 15 of the Appendix, we will assume that each party A of P either is an odd ring or a pair of mutually acceptable pair of agents or a singleton.

Let P be a stable partition of N. A matching μ is *P-stable* if: i) for each odd ring $A = (x_1, ..., x_k)$ of $P : \mu(x_i) = x_i$ for some $i \in \{1, ..., k\}$ and $\mu(x_j) = x_{j+1}$ for all $j = i+1, i+3, ..., i+k$ $(j \neq i)$, ii) for each pair of mutually acceptable agents $A = (x_1, x_2)$ of $P : \mu(x_1) = x_2$, and iii) for each singleton $A = (x_1)$ of $P : \mu(x_1) = x_1$.

We illustrate the notion of P-stable matching with an example.

Example 2 *Consider a roommate problem with* $N = \{1, 2, 3, 4, 5, 6\}$ *and the following preference relation:*

$$2 \succ_1 3 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6$$

$$3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6$$

$$1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6$$

$$5 \succ_4 4 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 6$$

$$4 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 6$$

$$6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 4 \succ_6 5$$

It is easy to verify that $P = \{(1, 2, 3), (4, 5), (6)\}$ *is a stable partition of* N *where* $A_1 = (1, 2, 3)$ *is an odd ring,* $A_2 = (4, 5)$ *is a pair of mutually acceptable agents and* $A_3 = (6)$ *is a singleton. The* P*-stable matchings associated with the stable partition* P *are:* $\mu_1 = [\{1\}, \{2, 3\}, \{4, 5\}, \{6\}]$, $\mu_2 = [\{2\}, \{1, 3\}, \{4, 5\}, \{6\}]$ *and* $\mu_3 = [\{3\}, \{1, 2\}, \{4, 5\}, \{6\}]$.

Remark 3 *Notice that there exists an unique* P*-stable matching, which is stable, if and only if* P *has no any odd ring .*

In the following lemma, we prove that if P contains some odd ring, then for any P-stable matching there exists a finite sequence of successive blocking pairs leading to any other.

Lemma 4 *Let P be a stable partition of N. If P contains some odd ring then, for any μ and μ' P-stable matchings, $\mu' >^T \mu$.*

Proof. Let $A_1, ..., A_r$ be the odd rings of P and $\mathcal{A} = \bigcup_{i=1}^{r} A_i$. Suppose that $A_1 = (x_1, ..., x_k)$. Since A_1 is a ring, then

$$x_{i+1} \succ_{x_i} x_{i-1} \succ_{x_i} x_i, \qquad \text{for all } i = \{1, ..., k\}. \qquad (2)$$

By definition, since μ and μ' are P-stable matchings, there exist agents x_l, $x_s \in A_1$ such that $\mu(x_l) = x_l$ and $\mu'(x_s) = x_s$. Then $\mu(x_i) = x_{i+1}$ for all $i = l+1,\ l+3,\ ...\ ,\ l+k\ \ (i \neq l)$ and $\mu'(x_j) = x_{j+1}$ for all $j = s+1,\ s+3,\ ...\ ,\ s+k\ \ (j \neq s)$. Now, since $\mu(x_l) = x_l$ and $\mu(x_{l-1}) = x_{l-2}$, by condition (2), the pair $\{x_l, x_{l-1}\}$ is a blocking pair of μ. Let $\mu_1$be the matching obtained from μ by satisfying this blocking pair. Then, we have $\mu_1(x_{l-2}) = x_{l-2}$ and $\mu_1(x_i) = x_{i+1}$ for all $i = (l-2)+1,\ (l-2)+3,\ ...\ ,\ (l-2)+k\ \ (i \neq l-2)$. Moreover, since $\mu_1 \mid_{(N \setminus A_1)} = \mu_1 \mid_{(N \setminus A_1)}$ [4], the matching μ_1 is P-stable. By repeating the process, we can construct, inductively, a sequence of matchings μ_0, μ_1, $\mu_{2,}$ ... as follows:
For $i = 0$, set $\mu_0 = \mu$.
For $i > 0$, let μ_i be the matching obtained from μ_{i-1} by satisfying the blocking pair $\{x_{l-2(i-1)}, x_{l-2(i-1)-1}\}$.

Then, $\mu_i(x_{l-2i}) = x_{l-2i}$ and $\mu_i(x_j) = x_{j+1}$ for all $j = (l-2i)+1,\ (l-2i)+3,\ ...\ ,\ (l-2i)+k\ \ (j \neq l-2i)$. As $\mu_i \mid_{(N \setminus A_1)} = \mu \mid_{(N \setminus A_1)}$, we have μ_i is a P-stable matching for all i.
Let $m_1 \in \{1, ..., k\}$ be such that $x_{l-2m_1} = x_s$. Then $\mu = \mu_0, \mu_1, ..., \mu_{m_1}$ is a finite sequence of P-stable matchings such that, for all $i \in \{1, ..., m_1\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair and $\mu_{m_1} \mid_{A_1} = \mu' \mid_{A_1}$.
Consider now the ring A_2. Reasoning in the same way as before for μ_{m_1} and μ' we obtain a finite sequence of P-stable matchings μ_{m_1}, μ_{m_1+1}, ..., $\mu_{m_1+m_2}$ such that, for all $i \in \{m_1+1, ..., m_1+m_2\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair and $\mu_{m_1+m_2} \mid_{(A_1 \cup A_2)} = \mu' \mid_{(A_1 \cup A_2)}$.

[4] $\mu \mid_A$ denoted the restriction of μ to A, when $\mu(x) \in A$ for all $x \in A$.

By repeating the same procedure to the remaining odd rings, we obtain eventually a finite sequence of P-stable matchings $\mu = \mu_0, \mu_1, ..., \mu_m$, where $m = \sum_{i=1}^{r} m_i$, and such that, for all $i \in \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair and $\mu_m \mid_{\mathcal{A}} = \mu' \mid_{\mathcal{A}}$. Now, as $\mu_m \mid_{(N \setminus \mathcal{A})} = \mu' \mid_{(N \setminus \mathcal{A})}$, then $\mu_m = \mu'$ and this verifies the assertion of the lemma. ■

In the following theorem we prove that if a matching is not P-stable matching for any P stable partition, then there exists a finite sequence of successive blocking pairs leading to a P'-stable matching. In particular, from Remark 3 this sequence of successive blocking pairs would lead to a stable matching. This result is analogous to the one established by Diamantoudi, Miyagawa and Xue [4] for solvable roommate problems which is a generalization of Roth and Vande Vate [9] for marriage problems.

Theorem 5 *If μ is not P-stable matching for any stable partition P then there exists a P'-stable matching μ' such that $\mu' >^T \mu$.*

Definition 6 *Let μ be a P-stable matching. Then, μ is P-strongly stable if there is no a P'-stable matching μ' $(P' \neq P)$ such that $\mu' >^T \mu$.*

The stable partition P associated to a P-strongly stable matching is called *strongly stable partition.*

Lemma 7 *The following asertions hold:*

a) If μ is *a P-strongly stable matching, then, for any P-stable matching μ' , not strongly stable, we have $\mu >^T \mu'$*

b) If a strongly stable matching no exists, then, for any P-stable and P'-stable matchings μ and μ', we have $\mu >^T \mu'$.

4 Absorbing sets for roommate problems

This section is devoted to the analysis of the structure of the absorbing sets for matching systems derived from roommate problems with strict preferences.

Let P be a stable partition of N. Denoted by S_p the set formed by the P-stable matchings and the matchings obtained from the P-stable matchings by satisfying a finite sequence of successive blocking pairs. Set $S^* = \bigcup_P S_P$

Theorem 8 *Let $(N, (\succ_i)_{i \in N})$ be a roommate problem in which preferences are strict and a strongly stable partition of N exists. Then, the following assertions hold:*

a) If P is a strongly stable partition, then S_P is an absorbing set.

b) If P and P' are any two distinct stable partitions, then $S_P \neq S_{P'}$.

c) If S is an absorbing set, then there exists a stable partition P such that $S = S_P$.

Proof. *a)* We prove that S_P satisfies the two following condictions: *i)* for all μ, $\mu' \in S_P$ $(\mu \neq \mu')$: $\mu' >^T \mu$, and *ii)* for all $\mu \in S_P$, there is no $\mu' \notin S_P$ such that $\mu' > \mu$.

If P does not contain any odd ring, by Remark 3, there exists an unique P-stable matching μ which is stable. Then $S_P = \{\mu\}$ and S_P is an absorbing set. Suppose that P contains some odd ring and set μ, $\mu' \in S_P$. Since μ, $\mu' \in S_P$, there exist P-strongly stable matchings $\overline{\mu}$ and $\overline{\mu}'$ such that $\mu >^T \overline{\mu}$ and $\mu' >^T \overline{\mu}'$. (If μ and μ' are P-stable matchings then, by Lemma 4, this is true). Now, if μ is no $P'-$stable maching for any stable partition P', by Theorem 5, there exists a $\widetilde{P}$-stable matching such that $\widetilde{\mu} >^T \mu$. Since $\mu >^T \overline{\mu}$ then $\widetilde{\mu} >^T \overline{\mu}$, and since $\overline{\mu}$ is P -strongly stable, by Definition 6, it follows that $\widetilde{P} = P$. Hence $\widetilde{\mu}$ is a P-stable matching. Then, by Lemma 4, $\overline{\mu}' >^T \widetilde{\mu}$ and since $\mu' >^T \overline{\mu}'$ and $\widetilde{\mu} >^T \mu$ this implies that $\mu' >^T \mu$ and condition i) is satisfied. Otherwise, μ is a $P'-$stable maching for some stable partition P'. Since $\mu >^T \overline{\mu}$, then, we have $P' = P$, and therefore, by Lemma 4, $\overline{\mu}' >^T \mu$. Now, as $\mu' >^T \overline{\mu}'$, then $\mu' >^T \mu$ and condition i) too is satisfied in this case.

Consider now any $\mu \in S_P$. Then, there exists a P-stable matching $\overline{\mu}$ such that $\mu >^T \overline{\mu}$. If there exists a $\mu' \notin S_P$ such that $\mu' > \mu$, then $\mu' >^T \overline{\mu}$. Hence $\mu' \in S_P$, contradicting that $\mu' \notin S_P$. Therefore, condition ii) is verified.

b) Suppose, by contradiction, that $S_P = S_{P'}$. Let μ' be a $P'-$stable maching. Then $\mu' \in S_{P'}$. As $S_P = S_{P'}$ we have $\mu' \in S_P$. Let μ be any P-stable matching. Then, $\mu \in S_P$. Now, since S_P is an absorbing set, then $\mu' >^T \mu$, which is impossible.

c) Set $\mu \in S$. If μ is a $P-$strongly stable maching, then $\mu \in S_P$, hence $S = S_P$. Otherwise, from Theorems 5 and Lemma 7, there exists a P-strongly stable matching $\overline{\mu}$ such that $\overline{\mu} >^T \mu$. Then $\overline{\mu} \in S$, and since $\overline{\mu} \in S_P$ we have $S = S_P$. ■

Theorem 9 *Let $(N, (\succ_i)_{i \in N})$ be a roommate problem in which preferences are strict and a strongly stable partition of N not exists. Then, the set S^* is an absorbing set and it is the unique.*

Proof. First we prove that S^* is an absorbing set by verifying that S^* satisfies the conditions *i)* and *ii)* as in the proof above. Let μ and $\mu' \in S^*$. Then, there exists a P-stable matching $\overline{\mu}$ and a P'-stable matching $\overline{\mu}'$ such that $\mu >^T \overline{\mu}$ and $\mu >^T \overline{\mu}'$. If μ is no $P'-$stable maching for any stable partition P', then, by Theorem 5, there exists a $\widetilde{P}$-stable matching $\widetilde{\mu}$ such that $\widetilde{\mu} >^T \mu$. Now, since a strongly stable partition not exists, by Lemma 7, we have $\overline{\mu}' >^T \widetilde{\mu}$. Then $\overline{\mu}' >^T \mu$ and since $\mu' >^T \overline{\mu}'$, this implies that $\mu' >^T \mu$, hence condition i) follows. Otherwise, μ is a $\widetilde{P}$-stable maching for some stable partition $\widetilde{P}$, then, by Lemma 7, we have $\overline{\mu}' >^T \mu$ and as $\mu' >^T \overline{\mu}'$, it follows that $\mu' >^T \mu$ and again condition i) is satisfied.

Consider now any $\mu \in S$. Then, there exists a P-stable matching $\overline{\mu}$ such that $\mu >^T \overline{\mu}$. If there exists $\mu' \notin S$ such that $\mu' > \mu$ then $\mu' >^T \overline{\mu}$ so $\mu' \in S$ and this is a contradiction. Thus condition *ii)* follows. Finally we have to prove that S^* is the unique absorbing set. Let S be any absorbing set and $\mu \in S$. If μ is a $P-$stable maching, then $\mu \in S^*$, hence $S = S^*$. Otherwise, from Theorem 5, there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} >^T \mu$. Then, $\overline{\mu} \in S$, and since $\overline{\mu} \in S^*$ we have $S = S^*$. ■

Notice that theorems 8 and 9 imply that if there is at most one P-strongly stable partition, then the absorbing set is unique and that the number of matchings of the absorbing set depends on the relationship among the players of the odd party of the P-stable partition.

The following examples illustrate these two theorems.

Example 10 *Consider a roommate problem with 4 agents and the following preferences*

$$3 \succ_1 2 \succ_1 4 \succ_1 1$$

$$1 \succ_2 3 \succ_2 4 \succ_2 2$$

$$4 \succ_3 1 \succ_3 2 \succ_3 3$$

$$2 \succ_4 1 \succ_4 3 \succ_4 4$$

There are two stable partitions $P = \{(1,2),(3,4)\}$ and $P' = \{(1,3),(2,4)\}$. Neither of them contains an odd ring. Then, by Remark 3, there is a P-stable matching $\mu = [\{1,2\},\{3,4\}]$ and a P'-stable matching $\mu' = [\{1,3\},\{2,4\}]$ that are stable. So the two partitions are strongly stables. Therefore, by Theorem 8, there are two absorbing sets for this roommate problem $S_P = \{\mu\}$ and $S_{P'} = \{\mu'\}$.

Example 11 *Consider a roommate problem with 8 agents and the following preference profile:*

$$2 \succ_1 3 \succ_1 8 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6 \succ_1 7$$

$$3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6 \succ_2 7 \succ_2 8$$

$$1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6 \succ_3 7 \succ_3 8$$

$$6 \succ_4 5 \succ_4 7 \succ_4 4 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 8$$

$$4 \succ_5 6 \succ_5 7 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 8$$

$$7 \succ_6 4 \succ_6 5 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 8$$

$$5 \succ_7 4 \succ_7 6 \succ_7 7 \succ_7 1 \succ_7 2 \succ_7 3 \succ_7 8$$

$$1 \succ_8 8 \succ_8 2 \succ_8 3 \succ_8 4 \succ_8 5 \succ_8 6 \succ_8 7$$

There are two stable partitions of N : $P = \{(1,2,3),(4,5),(6,7),(8)\}$ and $P' = \{(1,2,3),(4,6),(5,7),(8)\}$ which contain an odd ring. It is easy to verify that both are strongly stables. By Theorem 8, there are two absorbing sets. These are $S_P = \{\mu_1,\mu_2,\mu_3,\mu_4\}$ (where $\mu_1 = [\{1\},\{2,3\},\{4,5\},\{6,7\},\{8\}]$, $\mu_2 = [\{2\},\{1,3\},\{4,5\},\{6,7\},\{8\}]$, $\mu_3 = [\{3\},\{1,2\},\{4,5\},\{6,7\},\{8\}]$, $\mu_4 = [\{1,8\},\{2,3\},\{4,5\},\{6,7\}]$ (μ_1,μ_2,μ_3 are P-stable matchings and μ_4 is obtained form μ_1 by satisfying the blocking pair $\{1,4\}$)) and $S_{P'} = \{\mu'_1,\mu'_2,\mu'_3,\mu'_4\}$ (where $\mu'_1 = [\{1\},\{2,3\},\{4,6\},\{5,7\},\{8\}]$, $\mu'_2 = [\{2\},\{1,3\},\{4,6\},\{5,7\},\{8\}]$, $\mu'_3 = [\{3\},\{1,2\},\{4,6\},\{5,7\},\{8\}]$, $\mu'_4 = [\{1,8\},\{2,3\},\{4,6\},\{5,7\}]$ (μ'_1,μ'_2,μ'_3 are

P-stable matchings and μ'_4 is obtained form μ'_1 by satisfying the blocking pair $\{1,4\}$))

Example 12 *Consider a roommate problem with 7 agents and the following preference profile:*

$$2 \succ_1 3 \succ_1 4 \succ_1 1 \succ_1 5 \succ_1 6 \succ_1 7$$

$$3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6 \succ_2 7$$

$$1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6 \succ_3 7$$

$$1 \succ_4 6 \succ_4 5 \succ_4 7 \succ_4 2 \succ_4 3 \succ_4 4$$

$$4 \succ_5 6 \succ_5 7 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 5$$

$$7 \succ_6 4 \succ_6 5 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 6$$

$$5 \succ_7 4 \succ_7 6 \succ_7 1 \succ_7 2 \succ_7 3 \succ_7 7$$

There are two stable partitions of N: $P = \{(1,2,3),(4,5),(6,7)\}$ and $P' = \{(1,2,3),(4,6),(5,7)\}$. It is easy to verify that neither of them is strongly stable. By Theorem 9 there is an unique absorbing set with cardinality equal to 34 that is formed by all the $P-$stable matchings and those obtained from them by satisfying a blocking pair.

Denoted by $I = \{x \in A : A \in P$, for any strongly stable partition $P\}$ and by $S^*_P = \{\mu \mid_I : \mu \in S_P\}$[5].

Notice that there are as many absorbing sets as strongly stable partitions, except for the case in which there is no strongly stable partition (see theorem 9). When there are several strongly stable partitions, as in Example 11, the absorbing sets follow a property: all the agents of the set I of the strongly stable partitions, are matched identically in all the absorbing sets, as it is shown in the following theorem:

Theorem 13 *Let P and P' be any two distinct strongly stable partitions of N. Then $S^*_P = S^*_{P'}$.*

[5] The restriction of μ to I is possible by Lemma 16 of the appendix.

Proof. We first prove that $S_P^* \subseteq S_{P'}^*$. Set $\mu \in S_P$. We prove that there exists $\mu' \in S_{P'}$ such that $\mu' \mid_I = \mu \mid_I$. As $\mu \in S_P$, then μ is a P-stable matching or is obtained from a P-stable matching by satisfying a finite sequence of successive blocking pairs. If μ is a P-stable matching, by definition, there exists a P'-stable matching μ' such that $\mu' \mid_I = \mu \mid_I$. Then, we have $\mu' \in S_{P'}$ and $\mu' \mid_I = \mu \mid_I$. Suppose that μ is not a P-stable matching. Then, there exists a finite sequence of matchings $\overline{\mu} = \mu_0, \mu_1, ..., \mu_m = \mu$ such that $\overline{\mu}$ is a P-stable matching and, for all $i \in \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair $\{x_i, y_i\}$. Now, by Lemma 16 of the appendix, we have $\{x_i, y_i\} \subseteq I$. Since $\overline{\mu}$ is a P-stable matching, by the discussion above there exists a P'-stable matching $\overline{\mu}'$ such that $\overline{\mu}' \mid_I = \overline{\mu} \mid_I$. Let $\overline{\mu}' = \mu_0', \mu_1', ..., \mu_m' = \mu'$ be the finite sequence of matchings such that, for all $i \in \{1, ..., m\}$, μ_i' is obtained from μ_{i-1}' by satisfying the blocking pair $\{x_i, y_i\}$. Then, we have $\mu' \in S_{P'}$ and $\mu' \mid_I = \mu \mid_I$.
Conversely, if $\mu' \in S_{P'}$ then, reasoning as before, there exists $\mu \in S_P$ such that $\mu \mid_I = \mu' \mid_I$. It follows that $S_{P'}^* \subseteq S_P^*$. ■

5 Concluding remarks

Two additional results are of interest:

The distinction between unitary and cyclical absorbing sets proves to be very useful in this setting since for these problems there are *either* unitary stable matchings *or* matchings placed in cyclical absorbing sets, allowing to study the latter separetely.

Theorem 14 *Consider a roommate problem with strict preferences. Then either it has (core)-stable matchings or it has cyclical absorbing sets of cardinality equal or greater than 3.*

The following example in Chung [3] shows that this is not the case for roommate problems with weak preferences

$$4 \succ_1 2 \succ_1 3 \succ_1 1$$

$$3 \succ_2 1 \succ_2 2 \succ_2 4$$

$$1 \succ_3 2 \succ_3 3 \succ_3 4$$

$$1 \sim_4 4 \succ_4 3 \succ_4 2$$

There is a stable matching $\{14,23\}$ and an absorbing set formed by $\{12,3,4\}$, $\{1,23,4\}$, $\{13,2,4\}$.

6 Appendix

Lemma 15 *If P is a stable partition of N, then there exits a stable partition $P \subseteq P'$ such that for all $A \in P'$ either A is an odd ring or is a pair of muttually acceptable agents or a singleton.*

Proof. Let P be a stable partition of N. By definition, for all $A \in P$, A is a ring, a pair of mutually acceptable agents or a singleton. If P does not contain an even ring then $P = P'$ and the lemma follows.
Suppose that P contains an even ring $A = (x_1, ..., x_k)$. Set $A_i = (x_i, x_{i+1})$ for all $i = 1, ..., \frac{k}{2}$. Since A is a ring, it is verified that $x_{i+1} \succ_{x_i} x_{i-1} \succ_{x_i} x_i$ for all $i = 1, ..., k$. Therefore, the set A_i is a pair of mutually acceptable agents for all i. Let P' be the partition of N such that $P' = P \backslash \{A\} \bigcup_{i=1}^{k/2} \{A_i\}$. It is easy to verify that P' is also a stable partition. If P' does not contain an even ring then the lemma follows. Otherwise, by repeating this procedure, we will obtain eventually a stable partition which verifies the assertion of the lemma. ■

Lemma 16 *If P is a strongly stable partition then for any finite sequence of matchings $\mu = \mu_0, \mu_1, ..., \mu_m$, where μ is a P-stable matching and, for all $i = \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair $\{x_i, y_i\}$, we have x_i, $y_i \in I$.*

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