Cores for cooperative investment games

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1. Introduction

This paper deals with games where players invest fractions of their own resources to some joint projects and share the profit produced from the projects among them. In the case of linear production function, Molina and Tejada (2004) and Fukuda et al. (2005) studied a fuzzy game that incorporates players' partial investment into Owen's linear production game (LP game). However, in such games, players' utilities from the remaining resources were not taken into account.

Azrieli and Lehrer (2005) defined a cooperative investment game, which is similar to a fuzzy game. They focused on resources players leave in their hands after investment, and newly defined the comprehensive core of an investment game. While in the comprehensive core payoff allocations are assumed to be linear, Fukuda et al. (2005) showed that the core elements of a fuzzy game are linear only when the game is positively homogeneous of degree one. Then Muto et al. (2006) considered non-linear payoff schemes and generalized the core of Aubin.

In this paper, we model a cooperative investment situation where each player can privately gain utility from his/her own resources. Specifically we define a new game where all players can invest a fraction of their own resources and simultaneously make profits from their own remaining resources. This game can be interpreted as an extended fuzzy game (or multi-choice game) in which players may gain profits from their remaining resources. Note that in this game it is not necessarily the case that full investment yields the maximum profit.

Next we define efficiency and individual rationality for this game, and study its core. We also formulate this situation as a non-cooperative strategic form game and study

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its Nash equilibria. We find a condition under which efficient investment can be achieved as a Nash equilibrium. Finally we study relations between core elements and refinements of Nash equilibria.

2. Model and core

Let $N = \{1, 2, \ldots, n\}$ be the set of players and each player i has one type of resources which amounts to m_i units. Let M_i be the set of investment levels that player i can play where m_i means full investment. Denote $M_i = [0, m_i]$ when the resources are divisible, and $M_i = \{0, 1, 2, \ldots, m_i\}$ when the resources are indivisible. Suppose that there are k joint projects. Each player decides how to allocate his/her total amount m_i to each project. Denote player i's investment to project l by s_i^l. Note that, for each player i, the total quantity of investment does not exceed m_i, i.e., $\Sigma_{l=1}^k s_i^l \leq m_i$. Player i's action is an investment plan $s_i = (s_i^1, s_i^2, \ldots, s_i^k)$, and let the action space be A_i. Here each player's investment (profile) to project l can be denoted by $s^l = (s_1^l, s_2^l, \ldots, s_n^l)$. Each project l has a production function which assigns a real number to every s^l. Players share this value $v^l(s^l)$, that is the project l's profit from investment s^l.

For each $i \in N$, a function u_i: $A_i \to \mathbf{R}$ gives for every $s_i \in A_i$ the private utility of player i which is produced from his/her remaining resources $m_i - \Sigma_{l=1}^k s_i^l$.

Definition A *cooperative investment game* with k projects is given by $\mathcal{G}$ $= \left\langle N, m, (v^l)_{l=1}^k, (u_i)_{i \in N} \right\rangle$ in which

- N is the set of players,
- m is the vector describing the total quantity of resources for all players,
- v^l: $M_i \to \mathbf{R}$ is the production function of project l,
- u_i: $A_i \to \mathbf{R}$ is player i's private utility function.

In a cooperative investment game, $u_i(s_i)$ is non-transferable utility for each $s_i \in A_i$ and $i \in N$ whereas each $v^l(s)$ is transferable. Denote the investment that yields the maximum profit by $\hat{s}$, i.e., $\hat{s} \in \arg\max_s \{\Sigma_{l=1}^k v^l(s^l) + \Sigma_{i \in N} u_i(s_i)\}$, and call $\hat{s}$ an *efficient investment*. We now consider how to allocate $\Sigma_{l=1}^k v^l(s^l)$.

First we define imputations of this game in accordance with the commonly used definition of imputations for a multi-choice game (van den Nouweland et al. (1995)). For each l, a preimputation can be represented as a separable function x^l =

$(x_1^l, x_2^l, ..., x_n^l)$ whose element $x_i^l : A_i \to \mathbf{R}$ with $\Sigma_{i\in N} x_i^l(\hat{s}_i) = v^l(\hat{s}^l)$ that assigns to each s_i player i's gain x_i^l (s_i) when his/her investment level is s_i. Denote the set of preimputation by $P^l(\mathcal{G})$. We can define the preimputation set of cooperative investment game $\mathcal{G}$ by $P(\mathcal{G}) = \Pi_{l=1}^k P^l(\mathcal{G}) = \{x = (x^1, \ldots, x^k) \mid \Sigma_{l=1}^k \Sigma_{i\in N} x_i^l(s_i) = \Sigma_{l=1}^k v^l(\hat{s}^l)\}$.

A preimputation x is called *individually rational* if $\Sigma_{l=1}^k x_i^l(\hat{s}_i) - \Sigma_{l=1}^k x_i^l(s_i) \geq u_i(s_i) - u_i(\hat{s}_i)$ for all $s_i \in A_i$ and $i \in N$. In the case with $k = 1$, the individual rationality is quite similar to the level increase rationality of multi-choice games defined by van den Nouweland et al. (1995).

For each $x \in \Pi_{l=1}^k P^l(\mathcal{G})$, we can consider a non-cooperative game $\Gamma_x = \langle N, A, (\psi_i)_{i\in N} \rangle$ in which N is the set of players, $A = \Pi_{i\in N} A_i$ is the set of strategies, ψ_i is i's utility function defined by $\psi_i(s_i) = \Sigma_{l=1}^k x_i^l(s_i) + u_i(s_i)$.

Proposition *Let* $\mathcal{G} = \left\langle N, m, (v^l)_{l=1}^k, (u_i)_{i\in N} \right\rangle$ *be a cooperative investment game and* $\hat{s}$ *be an efficient investment of the game. If* $x \in \Pi_{l=1}^k P^l(\mathcal{G})$ *is individually rational then efficient investment* $\hat{s}$ *can be achieved as a dominant strategy equilibrium of* Γ_x.

The core for each game v^l can be defined by $C(v^l) = \{ x^l = (x_1^l, x_2^l, ..., x_n^l) \in P^l(\mathcal{G}) \mid \Sigma_{i\in N} x_i^l(s_i) \geq v^l(s^l)$ for all s^l }. We define the core of $\mathcal{G}$ by $C(\mathcal{G}) = \{ x \in P(\mathcal{G}) \mid$ x: individually rational and $x^l \in C(v^l)$ for all l=1, …, k }. It is shown that $C(\mathcal{G})$ is included in the set of undominated imputations (dominance core) of the game.

We can define a preimputation by a non-separable function $x^l = (x_1^l, x_2^l, ..., x_n^l)$ whose element $x_i^l : \Pi_{i\in N} A_i \to \mathbf{R}$ with $\Sigma_{i\in N} x_i^l(\hat{s}) = v^l(\hat{s}^l)$. We study relations between core elements and refinements of Nash equilibria such as strong equilibria, coalition-proof Nash equilibria.

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