

The Important Thing Is not (Always) Winning but Taking Part. Funding Public Goods with Contests

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Abstract

This paper considers a public good game with incomplete information affected by extreme free-riding. We overcome this problem through the implementation of a contest in which several prizes can be awarded. For any possible distribution of wealth we identify the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. At interior solutions the social planner sets the last prize equal to zero and the total expected welfare is independent of the distribution of the total prize sum among the other prizes. We prove that private provision via a contest Pareto-dominates both public provision and private provision via a lottery.

1 Introduction

This paper looks at contests as a way to overcome the free-riding problem. It is well known that simply asking each agent in an economy to contribute to the public good generally results in underprovision and might cause extreme free-riding (see for examples Bergstrom et al. 1986; Andreoni 1988). Different solutions have been offered. Groves and Ledyard (1977) and Walker (1981) proposed mechanisms to Nash implement first best public good allocations. Experimental evidence, though, does not support their results (Chen and Tang 1998). Harstad and Maresse (1982) examined an iterative version of the Groves and Ledyard mechanism, finding that Nash predictions perform badly. Further, contests as incentive mechanisms differ from the above solutions in that no tax or transfer power is required on the part of the organisation conducting the tournament. In the case of organisations with no coercive power such as charity or civic groups this difference may be fundamental.

Voluntary mechanisms with provision points (see Bagnoli and Lipman 1989; Admati and Perry 1991) can also lead to first-best outcomes. However, laboratory results show divergence from the equilibrium predictions (Ledyard 1995).

Morgan (2000) analyses lotteries as a means to solve the free-riding problem and Morgan and Sefton (2000) provide experimental results showing that contribution

under a lottery is generally higher than under the voluntary contribution mechanism. Both these papers only consider the case in which one prize is awarded. However, we will prove that a lottery with one prize performs worse than a contest, either with one or more prizes.

Contests are competitions in which agents spend resources in order to win one or more prizes. The main characteristic is that, independently of success, all participants bear some costs. Most of the economic literature on contests has focused on the case where only one prize is awarded. Moldovanu and Sela (2001) study a contest with several prizes and show that when costs are either linear or concave allocating only one prize maximises the effort exerted by the bidders; however, when costs are convex more prizes could be optimal. Although contests and tournaments have been largely analysed as incentive schemes¹, to my knowledge, they have not been studied as a means to overcome the free-riding problem and provide socially desirable public goods.

In this paper we present a theoretical model, which is different from the models presented in the literature and can be tested experimentally. We consider a public good game with incomplete information in which the agents' utility function is linear both in wealth and in the public good. Such a game is a modified version of the game with complete information which is typically employed in public good experiments. Each agent chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter, which takes a value between one and the number of players, and shared equally among all the agents. The unique Nash equilibrium is to contribute nothing, although it is socially optimum to contribute all the wealth. We overcome this extreme free-riding via a contest. We assume that the social planner has access to a budget, which can be allocated in form of prizes. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution and so on until all prizes are awarded. The social planner wants to maximise the expected total welfare net of the value of the total prize sum. For any possible distribution of wealth we identify the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. As we said, we assume the typical utility function that is used in laboratory experiments on public goods. Although it does not seem plausible that in real life people spend all their wealth in auctions or lotteries, in the framework we analyse we believe that wealth constraints are likely to be binding and studying interior solutions without identifying conditions for their existence might cause problems in the empirical application. We find that there exists a critical level of budget under which wealth constraints are non-binding for all agents. When the total prize sum is below such critical value the social planner sets the last prize equal to zero and the total expected contribution is independent of the distribution of the total prize sum among the other prizes. Interestingly, although the cost of providing the public good is linear, the result is strikingly different from Moldovanu and Sela's (2001). Further, provided interior solutions, we prove that private provision via a contest

¹Applications have been made to promotions in labour markets (Lazear and Rosen 1981), technological and research races (Wright 1983; Dasgupta 1986; Taylor 1995; Fullerton and McAfee 1999; Windham 1999), credit markets (Breocker 1990), and rent seeking (Tullock 1980) among others.

as the one we present Pareto-dominates both public provision and private provision via a lottery in which one prize is awarded.

In Section 2 we present a linear public good game with complete information. In Section 3 we present the model and identify the Nash equilibrium. In Section 4 we find necessary and sufficient conditions for interior solutions and we present the revenue equivalence result. Section 5 compares private provision via a contest with both public provision and private provision via a lottery. Section 6 concludes.

2 A Linear Public Good Game with Complete Information

In this Section we present the game that is typically used in public good experiments n subjects take part in the experiment. Each subject is endowed with the same amount of money z and simultaneously chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter α and shared equally among all the subjects. Agent i 's payoff can be described by

$$U_i = z - g_i + \alpha \frac{G}{n} \quad (1)$$

where g_i is i 's contribution to the public good and $G = \sum_{i=1}^n g_i$ is the total level of public good. If $\alpha \in (1, n)$ an individual opportunity cost of contributing to the public good exceeds the marginal return of investing in the public good. Thus, the unique Nash equilibrium of the game is to contribute nothing, while it is efficient to contribute z .

3 The Model

Let us consider n players. Each player i is assumed to have endowment z_i , which is private information. Endowments are drawn independently of each other from the interval $[0, 1]$ according to the distribution function $F(z)$, which is common knowledge, with mean $E[z]$. We assume that $F(z)$ has a continuous and bounded density $F'(z) > 0$. Players play a public good game in which each individual has to choose how much to contribute to the public good. At the same time they take part in a contest in which n prizes are awarded such that $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^n \pi_j = \Pi$. This assumption rules out the possibility of awarding n equal prizes and will enable us find an equilibrium. We will call $\pi = (\pi_1, \dots, \pi_n) \in \mathbb{R}^n$ the vector of prizes. The player with the highest contribution wins π_1 , the player with the second highest contribution wins π_2 , and so on until all the prizes are allocated. For each player, a strategy will be the contribution to the public good as a function of the player's endowment and the action space for player i will be the interval $[0, z_i]$. If player i , who has endowment

z_i and contributes g_i , wins prize j her payoff is

$$U_i = z_i - g_i + \alpha \frac{G}{n} + \pi_j \quad (2)$$

where $\alpha \in (1, n)$.

Each player i chooses her contribution in order to maximise expected utility (given the other players' contributions and given the values of the different prizes). We will assume that Π is exogenously determined. For a given value of Π , the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise the expected value of total welfare net of the value of Π (given the players' equilibrium strategy functions).

In this Section we will focus on the case in which the equilibrium strategy is less than z for any $z \in [0, 1]$.

In order to find the equilibrium of the game it is useful to present the function

$$K(F(z)) = \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \quad (3)$$

Given a vector of prizes π , $K(F(z))$ is a linear combination of n order statistics with weights equal to the prizes. If all agents adopt the same strictly increasing strategy $g(z)$, $K(F(z))$ represents the expected prize of the player with endowment z .

Lemma 1 *The function $K(F(z))$ is strictly monotonic increasing in z .*

Proof. Let's consider z_i and z_j such that $0 \leq z_i < z_j \leq 1$. Given that $F(z_i) < F(z_j)$, and given the assumption that $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$, $1 < m \leq n$, $K(F(z_j))$ assigns higher weights than $K(F(z_i))$ to higher prizes and lower weights than $K(F(z_i))$ to lower prizes. Therefore $K(F(z_i)) < K(F(z_j))$. ■

Proposition 1 *Given a vector π of prizes, at an interior solution for all players the game has a symmetric pure strategy equilibrium given by*

$$g(z) = \frac{n}{n-\alpha} (K(F(z)) - \pi_n)$$

Proof. The expected utility of a player from a choice g can be calculated as

$$E[U(z-g, \pi) \mid g, g_{-i}] = z - g + \alpha \frac{G}{n} + (\Pr[1 \mid g, g_{-i}] \pi_1 + \Pr[2 \mid g, g_{-i}] \pi_2 + \dots + \Pr[n \mid g, g_{-i}] \pi_n)$$

where $\Pr[j \mid g, g_{-i}]$ is the probability of a choice g being j -th highest conditional on the other strategies g_{-i} . If all agents adopt the same strictly increasing strategy $g(z)$, then the probability that a candidate with endowment z_i is higher ranked

than another randomly chosen candidate is $\Pr[g(z_i) > g(z)] = \Pr[z_i > z] = F(z_i)$. Therefore

$$\begin{aligned} & (\Pr[1 \mid g, g_{-i}]\pi_1 + \Pr[2 \mid g, g_{-i}]\pi_2 + \dots + \Pr[n \mid g, g_{-i}]\pi_n) = \\ K(F(z)) &= \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \end{aligned}$$

Now, given the common strategy $g(z)$, we suppose that an individual with endowment z chooses $g(\hat{z})$ for some \hat{z} , then her expected utility will be

$$z - g(\hat{z}) + \alpha \frac{G_{-i} + g(\hat{z})}{n} + K(F(\hat{z}))$$

where G_{-i} is the sum of the contributions of all the other players. Differentiating with respect to \hat{z} we obtain

$$\frac{\alpha - n}{n} g'(\hat{z}) + K'(F(\hat{z}))F'(\hat{z})$$

In equilibrium the individual with endowment z should choose $g(z)$ so that the above will be equal to zero when $\hat{z} = z$, and we have

$$g'(z) = \frac{n}{n - \alpha} K'(F(z))F'(z)$$

A player with the lowest possible endowment $z = 0$ does not contribute to the public good and wins the last prize. This yields the boundary condition $g(0) = 0$. Hence, the solution is

$$g(z) = \frac{n}{n - \alpha} (K(F(z)) - \pi_n)$$

From Lemma (1) we know that the candidate equilibrium function g is strictly monotonic increasing.

Assuming that all players rather than i play according to g , we finally need to show that, for any type z of player i , the contribution $g(z)$ maximises the expected utility of that type. Let us consider an individual with endowment z . If she plays $g(z) = \frac{n}{n-\alpha}(K(F(z)) - \pi_n)$ her expected utility is given by

$$E[U(z, g(z)) \mid g_{-i}] = z - \frac{\alpha}{n - \alpha} K(F(z)) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} G$$

If she deviates and plays $\frac{n}{n-\alpha}(K(F(\hat{z})) - \pi_n)$ for some $\hat{z} \neq z$ her expected utility will be

$$\begin{aligned} E[U(z, g(\hat{z})) \mid g_{-i}] &= \\ & z - \frac{n}{n - \alpha} K(F(\hat{z})) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} (G - \frac{n}{n - \alpha} K(F(z))) \\ & + \frac{n}{n - \alpha} \pi_n + \frac{n}{n - \alpha} K(F(\hat{z})) - \frac{n}{n - \alpha} \pi_n + K(F(\hat{z})) \\ & = z - \frac{\alpha}{n - \alpha} K(F(z)) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} G \end{aligned}$$

Therefore she is indifferent to play any other strategy $\frac{n}{n-\alpha}(K(F(\hat{z})) - \pi_n)$. If her action space $[0, z]$ is a subset of the set g_{-i} this rules out the possibility that she might be better off deviating from $g(z)$. If $z > g(1)$ it is easy to show that she would be worse off playing any strategy greater than $g(1)$. In fact, playing $g(1)$ would already guarantee π_1 and any higher contribution would result in a lower expected utility. ■

4 Interior Solutions and Revenue Equivalence

In this Section we are going to look for conditions that assure that the solution is interior for all players, given that the social planner wants to maximise the expected total welfare net of the value of the total prize sum. We will then analyse the expected total contribution when interior solutions are guaranteed.

First of all, let us present the social planner's maximisation problem, assuming that wealth constraints are non-binding for all players. Recall that Π is exogenously determined and the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise the expected value of total welfare net of the value of Π (given the players' equilibrium strategy functions). This means that, to analyse the maximisation problem we have let the vector of prizes π be variable, maintaining the assumptions that $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^n \pi_j = \Pi$, and we now have to study the family of functions

$$\begin{aligned} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \\ \dots = \pi_n \geq 0, 1 < m \leq n) = \\ \frac{n}{n-\alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \end{aligned} \quad (4)$$

Notice that, if π were fixed expression (4) would reduce to $K(F(z))$, as presented in (3).

Letting the vector of prizes π be variable, at an interior solution for all players, the equilibrium strategy is represented by the following²

$$\begin{aligned} g(z, \pi) &= \frac{n}{n-\alpha} (\phi(F(z), \pi) - \pi_n) = \\ &= \frac{n}{n-\alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} - \frac{n}{n-\alpha} \pi_n \end{aligned}$$

²For simplicity of notation, unless differently specified, from now on we will refer to $\phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0, 1 < m \leq n)$ as $\phi(F(z), \pi)$.

And the social planner's problem is given by

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n} G + \phi(F(z), \pi)) F'(z) dz - \Pi \quad (5)$$

Notice first that

$$n \int_0^1 \phi(F(z), \pi) F'(z) dz = \Pi \quad (6)$$

independently of the distribution of the total prize sum among the different prizes.

Further, notice that, at interior solutions, we have

$$G = n \int_0^1 g(z, \pi) F'(z) dz = \frac{n}{n - \alpha} \Pi - \frac{n^2}{n - \alpha} \pi_n$$

This means that the expected total contribution only depends on the total prize sum and the value of the last prize.

Therefore, we can rearrange expression (5) as

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n - \alpha} \Pi - \frac{\alpha n}{n - \alpha} \pi_n) F'(z) dz \quad (7)$$

and we can state the following result.

Proposition 2 *At an interior solution for all players the social planner will set $\pi_n = 0$.*

Proof. Expression (7) can be rewritten as

$$\begin{aligned} \max_{\pi} W &= n \int_0^1 (z - \frac{n}{n - \alpha} \phi(F(z), \pi) + \\ &\quad \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n - \alpha} \Pi - \\ &\quad \frac{\alpha n}{n - \alpha} \pi_n) F'(z) dz = n E[z] + \frac{n(\alpha - 1)}{n - \alpha} (\Pi - n \pi_n) \end{aligned} \quad (8)$$

It is obvious that $\pi_n = 0$ maximises the above expression. ■

Provided that at an interior solution for all players the social planner will set the last prize equal to zero, Proposition (9) in Appendix A provides necessary and sufficient conditions for the value of Π such that $g(z)$ is interior for any z on the interval $[0, 1]$ for any possible distribution of Π among the first $n - 1$ prizes. On the basis of this we can establish the following result.

Proposition 3 *Provided that the last prize is equal to zero, there exists $\bar{\Pi} > 0$ such that $g(z)$ is interior for all players independently of the distribution of the total prize sum among the first $n - 1$ prizes if and only if $\Pi \leq \bar{\Pi}$.*

Proof. See Appendix B. ■

As we noticed the expected total contribution only depends on the value of the total prize sum and on the last prize. We know, though, that if wealth constraints are non-binding for all players the social planner maximises the total expected welfare setting the last prize equal to zero. Therefore, at an interior solution the expected total contribution will be the same, independently of the distribution of the total prize sum among the first $n - 1$ prizes. This result is summarised by the following proposition.

Proposition 4 *If $\Pi \leq \bar{\Pi}$ the social planner will set the last prize equal to zero and the expected total contribution will be $G = \frac{n}{n-\alpha}\Pi$, independently of the distribution of the total prize sum among the first $n - 1$ prizes.*

Further, from the result above it is obvious that

Corollary 1 *If $\Pi \leq \bar{\Pi}$ the expected total contribution is strictly increasing in Π .*

5 Contest versus Public Provision and Lottery

In this Section, we will compare the result obtained through a contest as the one we described with both the result generated by public provision and the one obtained using a lottery.

When socially desirable public goods are not privately provided the obvious alternative is to publicly provide them. Let us imagine that the social planner has access to a budget equal to $\Pi \leq \bar{\Pi}$. Instead of allocating this sum in form of prizes the social planner provides an amount of public good equal to Π . We want to compare the expected total welfare generated by such public provision with the expected total welfare resulting from the use of a contest, where the social planner awards up to $n - 1$ prizes which sum is equal to Π .

Proposition 5 *Private provision of public good via a contest, in which the total sum prize $\Pi \leq \bar{\Pi}$ is distributed among the $n - 1$ players who contribute the most, Pareto-dominates public provision. If the social planner uses $\Pi \leq \bar{\Pi}$ to publicly provide the public good the expected total welfare net of the value of Π is $W^P = nE[z] + (\alpha - 1)\Pi$.*

Proof. If the social planner uses $\Pi \leq \bar{\Pi}$ to provide the public good the expected total welfare net of the value of Π is given by

$$W^P = n \int_0^1 (z + \frac{\alpha}{n}\Pi) F'(z) dz - \Pi = nE[z] + (\alpha - 1)\Pi \quad (9)$$

From expression (8) we know that, if the last prize is equal to zero, the expected total welfare generated by a contest is equal to

$$W = nE[z] + \frac{n(\alpha - 1)}{n - \alpha}\Pi$$

that is strictly greater than (9). ■

We now consider the case where the social planner resorts to a lottery to incentivize the contribution of the public good. To be able to compare the use of a lottery with the use of a contest we will have to restrict the analysis to interior solutions. To do this let us assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with \underline{z} strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that the social planner decides to award the sum Π via a lottery with the following properties. If a player i with endowment z_i contributes $g_i \in [0, z_i]$ and the sum of the contributions of all other players is equal to G_{-i} she wins Π with probability $\frac{g_i}{g_i + G_{-i}}$ and her expected utility is given by

$$E[U(z_i - g_i, \Pi) \mid g_i, G_{-i}] = z_i - g_i + \alpha \frac{G_{-i} + g_i}{n} + \frac{g_i}{g_i + G_{-i}} \Pi$$

Differentiating with respect to g and setting this equal to zero we have

$$\frac{\alpha - n}{n} + \frac{G_{-i}}{(g_i + G_{-i})^2} \Pi = 0$$

Assuming that the total contribution is different from zero³ and rearranging we obtain player i 's best response function

$$g_i^* = -G_{-i} + \sqrt[2]{\frac{n}{n - \alpha} \Pi G_{-i}} \quad (10)$$

From (10) we can write an expression for the total contribution when player i plays according to her best response

$$G(g_i^* \mid G_{-i}) = \sqrt[2]{\frac{n}{n - \alpha} \Pi G_{-i}}$$

Although the endowment is private information, notice that z does not enter the first order condition. Each player will have the same best response function and at interior solutions the contribution in equilibrium will be the same for any z . Therefore, assuming interior solutions, we know that g_i^* will be $\frac{G(g_i^* \mid G_{-i})}{n}$ and we can write

$$g_i^* = \frac{\sqrt[2]{\frac{n}{n - \alpha} \Pi G_{-i}}}{n} \quad (11)$$

Setting (10) and (11) equal we obtain an expression for G_{-i} when any player plays according to her best response function

$$G_{-i}^* = \frac{(n - 1)^2}{n(n - \alpha)} \Pi$$

³Notice that in equilibrium the total contribution will not be zero. In fact, if any other player different from i contributes zero, player i will contribute ε strictly positive and arbitrarily close to zero and win the prize.

Hence we know that in equilibrium, at an interior solution for all players, all agents will play

$$g^* = \frac{n-1}{n(n-\alpha)}\Pi$$

And the total contribution in equilibrium, at an interior solution for all players, will be

$$G^* = \frac{n-1}{n-\alpha}\Pi$$

It is easy to see that if $\Pi \leq \frac{n(n-\alpha)}{n-1}\underline{z}$ the solution will be interior for all players. These results are summarised in the following proposition.

Proposition 6 *Assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with \underline{z} strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that z is private information. If $\Pi \leq \frac{n(n-\alpha)}{n-1}\underline{z}$ the lottery has a symmetric pure strategy equilibrium in which any player contributes $g^* = \frac{n-1}{n(n-\alpha)}\Pi$ and the total contribution is $G^* = \frac{n-1}{n-\alpha}\Pi$*

It is interesting to notice that under such a lottery, unlike the contest, all players contribute the same amount in equilibrium.

Further, notice that in order to prove Proposition (8) in Appendix A we have not resorted to the support of z and that the same conditions guarantee that the solution will be interior for all players in the case in which endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with \underline{z} strictly positive, according to the distribution function $F(z)$, which is common knowledge, and which has a continuous and bounded density $F'(z) > 0$. Under a contest as the one described, provided that the social planner sets the last prize equal to zero, the total contribution is given by

$$\begin{aligned} G &= n \int_{\underline{z}}^{\bar{z}} \frac{n}{n-\alpha} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \\ &\cdots \geq \pi_n, \pi_n = 0) F'(z) dz = \frac{n}{n-\alpha} \Pi \end{aligned}$$

Hence, we can conclude that, for Π that guarantees interior solutions for all players under both mechanisms, the expected total contribution raised with a contest is greater than the total contribution under a lottery, for any finite n .

Proposition 7 *Assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with \underline{z} strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that z is private information and that $F(z)$ has a continuous and bounded density $F'(z) > 0$. If $\Pi \leq \min[\frac{n(n-\alpha)}{n-1}\underline{z}, \bar{\Pi}]$, the expected total provision of public good via a contest, in which the total sum prize is distributed among the $n-1$ players who contribute the most, is greater than the total contribution raised under a lottery.*

6 Conclusions

Different theoretical solutions to the free-riding problem have been offered and some of them are contradicted by experimental results. Although contests have been widely studied as incentive schemes they have not been analysed as a means to overcome free-riding. We presented a theoretical model that can be easily tested experimentally, in which a contest with multiple prizes is used to incentivate individuals to contribute to the public good.

We considered the case in which the social planner has access to a budget, and uses it to run a contest. We found that there exists a critical level of budget under which wealth constraints are non-binding for all agents. For any possible distribution of wealth we identified the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. When the total prize sum is below such critical value the social planner sets the last prize equal to zero and the total expected contribution is independent of the distribution of the total prize sum among the other prizes. Provided interior solutions, we proved that a contest Pareto-dominates public provision of the public good and performs better than a lottery. An interesting extension to the present work would be to test experimentally whether individuals actually contribute more in a contest than in a lottery and whether the revenue equivalence holds.

Appendix A: Necessary and Sufficient Conditions

We want to find necessary and sufficient conditions for the value of Π such that $g(z)$ is interior for any z on the interval $[0, 1]$ for any possible allocation of Π among the first $n - 1$ prizes. Notice in fact that, assuming interior solutions, Proposition (2) assures us that the social planner will set $\pi_n = 0$.

If we let the vector of prizes π be variable, provided that the last prize is equal to zero and that the sum of the first $n - 1$ prizes is equal to Π , $g(z)$ is represented by the following⁴

$$\frac{n}{n - \alpha} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0) = \frac{n}{n - \alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1 - F(z))^{i-1}$$

Let us define the following object.

Definition 1 *Define the envelope function*

$$V(z) = \max_{\pi} \{ \phi(F(z), \pi) \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0 \}$$

for any z on the interval $[0, 1]$.

⁴Notice that, unlike the rest of the paper, both in Appendix A and Appendix B, when writing $\phi(F(z), \pi)$ we will refer to $\phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0)$.

If we are able to provide necessary and sufficient conditions for $V(z)$ to be weakly less than z for any z on the interval $[0, 1]$, it will be easy to extend the result to $g(z)$. In order to do this we will define some useful concepts that will help us in the course of our analysis.

Definition 2 For any i such that $1 \leq i \leq n - 1$:

1) define the set $Q^i \subset \mathbb{R}^n$ such that for every $\pi \in Q^i$ it holds that $\pi_1 \geq \dots \geq \pi_i > \pi_{i+1} = \dots = \pi_n = 0$ and $\sum_{l=1}^i \pi_l = \Pi$.

2) call $\bar{\pi}^i$ the vector $\pi \in Q^i$ such that $\pi_1 = \dots = \pi_i = \frac{\Pi}{i}$.

Definition 3 For any i such that $2 \leq i \leq n - 1$ define the set $\tilde{Q}^i \subset Q^i$ such that for every $\pi \in \tilde{Q}^i$ it holds that $\pi_1 > \pi_i$.

Obviously $\bar{\pi}^1 \in Q^1$, characterised by $\pi_1^1 = \Pi, \pi_l^1 = 0$ for $2 \leq l \leq n$, is the only element of the set Q^1 and $\phi(F(z), \bar{\pi}^1) = \Pi(F(z))^{n-1}$.

The next Proposition presents necessary and sufficient conditions for $V(z)$ to be weakly less than z on the interval $[0, 1]$.

Proposition 8 $\phi(F(z), \bar{\pi}^i) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n - 1$ are necessary and sufficient conditions for $V(z) \leq z$.

Proof. The necessity of these conditions is obvious. In order to prove sufficiency we will have to present some technical results.

Lemma 2 Given a vector $\pi^R \in \mathbb{R}^n$ such that $\sum_{j=1}^n \pi_j^R = \Pi$ and $\pi_1^R \geq \dots \geq \pi_n^R, \pi_n^R = 0$, consider a redistribution of the type $-\Delta\pi_i^R = \Delta\pi_{i+1}^R$, with $1 \leq i \leq n - 1$ and $\Delta\pi_i^R > 0$, and call the resulting vector π^S . Then, $\phi(F(z), \pi^S) > \phi(F(z), \pi^R)$ for any z such that $F(z) < \frac{n-i}{n}$ and $\phi(F(z), \pi^S) < \phi(F(z), \pi^R)$ for any z such that $F(z) > \frac{n-i}{n}$.

Proof. Notice that $\frac{\partial\phi(F(z), \pi)}{\partial\pi_i} = \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}$. To see how a redistribution of the type $-\Delta\pi_i = \Delta\pi_{i+1}$ affects $\phi(F(z), \pi)$ we have to study the sign of

$$\begin{aligned} & -\frac{\partial\phi(F(z), \pi)}{\partial\pi_i} + \frac{\partial\phi(F(z), \pi)}{\partial\pi_{i+1}} \\ &= (F(z))^{n-i}(1-F(z))^{i-1} \left(-\binom{n-1}{i-1}(F(z)) \right. \\ & \quad \left. + \binom{n-1}{i}(1-F(z)) \right) \end{aligned} \tag{12}$$

It is the case that expression (12) > 0 for any z such that $F(z) < \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$ and (12) < 0 for any z such that $F(z) > \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$. Further, it is easy to show that

$$\frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}} = \frac{\frac{(n-1)!}{i!(n-1-i)!}}{\frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-1)!}{i!(n-1-i)!}} = \frac{n-i}{n}$$

■

Lemma 3 Assume $1 \leq i \leq n - 2$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$. If $2 \leq i \leq n - 2$ then $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$. If $i = 1$ then $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$.

Proof. Let us first consider the case in which $2 \leq i \leq n - 2$. The vector $\bar{\pi}^{i+1}$ can be obtained from vector π^B applying the following algorithm in i steps.

Algorithm 1 Step 1. From vector π^B construct vector π^{B1} such that $\pi_1^{B1} = \frac{\Pi}{i+1}, \pi_2^{B1} = \pi_2^B + \pi_1^B - \frac{\Pi}{i+1}, \pi_j^{B1} = \pi_j^B, 3 \leq j \leq i + 1$. Given that $\pi_2^B \geq \pi_3^B$ it will now be the case that $\pi_2^{B1} > \pi_3^{B1} \geq \dots \geq \pi_{i+1}^{B1}$. Therefore $\frac{\Pi}{i+1} + i\pi_2^{B1} > \Pi$. The last inequality can be rewritten as $\pi_2^{B1} > \frac{\Pi}{i+1}$, therefore we can move to the next step and repeat the process.

Step j , with $2 \leq j \leq i - 1$. From vector π^{Bj-1} construct vector π^{Bj} such that $\pi_j^{Bj} = \frac{\Pi}{i+1}, \pi_{j+1}^{Bj} = \pi_{j+1}^{Bj-1} + \pi_j^{Bj-1} - \frac{\Pi}{i+1}, \pi_l^{Bj} = \pi_l^{Bj-1}$ for $1 \leq l \leq j - 1$ and $j + 1 \leq l \leq i + 1$. Given that $\pi_{j+1}^{Bj-1} \geq \pi_{j+2}^{Bj-1}$ it will now be the case that $\pi_{j+1}^{Bj} > \pi_{j+2}^{Bj} \geq \dots \geq \pi_{i+1}^{Bj}$. Therefore it is the case that $2\frac{\Pi}{i+1} + (i + 1 - j) \pi_{j+1}^{Bj}$. Rearranging the last inequality we obtain $\pi_{j+1}^{Bj} > \frac{\Pi}{i+1}$. This means that we can move to the next step and repeat the process.

Step i . From vector π^{Bi-1} construct vector π^{Bi} such that $\pi_i^{Bi} = \frac{\Pi}{i+1}, \pi_{i+1}^{Bi} = \pi_{i+1}^{Bi-1} + \pi_i^{Bi-1} - \frac{\Pi}{i+1}, \pi_l^{Bi} = \pi_l^{Bi-1}$ for $1 \leq l \leq i - 1$. Notice that $\pi_l^{Bi-1} = \frac{\Pi}{i+1}$ for $1 \leq l \leq i - 1$. Therefore $\pi^{Bi} = \bar{\pi}^{i+1}$.

Notice that from Lemma 2 we know that $\phi(F(z), \pi^{Bj}) > \phi(F(z), \pi^{Bj-1})$ for any z such that $F(z) < \frac{n-j}{n}$ for $1 \leq j \leq i$. Therefore $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$, that contradicts our assumption.

Consider now the case in which $i = 1$. Notice that $\bar{\pi}_1^2 < \pi_1^B$ and $\bar{\pi}_2^2 > \pi_2^B$. Applying the same algorithm as above from π^B we will obtain $\bar{\pi}_2^2$ after the first step. Applying Lemma 2 we know that $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further, from Lemma 2 we also know that $\phi(F(z), \pi \mid \pi \in Q^2) > \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(F(z), \pi \mid \pi \in Q^2) < \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) > \frac{n-1}{n}$. Therefore, by continuity, we can conclude that $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$. ■

Lemma 4 Assume $2 \leq i \leq n - 2$. $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ and for $1 \leq j \leq i$.

Proof. The structure of this proof is in three parts.

First of all, from Lemma 3 we know that $\phi(F(z), \bar{\pi}^j) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-j-1}{n}$ and, given that $1 \leq j \leq i$, for any z such that $F(z) \leq \frac{n-i}{n}$.

For the second part of the proof, let us first assume $j = 1$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$. We want to show that $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) \leq \frac{n-i}{n}$.

If $2 \leq j \leq i$, consider a vector $\pi^B \in \tilde{Q}^{i+1}$ such that $\pi_l^B = \bar{\pi}_l^j$ for $1 \leq l \leq j - 1$. Notice that, obviously, $\pi_j^B < \bar{\pi}_j^j$. We want to show that

$\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ if $1 \leq j \leq i-1$ and for any z such that $F(z) < \frac{n-i}{n}$ if $j = i$.

Vector π^B can be obtained from $\bar{\pi}^j$ through the following algorithm in $i+1-j$ steps.

Algorithm 2 Step 1. If $j = 1$, from vector $\bar{\pi}^1$ construct vector $\pi^{A1} \in \tilde{Q}^2$ such that $\pi_1^{A1} = \pi_1^B$ and $\pi_2^{A1} = \Pi - \pi_1^B$. If $2 \leq j \leq i$, from vector $\bar{\pi}^j$ construct vector $\pi^{A1} \in \tilde{Q}^{j+1}$ such that $\pi_l^{A1} = \bar{\pi}_l^j = \frac{\Pi}{j}$ for $1 \leq l \leq j-1$, $\pi_j^{A1} = \pi_j^B$ and $\pi_{j+1}^{A1} = \bar{\pi}_j^j - \pi_j^B = \frac{\Pi}{j} - \pi_j^B$.

Step k , with $2 \leq k \leq i-j$. From vector π^{Ak-1} construct vector $\pi^{Ak} \in \tilde{Q}^{j+k}$ such that $\pi_l^{Ak} = \pi_l^{Ak-1}$ for $1 \leq l \leq j = k-2$, $\pi_{j+k-1}^{Ak} = \pi_{j+k-1}^B$ and $\pi_{j+k}^{Ak} = \pi_{j+k-1}^{Ak-1} - \pi_{j+k-1}^B$.

Step $i+1-j$. From vector π^{Ai-j} construct vector $\pi^{Ai+1-j} \in \tilde{Q}^{i+1}$ such that $\pi_l^{Ai+1-j} = \pi_l^{Ai-j}$ for $1 \leq l \leq i-2$, $\pi_i^{Ai+1-j} = \pi_i^B$ and $\pi_{i+1}^{Ai+1-j} = \pi_i^{Ai-j} - \pi_i^B$. Notice that $\pi_{i+1}^{Ai+1-j} = \pi_{i+1}^B$ and $\pi^{Ai+1-j} = \pi^B$ by construction.

From Lemma 2 we know that $\phi(F(z), \pi^{Ak}) > \phi(F(z), \pi^{Ak-1})$ for any z such that $F(z) < \frac{n+1-k}{n}$. Therefore if $1 \leq j \leq i-1$ then $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$. If $j = i$ then $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) < \frac{n-i}{n}$ and $\phi(\frac{n-i}{n}, \pi^B) = \phi(\frac{n-i}{n}, \bar{\pi}^j)$.

Finally, from Lemma 3 we know that $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$. Therefore $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-1}{n}$. ■

Lemma 5 Assume $2 \leq i \leq n-2$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$ such that $\pi_1^B > \pi_j^B$, with $2 \leq j \leq i$. Assume a vector $\pi^C \in \tilde{Q}^{i+1}$ such that $\pi_l^C = \pi_l^B$ for $j+1 \leq l \leq i+1$

and $\pi_1^C = \dots = \pi_j^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. If $3 \leq j \leq n-2$ then $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-j+1}{n}$. If $j = 2$ then $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$.

Proof. Notice that $\pi_1^B > \pi_1^C$ and $\pi_j^B < \pi_j^C$. Vector π^C can be obtained from vector π^B applying the following algorithm in $j-1$ steps.

Algorithm 3 Step 1. From vector π^B construct vector π^{B1} such that $\pi_1^{B1} = \pi_1^C$, $\pi_2^{B1} = \pi_2^B + \pi_1^B - \pi_1^C$, $\pi_l^{B1} = \pi_l^B$ for $3 \leq l \leq i+1$. Given that $\pi_2^B \geq \pi_3^B$ it will now be the case that $\pi_2^{B1} > \pi_3^{B1} \geq \dots \geq \pi_{i+1}^{B1}$. Therefore $\pi_1^{B1} + (j-1)\pi_2^{B1} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$. Since

$$\pi_1^{B1} = \pi_1^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}, \text{ the last inequality can be rearranged as } \pi_2^{B1} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}.$$

Therefore we can move to the next step and repeat the process.

Step k , $2 \leq k \leq j-2$. From vector π^{Bk-1} construct vector π^{Bk} such that $\pi_k^{Bk} = \pi_k^C$, $\pi_{k+1}^{Bk} = \pi_{k+1}^{Bk-1} + \pi_k^{Bk-1} - \pi_k^C$, $\pi_l^{Bk} = \pi_l^{Bk-1}$ for $1 \leq l \leq k-1$ and

$k+2 \leq l \leq i+1$. Notice that, by construction $\pi_l^{Bk} = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ for $1 \leq l \leq k$ and

$\pi_l^{Bk} = \pi_l^B$ for $k+2 \leq l \leq i+1$. Given that $\pi_{k+1}^{Bk-1} \geq \pi_{k+2}^{Bk-1}$ it will now be the case that $\pi_{k+1}^{Bk} > \pi_{k+1}^{Bk} \geq \dots \geq \pi_{i+1}^{Bk}$. Therefore $\frac{k}{j}(\Pi - \sum_{l=j+1}^{i+1} \pi_l^B) + (j-k)\pi_{k+1}^{Bk} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$.

The last inequality can be rearranged as $\pi_{k+1}^{Bk} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. Therefore we can move to the next step and repeat the process.

Step $j-1$. From vector π^{Bj-2} construct vector π^{Bj-1} such that $\pi_{j-1}^{Bj-1} = \pi_{j-1}^C, \pi_j^{Bj-1} = \pi_j^{Bj-2} + \pi_{j-1}^{Bj-2} - \pi_{j-1}^C, \pi_l^{Bj-1} = \pi_l^{Bj-2}$ for $1 \leq l \leq j-2$ and $j+1 \leq l \leq i+1$. Notice that $\pi^{Bj-1} = \pi^C$ by construction.

From Lemma 2 we know that $\phi(F(z), \pi^{Bk}) > \phi(F(z), \pi^{Bk-1})$ for any z such that $F(z) < \frac{n-k}{n}$ for $1 \leq k \leq j-1$. This means that if $3 \leq i \leq n-3$ then, by construction, we will have $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-j+1}{n}$. If $i=2$ then j will necessarily be equal to 3 and, by construction, we will have $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further it will be the case that $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$. ■

Lemma 6 Consider a vector $\pi^C \in \tilde{Q}^{i+1}$ such that $\pi_{i+1}^C = x, \pi_j^C = \frac{\Pi-x}{i}$ with $0 < x < \frac{\Pi}{i+1}$ for $1 \leq j \leq i$ and $2 \leq i \leq n-2$. If $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ then $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$.

Proof. The inequality $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ can be rewritten as

$$\begin{aligned} & \frac{\Pi-x}{i}((F(z))^{n-1} + \dots + \\ & \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}) + \\ & x \binom{n-i}{i}(F(z))^{n-i-1}(1-F(z))^i - \\ & \frac{\Pi}{i+1}((F(z))^{n-1} + \dots + (F(z))^{n-i-1}(1-F(z))^i) > 0 \end{aligned}$$

The above expression can be rearranged as

$$\begin{aligned} & \left(\frac{\Pi-x}{i} - \frac{\Pi}{i+1}\right)((F(z))^{n-1} + \dots + \\ & \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}) > \\ & \left(\frac{\Pi}{i+1} - x\right)(F(z))^{n-i-1}(1-F(z))^i \end{aligned} \tag{13}$$

Call A the expression $((F(z))^{n-1} + \dots + \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1})$ and call B the expression $(F(z))^{n-i-1}(1-F(z))^i$. Inequality (13) is satisfied for $\frac{A}{B} > i$.

The inequality $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$ can be rewritten as

$$\frac{\Pi}{i}A - \frac{\Pi-x}{i}A - xB > 0 \tag{14}$$

Inequality (14) is satisfied for $\frac{A}{B} > i$. ■

From Lemma (4) we know that $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ and for $2 \leq i \leq n-2$ and $1 \leq j \leq i$. In particular, this means that $V(z)$ will be equal to $\phi(F(z), \bar{\pi}^{n-1})$ for any z such that $0 \leq F(z) \leq \frac{2}{n}$. For those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$ we will have to check the family of functions $\phi(F(z), \pi \mid \pi \in Q^{n-1})$ and $\phi(F(z), \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n-3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$ we will have to check the families of functions $\phi(F(z), \pi \mid \pi \in Q^j)$ for $i+2 \leq j \leq n-1$ and the function $\phi(F(z), \bar{\pi}^{i+1})$.

Consider now a vector $\pi^C \in Q^{i+1}$ such that $\pi_1^C = \dots = \pi_i^C > \pi_{i+1}^C$, for $2 \leq i \leq n-2$. From Lemma (5) we know that, for those z such that $\frac{n-i}{n} < F(z) \leq \frac{n-i+1}{n}$, the function $\phi(F(z), \pi^C)$ is greater than any other function of the family $\phi(F(z), \pi \mid \pi \in Q^{i+1})$ with exclusion of $\phi(F(z), \bar{\pi}^{i+1})$.

From Lemma (6) though, we know that if $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ then it is the case that $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$.

Therefore, in order to find the envelope function $V(z)$ for those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$, it will be sufficient to check the two functions $\phi(F(z), \bar{\pi}^{n-1})$ and $\phi(F(z), \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n-3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$ we will have to check the functions $\phi(F(z), \bar{\pi}^j)$ for $i+1 \leq j \leq n-1$.

From this follows that $\phi(F(z), \bar{\pi}^i) \leq z$ for $1 \leq i \leq n-1$ are sufficient conditions for $V(z) \leq z$ on the interval $[0, 1]$. ■

Finally, given Proposition (8), by continuity we can establish the following result.

Proposition 9 *Provided that the last prize is equal to zero, $g(z)$ is interior for any z on the interval $[0, 1]$ independently of the distribution of Π among the first $n-1$ prizes if and only if $\frac{n}{n-\alpha} \phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n-1$.*

Appendix B

Proof of Proposition (3). Given a vector π^A such that $\pi_1^A \geq \dots \geq \pi_n^A, \pi_n^A = 0$ and $\sum_{j=1}^n \pi_j^A = \Pi^A$ consider π^B such that $\pi_i^B = c\pi_i^A$ for any $1 \leq i \leq n, c > 1$. Notice that $\sum_{j=1}^n \pi_j^B = \Pi^B = c\Pi^A$. Since $\phi(F(z), \pi^B) = c\phi(F(z), \pi^A)$ we conclude that $\phi(F(z), \pi^B) > \phi(F(z), \pi^A)$ for any $z \in (0, 1]$.

Consider Π such that $\frac{n}{n-\alpha} \phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) > z$ for $z \in (a, b)$ with $0 \leq a < b \leq 1$ for i such that $1 \leq i \leq n-1$. Given that $F(z)$ has a continuous and bounded density there exists $c > 1$ such that $\frac{n}{n-\alpha} \phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \frac{\Pi}{c}) \leq z$ for any $z \in [0, 1]$ and i such that $1 \leq i \leq n-1$. Therefore, by continuity there

exists $\bar{\Pi} > 0$ such that $\frac{n}{n-\alpha}\phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n - 1$ if and only if $\Pi \leq \bar{\Pi}$.

Given the result presented in Proposition (8) we can conclude that, provided that the last prize is equal to zero, $\Pi \leq \bar{\Pi}$ is a necessary and sufficient condition for the solution to be interior for all players independently of the distribution of Π among the first $n - 1$ prizes. ■

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