Ex Post Implementation in Environments with Private Goods

by

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Abstract

We prove by construction that ex post incentive compatible mechanisms exist in auctions when buyers have multi-dimensional signals and interdependent values. The mechanism shares features with the generalized Vickrey auction of single dimensional signal models; thus, ex post equilibrium in these models is robust to departures from a single dimensional information assumption. The construction implies that for environments with private goods, informational externalities (i.e., interdependent values) are compatible with ex post equilibrium in the presence of multi-dimensional signals.

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1 Introduction

In models of mechanism design with interdependent values, each player's information is usually modeled as a real number. While this is convenient, it might not capture a significant element of the setting. For instance, suppose that agent A's reservation value for an object is the sum of a private value, which is idiosyncratic to this agent, and a common value, which is the same for all agents in the model. Agent A's private information consists of an estimate of the common value and a separate estimate of his private value. As other agents care only about A's estimate of the common value, a single dimensional statistic will not capture all of A's private information that is relevant to every agent (including A).[1]

Therefore, it is essential to test whether insights from the literature are robust to relaxing the assumption that an agent's private information is a real number. Building on earlier work by Maskin (1992), Jehiel and Moldovanu (2001) show that if agents have multi-dimensional information, interdependent values, and independent signals then, unlike in models with single dimensional information, every Bayesian Nash equilibrium is (generically) inefficient.[2] Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2005) call into question the existence of ex post equilibrium when agents have multi-dimensional information and interdependent values. They show that ex post incentive compatible mechanisms do not generically exist (except, of course, trivial mechanisms which disregard the reports of players).

Our paper shows that non-trivial ex post incentive compatible mechanisms exist in auctions when buyers have interdependent values and multi-dimensional signals. To reconcile this with Jehiel et al.'s result, we note that their non-existence result depends on the assumption that for any pair of outcomes there exist at least two agents who are not indifferent between that pair of outcomes. If, as is usually assumed, buyers in an auction care only about their own allocation then this assumption is not satisfied. To see this, consider the auction of one indivisible object to two buyers, 1 and 2. There are three possible outcomes: a_i, the good is assigned to buyer i, $i = 1, 2$, and a_0, neither gets the good. Buyer 1 is indifferent between a_2 and a_0 and buyer 2 is

[1] A d-dimensional, $d \geq 2$, private signal s_A can be mapped, without any loss of information, into a single dimension using a one-to-one function $f : \Re^d \to \Re$. However, agents' values will not be non-decreasing or continuous in the signal $f(s_A)$. Hence, the assumption of single dimensional signals is a limitation only in conjunction with assumptions commonly made in the literature that a buyer's (one-dimensional) signal is ordered so that a higher realization is more favorable, or that the valuation is is a continuous function of the signal.

[2] See also Harstad, Rothkopf, and Waehrer (1996), who obtain sufficient conditions under which an efficient allocation is attained by common auction forms. Postlewaithe & McLean (2004) show that efficient Bayesian implementation is possible when signals are correlated.

indifferent between a_1 and a_0. There exist pairs of outcomes (namely (a_0, a_1) and also (a_0, a_2)) between which all agents except one is indifferent. Consequently, auctions are non-generic in the space of social choice settings considered by Jehiel et al. and their generic non-existence result does not apply. Therefore, even if buyers have multi-dimensional signals, the possibility that there exists a non-trivial selling mechanism in which truth-telling is ex post incentive compatible is not precluded. What is ruled out is the existence of a non-trivial ex post incentive compatible mechanism with outcomes a_1 and a_2 only, as neither buyer is indifferent between these two outcomes.

We prove by construction an existence result for ex post incentive compatible mechanisms for the sale of a single indivisible object to n buyers with multi-dimensional signals and interdependent values. In the construction, the rule for deciding whether buyer 1, say, should be assigned the object is as follows. Fix the other buyers' signals at some realization. Partition buyer 1's set of possible signal realizations into equivalence classes such that buyer 1's reservation value is constant on an equivalence class. These equivalence classes are completely ordered by buyer 1's value. If a generalization of the single crossing property is satisfied then there exists a pivotal equivalence class (of buyer 1 signals) with the property that it is ex post incentive compatible to award the object to buyer 1 if and only if buyer 1's signal realization is in an equivalence class which is greater than the pivotal class. If buyer 1 wins, the price paid by him is equal to his value in the pivotal equivalence class (which is also equal to the maximum of other buyers' values on buyer 1's pivotal equivalence class). This mechanism can be extended to multi-object auctions when buyer preferences over objects are subadditive. Moreover, when the efficient allocation rule is non-trivial, this mechanism is also non-trivial.

The mechanism shares the feature with the generalized Vickrey auction of single dimensional information models that the price paid by the winning buyer is equal to this buyer's value at the lowest possible signal (i.e., the pivotal equivalence class) at which this buyer would just win. Thus, ex post equilibria in auction models with one dimensional models are robust in that non-trivial ex post equilibria exist even when buyers have multi-dimensional signals.

In a multi-dimensional signal setting the pivotal equivalence class for a buyer consists of a continuum of this buyer's signal realizations whereas in a one dimensional setting there is exactly one pivotal signal realization for this buyer, for a given value of other buyers' signal realizations. Consequently, when the highest two buyer values are close to each other no buyer's signal is above his pivotal equivalence class. This ensures that the subset of buyers' signals in which one buyer gets the object does not share a common boundary with the (disjoint) subset of buyers' signals in which another buyer gets the object.[3] The social cost of incentive compatibility in our model is that the object is retained by the auctioneer and gains from trade are not realized when the highest two buyer valuations are close to each other.

There are only private goods in our model. Thus, in environments with private goods, informational externalities (i.e., interdependent values) alone do not preclude the existence of ex post equilibrium in the presence of multi-dimensional signals. One needs consumption externalities or public goods, in addition to information externalities, for non-existence. As ex post equilibrium has been employed mostly in auction models with private goods, this is not a significant limitation.[4]

The paper is organized as follows. A model with two buyers is presented in Section 2, together with preliminary results. An existence result for ex post incentive compatible mechanisms is proved in Section 3. This result is generalized to n buyers in Section 3.1. We show in Section 4 that non-trivial mechanisms exist in multi-object auctions when buyers have subadditive preferences. Section 5 concludes.

2 The model

The main idea can be seen in a model with two buyers and one indivisible object. Each buyer i, $i = 1, 2$, receives a $d^i \geq 2$ dimensional private signal, denoted $s_i = (s_{i1}, s_{i2}, ..., s_{id^i})$. The domain of s_i is $S_i = [0, 1]^{d^i}$, the unit cube in $\Re^{d^i}_+$. The buyers' signals are denoted $s = (s_1, s_2)$ with domain $S = S_1 \times S_2$. We also refer to s as an *information state*. At information state (s_i, s_j), buyer i's reservation value for the object is $V_i(s_i, s_j)$ [also denoted $V_i(s)$]. Buyers have quasilinear utility. If buyer i gets the object in state s and pays t, then his utility is $V_i(s) - t$; if he does not get the object and pays t, his utility is $-t$.

Denote by a_i, $i = 1, 2$, the outcome in which buyer i is allocated the object. The outcome in which no buyer gets the object is denoted a_0. A (deterministic) mechanism consists of an allocation rule h and two payment functions $\hat{t}_i$, $i = 1, 2$. The allocation rule $h : S \to \{a_0, a_1, a_2\}$ is a function from the buyers' reported signals to an allocation of the indivisible object to either no buyer or buyer 1 or buyer 2; the payment function $\hat{t}_i : S \to \Re$ is a function from the buyers' reported signals to a money payment by buyer i. A mechanism is *ex post incentive compatible* if for

[3] It is precisely the existence of such a common boundary that is used by Jehiel et al. to show the non-existence of ex post incentive compatible mechanisms in a setting in which auctions are non-generic.

[4] See, for example, Cremer and McLean (1985), Ausubel (1999), Dasgupta and Maskin (2000), Perry and Reny (2002), and Bergemann and Valimaki (2002).

$i = 1, 2,\ i \neq j$,

$$V_i(s_i, s_j)1_{\{h(s_i,s_j)=a_i\}} - \hat{t}_i(s_i, s_j) \ \geq\ V_i(s_i, s_j)1_{\{h(s'_i,s_j)=a_i\}} - \hat{t}_i(s'_i, s_j), \qquad \forall s_i,\ \forall s'_i,\ \forall s_j \tag{1}$$

where 1_A denotes the indicator function of the event A. In other words, at each information state if buyer j truthfully reports his signal then buyer i can do no better than truthfully report his signal.[5]

It is well-known that ex post incentive compatibility implies that the money payment made by buyer i depends only on (i) whether or not buyer i is assigned the object and (ii) buyer j's reported signal, $j \neq i$. We will restrict attention to mechanisms in which a buyer pays nothing if he does not get the object; that is, $\hat{t}_i(s_i, s_j) = 0$ if $h(s) \neq a_i$.[6] Consequently, we write the money payment function as

$$\hat{t}_i(s_i, s_j) \ \equiv\ \begin{cases} t_i(s_j), & \text{if } h(s_i, s_j) = a_i, \\ 0, & \text{otherwise.} \end{cases}$$

The function $t_i(s_j)$ is buyer i's payment conditional on getting the object. We interpret $t_i(s_j)$ as a *personalized price* at which the object is available to buyer i. For mechanisms in which losing buyers pay nothing, the requirement for ex post incentive compatibility, i.e., condition (1), may be rewritten as follows. For $i = 1, 2,\ i \neq j$,

$$\Big[V_i(s_i, s_j) - t_i(s_j)\Big]1_{\{h(s_i,s_j)=a_i\}} \ \geq\ \Big[V_i(s_i, s_j) - t_i(s_j)\Big]1_{\{h(s'_i,s_j)=a_i\}}, \qquad \forall s_i,\ \forall s'_i,\ \forall s_j. \tag{2}$$

A pair of personalized price functions $t_i(s_j)$, $t_j(s_i)$ is *admissible* if

$$V_i(s_i, s_j) \ >\ t_i(s_j) \ \Longrightarrow\ V_j(s_i, s_j) \ \leq\ t_j(s_i), \qquad \forall s_i, s_j. \tag{3}$$

That is, in each information state at most one buyer's value exceeds his personalized price. An allocation rule *implements* an admissible pair of prices t_1, t_2 if the rule assigns the object to a buyer if and only if the buyer's value exceeds his personalized price. That is, the allocation rule

$$h(s_1, s_2) \ \equiv\ \begin{cases} a_1, & \text{if } V_1(s_1, s_2) > t_1(s_2) \\ a_2, & \text{if } V_2(s_2, s_1) > t_2(s_1) \\ a_0, & \text{otherwise.} \end{cases} \tag{4}$$

implements the admissible pair t_1, t_2. Clearly, h is a feasible allocation rule in that it does not allocate more than one object. Observe that a buyer cannot change his

[5] Ex post incentive compatibility is the same as uniform equilibrium of D'Aspremont and Gerard-Varet (1979) and uniform incentive compatibility of Holmstrom and Myerson (1983).

[6] By adding to $\hat{t}_i(s_i, s_j)$ a lump sum payment $\phi_i(s_j)$ one can get ex post implementable mechanisms where this restriction does not hold. From (i) and (ii) it follows that relaxing this restriction on payment rules does not increase the set of allocation rules that are ex post implementable.

personalized price by lying about his private signal, as each buyer's price depends only on the other buyer's (reported) signal. If buyers report their signals truthfully, then at each information state (s_i, s_j) the mechanism (h, t) allocates the object to buyer i for a payment of $t_i(s_j)$ if and only if $V_i(s_i, s_j) > t_i(s_j)$. Suppose that the information state is (s_i, s_j) and buyer i reports $s_i' \neq s_i$. If he gets the same allocation at (s_i, s_j) and (s_i', s_j) then the lie is not profitable. Therefore, suppose that $h(s_i, s_j) \neq a_i$ and $h(s_i', s_j) = a_i$. But then $V_i(s_i, s_j) \leq t_i(s_j)$ and with a report of s_i' buyer i buys at a price at least as large as his value for the object. Similarly, if $h(s_i, s_j) = a_i$ and $h(s_i', s_j) \neq a_i$ then with a report of s_i' he ends up not buying the object at a price strictly less than his value. Thus, (h, t) satisfies (2) [and (1)] and is ex post incentive compatible. We have:

Lemma 1: *If an allocation rule h implements an admissible pair of personalized prices $t = (t_1, t_2)$ then the mechanism (h, t) is ex post incentive compatible.*

A mechanism is *non-trivial* if there exist two distinct outcomes, each of which is implemented at a positive (Lebesgue) measure of information states by the mechanism.

It is possible to satisfy (3) by choosing personalized prices so high that each buyer's valuation is always less than his personalized price. Such prices lead to the trivial ex post incentive compatible mechanism in which $h(s) = a_0$, $\forall s$. In Section 3, we show that under reasonable conditions on buyers' information, there exists an admissible pair of personalized prices which is implemented by a non-trivial ex post incentive compatible mechanism. In particular, each of the outcomes a_0, a_1, and a_2 occur at a positive measure of information states. First, we illustrate a non-trivial mechanism in an example from Jehiel et al. (2005).

Example 1: Two buyers compete for a single indivisible object. Each gets a pair of signals (p_i, c_i), $i = 1, 2$. Buyer i's valuation for the object is $V_i(p_i, c_i, p_j, c_j) = p_i + c_i c_j$, $j \neq i$. Further, each buyer's signal lies in the unit square: $(p_i, c_i) \in [0, 1]^2$, $i = 1, 2$.

Consider personalized prices $t_i(p_j, c_j) \equiv p_j + c_j^2,\ t_j(p_i, c_i) \equiv p_i + c_i^2$. Suppose that

$$V_i(p_i, c_i, p_j, c_j) \ = \ p_i + c_i c_j \ > \ p_j + c_j^2 \ = \ t_i(p_j, c_j).$$

Then

$$p_i - p_j \ > \ c_j^2 - c_i c_j \ \geq \ c_i c_j - c_i^2,$$

where we use the fact that $c_i^2 + c_j^2 - 2c_i c_j \geq 0$. Therefore,

$$V_j(p_j, c_j, p_i, c_i) \ = \ p_j + c_i c_j \ < \ p_i + c_i^2 \ = \ t_j(p_i, c_i).$$

Consequently, personalized prices $p_2 + c_2^2$ for buyer 1 and $p_1 + c_1^2$ for buyer 2 are admissible; that is they satisfy (3).

Using (4), define an allocation rule which implements these prices. In this mechanism, the buyers report their private signals to the mechanism designer. The mechanism designer allocates the object to buyer i for a payment equal to his personalized price $t_i(p_j, c_j) = p_j + c_j^2$ if and only if $V_i(p_i, c_i, p_j, c_j) = p_i + c_i c_j$ exceeds $t_i(p_j, c_j)$. By Lemma 1, this mechanism is ex post incentive compatible.

Let $h^{-1}(a_i)$ be the set of information states which are mapped on to a_i by this allocation mechanism. Each of the sets

$$\begin{aligned} h^{-1}(a_i) &= \{(p_i, c_i, p_j, c_j) \in [0,1]^4 \,|\, p_i - p_j > c_j^2 - c_i c_j\}, \quad i = 1, 2 \\ h^{-1}(a_0) &= \{(p_i, c_i, p_j, c_j) \in [0,1]^4 \,|\, c_i c_j - c_i^2 \leq p_i - p_j \leq c_j^2 - c_i c_j\} \end{aligned}$$

is of positive measure. Hence, the mechanism is non-trivial.[7]

The boundary between the sets $h^{-1}(a_1)$ and $h^{-1}(a_2)$ is:

$$\overline{h^{-1}(a_1)} \cap \overline{h^{-1}(a_2)} = \{(p_1, c_1, p_2, c_2) \in [0,1]^4 \,|\, p_1 = p_2, \; c_1 = c_2 = 0\}$$

where $\overline{A}$ is the closure of set A. This boundary is a one dimensional set and the projection (onto buyer i's signal space) of boundary points with a fixed value of buyer j signal $(p_j, c_j) = (p, 0)$ is the single point $(p_i, c_i) = (p, 0)$. We shall return to this below.

There exists a continuum of non-trivial ex post incentive compatible mechanisms in this example. Consider personalized prices $t'_i(p_j, c_j) = p_j + c_j^2 + \epsilon_i(p_j, c_j)$ where $\epsilon_i(p_j, c_j)$ is non-negative. Since $t_i(p_j, c_j) = p_j + c_j^2$ satisfy (3), so do the prices $t'_i(p_j, c_j)$. An allocation rule that implements t'_1, t'_2 is ex post incentive compatible. For small enough $\epsilon_i(p_j, c_j)$, this mechanism is non-trivial. $\triangle$

We summarize the main result of Jehiel et al. (2005). Consider a setting with two agents, 1 and 2, and two outcomes, a_1 and a_2. Each agent i gets a d^i-dimensional signal. Define $\mu^i(s)$, $i = 1, 2$, to be the difference between i's utility for outcomes a_1 and a_2.[8] The domain of signals, S, is shown schematically in Figure 1a. Any allocation rule partitions S into two subsets, depending on whether $h(s) = a_1$ or $h(s) = a_2$. The boundary between these two sets is the broken line in Figure 1a. Jehiel et al. show that for any non-trivial allocation rule (i) the projection of points on this boundary

[7] In fact, if $(p_i, c_i) \neq (0, 0)$ and $(p_i, c_i) \neq (1, 1)$ then each of the outcomes a_0, a_1, and a_2 is implemented for a positive Lebesgue measure of buyer j signals.

[8] Thus, in the above example, we restrict attention to mechanisms which allocate the object to a buyer at every information state s and $\mu^1(s) = V_1(s)$, $\mu^2(s) = -V_2(s)$.

with fixed value of s_j onto the domain of i's signals, S_i, $i = 1, 2$, is a $d^i - 1$ dimensional submanifold[9] and (ii) for any ex post incentive compatible allocation rule the gradients of $\mu^1(s)$ and $\mu^2(s)$ must be, roughly speaking, co-directional on this submanifold. For generic $\mu^i(s)$ it is impossible to satisfy (i) and (ii). Thus, when there are two agents and two outcomes, non-trivial ex post incentive compatible mechanisms do not exist for generic utilities.

The argument summarized in the preceding paragraph depends on the assumption that each agent is not indifferent between the two outcomes a_1 and a_2. Suppose we add a third outcome, a_0. For this argument to extend it must be case that each agent is not indifferent between any two of the three outcomes. This assumption is violated in Example 1. Therefore, consider a setting with two agents and three outcomes a_0, a_1, and a_2 such that $a_0(s) \sim_1 a_2(s)$, $\forall s$ (i.e., agent 1 is indifferent between a_0 and a_2 in every information state) and $a_0(s) \sim_2 a_1(s)$, $\forall s$. This condition is satisfied in Example 1.[10] Now consider a non-trivial allocation rule h that yields each of the three outcomes a_0, a_1, and a_2. Jehiel et al.'s theorem implies that if h is ex post incentive compatible then the boundary between the $h^{-1}(a_1)$ and $h^{-1}(a_2)$ has less than full dimension. Their theorem does not impose any restriction on the dimensionality of the boundary between a_0 and a_i, $i = 1, 2$. In particular, the possibility that h partitions S as shown in Figure 1b is not ruled out. In fact, this figure is a schematic representation of Example 1 where we demonstrated existence of an ex post incentive compatible mechanism in which the boundary between $h^{-1}(a_1)$ and $h^{-1}(a_2)$ is a one dimensional set in S and the projection onto S_i of points on the boundary with a fixed value of s_j is a single point.

More generally, suppose there are $i = 1, 2, ..., n$ agents and L outcomes labeled a_ℓ, $\ell = 1, 2, ..., L$. Suppose that for some outcome a_ℓ there exists an outcome a_k and an agent j (where a_k may depend on a_ℓ and j may depend on a_ℓ and a_k) such that

$$a_\ell(s) \sim_i a_k(s), \quad \forall s, \ \forall i \neq j. \tag{5}$$

Then, for any ex post incentive compatible allocation rule h, the Jehiel et al. theorem places no restriction on the boundary between $h^{-1}(a_\ell)$ and $h^{-1}(a_k)$. Generic existence of a non-trivial ex post incentive compatible mechanism with outcomes a_ℓ and a_k is not precluded when (5) is satisfied.

Consider the allocation of a bundle of private goods to n buyers. Each outcome is an assignment of objects among the n buyers, where we allow the possibility that not all objects are allocated to the buyers. Let a_ℓ be any assignment and let a_k be

[9] Hereafter, this condition is referred to as the boundary is of full dimension.

[10] In Example 1, apart from mechanisms with three outcomes described above, there also exist ex post incentive compatible mechanisms with two outcomes a_0 and, say, a_1.

another assignment which differs from a_ℓ only in the allocation that buyer j receives.[11] Condition (5) is satisfied for each assignment a_ℓ. A full range ex post incentive compatible mechanism is a possibility.[12]

In Section 4 we show that non-trivial ex post incentive compatible mechanisms exist in multi-object auctions. First, we prove a general existence result for such mechanisms in single object auctions.

3 The main result

We prove an existence theorem for non-trivial ex post incentive compatible mechanisms for the allocation of a single object when buyers have multi-dimensional signals. The starting point is the model with two buyers described in Section 2. The extension to n buyers is straightforward and sketched out in Section 3.1.

We assume that higher signals correspond to better news. That is, players' reservation values do not decrease as buyer signals increase.[13] In order to simplify the proofs, we also assume that buyers' reservation values are continuous.

Assumption 1a: *$V_i(s)$ is non-decreasing in s, $i = 1, 2$.*
1b: *$V_i(\cdot)$ is continuous in all its arguments.*

The next assumption is a generalization of the single crossing property.

Assumption 2: *For any s_j we have*[14]

$$V_i(s_i', s_j) - V_i(s_i, s_j) \;\; \geq \;\; V_j(s_j, s_i') - V_j(s_j, s_i), \qquad \forall s_i' > s_i.$$

As buyer i's signal increases from s_i to s_i', the increase in i's value is greater than the increase in buyer j's value. That is, buyer i's value is more sensitive than buyer j's value to changes in buyer i's signal. In a model with one dimensional signals, Assumption 2 is a version of the single crossing property, which is a sufficient condition for existence of an efficient mechanism in such models (see Maskin 1992).

[11] Thus, not all the objects are allocated in at least one of the two assignments a_ℓ, a_k.

[12] A mechanism has full range if each outcome is implemented at a positive measure of information states.

[13] The following terminology regarding monotonicity of a function $f : \Re^n \to \Re$ is adopted. For $x,\ x' \in \Re^n$, $x' > x$ denotes that x' is at least as large as x in every co-ordinate and $x' \neq x$. If $f(x') \geq f(x)$ whenever $x' > x$ then f is *non-decreasing*.

[14] An equivalent assumption is that for each s_j, $V_i(s_i, s_j) - V_j(s_j, s_i)$ is a non-decreasing function of s_i.

The next assumption rules out the uninteresting case where the efficient rule is trivial. As any trivial rule is ex post incentive compatible, if assumption 3 is violated then the efficient rule is ex post incentive compatible.

Assumption 3: *For each buyer, there exists a positive measure of information states at which this buyer's valuation is strictly greater than the other buyer's valuation.*

Using Assumptions 1 and 2, we construct a personalized price function for each buyer. This pair of price functions is shown to be admissible. Assumption 3 is then employed to show that the ex post incentive compatible mechanism which implements the pair of admissible prices is non-trivial.

Fix buyer j's signal at some level s_j. The domain of s_i, $i \neq j$, is the unit cube in $\Re_+^{d^i}$ and each buyer's valuation is non-decreasing in s_i. Therefore, with buyer j's signal fixed at s_j, the maximum of either buyer's reservation value as a function of buyer i's signal is attained when $s_i = \mathbf{1}$, where $\mathbf{1}$ denotes the point $(1, 1, ..., 1)$ in $\Re_+^{d^i}$. Similarly, the minimum of either buyer's value as a function of s_i is attained at $s_i = \mathbf{0} \equiv (0, 0, ..., 0)$. Define $S_i(\lambda, s_j)$, the set of signals of buyer i which lead to the same reservation value for buyer i as the signal $\hat{s}_i = \lambda\mathbf{1}$, where $\lambda \in [0, 1]$. That is,

$$S_i(\lambda, s_j) \;\equiv\; \{s_i \in S_i \,|\, V_i(s_i, s_j) = V_i(\lambda\mathbf{1}, s_j)\,\}, \qquad 0 \leq \lambda \leq 1.$$

Thus, for a fixed s_j, buyer i's signal space partitions into equivalence classes or "indifference" curves, $S_i(\lambda, s_j)$, one for each $\lambda \in [0, 1]$. These equivalence classes are naturally ordered by λ as larger λ leads to larger buyer i reservation values.

While buyer i's value (as a function of s_i) is constant on $S_i(\lambda, s_j)$, buyer j's value will, in general, not be constant on this set. The maximum of buyer j's value on buyer i's equivalence class $S_i(\lambda, s_j)$ is

$$V_{ij}^m(\lambda, s_j) \;\equiv\; \max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i).$$

As S_i is compact and $V_i(\cdot, s_j)$ is continuous, $S_i(\lambda, s_j)$ is compact. This, together with the continuity of $V_j(s_j, \cdot)$ implies that $V_{ij}^m(\lambda, s_j)$ exists. Further, the continuity of $V_i(\cdot, \cdot)$ and $V_j(\cdot, \cdot)$ implies that $V_{ij}^m(\lambda, s_j)$ is continuous in λ and s_j. Let $s_{ij}^m(\lambda, s_j)$ be

$$s_{ij}^m(\lambda, s_j) \;\in\; \arg\max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i).$$

Thus, $V_{ij}^m(\lambda, s_j) = V_j(s_j, s_{ij}^m(\lambda, s_j))$ and $V_i(s_{ij}^m(\lambda, s_j), s_j) = V_i(\lambda\mathbf{1}, s_j)$. Observe that

$$V_j(s_j, s_i) \;\leq\; V_{ij}^m(\lambda, s_j), \qquad \forall s_i \in S_i(\lambda, s_j). \tag{6}$$

For a fixed realization of s_j, Figure 2 depicts indifference curves (i.e., equivalence classes) of buyers i and j in buyer i's (two dimensional) signal space. By Assumption 1a, indifference curves will be negatively sloped. However, (i) the indifference curves need not be convex; (ii) indifference curves may touch the axes (as is the case in Example 1); and (iii) $V_{ij}^m(\lambda, s_j)$, the maximum value of buyer j in buyer i's indifference curve $S_i(\lambda, s_j)$, may be attained at more than one point in $S_i(\lambda, s_j)$.

Ex post incentive compatibility imposes the following necessary condition. If buyer 1, say, is allocated the object at information state (s_1, s_2), then he should also be allocated the object at any information state (s_1', s_2) such that $V_1(s_1', s_2) > V_1(s_1, s_2)$. Otherwise, buyer 1 would have an incentive to report s_1 instead of s_1' at the information state (s_1', s_2). That is, an incentive compatible allocation rule must be weakly monotone.[15] Or in the terminology of equivalence classes, if $s_1 \in S_1(\lambda, s_2)$ and buyer 1 is allocated the object at (s_1, s_2), then buyer 1 must be allocated the object at all $s_1' \in S_1(\lambda', s_2)$ where $\lambda' > \lambda$.

We construct an ex post incentive compatible mechanism in which, for each value of s_j, there exists a $\lambda_{ij}^*(s_j) \in [0, 1]$ such that buyer i wins if his signal is in an equivalence class greater than $\lambda_{ij}^*(s_j)$, and buyer i loses otherwise. Clearly, this allocation rule satisfies weak monotonicity. Call $S_i(\lambda_{ij}^*(s_j), s_j)$ the *pivotal* equivalence class for buyer i at s_j. Any s_i in the pivotal equivalence class is a *pivotal signal* for buyer i. Buyer i's personalized price is defined to be $V_{ij}^m(\lambda_{ij}^*(s_j), s_j)$, the maximum of buyer j's value in buyer i's pivotal equivalence class (and this is usually equal to buyer i's value on the pivotal equivalence class). These personalized prices will be admissible if $V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda, s_j)$ is non-decreasing in λ. This is shown in the next lemma.

Lemma 2: *If Assumptions 1 and 2 are satisfied then for any s_j and $1 \geq \lambda' > \lambda > 0$,*

$$V_i(\lambda'\mathbf{1}, s_j) - V_{ij}^m(\lambda', s_j) \geq V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda, s_j).$$

Proof: To simplify notation, we write $s_{ij}^m(\lambda')$, $s_{ij}^m(\lambda)$ for $s_{ij}^m(\lambda', s_j)$, $s_{ij}^m(\lambda, s_j)$.

Let $\lambda^m \in [0, 1]$ be such that $\lambda^m s_{ij}^m(\lambda') \in S_i(\lambda, s_j)$. To see that λ^m exists, define $f(x) \equiv V_i(x s_{ij}^m(\lambda'), s_j)$, where $x \in [0, 1]$ and note that $f(1) = V_i(s_{ij}^m(\lambda'), s_j) = V_i(\lambda'\mathbf{1}, s_j) \geq V_i(\lambda\mathbf{1}, s_j) \geq V(0, s_j) = f(0)$. By Assumption 1b, $f(x)$ is a continuous function of x, and therefore there exists λ^m such that $f(\lambda^m) = V_i(\lambda^m s_{ij}^m(\lambda'), s_j) = V_i(\lambda\mathbf{1}, s_j)$. (As shown in Figure 2, λ^m is on the line joining $S_i^m(\lambda')$ to the origin.) Hence,

$$V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda, s_j) \quad = \quad V_i(\lambda^m s_{ij}^m(\lambda'), s_j) - V_{ij}^m(\lambda, s_j)$$

[15] See Bikhchandani et al. (2005) for conditions under which weak monotonicity is also sufficient for incentive compatibility.

$$\begin{aligned} &\leq V_i(\lambda^m s^m_{ij}(\lambda'), s_j) - V_j(s_j, \lambda^m s^m_{ij}(\lambda')) \\ &\leq V_i(s^m_{ij}(\lambda'), s_j) - V_j(s_j, s^m_{ij}(\lambda')) \\ &= V_i(\lambda'\mathbf{1}, s_j) - V^m_{ij}(\lambda', s_j) \end{aligned}$$

where the first inequality follows from the fact that $\lambda^m s^m_{ij}(\lambda') \in S_i(\lambda, s_j)$ and (6), and the second inequality from Assumption 2. ∎

For $\lambda \in [0,1]$, define

$$g_{ij}(\lambda; s_j) \equiv V_i(\lambda\mathbf{1}, s_j) - V^m_{ij}(\lambda, s_j).$$

Lemma 2 implies that $g_{ij}(\lambda; s_j)$ is a non-decreasing function of λ. The continuity of V_i and of V^m_{ij} implies that $g_{ij}(\lambda; s_j)$ is continuous (in λ). Thus, the following is well-defined:

$$\lambda^*_{ij}(s_j) \equiv \begin{cases} 1, & \text{if } g_{ij}(1; s_j) < 0, \\ \max\{\lambda \in [0,1] \mid g_{ij}(\lambda; s_j) = 0\}, & \text{if } g_{ij}(1; s_j) \geq 0 \geq g_{ij}(0; s_j), \\ 0^-, & \text{if } g_{ij}(0; s_j) > 0, \end{cases}$$

where 0^- is a negative number arbitrarily close to 0. Hence, $V_i(\lambda\mathbf{1}, s_j) > V^m_{ij}(\lambda, s_j)$ if and only if $\lambda > \lambda^*_{ij}$.[16] Define $V^m_{ij}(0^-, s_j) = V^m_{ij}(0, s_j)$. Then, as $V^m_{ij}(\lambda, s_j)$ is non-decreasing in λ, we have

$$V_i(\lambda\mathbf{1}, s_j) > V^m_{ij}(\lambda^*_{ij}, s_j) \text{ if and only if } V_i(\lambda\mathbf{1}, s_j) > V^m_{ij}(\lambda, s_j) \text{ if and only if } \lambda > \lambda^*_{ij}. \tag{7}$$

Let

$$t^*_i(s_j) \equiv V^m_{ij}(\lambda^*_{ij}, s_j) \tag{8}$$

be buyer i's personalized price as a function of s_j.[17]

Let $\lambda_i(s_i, s_j)$ be the index of the equivalence class that s_i belongs to at s_j. That is, $s_i \in S_i(\lambda_i(s_i, s_j), s_j)$. The main result shows that the following allocation rule is non-trivial and ex post incentive compatible: buyer i wins if and only if $V_i(s_i, s_j) > V^m_{ij}(\lambda_i(s_i, s_j), s_j)$. This rule is implemented through the personalized prices defined above.

Theorem 1: *The personalized prices $t^* = (t^*_1, t^*_2)$ defined in (8) are admissible. The mechanism (h^*, t^*), where h^* implements t^*, is non-trivial and ex post incentive compatible.*

[16] Hereafter, the dependence of λ^*_{ij} on s_j is usually suppressed to simplify the notation.

[17] Note that if $\lambda^*_{ij} \in [0,1)$ then $V_i(\lambda^*_{ij}\mathbf{1}, s_j) = V^m_{ij}(\lambda^*_{ij}, s_j)$ and therefore $t^*_i(s_j) = V_i(\lambda^*_{ij}\mathbf{1}, s_j)$. If $\lambda^*_{ij} = 0^-$ then $V_i(\mathbf{0}, s_j) > V^m_{ij}(0, s_j) = t^*_i(s_j)$ and if $\lambda^*_{ij} = 1$ then $V_i(\mathbf{1}, s_j) \leq V^m_{ij}(1, s_j) = t^*_i(s_j)$.

Proof: Suppose that the information state is (s_1, s_2). We write λ_i instead of $\lambda_i(s_i, s_j)$ to simplify the notation. That is, $V_i(s_i, s_j) = V_i(\lambda_i \mathbf{1}, s_j)$. Note that (6) implies

$$V_2(s_2, s_1) \ \leq \ V_{12}^m(\lambda_1, s_2), \qquad V_1(s_1, s_2) \ \leq \ V_{21}^m(\lambda_2, s_1). \tag{9}$$

Suppose that $V_1(s_1, s_2) > t_1^*(s_2) = V_{12}^m(\lambda_{12}^*, s_2)$. By (7), $\lambda_1 > \lambda_{12}^*$ and $V_1(s_1, s_2) = V_1(\lambda_1 \mathbf{1}, s_2) > V_{12}^m(\lambda_1, s_2)$. Hence, (9) implies that $V_{21}^m(\lambda_2, s_1) > V_2(s_2, s_1) = V_2(\lambda_2 \mathbf{1}, s_1)$. From (7) we have $\lambda_{21}^* \geq \lambda_2$ and, therefore, $V_2(s_2, s_1) = V_2(\lambda_2 \mathbf{1}, s_1) \leq V_{21}^m(\lambda_{21}^*, s_1) = t_2^*(s_1)$.

An identical argument implies that if, instead, $V_2(s_1, s_1) > t_2^*(s_1)$, then $V_1(s_1, s_2) \leq t_1^*(s_2)$. Thus, the personalized prices satisfy (3) and, by Lemma 1, the allocation rule

$$h^*(s_1, s_2) \ \equiv \ \begin{cases} a_1, & \text{if } V_1(s_1, s_2) > t_1^*(s_2) \\ a_2, & \text{if } V_2(s_2, s_1) > t_2^*(s_1) \\ a_0, & \text{otherwise} \end{cases}$$

which implements admissible prices $t_1^*(s_2)$, $t_2^*(s_1)$ is feasible and ex post incentive compatible.

To complete the proof, we show that (h^*, t^*) is non-trivial. Let information state $s^1 = (s_1^1, s_2^1)$ be such that $V_1(s_1^1, s_2^1) > V_2(s_1^1, s_2^1)$. Assumption 3 guarantees that there exists a positive measure of such information states. By Assumption 2, $V_1(\mathbf{1}, s_2^1) > V_2(s_2^1, \mathbf{1})$ and by Assumption 1a, $V_2(s_2^1, \mathbf{1}) = V_{12}^m(1, s_2^1)$. Thus, $V_1(\mathbf{1}, s_2^1) > V_{12}^m(1, s_2^1)$ and $\lambda_{12}^*(s_2^1) < 1$. Hence buyer 1 gets the object at (s_1, s_2^1) for all $s_1 \in S_1(\lambda, s_2^1)$, $\lambda \in (\lambda_{12}^*(s_2^1), 1]$. As there are a positive measure of information states at which buyer 1's value is strictly greater than buyer 2's value, there is a positive of information states at which buyer 1 is allocated the object. A similar argument establishes that buyer 2 is allocated the object at a positive measure of information states. Hence, the mechanism is non-trivial. ∎

If the mechanism assigns the object to a buyer then this buyer must have the highest reservation value. To see this, suppose that buyer i is allocated the object at information state $s = (s_i, s_j)$. Let $\lambda_i(s_i, s_j)$ and $\lambda_{ij}^*(s_j)$ be defined at this state in the usual manner. Then, from the proof of Theorem 1 it is clear that $\lambda_i(s_i, s_j) > \lambda_{ij}^*(s_j)$ and therefore $V_i(s_i, s_j) > V_{ij}^m(\lambda_i(s_i, s_j), s_j) \geq V_j(s_j, s_i)$.

Neither buyer is allocated the object when their valuations are close to each other. This occurs at information states $s = (s_i, s_j)$ such that $\lambda_i(s) < \lambda_{ij}^*(s_j)$ and $\lambda_j(s) < \lambda_{ji}^*(s_i)$. Such information states have positive measure unless either (i) buyers' indifference curves in S_i space for each fixed s_j, $i = 1, 2$, $i \neq j$ are identical or (ii) if Assumption 3 is not satisfied. From Jehiel and Moldovanu (2001) we know that any incentive compatible mechanism, including this one, is (generically) inefficient. The source of the inefficiency in the current mechanism is the cost of enforcing incentives when buyer valuations are close.

Recall that any signal s_i in the pivotal equivalence class $S_i(\lambda^*_{ij}, s_j)$ is a pivotal signal for buyer i at s_j. With buyer j's signal fixed at s_j, buyer i wins (loses) at signals greater (less) than a pivotal signal. Thus a pivotal signal is an infimum of winning signals. The price paid by a winning buyer i equals the valuation of this buyer at a pivotal signal (provided that $\lambda^*_{ij} \in [0, 1)$). This is similar to the efficient mechanisms in Ausubel (1999) and Dasgupta and Maskin (2000), where buyers have one dimensional signals.[18] However, unlike in these models, in the mechanism of Theorem 1 the valuations of buyers i and j need not be equal at a pivotal signal of buyer i; at a pivotal signal for buyer i, buyer i's valuation equals the most that buyer j's valuation can be in the pivotal equivalence class of buyer i. The difference arises because in one dimensional models, equivalence classes (or indifference curves) of buyer i signals are singletons and hence for a given realization of s_j there can be only one pivotal signal for buyer i. A second difference is the role of the single crossing property or Assumption 2. With one dimensional signals, the single crossing property is sufficient for existence of an efficient ex post incentive compatible mechanism whereas with multi-dimensional signals Assumption 2 is sufficient for the existence of an ex post incentive compatible mechanism mechanism (which is inefficient).

3.1 Extension to many buyers

We outline the minor changes in notation, assumptions, and analysis required to extend Theorem 1 to many buyers.

Each buyer's valuation depends on the (possibly multi-dimensional) signals of all n buyers. The information states are denoted $s = (s_1, s_2, ..., s_n) = (s_i, s_{-i})$. Change s_j to s_{-i} in Assumption 2, and require the assumption to hold for every V_i and V_j. Assumption 3 is required to hold for two distinct buyers, i.e., there exist two sets of information states, A_i and A_j, each set of positive measure, such that buyer i's [j's] value is strictly greater than all other buyers' values on the set A_i [A_j].

We write $V_i(s_i, s_{-i})$, $V^m_{ij}(\lambda, s_{-i})$, $g_{ij}(\lambda; s_{-i})$ instead of $V_i(s_i, s_j)$, $V^m_{ij}(\lambda, s_j)$, $g_{ij}(\lambda; s_j)$,

[18] There is one difference in Dasgupta and Maskin (2000). The mechanism designer (auctioneer) does not know the mapping from buyer signals to valuations. Hence, buyers submit contingent bids rather than report their private signals.

etc. The definition of $\lambda^*_{ij}(s_{-i})$ is:

$$\lambda^*_{ij}(s_{-i}) \quad \equiv \quad \begin{cases} 1, & \text{if } g_{ij}(1;s_{-i}) < 0, \\ \max\{\lambda \in [0,1] \mid g_{ij}(\lambda;s_{-i}) = 0\}, & \text{if } g_{ij}(1;s_{-i}) \geq 0 \geq g_{ij}(0;s_{-i}), \\ 0, & \text{if } g_{ij}(0;s_{-i}) > 0. \end{cases}$$

where $g_{ij}(\lambda;s_{-i}) \equiv V_i(\lambda\mathbf{1}, s_{-i}) - V^m_{ij}(\lambda, s_{-i})$. Buyer i's personalized price is

$$t^*_i(s_{-i}) \quad \equiv \quad \max_{j \neq i} V^m_{ij}(\lambda^*_{ij}(s_{-i}), s_{-i}) \tag{10}$$

Once again, buyer i's personalized price equals the maximum valuation of all other buyers on the pivotal equivalence class (which equals i's valuation at a pivotal signal whenever $\max_{j\neq i} \lambda^*_{ij}(s_{-i}) \in [0,1)$).

4 Many buyers and many objects

There are n buyers indexed by i or j, and K objects indexed by k or ℓ. Each buyer i receives a d^i dimensional signal $s_i \in [0,1]^{d^i}$. Buyer i's valuation for object k alone is $V^k_i(s_i, s_{-i})$; his valuation for a subset $L \subseteq \{1, 2, ..., K\}$ is denoted $V^L_i(s_i, s_{-i})$. Each buyer's preferences over subsets of objects are subadditive (defined in (13) below).

We write $S^k_i(\lambda, s_{-i})$, $V^{m,k}_{ij}(\lambda, s_{-i})$, $g^k_{ij}(\lambda; s_{-i})$, $\lambda^k_{ij}(s_{-i})$ instead of $S_i(\lambda, s_j)$, $V^m_{ij}(\lambda, s_j)$, $g_{ij}(\lambda; s_j)$, $\lambda^*_{ij}(s_{-i})$, etc. Note that only $V^k_i(s_i, s_{-i}) = V^k_i(\lambda\mathbf{1}, s_{-i})$ for $s_i \in S^k_i(\lambda, s_{-i})$; in general, $V^\ell_i(s_i, s_{-i}) \neq V^\ell_i(\lambda\mathbf{1}, s_{-i})$ when $s_i \in S^k_i(\lambda, s_{-i})$, $\ell \neq k$.

The following generalizations of Assumptions 1 and 2 of Section 3 will be sufficient for existence of admissible prices:

Assumption 1a*: *For all i, k, $V^k_i(s)$ is non-decreasing in s.*
1b*: *For all i, k, $V^k_i(\cdot)$ is continuous in all its arguments.*

Assumption 2*: *For all i, j, and k, for any $s'_i > s_i$, for any s_{-i} we have*

$$V^k_i(s'_i, s_{-i}) - V^k_i(s_i, s_{-i}) \quad \geq \quad V^k_j(s'_i, s_{-i}) - V^k_j(s_i, s_{-i}).$$

Personalized prices (which we shall show to be admissible for subadditive preferences over subsets of objects) for each object are defined by:

$$t^k_i(s_{-i}) \quad \equiv \quad \max_{j \neq i} V^{m,k}_{ij}(\lambda^k_{ij}(s_{-i}), s_{-i}), \qquad \forall s_{-i}, \ \ \forall k, \ \ i. \tag{11}$$

Note that $t^k_i(s_{-i}) \geq 0$. With Assumptions 1* and 2*, the results of Section 3 generalize so that for any s:

$$\text{If} \ \ V^k_i(s) \ > \ t^k_i(s_{-i}) \ \ \text{then} \ \ V^k_j(s) \ \leq \ t^k_j(s_{-j}) \ \ \text{for all} \ \ j \neq i. \tag{12}$$

Consider the following mechanism, which gives each buyer a surplus maximizing bundle at personalized prices t_i^k which satisfy (12) (for instance, the prices defined in (11)). Buyers report their signals. Personalized prices t_i^k are computed for each buyer and each object. Every buyer gets a minimal element in his demand set at the reported signals at these personalized prices. Thus, if buyers report $s = (s_1, s_2, ..., s_n)$ then buyer i gets $L_i \subseteq \{1, 2, ..., K\}$ such that

$$V_i^{L_i}(s) - \sum_{k \in L_i} t_i^k(s_{-i}) \;\; \geq \;\; V_i^L(s) - \sum_{k \in L} t_i^k(s_{-i}), \quad \forall L \subseteq \{1, 2, ..., K\},$$

and if $V_i^{L_i}(s) - \sum_{k \in L_i} t_i^k(s_{-i}) = V_i^L(s) - \sum_{k \in L} t_i^k(s_{-i})$ for some L then $L \not\subset L_i$. Call this mechanism the *demand mechanism* (as each buyer gets an element of his demand set).

No matter what preferences buyers have over subsets of objects, because at each information state each buyer gets an element of his demand set, the demand mechanism "satisfies" the ex post incentive compatibility constraints. However, in general the demand mechanism may not be feasible as demand for an object may exceed its supply (of one unit). We show that for subadditive preferences (defined below), the demand mechanism is feasible: each object is allocated to at most one buyer. Moreover, we exhibit examples of subadditive preferences in which this mechanism is non-trivial.

Subadditive preferences: The value of the union of two disjoint subsets is no greater than the sum of the values of the two subsets. That is, for all s

$$V_i^{L \cup L'}(s) \;\leq\; V_i^L(s) + V_i^{L'}(s), \qquad \forall L,\; L' \subset \{1, 2, ..., K\},\; L \cap L' = \emptyset. \tag{13}$$

We mention two special cases of subadditive preferences. Clearly, *additive preferences*, where the valuation of a subset is the sum of the valuations of objects in the subset, satisfy (13) with equality. A second example is that of *unit demand preferences*, i.e., the preferences of the assignment model. Each buyer has utility for at most one object. If a buyer is given a subset of objects L, he will select an object with the highest valuation and throw away the rest. Thus, his reservation value for any subset $L \subseteq \{1, 2, ..., K\}$ is:

$$V_i^L(s) \;\equiv\; \max_{k \in L} \{V_i^k(s)\}.$$

Note that the object which attains the max may vary with s. It is easily verified that unit demand preferences also satisfy (13).

Lemma 3: *If Assumptions* 1^* *and* 2^* *are satisfied and each buyers' preferences are subadditive then the demand mechanism is feasible and ex post incentive compatible.*

Proof: Let L_i and L_j be the subsets allocated by the demand mechanism to buyers i and j, $j \neq i$, at some information state s. We show that $L_i \cap L_j = \emptyset$. If $L_i = \emptyset$ there is nothing to prove. Therefore, suppose that $L_i \neq \emptyset$. If $L_i = \{k\}$ for some k, then minimality of L_i implies that $V_i^k(s) - t_i^k(s_{-i}) > 0$. Next, suppose that $|L_i| \geq 2$. Then for any $k \in L_i$,

$$\begin{aligned} V_i^{L_i}(s) - \sum_{\ell \in L_i} t_i^\ell(s_{-i}) &> V_i^{L_i \setminus k}(s) - \sum_{\ell \in L_i \setminus k} t_i^\ell(s_{-i}) \\ \implies \quad V_i^{L_i}(s) - V_i^{L_i \setminus k}(s) &> t_i^k(s_{-i}) \end{aligned}$$

where the first inequality follows from the fact that L_i is minimal. By subadditivity,

$$V_i^k(s) \geq V_i^{L_i}(s) - V_i^{L_i \setminus k}(s) > t_i^k(s_{-i}).$$

Therefore, if $|L_i| \geq 1$ then for any $k \in L_i$ we have $V_i^k(s) > t_i^k(s_{-i})$; (12) implies that $V_j^k(s) \leq t_j^k(s_{-j})$. This, together with subadditivity, implies that for any subset of objects L' such that there exists $k \in L' \cap L_i$,

$$\begin{aligned} V_j^{L'}(s) - \sum_{\ell \in L'} t_j^\ell(s_{-j}) &\leq V_j^{L' \setminus k}(s) + V_j^k(s) - \sum_{\ell \in L' \setminus k} t_j^\ell(s_{-j}) - t_j^k(s_{-j}) \\ &\leq V_j^{L' \setminus k}(s) - \sum_{\ell \in L' \setminus k} t_j^\ell(s_{-j}). \end{aligned}$$

Hence, L' cannot be a minimal element of j's demand set. Therefore, $L_j \cap L_i = \emptyset$. The demand mechanism is feasible for subadditive preferences.

Under truth-telling, each buyer gets an element of his demand set at the realized information state. Therefore, the mechanism is ex post incentive compatible. ∎

It may be verified that the demand mechanism is non-trivial for additive preferences, provided that the analog of Assumption 3 of Section 3 holds. The following example, which builds on Example 1, exhibits two examples of strictly subadditive preferences for which the demand mechanism is non-trivial.

Example 2: There are three buyers, 1, 2, and 3, and two objects, a and b. Buyer i gets signal (p_{ai}, p_{bi}, c_i), $i = 1, 2, 3$. Buyer i's valuation for the object k as a function of buyer signals is[19]

$$V_i^k \equiv p_{ki} + w_k c_i \max\{c_j, c_{j'}\}, \qquad k = a, b,$$

[19] We assume that $i, j, j' = 1, 2, 3$ are three distinct buyers, that is $j \neq j' \neq i \neq j$.

where $w_a,\ w_b \geq 0$ are constants. We specify buyer i's reservation value for the bundle $\overline{ab}$ later.

Define

$$t_i^k \equiv \max\{p_{kj} + w_k c_j^2,\ p_{kj'} + w_k c_{j'}^2\}, \qquad k = a, b. \tag{14}$$

Let $t_i = (t_i^a, t_i^b)$ be the personalized prices at which the two objects are available to buyer i, $i = 1, 2$. Suppose that at some information state, buyer i's value for object k exceeds his personalized price. This implies that $V_i^k = p_{ki} + w_k c_i c_j > t_i^k \geq p_{kj} + w_k c_j^2$, $j \neq i$. Then, mimicking the steps in Example 1, it is easily shown that $V_j^k < p_{ki} + w_k c_i^2 \leq t_j^k$. Thus, the personalized prices defined in (14) satisfy (12). By Lemma 3, the demand mechanism at these personalized prices is ex post incentive compatible for any specification of V_i^{ab} which satisfies the subadditive inequality (13).[20]

In the rest of this example we assume that $w_a = 1$ and $w_b = 1.5$. Thus, buyer i's reservation values for exactly one object are $V_i^a = p_{ai} + c_i \max\{c_j, c_{j'}\}$ and $V_i^b = p_{bi} + 1.5c_i \max\{c_j, c_{j'}\}$. We give two sets of preferences over the bundle $\overline{ab}$ for which this mechanism is non-trivial.

Subadditive preferences: Buyer i's valuation for the bundle $\overline{ab}$ is

$$V_i^{ab} \equiv p_{ai} + p_{bi} + 2c_i \max\{c_j, c_{j'}\}.$$

These preferences are subadditive. In the demand mechanism, buyer i is allocated $L_i = \emptyset$ if none of the subsets $\overline{a}$, $\overline{b}$, or $\overline{ab}$ yields a strictly positive surplus at the personalized prices at the realized (i.e., reported) information state. Otherwise he is allocated the smallest $L_i \in \{\overline{a}, \overline{b}, \overline{ab}\}$ that maximizes his surplus.

To see that the mechanism is non-trivial, fix buyer 2's signals at $(p_{a2}, p_{b2}, c_2) = (0.4, 0.1, 0.4)$ and buyer 3's signals at $(p_{a3}, p_{b3}, c_3) = (0.1, 0.4, 0.4)$. Thus, $t_1^a = \max\{0.4 + (0.4)^2, 0.1 + (0.4)^2\} = 0.56$, and $t_1^b = \max\{0.1 + 1.5 \times (0.4)^2, 0.4 + 1.5 \times (0.4)^2\} = 0.64$. Further, fix $c_1 = 0.5$. Hence, $V_2^a = 0.6$, $V_2^b = 0.4$, and $V_2^{ab} = 0.9$, and $V_3^a = 0.3$, $V_3^b = 0.7$, and $V_2^{ab} = 0.9$. In Table 1, we obtain the allocations achieved by the demand mechanism at three different values of buyer 1's private signals (p_{a1}, p_{b1}).

With (p_{a2}, p_{b2}, c_2) fixed at (0.4,0.1,0.4), and (p_{a3}, p_{b3}, c_3) at (0.1,0.4,0.4), we see that (i) buyer 1 gets object a, 2 gets nothing, and 3 gets b when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.1, 0.5)$, (ii) buyer 1 gets object b, 2 gets a, and 3 gets nothing when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.1, 0.5)$, and (iii) buyer 1 gets both the objects when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.9, 0.5)$. Moreover, at each of these three information states, buyers' demand sets are singletons. Hence, the allocation attained by the demand mechanism is constant in a

[20]These prices of (14) have not been defined as in (11). All that the proof of Lemma 3 requires is that the personalized prices used in the demand mechanism satisfy (12).

positive measure neighborhood around each information state. Thus, the demand mechanism is non-trivial. By symmetry, the mechanism has full range.

Table 1

(p_{a1}, p_{b1}, c_1)	$(0.9, 0.1, 0.5)$	$(0.1, 0.9, 0.5)$	$(0.9, 0.9, 0.5)$
V_1^a	1.1	0.3	1.1
V_1^b	0.4	1.2	1.2
V_1^{ab}	1.4	1.4	2.2
t_1^a	0.56	0.56	0.56
t_1^b	0.64	0.64	0.64
Buyer 1's allocation	$\overline{a}$	$\overline{b}$	$\overline{ab}$
V_2^a	0.6	0.6	0.6
V_2^b	0.4	0.4	0.4
V_2^{ab}	0.9	0.9	0.9
t_2^a	1.15	0.35	1.15
t_2^b	0.64	1.275	1.275
Buyer 2's allocation	$\emptyset$	$\overline{a}$	$\emptyset$
V_3^a	0.3	0.3	0.3
V_3^b	0.7	0.7	0.7
V_3^{ab}	0.9	0.9	0.9
t_3^a	1.15	0.56	1.15
t_3^b	0.475	1.275	1.275
Buyer 3's allocation	$\overline{b}$	$\emptyset$	$\emptyset$

Unit demand preferences: Buyer i's valuation for the bundle $\overline{ab}$ is

$$V_i^{ab} \equiv \max\{V_i^a, V_i^b\}, \qquad i = 1, 2, 3.$$

The demand mechanism allocates to buyer i the object that maximizes his surplus, $V_i^k - t_i^k$, $k = a, b$, provided this surplus is positive; if $V_i^k - t_i^k \leq 0$ for $k = a, b$ then buyer i gets nothing. To check that the mechanism is non-trivial, fix buyer 2's signals at $(p_{a2}, p_{b2}, c_2) = (0.4, 0.1, 0.4)$, and buyer 3's signals at $(p_{a3}, p_{b3}, c_3) = (0.1, 0.4, 0.4)$. Table 2 shows that when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.1, 0.5)$ buyer 1 gets object a and 3 gets b, and when $(p_{a1}, p_{b1}, c_1) = (0.1, 0.9, 0.5)$, buyer 1 gets b and buyer 2 gets a. Moreover, as the demand set of each buyer is a singleton at each of the two information states, the two allocations are implemented at a positive measure of information states. Hence, the mechanism is non-trivial, and, by symmetry, full range.

Table 2

(p_{a1}, p_{b1}, c_1)	$(0.9, 0.1, 0.5)$	$(0.1, 0.9, 0.5)$
V_1^a	1.1	0.3
V_1^b	0.4	1.2
t_1^a	0.56	0.56
t_1^b	0.64	0.64
Buyer 1's allocation	$\bar{a}$	$\bar{b}$
V_2^a	0.6	0.6
V_2^b	0.4	0.4
t_2^a	1.15	0.35
t_2^b	0.64	1.275
Buyer 2's allocation	$\emptyset$	$\bar{a}$
V_3^a	0.3	0.3
V_3^b	0.7	0.7
t_3^a	1.15	0.56
t_3^b	0.475	1.275
Buyer 3's allocation	$\bar{b}$	$\emptyset$

$\triangle$

5 Concluding remarks

Jehiel et al. (2005) question the existence of ex post equilibrium in models with multi-dimensional signals. Our paper shows that ex post equilibrium exists in auctions of private goods. Existence is proved under the assumption that buyers' information satisfies a generalization of the single crossing property. The mechanism shares the feature with the generalized Vickrey auction of single dimensional information models that the price paid by the winning buyer is equal to this buyer's value at the lowest possible signal (equivalence class) at which this buyer would just win. Thus, ex post equilibria in auction models with one dimensional models are robust in that non-trivial ex post equilibria exist even when buyers have multi-dimensional signals.

To reconcile our positive result with the negative result of Jehiel et al., observe that selfish preferences are a natural assumption in auction models. But in the space of all preferences (selfish or not), selfish preferences are non-generic. At a tiny perturbation away from selfish preferences, (5) is not satisfied by any pair of outcomes. Jehiel et al.'s theorem would then imply generic non-existence of ex post incentive compatible mechanisms. However, as non-trivial mechanisms that are "approximately" ex post incentive compatible still exist at these tiny perturbations away from selfish preferences, ex post incentive equilibrium is a robust equilibrium concept for auctions of private goods. Under a small departure from the usual assumption of selfish preferences in private goods models, many results in economics would be only approximately true.

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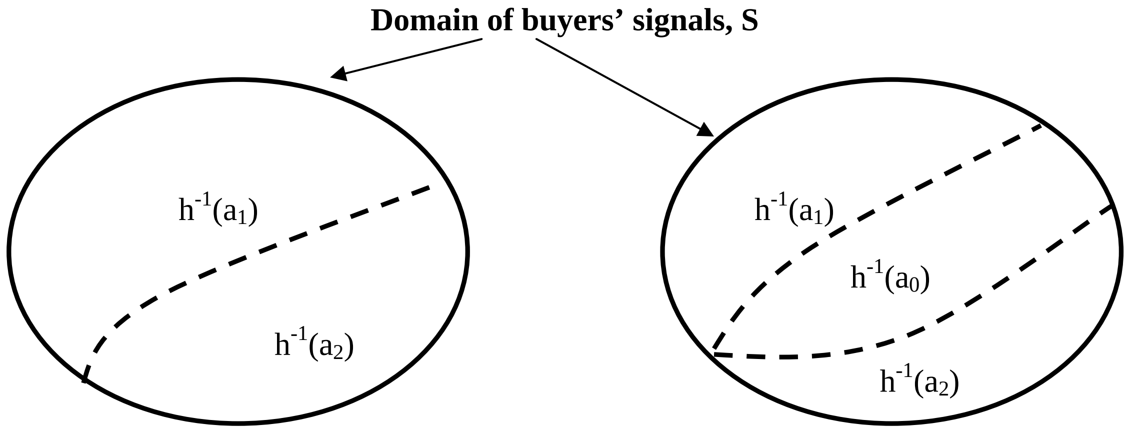

Two alternatives
Figure 1a

Three alternatives
Figure 1b

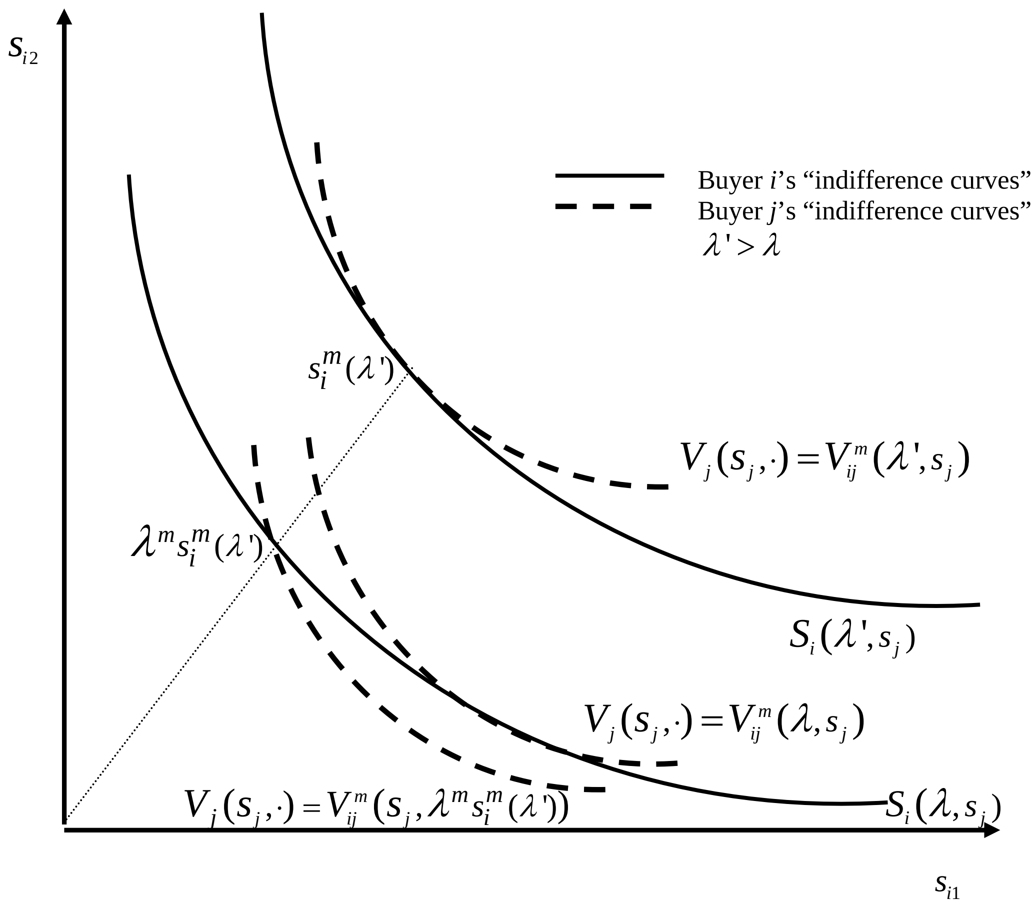

Figure 2