Imminent Nash Implementation. Extended Abstract

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This paper studies the complete information simultaneous-move implementation problem on the domain extended by time: $\Omega \times \Re_+$, where ω is a physical outcome and a positive real number is interpreted as a delay in the delivery of that outcome. The designer implementing a social choice correspondence F is allowed to approximate F by delaying the outcome. The delay is infinitesimal; hence the name: imminent implementation. This extension is different from the lottery-based extension of virtual implementation in that it does not allow mixing of the outcomes but similar in that it makes the set of outcomes dense and makes approximation possible. The result is, though, almost the opposite of universal implementability of virtual implementation: impossibility results of Nash implementation extend to imminent implementation with little modification. It is, thus, suggesting the crucial role of mixing in the virtual implementation success. The paper also provides characterization of the imminently implementable SCC and shows that there are some SCC, including a few famous ones, that are not Nash implementable but imminently implementable.

Suppose that a designer wishes to implement some social choice correspondence F. The players' preferences, although known to the players themselves, are not known to the designer. The designer's task is to create a mechanism that would pick the right outcome for each preference profile based on the reports submitted by agents. If designer intends to use simultaneous-move game forms and considers Nash equilibrium to be the predictor of the player's behaviour, the set of implementable social choice correspondences is characterized in (Maskin 1999). Although many social choice correspondences are implementable, there are prominent SCC, which are not. Moreover, Maskin (1999) and, e.g., Serrano (2004) show that, under some conditions, no unanimous SCF or, for two players, no Paretian SCC can be implemented.

One possible way out of those negative results is to use virtual implementation, suggested by Abreu and Matsushima (1992). In their approach the set of physical outcomes, Ω, is replaced by the lotteries over those outcomes Δ. It is shown

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that, under very mild restrictions, any SCC can be implemented approximately, delivering the desired outcome with an arbitrarily high probability.

We propose another extension, $\Omega \times \Re_+$. We interpret the second argument of the (ω, t) pair as time when the physical outcome ω is delivered. With time interpretation in mind, we make assumptions of time consistency (as in (Fishburn and Rubinstein 1982)) and continuity at infinity. The definition of approximate implementation in this domain, called imminent Nash implementation, is the following:

Definition 1 *The social choice correspondence $F : \Theta \to \Omega \times T$ is imminently Nash implementable if $\forall \varepsilon$ there exists $F^\varepsilon : \Theta \to \Omega \times T$ such that $\forall (\omega_\theta, t_\theta) \in F(\theta)$ there is $(\omega_\theta, t'_\theta(\omega_\theta)) \in F^\epsilon(\theta)$ and $\forall (\omega_\theta, t'_\theta(\omega_\theta)) \in F^\epsilon(\theta)$ there is $(\omega_\theta, t_\theta) \in F(\theta)$ and $t'_\theta(\omega_\theta)$ is such that $0 \le (t'_\theta(\omega_\theta) - t_\theta) \le \varepsilon$ and F^ε is Nash implementable*

Note the difference of time extension from the lottery extension. The latter changes the domain in three ways at once. First, it makes the domain dense, since, for any given outcome $a \in \Omega$, we now have an outcome, $(1-\epsilon)a + \epsilon b, b \in \Omega$, that is arbitrarily close to a. Second, a SCC can be approximated on the new domain. Third, there are mixed outcomes as above.

Time extension, in turn, introduces only the first two changes into the domain but does not allow mixed outcomes. Therefore, the results of this paper can be seen as breaking down the success of virtual implementation into two parts: one that is due to mixing outcomes and one that is due to approximation and denseness of the set per se.

Chambers (2004), in a paper that build on the model of repeated implementation, developed by Kalai and Ledyard (1998), shows that his time extension, that allow for mixing the outcomes, is equivalent to the model of virtual implementation and, thus, enjoys its permissive results; his model is, however, quite different from the one presented here.

Aside from theoretical interest of separating the different aspects of virtual implementation, imminent implementation may sometime be more desirable on practical grounds. For example, in King Solomon's dilemma, a problem of allocating indivisible prize to two players, virtual implementation calls for destruction of the award (baby) with some small probability. Practically, delaying the award infinitesimally, as suggested in (Artemov 2005), seems to be a better option.

The social correspondences that are imminently Nash implementable are characterized by the conditions similar to (Maskin 1999):

Theorem 1 *If SCC is imminently Nash implementable, the following holds: if $(a, t) \in F(\theta)$ and $(a, t) \notin F(\theta')$, then there exist $\bar{t}$, $0 < (\bar{t} - t) < \epsilon$, and t' such that $(a, \bar{t}) \succeq_i^\theta (b, t')$, $(b, t') \succ_i^{\theta'} (a, \bar{t})$.* (INI)

Definition 2 *Environment satisfies **Impatience** if, for any preference profile θ, there are at least two individuals for whom neutral outcome is not a top-ranked alternative.*

Theorem 2 *In the environments satisfying Impatience, with $N \geq 3$, SCC is imminently Nash implementable if it satisfies* `INI`

We also show, by the means of examples, that the set of SCC that are imminently implementable is different from the set of Nash implementable and the set of virtually implementable correspondences. In particular, example of majority voting from (Abreu and Sen 1990) can be implemented in an appropriate time extension. Yet, the set of SCC that are imminently implementable is far from being universal: the impossibility results, mentioned above, extend into this domain with little modifications:

Definition 3 *The agent i is an allocation-dictator if, for every preference profile θ such that $(a,t) \succ_i^\theta (b,t')$, $F(\theta) = (a,\tau)$, for some τ.*

Theorem 3 *Suppose that the domain of preferences over $\Omega \times T$ is universal and time consistent and the number of pure alternatives, $|\Omega| \geq 3$. Then if SCF F is Maskin monotonic and unanimous, it is allocation-dictatorial.*

Moreover, if we restrict the domain, we could have a traditional dictator, which gets the desired outcome without a delay ($\tau = 0$). For Paretian SCC, Maskin (1999) result also extends.
Those two theorems give us the answer to the question posed at the beginning: most of the power virtual implementation has comes from the ability to mix outcomes rather than the denseness or approximation. On the other hand, examples suggest that there are cases when imminent implementation is more powerful than Nash implementation and may be preferred to virtual implementation on practical grounds.

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