Coalitional games with veto players: sequential proposals, nucleolus and Nash outcomes*

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Abstract

The paper is based on a mechanism presented by Dagan, Serrano and Volij (1997) for bankruptcy problems. According to this mechanism a player, the proposer, makes a proposal that the rest of the players should responde saying yes or not. In this paper we present a model where the proposer can make sequential proposals. that is, the game is played in n stages and each stage starts with a proposal. We investigate the subgame perfect equilibria of this game and we relate them to a cooperative solution concept: the nucleolus of TU veto balanced games.

1. Introduction

In 1997, Dagan, Serrano and Volij present a simple tree game for bankruptcy problems. In the game a special player, the one with highest claim, has a special role. He makes a proposal and the rest of the players, given an order, accept or reject this proposal sequentially. In case of rejection the conflict is solved bilaterally, applying a normative solution concept to a special two-claimant bankruptcy problem. In their conclusions, Dagan, Serrano and Volij (1997) write:

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> Thus, constructing consistency based noncooperative models that support consistent cooperative solutions concepts which are not monotonic seems to us a difficult task. Therefore there might be problems in supporting the nucleolus or the Nash bargaining solution on general pies by means of a noncooperative model.

The main aim of this paper is to check the validity of this comment. This paper is based on a similar simple mechanisms adapted to the context of coalitional games with veto players. In our models, a veto player is the proposer and similarly to the case of Dagan, Serrano and Volij, in the case of negative answer of some responder a bilateral resolution is formulated.

In this bilateral resolution a two-person Davis-Maschler reduced game is defined. The solution applied in those two-person games is the *standard solution*[1] whenever this solution provides non negative payoff to the players (this requirement of non negativity is not used to obtain the main results of the paper but is maintained in order to guarantee the simplicity of the models). Under this resolution of the conflict the nucleolus appears as a candidate to be a Nash outcome of those models. The reason is that in the class of veto balanced games the nucleolus and the prekernel coincide (Arin and Feltkamp, 1997) and the prekernel is the maximal set satisfying Davis-Maschler reduced game property and *standard solution* (Peleg, 1986)[2].

Arin and Feltkamp (2005) study this mechanism when the proposer is allowed to make one proposal. This paper studies a more complex game where a veto player, the proposer, could make sequential proposals whenever there is a positive value to divide among the players. The proposer is allowed to make at most n proposals. Each proposal is answered by the responders pn each stage and in case of rejections again a two-person game is formed.

Relatef to this second model we define a cooperative solution concept on the class of veto balanced games. This solution satisfies several monotonicity properties. We prove that the solutio id the unique outcome of a special profile of strategies, a profile where the responders behave as myopic maximizers while the proposer is a rational player knowing the myopic behavior of the responders.

These results can be interpreted as a support of the comment of Dagan, Serrano and Volij. Monotonicity requirements of the solutions are important in these simple non cooperative models.

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[1] Given a two-person game $(\{1,2\},v)$ we called standard solution the following vector: $(v(\{1\})+d, v(\{2\})+d)$ where $d=\frac{v(\{1,2\})-v(\{1\})-v(\{2\})}{2}$.

[2] See also Maschler (1992).

The paper is organized as follows: Section 2 introduces preliminaries on TU games and the noncooperative game. Section 3 studies Myopic Best Response equilibrium. Section 4 shows by means of one example that, in general, the MBRE is not a Subgame Perfect Equilibrium.. The last section deals with the relation between rational behavior and myopic strategies..

2. Preliminaries

2.1. TU games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of the player set N is called a *coalition*. The number of players in a coalition S is denoted by $|S|$. Given a set of players N and a coalition $S \subset N$ we denote by S^c the set of players of N that are not in S. Generally we shall identify the game (N, v) by its characteristic function v. In this work we only consider games where the worth of all coalitions are non negative.

A distribution among the players is represented by a real valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution of an amount lower or equal to v is called a feasible distribution. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. A distribution satisfying $x(N) = v(N)$ is called an efficient allocation. An efficient allocation satisfying $x_i \geq v(i)$ for all $i \in N$ is called an *imputation* and the set of imputations is denoted by $I(N, v)$. The set of non negative feasible allocations is denoted by $D(N, v)$ and defined as follows

$$D(N, v) = \left\{ x \in \mathbb{R}^N : x(N) \leq v(N) \text{ and } x_i \geq 0 \text{ for all } i \in N \right\}.$$

The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{ x \in I(v) : x(S) \geq v(S) \text{ for all } S \subset N \}.$$

A game with a nonempty core is called a balanced game. A game v is a *veto-rich game* if it has at least one veto player and the set of imputations is nonempty. *A player i is a veto player if* $v(S) = 0$ for all coalitions where player i is not present. A balanced game with at least one veto player is called a veto balanced game. In this work we only consider games where the worth of all coalitions is non negative.

A solution ϕ on a class of games Γ_0 is a correspondence that associates with every game (N, v) in Γ_0 a set $\phi(N, v)$ in $\mathbb{R}^N$ such that $x(N) \leq v(N)$ for all $x \in \phi(N, v)$. If there is no confusion with the set of players we write (v) instead of (N, v). This solution is called *efficient* if this inequality holds with equality. The solution is called it *single-valued* if for every game in the class the set contains a unique element.

We introduce the most simple requirement of monotonicity that we ask for to a solution. Let ϕ be a single-valued solution on a class of games Γ_0. We say that solution ϕ satisfies aggregate-monotonicity property (Meggido, 1974) if the following holds: for all $v, w \in \Gamma_0$, such that for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i \in N$, $\phi_i(v) \leq \phi_i(w)$.

Given a vector $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game v is defined as $e(S, x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order of magnitude. The lexicographic order $\prec_L$ between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$ and the weak lexicographic order $\preceq_L$by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a game v, denoted by $\nu(N, v)$, as the unique imputation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of imputations. In formula:

$$\{\nu(N, v)\} = \{x \in I(N, v) \,|\theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N, v)\}.$$

For any game v with a nonempty imputation set, the nucleolus is a single-valued solution, is contained in the kernel and lies in the core provided that the core is nonempty.

In the class of veto balanced games the kernel, the prekernel and the nucleolus coincide (see Arin and Feltkamp (1997).

2.2. A noncooperative game

Given a veto balanced game (N, v) and an order of players, we will define a tree game associated to the TU game and denote it by $G(N, v)$. The game has n stages and in each stage only one player is playing. In the first stage a veto player is playing and he announces a proposal x^1 that belongs to the set of feasible and non negative allocations of the game (N, v). In the next stages the responders are playing, each one once at one stage. They have two actions. To accept or to reject. If a player, say i, accepts the proposal x^{t-1} at stage t, he leaves the game with the payoff x_i^{t-1} and for the next stage the proposal x^t coincides with the proposal at $t-1$, that is x^{t-1}. If player i rejects the proposal then a two-person TU game is formed with the proposer and the player i. In this two-person game the value of the grand coalition is $x_1^{t-1} + x_i^{t-1}$ and the value of the singletons is obtained by applying the Davis-Maschler reduced game[3] (Davis and Maschler (1965)) given the game (N, v) and the allocation x^{t-1}. The player i will receive as payoff the result of some restricted *standard solution* applied in the two-person game. Once all the responders have played and consequently have received their payoffs the payoff of the veto player is also determined.

Formally, the resulting outcome of playing the game can be described by the following algorithm.

Input : a veto balanced game (N, v) with a veto player, the player 1, and an order in the set of the rest of the players (responders)
Output : a feasible and non negative distribution x.

1. Start with stage 1. The veto player makes a feasible and non negative proposal x^1 (not necessarily an imputation). The superscript denotes at which stage the allocation is considered as the proposal in force.

2. In the next stage the first responder says yes or no to the proposal. If he says yes he receives the payoff x_2^1, leaves the game, and $x^2 = x^1$.

   If he says no he will receive the payoff

$$y_2 = \max\left\{0, 1/2(x_1^1 + x_2^1 - v_{x^1}(\{1\}))\right\} \text{ where}$$

[3]Let (N, v) be a game, $T \subset N$, and consider $T \neq N, \emptyset$ and a feasible allocation x. Then the *Davis-Maschler reduced game* with respect to $N \setminus T$ and x is the game $(N \setminus T, v_x)$ where

$$v_x^{N\setminus T}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ v(N) - x(T) & if \;\; S = N \setminus T \\ \max\limits_{Q \subset T} \{v(S \cup Q) - x(Q)\} & \text{for all } S \subset N \setminus T\,. \end{cases}$$

We also denote the game $(N \setminus T, v_x)$ by $v_x^{N \setminus T}$. Note that we define a modified Davis-Maschler reduced game where the value of the grand coalition of the reduced game is obtained in a different way. In our case, $v(N \setminus T) = x(N \setminus T)$. If x is efficient both reduced games coincide.

$$v_{x^1}(\{1\}) = \max_{1\in S\subseteq N\setminus\{2\}} \left\{v(S) - x^1(S\setminus\{1\})\right\}$$

Now, $x^2 = \begin{cases} x_1^1 + x_2^1 - y_2 & \text{for player 1} \\ y_2 & \text{for player 2} \\ x_i^1 & \text{if } i \neq 1, 2. \end{cases}$

3. Let the stage t where the k responder plays, given the allocation x^{t-1}. If he says yes he receives the payoff x_k^{t-1}, leaves the game, and $x^t = x^{t-1}$.

   If he says no he will receive the payoff

$$y_k = \max\left\{0, 1/2(x_1^{t-1} + x_t^{t-1} - v_{x^{t-1}}(\{1\}))\right\} \text{ where}$$

$$v_{x^{t-1}}(\{1\})) = \max_{1\in S\subseteq N\setminus\{t\}} \left\{v(S) - x^{t-1}(S\setminus\{1\})\right\}.$$

   Now, $x^t = \begin{cases} x_1^{t-1} + x_k^{t-1} - y_t & \text{for player 1} \\ y_k & \text{for player } k \\ x_i^{t-1} & \text{if } i \neq 1, k \end{cases}$.

4. The game ends when the stage n is played and we define $x^n(N, v)$ as the vector with coordinates $\left(x_j^n\right)_{j\in N}$.

In this game we assume that the conflict between the proposer and a responder is solved bilaterally. In the case of conflict, the players face a two-person TU game that shows the strength of the players given the fact that the rest of the responders do not play. Once the game is formed the allocation proposed for the game is a normative proposal, a kind of restricted *standard solution*[4]. It is restricted because negative payoffs are not allowed. If the formed two-person game is balanced, the solution will be the *standard solution* that coincides with the prekernel and the nucleolus.

[4] The main results of the paper do not change if we use the standard solution instead of the restricted standard solution as the concept with which we solve the bilateral conflict. Since our main idea is to discuss simple mechanisms we think is more credible to assume that no player will accept a negative payoff, a payoff lower than his individual worth.

2.3. The Nash outcomes of the game

Given a game (N,v) and a feasible allocation x we define the complaint of the player i against the player j as follows:

$$f_{ij}(x) = \min_{i\in S\subseteq N\setminus\{j\}} \{x(S) - v(S)\}.$$

The set of *bilaterally balanced allocations for player* i is

$$F_i(N,v) = \{x \in D(N,v) : f_{ji}(x) \geq f_{ij}(x) \text{ for all } j \neq i\}$$

while the *set of optimal allocations for player* i in the set $F_i(N,v)$ is defined as follows:

$$B_i(N,v) = \underset{x\in F_i(N,v)}{\arg\max} x_i.$$

Note that since $F_i(N,v)$ is a nonempty (it contains the prekernel[5]) compact set the set $B_i(N,v)$ is nonempty.

Theorem 2.1. *(Arin and Feltkamp, 2005) Let (N,v) be a veto balanced TU game and let $G(N,v)$ be its associated tree game. Let z be a feasible and non negative allocation. Then z is a Nash outcome if and only if $z \in B_1(N,v)$.*

3. A new game: sequential proposals

3.1. The model

We model a new game where the proposer can make sequential proposals, and each proposal is answered by the responders as in the previous model. Again, given a veto balanced game with a proposer and an order in the set of responders we will construct a tree game, denoted by $G^2(N,v)$.

Formally, the resulting outcome of playing the game can be described by the following algorithm.

Input : a veto balanced game (N,v) with a veto player, player 1, and an order in the set of the rest of the players (responders)

Output : a feasible and non negative distribution x.

[5]If we denote by PK the prekernel and (N,v) is a veto balanced games then $PK(N,v) = \bigcap_{i\in N}(F_i(N,v) \cap I(N,v))$. In general, the result is not valid and there exist TU games for which some sets F_i are empty.

1. Start with stage 1. Given a veto balanced TU game (N, v) and an order in the set of responders (the order is not fixed for all the stages) we play the game $G(N, v)$ and define the veto balanced TU game (N, v^{2,x_1}) where $v^{2,x_1}(S) = \max\{0, v(S) - x^1(S)\}$ and x_1 is the final outcome obtained in the first stage. Then go to the next step. The superscripts in the characteristic function denote at which stage and after which outcome the game is considered as the game in force. If no confusion we write v^2 instead of v^{2,x_1}.

2. Let be the stage t $(t < n + 1)$ and the TU game $(N, v^{t,x_{t-1}})$. We play the game $G(N, v^t)$ and define the veto balanced TU game (N, v^{t+1,x_t}) where $v^{t+1}(S) = \max\{0, v^t(S) - x^t(S)\}$ and x_t is the final outcome obtained in the previous stage. Then go to the next step.

3. The game ends after stage n. (If at some stage before n the proposer makes an efficient proposal (efficient according to the TU game underlying at this stage) the game is trivial for the rest of the stages).

4. The outcome is the sum of the outcomes generated at each stage.

3.2. A serial rule

We introduce now a solution concept defined on the class of veto balanced games and denoted by ϕ. This solution concept will be related to the tree game we have presented. Let (N, v) a veto balanced game. Define for each player i a value d_i as follows $d_i = \max_{S \subseteq N\backslash\{i\}} v(S)$. Then $d_1 = 0$. Let $d_{n+1} = v(N)$ and rename players according to the nondecreasing order of those values. That is, player 2 is the player with the lowest value and so on. Define the solution ϕ as follows:

$$\phi_l = \sum_{i=l}^{n} \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1, ..., n\}.$$

It is clear that in the class of veto balanced games the solution ϕ satisfies some well-known properties as nonemptiness, efficiency, anonymity[6] and equal treatment property among others. It also satisfies aggregate monotonicity. We will see that the monotonic solution we have introduced can be considered as the

[6]A solution ϕ satisfies **anonymity** if for each (N, v) in Γ_0 and each bijective mapping τ : $N \longrightarrow N$ such that $(N, \tau v)$ in Γ_0 it holds that $\phi(N, \tau v) = \tau(\phi(N, v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ $(x \in R^N, j \in N, T \subseteq N)$). In this case v and τv are equivalent games.

unique outcome of a special equilibrium, a equilibrium where all the responders play myopically. Preciously we show that ϕ is a core allocation.

Lemma 3.1. *Let (N,v) be a veto balanced TU game. Then $\phi \in C(N,v)$.*

Proof. Note that $\sum_{l=1}^{k}(\phi_l - \phi_k) = d_k$. Let S a coalition and let d_j such that $d_j = \max_{i \notin S} d_i$. Let k the first player for which $d_k = d_j$. Therefore $\{1, 2, ..., k-1\} \subseteq S$. By definition $v(S) \leq d_k = \sum_{l=1}^{k}(\phi_l - \phi_k) \leq \sum_{l=1}^{k}\phi_l \leq \sum_{i \in S}\phi_l$. ■

3.3. Myopic Best Response Equilibrium

In the following we study the strategies of the responders. A special strategy is the one in which the responders behave optimally at each subgame independently of the influence of such behavior in the following subgames. If all the responders behave as myopic maximizers and the proposer is playing optimally the resulting outcome is unique. The next result solves the following question: Suppose all the responders answer optimally but myopically in each subgame and that the proposer is playing optimally the game (knowing that the responders are myopic maximizers), which outcomes we can expect? We call myopic best response strategies (MBRE) such profile of strategies.

Definition 3.2. *Let (N,v) be at veto balanced TU game.Consider the associated games $G^2(N,v)$. Given a stage k, a proposal x is balanced if the proposal x results the final outcome of the stage k independently of the responses of the players.*

Lemma 3.3. *Let (N,v) be at veto balanced TU game.Consider the associated games $G^2(N,v)$. Given a stage k, a proposal x is balanced if and only if the proposal x results the nucleolus of the game $(N.w)$ where $w(S) = v^k(S)$ for all $S \neq N$ and $w(N) = x(N)$.*

Proof. Assume that x is a balanced proposal on the stage k with the game (N, v^k).

a) Let l be a responder for which $x_l = 0$. If whatever is the response of player l the proposal does not change then $f_{1l}(x) \leq 0 = x_l = f_{l1}(x)$.

b) Let m be a responder for which $x_m > 0$. If whatever is the response of player m the proposal does not change then $f_{1m}(x) = x_m = f_{m1}(x)$.

Therefore for the veto player are satisfied the bilateral kernel conditions. In Arin and Feltkamp (2005) it is shown that if the bilateral kernel conditions are satisfied between the veto player and the rest of the players then the bilateral kernel conditions are satisfied between any pair of players.

Therefore x is the kernel (nucleolus) of the game (N,w). ■

Lemma 3.4. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. Let z be a Nash outcome of the game $G^2(N,v)$. Then $z_1 \geq \phi_1 = \sum\limits_{i=1}^{n} \frac{d_{i+1}-d_i}{i}$.*

Proof. The result is based on the fact that the proposer has a strategy with which he will guarantee the payoff $\sum\limits_{i=1}^{n} \frac{d_{i+1}-d_i}{i}$ independently of the strategies of the rest of the players by making sequentially balanced proposals.

The strategy is the following. At each stage t, $(t \in \{1,...,n\})$ consider the set $S_t = \{l;\ d_l \leq d_t\}$ and the proposal x^t, defined as follows:

$$x_l^t = \begin{cases} \frac{d_{t+1}-d_t}{t} & \text{for all } l \in S_t \\ 0 & \text{otherwise.} \end{cases}$$

whenever x^t is feasible and propose the 0 vector otherwise.

It can be checked immediately that in each stage the proposed allocation will be the final allocation independently of the answers of the responders and independently of the order of those answers. The proposals are balanced proposals. Therefore this strategy of the proposer determines the total payoff of all the players, that is, the final outcome of the game $G^2(N,v)$. This final outcome coincides with the solution ϕ. ■

The proof suggests a new interpretation of the solution concept ϕ. At each stage the proposal coincides with the nucleolus of a veto rich game. Formally,

$$\phi(N,v) = \sum_{i=2}^{n+1} \nu(N,v^i)$$

where the games (N,v^i) are defined as follows:

(N,v^0) is the zero game, i.e., all the coalitions have the same worth, 0. (N,v^1) is the original game, i.e., (N,v). And for $i \in \{2,...,n+1\}$

$$v^i(S) = \begin{cases} d_i - d_{i-1} & \text{if } S = N \\ \max\left\{0, v^{i-1}(S) - \sum\limits_{l\in S} \nu_l(N,v^{i-2})\right\} & \text{otherwise.} \end{cases}$$

The solution ϕ is a kind of monotonic extension of the nucleolus.

Lemma 3.5. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. Let i be a responder and w be an outcome resulting from some MBRE of the game $G^2(N,v)$. Then* $w_i \geq w_1 - \sum_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1}$ *for all* $i \in \{2,...,n\}$.

Proof. By induction on the players index, where the index is given by the d-values and increasing index corresponds to nondecreasing d-value.

We denote by z_i^t the accumulated payoff of player i after t completed stages of the game $G^2(N,v)$. We need to prove that

$z_i^t \geq z_1^t - \sum_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1}$ for all $i \in \{2,...,n\}$ and for all $t \in \{1,...,n\}$.

The case $k=1$ is trivial if we take the convention that the empty sum is 0.

Suppose that the lemma is true for players $2,3,...,k-1$. We need to prove that $z_k^t \geq z_1^t - \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$ for all $t \in \{0,...,n\}$. Suppose on the contrary that $z_k^t < z_1^t - \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$ for some $t \in \{2,...,n\}$. That means that there exists a subgame after which the difference of accumulated payoffs between player i and player k is higher than $\sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. Let $G(N,v^t)$ be the subgame played at stage t in which subgame we have for the first time that $z_1^{t-1} + x_1^t - (z_k^{t-1} + x_k^t) > \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$ where x^t is the proposal faced by player k in the subgame. If player k rejects the proposal x^t we have the following two person game:

$$v_{x^t}(\{1\}) = \max(0, v(\{1\}) - z_1^{t-1}, \max_{p \in \{2,...,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l - z_1^{t-1} \right\})$$

$$v_{x^t}(\{k\}) = 0 \text{ and } v_{x^t}(\{1,k\} = x_1^t + x_k^t,$$

where we denote by q_l the accumulated payoff obtained by player l up in the stage k of the subgame played at the period t. The proposer has different coalitions with which he can get his value in the bilateral game. The highest value that a coalition could have without player k is d_k. It is also immediate that the coalitions with value d_k should contain all the responders preceding[7] player k. Therefore

[7]By preceding we mean players that have a value d_i lower than d_k.

the payoffs of all this players should be taken into account. The same occurs if the proposer decides to use a coalition with a value of d_{k-1}. In this case all the players preceding player $k-1$ belong to such coalitions. Therefore we get

$$v_{x^t}(\{1\}) = \max(0, v(\{1\}) - z_1^{t-1}, \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l - z_1^{t-1} \right\}).$$

The case $v_{x^t}(\{1\}) = 0$ implies that after rejection of player k, $z_1^t - z_k^t = z_1^{t-1} - z_k^{t-1}$ and by assumption in the stage $t-1$ the difference of accumulated payoffs is no higher than $\sum\limits_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. The case $v_{x^t}(\{1\}) = v(\{1\}) - z_1^{t-1}$ implies that after rejection of player k, $z_1^t - z_k^t = v(\{1\}) \leq d_2 \leq \sum\limits_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$.

Therefore we focus in the case

$$v_{x^t}(\{1\}) = \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l - z_1^{t-1} \right\}$$

.

Note that since all the accumulated payoffs are non negative if $z_k^t < z_1^t - \sum\limits_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1})$ then $z_1^t > \sum\limits_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. And for all $i \in \{2, ..., k-1\}$ has been proved that $z_i^t \geq z_1^t - \sum\limits_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1}$ for all $i \in \{2, ..., n\}$ and for all $t \in \{2, ..., n\}$.

Therefore combining both inequalities $z_m^t \geq \sum\limits_{l=m+1}^{k} \frac{d_l - d_{l-1}}{l-1}$ for all $m \in \{2, ..., k-1\}$.

Computing the difference between the accumulated payoffs of the proposer and player k after the response of player k in the stage t facing the proposal x^t we get

$$q_1 - z_k^t \leq z_1^{t-1} + x_1^t + x_k^t - (x_1^t + x_k^t - v_{x^t}(\{1\}) =$$

$$= \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l \right\} = d_j - \sum_{l=2}^{j-1} q_l$$

where j is the index with which the maximun is attained.

This difference is obtained after the response of player k. But this difference is going to change after the response of all the players in the stage. The final accumulated payoff of the responders is $z_l^t \geq q_l$. Note that if for some players hold that $z_l^t > q_l$ the reason is that the proposer has transferred this difference to this responder.

That is, $q_1 - z_1^t \geq \sum\limits_{l=2}^{j-1} z_l - \sum\limits_{l=2}^{j-1} q_l$.

Therefore the final difference between the proposer and the responder k will be:

$$z_1^t - z_k^t \leq d_p - \sum_{l=2}^{d-1} z_l^t \leq \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} z_l^t \right\} \leq$$

$$\leq \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} \frac{l-1}{l}(d_{l+1} - d_l) \right\} = \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}.$$

The last inequality is a direct consequence of the following two facts:

1) $z_p \geq \sum\limits_{l=p+1}^{k} \frac{d_l - d_{l-1}}{l-1}$ for any responder p preceding player k.

2) $\sum\limits_{l=2}^{p-1} \sum\limits_{j=l+1}^{p} \frac{d_l - d_{l-1}}{l-1} = \sum\limits_{l=2}^{p-1} \frac{l-1}{l}(d_{l+1} - d_l)$.

Therefore even if the difference is not established immediately after the decision of rejecting or accepting, this difference is going to appear after the response of all the players in the subgame k.

Note that the result holds for any order in the set of responders. ■

Combining lemma 3.1 and lemma 3.2 we get the main result.

Theorem 3.6. *Let (N, v) be a veto balanced TU game and $G^2(N, v)$ a tree game. Let z be an outcome resulting from a MBRE of the game $G^2(N, v)$. Then $z = \phi$.*

Proof. By Lemma 3.1 $z_1 \geq \phi_1$. And by Lemma 3.2 for all $l \in \{2, ..., n\}$ it holds

$$z_l \geq z_1 - \sum_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1} \geq \phi_1 - \sum_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1} = \sum_{l=2}^{n+1} \frac{d_l - d_{l-1}}{l-1} - \sum_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1} = \phi_l.$$

The unique feasible allocation satisfying those inequalities is ϕ. ■

4. An example

The next example illustrates that, in general, the profile of strategies that form a MBRE is not subgame perfect equilibrium.

Example 4.1. *Let* $N = \{1, 2, 3, 4, 5\}$ *a set of players and consider the following 5-person veto balanced game* (N, v) *where*

$$v(S) = \begin{cases} 36 & \text{if } S \in \{\{1,2,3,5\}, \{1,2,3,4\}\} \\ 31 & \text{if } S = \{1,2,4,5\} \\ 51 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the outcome associated to any MBRE we see that the proposer receives as payoff the amount $\phi_1(N, v) = 121/6$. As we know this result is true for any order. Suppose now the following order in the set of responders: The first responder is player 2, the second player 3 and the last one is player 5. The following result holds given this order.

If the responders play optimally the game (not necessarily as myopic maximizers) the proposer can get a payoff higher than the one provided by the MBRE outcome. Therefore MBRE outcome and SPE outcomes do not necessarily coincide.

The strategy is the following: The proposer offers nothing in the first three stages. In the 4th stage the proposal is: $(10, 10, 5, 0, 0)$.

The answer of player 2, 4 and 5 does not change the proposal (even if the proposal faced by player 4 and 5 is a new one resulting from a rejection of player 3). If player 3 accepts the TU game for the last stage will be:

$$w(S) = \begin{cases} 11 & \text{if } S \in \{\{1,2,3,5\}, \{1,2,3,4\}\} \\ 11 & \text{if } S = \{1,2,4,5\} \\ 26 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

In the last stage everybody play as myopic maximizer and the outcome should be an element of $B_1(N, w)$. It can be checked that $B_1(N, w) = \{(5.5, 5.5, 0, 0, 0)\}$. Therefore after accepting the proposal player 3 gets a total payoff of 5.

In case of rejection, the TU game for the last stage will be:

$$u(S) = \begin{cases} 11 & \text{if } S \in \{\{1,2,3,5\}, \{1,2,3,4\}\} \\ 6 & \text{if } S = \{1,2,4,5\} \\ 26 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

As before, in the last stage everybody play as myopic maximizer and the outcome should be an element of $B_1(N,u)$. It can be checked that $B_1(N,u) = \{(5.2, 5.2, 5.2, 5.2, 5.2)\}$. Therefore after rejecting the proposal player 3 gets a total payoff of 5.2.

Therefore a rational behavior of player 3 implies a rejection of the proposal in the 4th stage. This rejection is not a myopic maximizer´s behavior. After rejection of player 3 the proposer gets a payoff of 20.2, higher than $121\backslash 6$.

In this game the outcome associated to MBRE is not the result of a SPE.

If we check the example we see that the proposer *finds a credible way to co-operate* with player 3 to get a payoff higher than the one obtained by player 2 (a veto player). Player 2 can not avoid this *cooperation* since he is playing before player 3. If he would play after player 3 the *cooperation* between player 1 and 3 (at least as in the example) is not profitable anymore. This observation results crucial and we exploit it in detail in the next section.

Remark 1. *If we replace restricted standard solution by pure standard solution in the model it is still true that we can find games where the MBRE are not a SPE.*

5. From Myopic behavior to rational behavior

In the following we assume that the number of proposals is equal to the number of players. And more important, we assume that at each stage the order of the responders is given by the nondecreasing order of the $d-$values of the corresponding game. That is, the order of the responders is not fixed and can change in different stages.

5.1. The behavior of the veto players

Lemma 5.1. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. The optimal behavior of a veto player responder is to be myopically best responder.*

Proof. Let k be a responder veto player playing in stage t and facing the proposal x^t. The game (N, v^{t+1}) resulting after playing stage t does not depend at all on the response of player k. Therefore the best option of player k is to maximize his payoff at stage $\dot{t}$. That is, to behave as myopically best responder at stage t. ■

Corollary 5.2. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. If in a SPE there is an stage t where the optimal response of a responder does not coincide with his myopic best response then this responder is not a veto player in stage $k+1$.* .

Corollary 5.3. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. Let $z = \sum_{1}^{k} x^t$ be an outcome resulting from some SPE of the game $G^2(N,v)$.If in the stage k a responder l is a veto player then the optimal behavior of responder l implies that in the game (N,v^{k-1}), $x_i^{k-1} \geq f_{i1}(x^{k-1})$.*

Lemma 5.4. *Let (N,v) be a veto balanced TU game and $G^2(N,v)$ its associated tree game. Let $z = \sum_{1}^{k} x^t$ be an outcome resulting from some SPE of the game $G^2(N,v)$. Then $z_1 = z_i$ for all $i \in T$ where T is the set of veto players.*

Proof. Clearly, $z_1 \geq z_i$ for all $i \in T \setminus \{1\}$ .And the optimal behavior of the veto players responding after no veto players implies that $z_i \geq z_1$ for all $i \in T \setminus \{1\}$. ■

5.2. The behavior of the proposer

Lemma 5.5. *Let (N,v) be a veto balanced TU game.Consider the associated games $G^2(N,v)$. Let $z = \sum_{1}^{k} x^t$ be an outcome resulting from some SPE of the game $G^2(N,v)$.Then there exists stage $k \leq n$ such that $\sum_{k}^{n} x^t$ is the outcome of some MBRE of the game $G^2(N,v^k)$.*

Lemma 5.6. *Let (N,v) be a veto balanced TU game. Consider the associated games $G^2(N,v)$. Let $z = \sum_{1}^{k} x^t$ be an outcome resulting from some SPE of the game $G^2(N,v)$. Assume that the final outcome of stage k, x^k, is such that there exists player l such that $x_l^k > f_{1k}(x^k, v^k)$ and $x_l^k > 0$. Then there exists y such that $y_1 = z_1$ $y = \sum_{1}^{k-1} x^t + \sum_{k} q^t$ where $\sum_{k}^{n} q^t$ is an outcome of the game $G^2(N,v^k)$ obtained by making* ***balanced*** *proposals.*

Proof. Consider the game (N,v^k) and the payoff x^k

Assume that $x_i^k < f_{i1}(x^k)$ for some $i \neq 1$ and $x_i^k > 0$. Since $f_{i1}(x^k) = x_i^k$ by decreasing the payoff of player i we can construct a new allocation y such that $f_{1i}(y) = f_{i1}(y)$ or $f_{1i}(y) < f_{i1}(y)$ and $y_i = 0$. In any case, $x_1^k = y_1$ and the payoff of the proposer does not change.

Now if there exists another player l such that $f_{1l}(y) < f_{l1}(y)$ and $y_l > 0$ we construct a new allocation z such that $f_{1l}(z) = f_{l1}(z)$ or $f_{1i}(z) < f_{i1}(z)$ and $z_i = 0$. Note that $z_1 = y_1$. Repeating this procedure we will end[8] with an allocation that is the kernel (nucleolus) of a game where the only change with respect to the game (N, v^k) is the fact that we have decreased the worth of the grand coalition. If q is the final outcome of this procedure, q is the nucleolus of the game (N, w^k) where $w(N) = q(N)$ and $w(S) = v^k(S)$ for all $S \neq N$. This is so because of the previous lemma; once the *kernel bilateral conditions* hold between the veto player and the rest of the players, those *kernel bilateral conditions* hold for any pair of players.

The TU game, (N, w^{k+1}) resulting after proposing q satisfies that $w^{k+1}(S) \geq v^{k+1}(S)$ for all S, $1 \in S$. Therefore $f_{i1}(x, (N, w^{k+1})) \leq f_{i1}(x, (N, v^{k+1}))$ for any feasible allocation x.

Consider the game (N, w^{k+1}) and the payoff x^{k+1}

Assume that $x_i^{k+1} < f_{i1}(x^{k+1})$ for some $i \neq 1$ and $x_i^{k+1} > 0$. By decreasing the payoff of player i we can construct a new allocation y such that $f_{1i}(y^{k+1}) = f_{i1}(y^{k+1})$ or $f_{1i}(y^{k+1}) < f_{i1}(y^{k+1})$ and $y_i^{k+1} = 0$. In any case, $x_1^{k+1} = y_1^{k+1}$.

Now if there exists another player l such that $f_{1l}(y^{k+1}) < f_{l1}(y^{k+1})$ and $y_l^{k+1} > 0$ we construct a new allocation z^{k+1} such that $f_{1l}(z^{k+1}) = f_{l1}(z^{k+1})$ or $f_{1i}(z^{k+1}) < f_{i1}(z^{k+1})$ and $z_l^{k+1} = 0$. Repeating this procedure we will end with an allocation that is the kernel (nucleolus) of a game where the only change with respect to the game (N, w^{k+1}) is the fact that we have decreased the worth of the grand coalition. If q^{k+1} is the final outcome of this procedure, q^{k+1} is the nucleolus of the game $(N, w_{q^{k+1}}^{k+1})$ where $w_{q^{k+1}}^{k+1}(N) = q^{k+1}(N)$ and $w_{q^{k+1}}^{k+1}(S) = w^{k+1}(S)$ for all $S \neq N$.

INote that q^{k+1} is a balanced proposal in the stage k'1. The TU game, $(N, w_{q^{k+1}}^{k+1})$ resulting after proposing q^{k+1} satisfies that $w_{q^{k+1}}^{k+1}(S) \geq w^{k+1}(S)$ for all S, $1 \in S$.Therefore $f_{1i}(x, (N, w_{q^{k+1}}^{k+1})) \leq f_{1i}(x, (N, w^{k+1}))$ for any feasible allocation x.

This procedure can be repeated till the last stage of the game ending with an outcome equals to $\sum_{k}^{n} q^t$.. ■

[8] In a finite number of steps.

We call a profile of strategies *veto efficient*[9] if the resulting outcome satisfies that the final outcome of any stage k, x^k, is such that there is no player l for which $x_l^k > f_{1k}(x^k, v^k)$ and $x_l^k > 0$.

A direct implication of the above lemma is that always there exist SPE that are *veto efficient*. This is so because for any SPE no veto efficient we can construct a veto efficient profile of strategies where the payoff of the proposer does not change. And can do such a construction by using balanced proposals.

Corollary 5.7. *Let (N, v) be a veto balanced TU game. Consider the associated games $G^2(N, v)$. Let $z = \sum\limits_1^k x^t$ be an outcome resulting from profile of strategies* veto efficient. *If player l is no veto in the stage k then $x_{t,l} = 0$ for all $t < k$.*

Proof. We need to prove that $f_{1k}(x^k, (N, v^{k-1})) \leq 0$. If player k is no veto in stage $k+1$ after the final outcome x^k we know that there exists a coalition T such that $k \notin T$ and $x^k(T) < v(T)$. Therefore $f_{1k}(x^k, (N, v^{k-1})) \leq x^k(T) - v(T) < 0$. ■

We focus now in SPE that are *veto efficient*.

Lemma 5.8. *Let (N, v) be a veto balanced TU game and $G^2(N, v)$ its associated tree game. Let $z = \sum\limits_1^k x^t$ be an outcome resulting from some SPE veto efficient of the game $G^2(N, v)$. Assume that there exists an stage k and a responder l behaving as no best myopic responder at this stage. Then $x_l^k = 0$. And x^k is a balanced proposal.*

Proof. It is a consequence of the following facts. Player l/is no veto in stage $k+1$ and veto efficiency implies that $x_l^k = 0$.

Assume now that x^k is no balanced.

a) Players preceding player l. Since player l is no veto in stage k+1 we know that $f_{1l}(x^k, (N, v^k)) \leq 0$. Let p be a player preceding player l at this stage. We will show that $f_{1p}(x^k, (N, v^k)) \leq 0$. Assume on the contrary that $f_{1p}(x^k, (N, v^k)) > 0$ and let $Q = \arg\min\limits_{p \notin S}(x^k(S) - v(S))$.Therefore $x^k(S) > v(S)$ for all S such that $p \notin S$ and $1 \in S$. Let $Q = \max\limits_{p \notin S} v(S)$. Since $d_p \geq d_l > 0$ we know that $x_k(Q) > v(Q) > 0$. Respect to player l let $T = \arg\min\limits_{l \notin S}(x^k(S) - v(S))$. Therefore $0 \leq x^k(T) \leq v(T)$.

[9]Veto efficiency implies that the no veto responders receive a positive payoff if and only if they are veto responders in the next stage,

Since $d_p \geq d_l > 0$ we know that $x^k(Q) > v(Q) \geq v(T) \geq x^k(T) > 0$.Since the set of veto players is included in $T \cap Q$ there exists a no veto player m such that $m \in Q, m \notin T$ and $x^k_m > 0$. But if $m \notin T$ then $f_{1p}(x^k, (N, v^k) \leq x^k(T) - v(T) \leq 0$. That contradicts that x^k is veto efficient. Therefore $f_{1p}(x^k, (N, v^k)) \leq 0$ and $x^k_p = 0$ by veto efficiency.

b) Players receiving a positive payoff at stage k : let r a responder receiving a positive payoff at stage k. Player r is veto player in stage k+1. Therefore veto efficiency implies that for all those players $f_{1r}(x^k, (N, v^k)) = x^k_r > 0$. ■

Therefore, from any SPE veto efficient we can construct a SPE veto efficient where all proposals are balanced.

Theorem 5.9. *Let (N, v) be a veto balanced TU game and $G^2(N, v)$ its associated tree game. There exists a SPE where the proposer makes balanced proposals.*

Theorem 5.10. *Let (N, v) be a veto balanced TU game and $G^2(N, v)$ its associated tree game. Then ϕ is the outcome of some SPE.*

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