# Contract and Mechanism Design in Settings with Multi-Period Trade

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**Abstract**

This paper presents analysis of contractual settings with complete but unverifiable information and where trade consists of a sequence of verifiable productive actions, between which renegotiation can occur. The main result identifies a simplified model that is equivalent for calculating the set of implementable value functions; the result also shows that the detrimental effects of renegotiation between productive action can be counteracted when the parties can sufficiently communicate with the external enforcer over time. Corollary results address the form of optimal contracts and the effect of irreversible productive actions.

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In any contractual setting, details of the trading technology may have a significant impact on the optimal design of contracts and on the prospects for cooperation and efficient exchange. The trade of a good or service often involves a nontrivial period of time in which a sequence of productive actions can be taken. For example, delivery of an industrial product (such as a specialized computer system) may require, in order, (i) construction at the seller's plant, (ii) transportation to the buyer's facility, and (iii) configuration of the equipment; these actions may take place over weeks or months. Another example is a durable trading opportunity, where a product (say, a software package) can be provided at any point in a window of time and where trade can possibly be reversed (the software uninstalled).

In this paper, I examine a contracting model in which verifiable productive actions take place over multiple periods of time. My goals are to explore the effect of renegotiation between productive actions and to develop a simple way of analyzing these contractual settings, so one can better understand how the technology of trade affects contracts and behavior. The main result in this paper shows that the scope of contracting in a large class of models with multi-period trade can be exactly characterized by examining related simplified models that are standard, static mechanism-design problems. Corollary results address the form of optimal contracts and the effect of irreversible productive actions.

In relation to the literature, the starting point for the modeling exercise herein is the standard contracting model with complete, but unverifiable information and "ex ante renegotiation" (Moore and Repullo 1988, Maskin 1999). The timing in this model is: two contracting parties form a contract, they making unverifiable investments, they have an opportunity to renegotiate their contract, they make announcements, and then an external enforcer compels a trade action and transfer (modelled as a public action). The standard contracting model lends itself to mechanism-design analysis, whereby one calculates the set of implementable state-contingent (as a function of investments) outcomes.

In terms of enriching the standard model to allow more dynamic elements, while maintaining the structure of unverifiable actions followed by verifiable ones, there are two obvious directions for study: (1) examine situations in which the unverifiable investments can take place over time, keeping the verifiable trade action as one-shot, and (2) examine situations in which trade involves a sequence of verifiable productive actions, keeping the unverifiable investment as one-shot. Che and Sakovics (2004a,b) work in the first direction by examining durable investment opportunities in an infinite-horizon setting. Herein, I work in the second direction.

Figure 1 displays the time line of the model that I analyze. Relative to the standard contracting model with complete/unverifiable information and ex ante renegotiation, the model here has $T$ periods in which verifiable productive actions are taken; the standard model is the special case of $T = 1$. I refer to the case of $T > 1$ as the "multiple-trading-periods model."

The complication for contracting in the case of multiple trading periods is that the players can renegotiate their contract between productive actions in successive periods of time (that is, at the beginning of each period $2, 3, \ldots, T$). Renegotiation causes distortions that generally interfere with contractual objectives. For example, suppose the parties send an

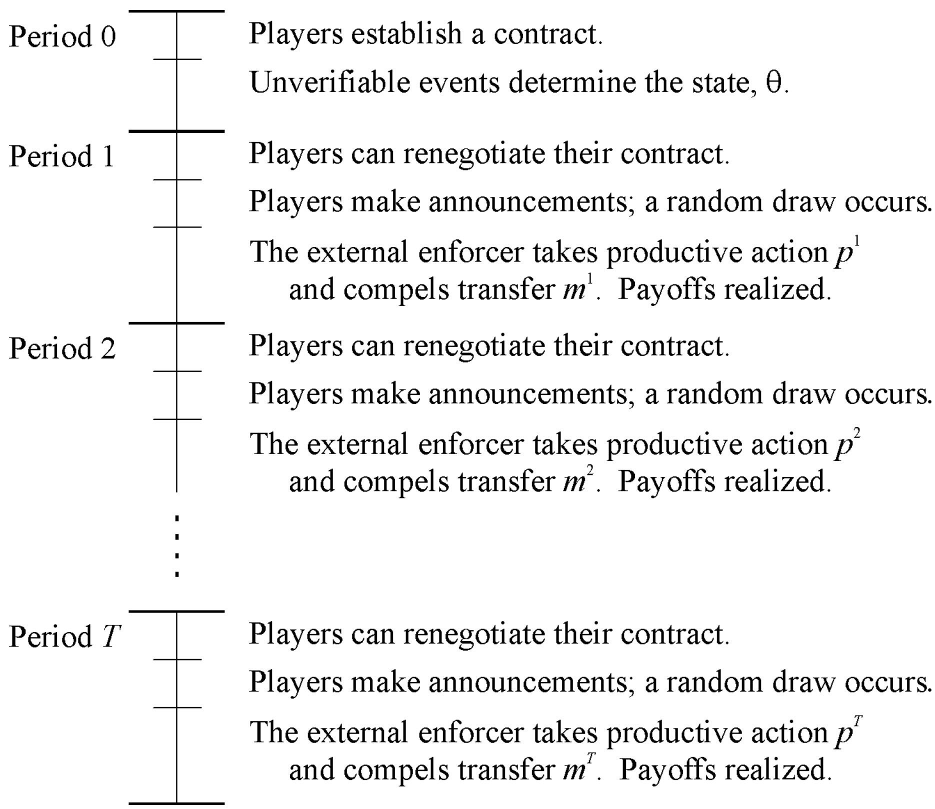


Figure 1: Time line of a contractual relationship with $T$ periods of productive actions.

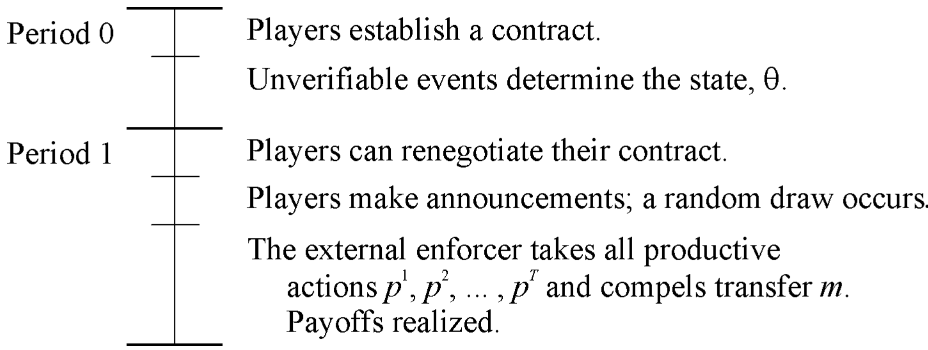


Figure 2: Time line of the simplified model.

out-of-equilibrium announcement profile in period 1. Then implementation relies on specifying for this message profile a sequence of productive actions $p^1, p^2, \ldots, p^T$ that sufficiently punishes the players for deviating in period 1. Such a sequence may be inefficient in the actual state, in which case the players will want to renegotiate. Although the players are not able to renegotiate productive action $p^1$ following their first-period messages, they will be able to renegotiate the productive actions specified for later periods. Thus, at least for productive actions $p^2, p^3, \ldots, p^T$, there is a sense in which renegotiation has an "interim" (after messages, before productive actions) flavor. This suggests that settings with multiple trading periods may have tighter constraints on implementation than would be present if commitment to a sequence of productive actions were possible.

Calculating the implementable set for the multiple-trading-periods model can be difficult because it involves quite a few incentive conditions applied at different points in time. Essentially, one must perform a mechanism-design analysis for each period, using as continuation values some state-contingent value functions that can be implemented from the start of the next period.

The main result in this paper addresses both the effect of renegotiation in settings with multiple trading periods and the issue of how to calculate the implementable set. The analysis shows that, if the players can communicate with the external enforcer before each productive action, then the distorting effects of renegotiation in periods $2, 3, \ldots, T$ can be counteracted. To be precise, I prove that implementation of state-contingent outcomes in the multiple-trading-periods model is equivalent to implementation in the simplified and standard mechanism-design model described in Figure 2. In the simplified model, the external enforcer takes the sequence of productive actions all at once and there is no opportunity for renegotiation between productive actions. Whereas the multiple-trading-periods model is quite difficult to analyze directly (in terms of added incentive conditions to check), the simplified model entails quite straightforward analysis.

The next section discusses aspects of an example to generate the intuition behind the main result and the other results in this paper. Section 2 presents the general model in detail and contains the main result. Section 3 contains the supporting analysis, including how the contracting model can be formulated as a recursive mechanism-design problem.

In Section 4, I discuss contractual form and the special case in which there is a durable trading opportunity. I show how optimal contracts can be structured to utilize messages in a minimal way. I also prove that, in stationary environments—in which there are no transitory costs of reversing a productive action—it is optimal to use stationary contracts that treat periods independently. Further, reversal costs have a positive effect on implementability. Concluding remarks are in Section 5, and the Appendix contains proofs of the lemmas.

By analyzing how trade involves specific sequences of productive actions, this paper furthers my general research agenda, part of which is to (i) discover how the technological details of contractual settings influence outcomes, (ii) demonstrate the importance of carefully modeling these details, and (iii) provide a flexible framework that facilitates the analysis of various applications. By "technological details," I mean the nature of productive actions, the actions available to external enforcers, the manner in which agents communicate and negotiate with one another, and the exact timing of these various elements in a given contractual relationship.[1] The results herein show that it is instructive to model the sequence of productive actions that compose trade, but that some aspects of the technology can be safely abstracted from (in particular, renegotiation between productive actions when players can send messages).

# 1 Example

This section contains partial analysis of an example to illustrate the issues addressed in the rest of the paper. Several variants of the example are presented and related. I do not analyze the variants completely, but try to provide enough details to yield intuition.

## A Contractual Relationship with Two Trading Periods

Consider a setting in which a buyer (player 1), a seller (player 2), and an external enforcer (who is not a strategic player) interact over three periods of time, numbered 0, 1, and 2. Here is a description of the production technology and external enforcement in the contractual relationship.

At the beginning of period 0, the contracting parties (players 1 and 2) form a contract that includes a specification of how the external enforcer should take public actions (described below) as a function of announcements that the players will make in subsequent periods. At the end of period 0, the seller chooses a level of investment, which is either "good" (denoted g) or "bad" (denoted b). This investment decision determines the *state* of the relationship: g or b. The good investment entails an immediate cost $\eta \in [0, 4]$ borne by the seller; the bad investment costs nothing.

[1] In other papers, I focus on the nature of productive actions (Watson 2005a), the mechanics of evidence production (Bull and Watson 2004a,b), and contract writing and renegotiation costs (Schwartz and Watson 2004, Brennan and Watson 2002). These papers and the present one show that the technological details can matter significantly.

At the beginning of period 1, the players have an opportunity to renegotiate their contract. Then they simultaneously make announcements. Finally, an external enforcer compels the parties to take a productive action and compels a monetary transfer between the parties. The productive action is either "trade" or "not." Formally, the productive action and monetary transfer are modelled as a public action taken by the external enforcer.[2] At the end of period 1, the players receive payoffs. The following table describes the players' first-period payoffs as a function of the state and the productive action, holding aside the seller's investment cost and the externally enforced transfer.

| | Trade | Not |
|---|---|---|
| State g | $2, 0$ | $1, 0$ |
| State b | $0, 0$ | $0, 0$ |

The buyer's payoff is listed first. Assume that monetary transfers affect the players' payoffs in an additive fashion.

In period 2, the players interact as they did in period 1: they have an opportunity to renegotiate their contract and then they make announcements. The external enforcer observes the announcements and takes the period-2 public action, which, as in period 1, is a transfer and whether to trade. Payoffs in period 2 are just as described for period 1—that is, they are given by the table shown in the previous paragraph—except that the seller bears an additional cost $\kappa$ if the productive action in period 1 was "trade" and the productive action in period 2 is "not."

To interpret the model, think of the buyer's value of trade as a flow over two periods. For example, the transaction involves a piece of software that can be installed on the buyer's computer and is valuable to the buyer in both periods 1 and 2. If "trade" occurs in the first period, then the buyer enjoys the benefit of the software in this period. If "trade" in the first period is followed by "trade" in the second period, then the latter merely means that the software remains installed so that the buyer gets the benefit of the software also in period 2. However, if "trade" were selected in the first period and "not" is chosen in the second period, then the latter means that the software is uninstalled; in this case, not only does the buyer miss the value of the software in the second period, but the seller pays $\kappa$ to perform the uninstallation. Another possibility is that "not" is chosen in period 1 and then "trade" is chosen in period 2; in this case, the buyer obtains the value of the software only in the second period. The buyer's gain of 1 when "not" is selected in a given period is interpreted as a direct benefit of the seller's investment that is not contingent on trade.

The parameter $\kappa$ measures the degree to which the decision "trade" in the first period is reversible in the second period. The case of $\kappa = 0$ has frictionless reversibility. Note that, regardless of $\kappa$, the efficient outcome of the contractual relationship is good investment

[2]The interpretation is that, in reality, the players themselves take the productive action. However, the productive action is verifiable and so the external enforcer can force any particular public action to be chosen. In the setting studied herein, where there is no renegotiation between the time messages are sent to the external enforcer and when the trade action occurs, there is no distortion in modeling the productive action as public. See Watson (2005a) on this point.

followed by trade in both periods, which generates a joint value of $4 - \eta$ (nonnegative by assumption).

A contract may be fairly complicated in this two-trading-periods model. Specifically, it specifies two functions, one for each period. For period 1, the contract specifies the first-period public action as a function of the announcements made in period 1. For period 2, the contract specifies the second-period public action as a function of announcements made in *both* periods. The contract, along with equilibrium behavior in the message phase in every state and in both periods, implies the implementation of a *value function* $v$ that gives the players' payoff vector from the beginning of period 1 as a function of the state. To understand what can be achieved in the contractual relationship, we calculate the set of implementable value functions. Obtaining the efficient outcome relies on giving the seller the incentive to invest at level g. That is, from period 1, the difference between his payoff in states g and b must be at least $\eta$, meaning that the value function satisfies

$$v_2(\mathrm{g}) - v_2(\mathrm{b}) \geq \eta. \tag{1}$$

## A Simplified Model

We convert the two-trading-periods model into a one-trading-period "simplified model" by letting the external enforcer take both of the productive actions (formerly taken in periods 1 and 2) at the end of the first period. That is, the simplified model is defined by consolidating the productive actions into one trading period. The players' payoffs are otherwise as described in the two-trading-periods model.

To be more precise, the simplified model runs for two periods, 0 and 1. Period 0 interaction works as in the original model: the players form a contract and then the seller chooses either good or bad investment, paying $\eta$ if he makes the good investment. The set of states is $\{\mathrm{g}, \mathrm{b}\}$ as before. In period 1, the players have an opportunity to renegotiate their contract and then they make announcements. At the end of period 1, the external enforcer takes a public action, which is to compel a transfer and a sequence of two productive actions (each either "trade" or "not"). Payoffs are determined as in the original model. For example, in state g if the public action is no transfer and the productive-action sequence is (trade, not), then the buyer obtains 3—that is, 2 from the first productive action and 1 from the second—and the seller gets $-\kappa$. As another example, if in state g the public action is a transfer of 2 from the buyer to the seller and the productive-action sequence (trade, trade), then the buyer obtains 2 and the seller gets 2. The seller's cost of good investment is not included here.

The simplified model is basically the original model without the renegotiation and message phases that occur between the public action in period 1 and the public action in period 2. A contract in the simplified model specifies the sequence of public actions as a function of the announcement profile in the single trading period. The contract, along with equilibrium behavior in the message phase in every state, implies the implementation of a value function $v$ described earlier. Obtaining the efficient outcome relies on implementing a value function that satisfies condition 1, the same as in the original two-trading-periods model.

## Notes on Implementation for $\kappa = 0$: Stationary Contracts

In this subsection, I partially analyze the original and simplified models for the case of $\kappa = 0$. Intuition emerges suggesting that the simplified model characterizes the scope of contracting in the original model and that stationary contracts are optimal in the original model. I start by sketching some of the conditions for achieving efficient investment and trade in the simplified model.

**Simplified Model:**

The simplified model is essentially a static, complete-information mechanism-design problem and, as such, is quite easy to analyze. The mechanism-design problem is defined by the set of states $\{g, b\}$, the set of feasible public actions (transfers, whether to trade at the first opportunity, and whether to trade at the second opportunity), and the payoffs as a function of the state and public action. The players' contract is a mechanism, specifying message spaces for the players and a function relating the public action to message profiles. The mechanism, along with Nash equilibrium behavior in the message phase in each state, implies a value function.[3]

Consider whether it is possible to implement a value function satisfying condition 1. One way to achieve this is to have the following state-contingent public action:

*State g:* (Trade, trade) and transfer at least $\eta$ from the buyer to the seller.

*State b:* (Not, not) and no transfer.

Unfortunately, this cannot be directly imposed by the external enforcer because the external enforcer cannot observe the state. State-contingent public actions require the players to reveal the state through their messages, but the players' may not have the incentive to do so.

To determine the value functions that can be implemented in the simplified model, we can perform a standard mechanism-design analysis. The revelation principle applies, so we can focus on direct-reporting mechanisms (where each player's message space is $\{g, b\}$) and truthful reports in equilibrium. With this restriction, a mechanism is given by a table that specifies public actions for each of the four message profiles.[4] Further, we look for mechanisms for which the message profile $(g, g)$ is a Nash equilibrium in state g and the message profile $(b, b)$ is a Nash equilibrium in state b. The players can renegotiate their contract at the beginning of period 1, so this is a setting of "ex ante renegotiation." As discussed herein and in the related literature, the effect of ex ante renegotiation is represented by constraining attention to mechanisms that yield efficient outcomes in equilibrium in both

[3]This is "weak Nash implementation," because we look for Nash equilibria in the message phase and will not require uniqueness.

[4]It will not be necessary to consider randomization over public actions.

states. This means that we constrain attention to mechanisms that specify the productive-action (trade, trade) when the message profile is (g, g).

Here is a candidate mechanism:

| B \ S | g | b |
|---|---|---|
| g | (trade, trade) and transfer $k$ from B to S | (not, not) and transfer $k'''$ from B to S |
| b | (not, not) and transfer $k'$ from B to S | (not, not) and transfer $k''$ from B to S |

Table $\ast$

This mechanism implies the following static message games for the two states.

| B \ S | g | b |
|---|---|---|
| g | $-k,\ k$ | $-k''',\ k'''$ |
| b | $-k',\ k'$ | $-k'',\ k''$ |

Message game in state b

| B \ S | g | b |
|---|---|---|
| g | $4-k,\ k$ | $2-k''',\ k'''$ |
| b | $2-k',\ k'$ | $2-k'',\ k''$ |

Message game in state g

Suppose this mechanism implements value function $v$. Then we have $v(\mathrm{g}) = (4-k, k)$ and $v(\mathrm{b}) = (-k'', k'')$. Regarding incentives to truthfully report the state, note that the off-diagonal cells of the mechanism must serve to punish the players. For instance, the public action specified for message profile (b, g) must be sufficient to simultaneously (i) dissuade the buyer from declaring the state to be b when the state is actually g and (ii) discourage the seller from declaring "g" in state b. Thus, we need $v_1(\mathrm{g}) \geq 2 - k'$ and $v_2(\mathrm{b}) \geq k'$. Summing these inequalities and substituting for $v_1(\mathrm{g})$ using the fact that $v_1(\mathrm{g}) + v_2(\mathrm{g}) = 4$, we obtain

$$v_2(\mathrm{g}) - v_2(\mathrm{b}) \leq 2, \qquad (2)$$

which contradicts inequality 1 for values of $\eta$ above 2. It is easy to also check that specifying any other sequence of productive actions—such as (trade, trade)—for message profile (b, g) leads to an even tighter bound. Thus, in the case in which $\eta \in (2, 4]$, it is efficient for the seller to make the good investment but there is no contract that gives him the incentive to do so.

**Original Two-Trading-Periods Model:**

Implementation in the original two-trading-periods model is not as straightforward as is the case in the simplified model. Consider, for instance, a contract like that discussed above, where the public action is prescribed by Table $\ast$. Specifically, suppose the players use a contract in which the external enforcer notes the announcements make in period 1 but ignores any announcements made in period 2. The external enforcer takes the sequence of productive actions prescribed in the table (now taken over periods 1 and 2) and compels the prescribed transfer in both periods 1 and 2.

For this contract, examine what would happen in state g if the message profile in period 1 is $(b, g)$. In this contingency, the pubic action in period 1 is "not" with the transfer $k'$. However, the players have an opportunity to renegotiate the contract at the beginning of period 2 to avoid the inefficient productive action "not" from being taken in the second period. For simplicity, suppose that the buyer (player 1) has all of the bargaining power and gets the full surplus of renegotiation. The surplus is the difference in the joint value of "trade" and "not" in the second period, which is 1. Then, from the message phase of period 1, the buyer's payoff of reporting "b" when the state is actually g is

$$[1 - k'] + [(1 - k') + 1] = 3 - 2k'.$$

The first bracketed term is the buyer's period 1 payoff from the prescribed public action. The second bracketed term is the same amount, which is now the buyer's disagreement value for renegotiation, plus the renegotiation surplus. The seller's payoff of reporting "g" when the state is actually b is $2k'$.

Suppose this contract implements value function $v$. As before, the renegotiation opportunities imply efficiency in each state, so $v_1(g) + v_2(g) = 4$. Without fully analyzing behavior, we see at least that equilibrium conditions for messages in the first period include $v_1(g) \geq 3 - 2k'$ and $v_2(b) \geq 2k'$. Summing these and using the efficiency condition for state g, we obtain

$$v_2(g) - v_2(b) \leq 1.$$

Comparing this with inequality 2, we conclude that a simple translation of the contract studied for the one-trading-period case fares worse in the two-period setting.

Utilizing second-period messages allows for a greater scope of implementation. To see this, consider a stationary contract that treats periods 1 and 2 separately. In period 1, the public action is a function of the first-period messages, as prescribed by the following table.

| B \ S | g | b |
|---|---|---|
| g | trade and transfer $k$ from B to S | not and transfer $k'''$ from B to S |
| b | not and transfer $k'$ from B to S | not and transfer $k''$ from B to S |

Table $**$

In period 2, the public action is again as prescribed in Table $**$, but as a function of the second-period messages (and ignoring first-period messages).

This stationary contract creates two separate instances of the same single-period mechanism. Let $v^2$ denote the value function from the start of period 2 that is implemented by this contract. Performing the kind of analysis that was done for the one-trading-period case, we find that

$$v_2^2(g) - v_2^2(b) \leq 1,$$

with no renegotiation at the start of period 2. Because the implemented value from the start of period 2 does not depend on behavior in period 1, incentives in period 1 can be determined without considering the period 2 continuation. The equilibrium conditions therefore support transfers in period 1 that give the seller 1 more in state g than in state b. Adding the margins supported in the two periods implies

$$v_2(\mathrm{g}) - v_2(\mathrm{b}) \leq 2,$$

which coincides with inequality 2 for the implementable value functions.

## Notes on Implementation for $\kappa > 0$: The Effect of Irreversibility

Next consider the case of $\kappa > 0$, so that there is a cost of reversing the productive action. Relative to the sketch in the previous subsection, it is not clear what one should expect with irreversibility, where stationary contracts are generally not optimal and the analysis of contracts is quite a bit more complicated. In fact, the example gives reason to believe that the connection between multiple-trading-periods models and one-trading-period models holds beyond the case of frictionless reversibility and that reducing reversibility (raising $\kappa$) has a positive effect on the scope of implementation.

**Simplified Model:**

Consider first the simplified model and focus on the implications of a contract that specifies for message profile $(\mathrm{b}, \mathrm{g})$ the productive-action sequence (trade, not) with no transfer. Suppose value function $v$ is implemented. In the first-period message phase, the buyer's equilibrium condition in state g is thus

$$v_1(\mathrm{g}) \geq 2 + 1,$$

whereas the seller's equilibrium condition in state b is

$$v_2(\mathrm{b}) \geq 0 - \kappa.$$

Summing these and using $v_1(\mathrm{g}) + v_2(\mathrm{g}) = 4$, we get

$$v_2(\mathrm{g}) - v_2(\mathrm{b}) \leq 1 + \kappa. \tag{3}$$

**Original Two-Trading-Periods Model:**

Next consider the original two-trading-periods model and again focus on the message profile $(\mathrm{b}, \mathrm{g})$ sent in the first period. Let us ask whether we can find a contractual specification for this message profile that can match what was achieved in the simplified model with productive-action sequence (trade, not). In the simplified model, the players could commit to the sequence (trade, not) contingent on the message profile $(\mathrm{b}, \mathrm{g})$. However, in the two-trading-periods model, they cannot commit to this sequence and would in fact renegotiate "not" to "trade" in the second period. The key issue is whether a contract can be designed so that, even with the efficient productive action in each state, the relevant payoffs can be held down to the levels that would be achieved by committing to (trade, not). Here, the relevant payoffs are player 1's in state g and player 2's in state b, because these are the critical ones for testing whether the players want to unilaterally deviate to message profile $(\mathrm{b}, \mathrm{g})$.

We begin by analyzing the value functions that can be implemented from the start of period 2. Suppose that productive action "trade" was taken in period 1. Then, the payoffs in period 2, in terms of the state and period-2 productive action (holding aside the transfer) are given by:

| | Trade | Not |
|---|---|---|
| State g | $2, 0$ | $1, -\kappa$ |
| State b | $0, 0$ | $0, -\kappa$ |

Let $v^2$ be a value function from the start of period 2 that can be implemented by a contract, conditional on "trade" chosen in the first period. Assume efficiency in each state, to capture renegotiation at the start of period 2. Analyzing the message game in period 2 using the same techniques described earlier, we find that the best way to punish the players for sending the message profile $(\mathrm{b}, \mathrm{g})$—that is, punishing the buyer in state g and the seller in state b—is to specify productive action "not" for this message profile. Summing the buyer's equilibrium condition for the message phase in state g with the seller's condition in state b, we obtain

$$v_1^2(\mathrm{g}) + v_2^2(\mathrm{b}) \geq 1 - \kappa. \tag{4}$$

Moving to period 1, consider a contract that has the following features. If the message profile in period 1 is $(\mathrm{g}, \mathrm{g})$ then the second-period announcements are ignored, "trade" is selected in both periods, and some transfer is made. If the message profile in period 1 is $(\mathrm{b}, \mathrm{b})$, the same outcome is specified, except with a possibly different transfer. If the message profile is $(\mathrm{b}, \mathrm{g})$, then "trade" is selected in the first period and, in the second-period continuation, the contractual terms are arranged to support the continuation value $v^2$ that achieves the bound of inequality 4. Suppose value function $v$ is implemented. In the first-period message phase, the buyer's equilibrium condition in state g is thus

$$v_1(\mathrm{g}) \geq 2 + v_1^2(\mathrm{g}),$$

whereas the seller's equilibrium condition in state b is

$$v_2(\mathrm{b}) \geq 0 + v_2^2(\mathrm{b}).$$

Summing these, using the bound of inequality 4 and that $v_1(\mathrm{g}) + v_2(\mathrm{g}) = 4$, we get

$$v_2(\mathrm{g}) - v_2(\mathrm{b}) \leq 1 + \kappa.$$

Note that this is the same condition derived by using productive-action sequence (trade, not) in the simplified model (equation 3). The idea is that, for the out-of-equilibrium message profile $(\mathrm{b}, \mathrm{g})$ in period 1, we do not need to commit to "not" in period 2. Rather, we just need sum of period 2 continuation values $v_1^2(\mathrm{g}) + v_2^2(\mathrm{b})$ to be the same as if "not" were chosen. These continuation values can be implemented in the example, while still having efficient productive actions in equilibrium so that, for example, $v_1^2(\mathrm{g}) + v_2^2(\mathrm{g}) = 2$.

Also note that, as $\kappa$ increases, so does the extent to which the seller can be motivated to make the good investment in period 0. If $\kappa$ exceeds 1, then implementability strictly improves over the case of $\kappa = 0$. Furthermore, if $\kappa \geq 3$ then efficient investment can be induced, regardless of the cost $\eta$.

### Insights From the Example

The analysis performed earlier in this section is incomplete because it examined only some of the required incentive conditions for implementation. However, the example hints at several results that may hold generally. First, the example suggests a connection between any multiple-trading-periods model and a related one-trading-period model. The main result of this paper is that there is a formal connection; indeed, implementability conditions in any given multiple-trading-periods model are equivalent to those of a related one-trading-period model. Second, the example suggests that stationary contracting environments, with frictionless reversibility of productive actions, are special in that one can focus on simple stationary contracts. Third, the example suggests that irreversibility of productive actions serves to enhance implementability and improve the prospects for efficient investment and trade. These claims are also investigated and validated in the general modeling exercise of this paper.

## 2 The Contracting Model and Main Result

The model features a long-term contractual relationship with complete but unverifiable information and external enforcement. Interaction occurs as shown in Figure 1. Here is a description of the basic technology of the relationship. The players interact over $T + 1$ periods of time, starting in period 0 and ending in period $T$. In period 0, individual actions of the players, and possibly random events, determine the state $\theta \in \Theta$, which is commonly observed by the players but not verifiable to the external enforcer. Most of the analysis does not concern how the state is determined, rather concentrating on what state-contingent continuation values can be implemented from period 1. In each period $t \in \{1, 2, \ldots, T\}$, which is called a *trading period*, the external enforcer takes a public action and the players receive payoffs. The public action includes a specification of monetary transfers induced by the external enforcer.

The public action in period $t$ is written as $(p^t, m^t)$, where $m^t$ denotes the monetary transfer between the players (the "transfer") and $p^t$ is the non-monetary part (call it the "productive action"). The transfer is a vector $m^t = (m_1^t, m_2^t)$, where $m_i^t$ denotes the amount of money received by player $i$. Assume that transfers are balanced, in that

$$m^t \in \mathbf{R}_0^2 \equiv \{(m_1', m_2') \mid m_1' + m_2' = 0\}.$$

[5] The productive action is an element of a given set $P^t$, assumed compact. Let $\lambda^t = (p^1, p^2, \ldots, p^t)$ describe the list of public actions taken from period 1 to period $t$. Let

[5]The assumption of balanced transfers plays an important role in models like the one studied here. It

$\lambda = (p^1, p^2, \ldots, p^T)$ describe the sequence of productive actions for all $T$ periods. Define

$$\Lambda^t = \{(p^1, p^2, \ldots, p^t) \mid p^\tau \in P^\tau \text{ for } \tau = 1, 2, \ldots, t\}$$

and write $\Lambda \equiv \Lambda^T$.

Payoffs in a given period are a function of the state and of the history of public actions to this point in time. Assume that payoffs are linear in money and that the payoff in a given period does not depend on the monetary transfers made in earlier periods. Thus, for any $t \geq 1$, the period-$t$ payoff vector is given by a function $u^t : \Lambda^t \times \Theta \to \mathbf{R}^2$, so that if $\lambda^t$ is the history of productive actions through period $t$ and $m^t$ is the monetary transfer in period $t$, then the payoff vector in period $t$ is $u^t(\lambda^t, \theta) + m^t$. Payoffs for the entire game are the sum of payoffs in the individual periods. Assume that the functions $u^t$ are continuous in $\lambda^t$.

Players can communicate with the external enforcer in each period by sending public messages. The external enforcer also manages a public randomization device as directed by the contracting parties. Players contract on how they will communicate with the external enforcer and on how the public actions will be selected as a function of messages they send and the realization of the public randomization device.

The contract specifies, for each period $t \geq 1$, message (announcement) spaces $A_1^t$ and $A_2^t$ for the players, as well as the space $A_0^t$ and distribution $\alpha_0^t \in \Delta A_0^t$ for the public random variable. Call $a^t = (a_0^t, a_1^t, a_2^t)$ the "message profile" or "announcement profile." Let $A^t \equiv A_0^t \times A_1^t \times A_2^t$ and let $H^t \equiv A^1 \times A^2 \times \cdots \times A^t$ be the set of $t$-period histories of message profiles. Also, write $H^0 = \{h^{\text{null}}\}$, where $h^{\text{null}}$ is the null history at the start of the first period. Assume that there is a given set $\mathcal{A}$ such that $A^t$ is restricted to be a subset of $\mathcal{A}$.

The contract also specifies functions $\rho^t : H^t \to P^t$ and $\mu^t : H^t \to \mathbf{R}_0^2$ that prescribe, respecively, the productive action and transfer in period $t$ as a function of history of message profiles. Summarizing, the externally enforced component of the players' contract is

$$c = \langle A^t, \alpha_0^t, \rho^t, \mu^t \rangle_{t=1}^T.$$

The externally enforced component is one part of the players' contract (the other part being the self-enforced part described below), but for brevity I often describe $c$ as simply a "contract." Let $C$ be the space of such contracts. Note that $H^t$ is defined relative to a given contract $c$; this dependence can be highlighted by writing $H^t(c)$.

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is appropriate as a representation of one or both of the following realistic features of real contractual settings. First, holding aside litigation costs, courts do not generally perform "money burning." Second, court enforcement typically occurs at the end of productive relations (period $T$ here), at which point all productive actions have been taken and the court merely compels monetary transfers. If a contract specified that money be burned, or given to a third party, the contracting parties would renegotiate the contract—and retain the money—just before going to court at the end of their relationship. Remember that the modeling choice of treating verifiable productive actions as public is for mathematical convenience and is innocuous so long as renegotiation is not possible between messages and productive actions in each period. In reality, productive actions are taken by the contracting parties themselves, but they are "forced" in that the players anticipate punishing transfers if they fail to perform.

In each period, the players have an opportunity to renegotiate their contract. Note that, at a given period $t \geq 1$, the messages, random draws, and public actions in preceding periods have already occurred; further, these are recorded in the history $h^{t-1}$ and in the externally enforced contract in place. Thus, there is no loss in assuming that the players can renegotiate over only the message spaces and specification of public actions for periods $t, t+1, \ldots, T$. Let $C^t(c)$ denote the set of externally enforced contracts that are feasible in period-$t$ renegotiation when $c$ is the outstanding contract.[6]

To describe payoffs in each period in terms of the contract and announcements, a bit more notation is helpful. For a given contract $c$, a $t$-period message history $h^t = (a^1, a^2, \ldots, a^t) \in H^t(c)$ implies the sequence of public actions from period 1 to period $t$. Let $\phi^t(c, h^t)$ denote this sequence of public actions. That is,

$$\phi^t(c, h^t) = (\rho^1(a^1), \rho^2(a^1, a^2), \ldots, \rho^t(a^1, a^2, \ldots, a^t)).$$

The payoff vector at the end of period $t$, in terms of the contract and message history, can then be written

$$U^t(c, h^t, \theta) \equiv u^t(\phi^t(c, h^t), \theta) + \mu^t(h^t).$$

On the right side of this expression, $\phi^t$ is defined by the $\rho^\tau$'s, which, along with $\mu^t$, are as specified by the contract $c$.

Writing the payoffs as a function of the contract and message history, here is a more detailed version of the time line shown in Figure 1:

In period 0:

- Contract formation, with externally enforced component $c$.
- Realization of the state, $\theta \in \Theta$.

In each period $t \in \{1, 2, \ldots, T\}$:

- Renegotiation to contract $c'$. If no renegotiation occurs, then $c'$ is the contract outstanding from the previous period.
- Players send messages $a_1^t \in A_1^t$ and $a_2^t \in A_2^t$; public random draw $a_0^t \in A_0^t$ is realized.
- Public action $(p^t, m^t)$ is taken by the external enforcer, as directed by the contract $c'$, and the players received payoffs $u^t(\lambda^t, \theta) + m^t = U^t(c', h^t, \theta)$ for the period.

## Equilibrium and Implementation

As described above, the externally enforced component of the players' contract is given by $c$. There is also a self-enforced component, which refers to how the players coordinate their behavior in the message phase of each period. Rational behavior is analyzed

[6] That is, for any $\hat{c} = \langle \hat{A}^\tau, \hat{\alpha}_0^\tau, \hat{\rho}^\tau, \hat{\mu}^\tau \rangle_{\tau=1}^T$, we have $\hat{c} \in C^t(c)$ if and only if $\hat{A}^\tau = A^\tau$, $\hat{\alpha}_0^\tau = \alpha_0^\tau$, $\hat{\rho}^\tau = \rho^\tau$, and $\hat{\mu}^\tau = \mu^\tau$ for $\tau = 1, 2, \ldots, t-1$.

using the notion of *contractual equilibrium*, which combines (i) a bargaining solution to describe how the players renegotiate at the beginning of each period with (ii) a Nash equilibrium (that is, individual best response behavior) in the message phase of each period.[7] Regarding item (i), consider the standard bargaining solution in which the players negotiate to the Pareto frontier and divide the surplus of negotiation by the bargaining weights $\pi = (\pi_1, \pi_2)$ that are fixed over time.[8] The surplus is defined relative to a disagreement point, which is a continuation value that the players expect to be realized if they fail to reach an agreement. Assume that this disagreement point is an arbitrary selection from the supportable continuation values with the existing contract.

Contractual equilibrium is represented by sets of continuation values for the various contingencies that can arise during the game. Let

$$\Xi^t \equiv \{(c, h^{t-1}) \mid c \in C,\ h^{t-1} \in H^{t-1}(c)\}$$

be the set of consistent pairs of contracts and $(t-1)$-period message histories. For each period $t \geq 1$, the continuation values are given by a correspondence $V^t : \Xi^t \times \Theta \rightrightarrows \mathbf{R}^2$. That is, $V^t(c, h^{t-1}, \theta)$ is the set of continuation values that are consistent with the behavioral theory from the start of period $t$, with contract $c$ outstanding, the history $h^{t-1}$ of messages in previous periods, and in state $\theta$.

Contractual equilibrium is derived inductively, by calculating supportable continuation values from period $t$ as a function of the posited sets of continuation values from period $t+1$. To write the conditions, take as given the function $V^{t+1}$. Suppose that, in state $\theta$ at the end of period $t$, externally enforced contract $c$ is in force and the history of messages is $h^t$. Then the continuation value from period $t+1$ is some vector $v^{t+1}(h^t, \theta) \in V^{t+1}(c, h^t, \theta)$. Here, the function $v^{t+1} : H^t(c) \times \Theta \to \mathbf{R}^2$ describes the continuation value that the players anticipate selecting, as a function of the state and history.

In the message phase of period $t$, then, player $i$ expects the (period plus continuation) payoff

$$E\,[U_i^t(c, (h^{t-1}, a_0, a_i, a_j), \theta) + v_i^{t+1}((h^{t-1}, a_0, a_i, a_j), \theta) \mid \alpha_0^t]$$

when he chooses message $a_i$ and the other player sends message $a_j$. Here, the expectation is taken with respect to the random draw $a_0$, which has distribution $\alpha_0^t$. A mixed message profile $(\alpha_1^*, \alpha_2^*) \in \Delta A_1^t \times \Delta A_2^t$ is a Nash equilibrium of the period-$t$ message phase in state $\theta$ if

$$\begin{aligned} y_i &= E\,[U_i^t(c, (h^{t-1}, a_0, a_i, a_j), \theta) + v_i^{t+1}((h^{t-1}, a_0, a_i, a_j), \theta) \mid \alpha_0^t, \alpha_i^*, \alpha_j^*] \\ &\geq E\,[U_i^t(c, (h^{t-1}, a_0, a_i', a_j), \theta) + v_i^{t+1}((h^{t-1}, a_0, a_i', a_j), \theta) \mid \alpha_0^t, \alpha_j^*] \\ &\qquad \text{for every } a_i' \in A_i^t \text{ and } i = 1, 2. \end{aligned} \tag{5}$$

[7] This is a natural equilibrium concept when the renegotiation phase is modelled cooperatively. See Watson (2005b) for details on the general notion of contractual equilibrium, including a full discussion of the relation between cooperative and noncooperative models of negotiation.

[8] The assumption of fixed $\pi$ is not important to the analysis herein; in fact, the characterization results are unchanged even if $\pi$ is history-dependant.

In terms of play from period $t$, the self-enforced component of the players' contract is the selection of continuation values for the start of the next period and the selection of a Nash-equilibrium message profile for the current period.

Write $Y^t(c, h^{t-1}, \theta)$ as the set of supportable continuation values in state $\theta$ from the start of period $t$, with the externally enforced contract $c$ and the history of messages $h^{t-1}$. That is,

$$\begin{aligned} Y^t(c, h^{t-1}, \theta) \;\equiv\; \{ & y \in \mathbf{R}^2 \mid \text{there exist } v^{t+1} : H^t(c) \times \Theta \to \mathbf{R}^2, \alpha_1^* \in \Delta A_1^t, \\ & \text{and } \alpha_2^* \in \Delta A_2^t, \text{ such that condition 5 holds and} \\ & v^{t+1}(h^t, \theta) \in V^{t+1}(c, h^t, \theta) \text{ for all } h^t \in H^t(c) \text{ and } \theta \in \Theta \}. \end{aligned} \tag{6}$$

Suppose that the state is $\theta$, the history of messages through period $t-1$ is $h^{t-1}$, and that externally enforced contract $c$ in force at the end of period $t-1$. Consider contract renegotiation at the start of period $t$. If the players fail to renegotiate, they will expect some continuation value $\underline{y}$ from the set $Y^t(c, h^{t-1}, \theta)$, which is their disagreement point for the renegotiation phase.[9] Because the players can renegotiate to any externally enforced contract $c'$, the set of continuation values over which the players are negotiating is

$$\bigcup_{c' \in C^t(h^{t-1})} Y^t(c', h^{t-1}, \theta).$$

The players will select a continuation value that maximizes their joint value from period $t$, which is

$$\gamma^t(\lambda^{t-1}, \theta) \equiv \max_{\lambda \in \Lambda} \sum_{\tau=t}^{T} [u_1^\tau(\lambda^\tau, \theta) + u_2^\tau(\lambda^\tau, \theta)]$$

subject to $\lambda^{t-1} = \phi^{t-1}(c, h^{t-1})$. In the maximand above, for each $\tau \in \{t, t+1, \ldots, T\}$, $\lambda^\tau$ is defined as the first $\tau$ elements of $\lambda$. Regarding the case of $t = 1$, recall that there is only the null 0-length history, so $\gamma^1$ is a function of only $\theta$; in this case, there are no restrictions on the choice of $\lambda$.

The reason $\gamma^t$ identifies the maximum joint value from period $t$ is that (i) players have common knowledge of the state and (ii) they can always write a new contract that prescribes any desired sequence of public actions from period $t$. Thus, they are selecting $\lambda$ constrained only by the sunk public actions from periods 1 through $t-1$ (hence the condition that $\lambda^{t-1} = \phi^{t-1}(c, h^{t-1})$). This specification yields the maximum joint value from period $t$. Furthermore, it would not be renegotiated in future periods because there is no more surplus to be achieved.

The theory of negotiation implies the following relation between the sets $Y^t$ and the sets of renegotiated continuation values $V^t$.

$$\begin{aligned} V^t(c, h^{t-1}, \theta) \;=\; \{ & y \in \mathbf{R}^2 \mid \text{there exists } \underline{y} \in Y^t(c, h^{t-1}, \theta) \text{ such that} \\ & y = \underline{y} + \pi[\gamma^t(\phi^{t-1}(c, h^{t-1}), \theta) - \underline{y}_1 - \underline{y}_2] \} \end{aligned} \tag{7}$$

[9] Note that this is a selection from possibly multiple continuation values that can be supported.

**Definition 1** *A collection of value correspondences* $\langle V^t : \Xi^t \times \Theta \rightrightarrows \mathbf{R}^2 \rangle_{t=1}^{T+1}$ *represents* **contractual equilibrium** *in the game from period 1 if (i) for every* $\theta \in \Theta$ *and each* $t \in \{1, 2, \ldots, T\}$*, expression 7 holds and* $V^t$ *is nonempty valued; and (ii)* $V^{T+1}(c, h^T, \theta) = \{(0, 0)\}$ *for every* $\theta$*, every* $c \in C$*, and every* $h^T \in H^T(c)$*. If such a collection of value correspondences exists then I say that* **contractual equilibrium exists***.*

**Theorem 1:** *If contractual equilibrium exists then the value correspondences are uniquely defined. If* $\mathcal{A}$ *is finite then contractual equilibrium exists.*

This result is a special case of Theorem 2 in Watson (2005b). Hereinafter, I assume that contractual equilibrium exists.

I use term *value function* (or *continuation value function*) for any function $v : \Theta \to \mathbf{R}^2$ that gives the continuation payoff from the start of a given period as a function of the state. The key value functions to be considered are those that relate to period 1 and arise in contractual equilibrium.

**Definition 2** *An externally enforced contract component* $c$ *is said to* **implement value function** $v : \Theta \to \mathbf{R}^2$ *if* $v(\theta) \in V^1(c, h^{\text{null}}, \theta)$ *for every* $\theta \in \Theta$*. If there is a contract that implements a given value function* $v$ *then* $v$ *is said to be* **implementable***.*

It will also be appropriate to speak of implementing a value function from a period other than period 1, as the analysis in the next section requires. In this case, I will say "implement from period $t$."

## Characterization Result

Consider any contractual relationship, which is defined by $T$, $\Theta$, and $P^t$ and $u^t$ for $t \in \{1, 2, \ldots, T\}$. Then define the related *simplified model* as the contractual relationship given by $\hat{T} \equiv 1$, $\hat{\Theta} = \Theta$, $\hat{P}^1 \equiv P^1 \times P^2 \times \cdots P^T$, and

$$\hat{u}^1(\lambda, \theta) \equiv \sum_{\tau=1}^{T} u^\tau(\lambda^\tau, \theta),$$

where $\lambda = (p^1, p^2, \ldots, p^T)$ is as previously defined, and $\lambda^\tau$ is the first $\tau$ elements of $\lambda$. Note that the simplified model has a single trading period, where all of the productive actions from the original model are consolidated. The payoffs are just as in the original model as a function of the sequence of productive actions, except they accrue in the new single trading period. There is a single transfer in the trading period.

**Theorem 2:** *Consider any contractual relationship with multiple trading periods—the original model. A value function* $v$ *is implementable in the original model if and only if it is implementable in the simplified model.*

Thus, however complicated is the original model, it can be analyzed by examining a particularly simple, standard mechanism-design problem.

# 3 Supporting Analysis

This section contains details on how the contracting problem can be written as a recursive mechanism-design problem, a formulation used to prove the main result.

## Mechanism-Design Preliminaries

In this subsection, I review and add a bit to the tools of static mechanism-design theory, which will be helpful in analyzing the dynamic contractual settings on which this paper is focused.[10] Note that a complete-information mechanism-design problem can be expressed in terms of the state space $\Theta$ and a set of *outcome functions* $W$, where each element of $W$ is a function $w : \Theta \to \mathbf{R}^2$ that gives a payoff vector contingent on the state. An outcome $w$ captures the payoff consequences of whatever is physically specified for the continuation following messages. For example, consider the one-period case of the model developed in Section 2, where the public action is $(p^1, m^1)$ and the payoffs are given by $u^1(p^1, \theta) + m^1$. Then a given public action $(\overline{p}^1, \overline{m}^1)$ is represented by the value function $w$, where

$$w(\theta) \equiv u^1(\overline{p}^1, \theta) + \overline{m}^1$$

for every $\theta \in \Theta$. Thinking of outcomes in terms of state-contingent payoffs is more general (in a useful way) than is thinking in terms of physical outcomes such as public actions.

A mechanism specifies message spaces $A_1$ and $A_2$ for the players, a space $A_0$ with probability distribution $\alpha_0$, and a function $f : A_0 \times A_1 \times A_2 \to W$ that gives the outcome as a function of the players' announcements. The revelation principle applies, so one can focus on direct-reporting mechanisms (where $A_1 = A_2 = \Theta$) and truthful reports in equilibrium. With this restriction, a mechanism is given by $A_0$, $\alpha_0$, and $f : A_0 \times \Theta \times \Theta \to W$. For such a mechanism, the strategies of reporting truthfully form a Nash equilibrium in each state $\theta$ if

$$v_i(\theta) = E[f(a_0, \theta, \theta)_i(\theta) \mid \alpha_0] \geq E[f(a_0, \theta_i', \theta)_i(\theta) \mid \alpha_0] \quad \text{for all } \theta_i' \in \Theta_i \text{ and } i = 1, 2 \tag{8}$$

A value function $v : \Theta \to \mathbf{R}^2$ is implementable if there exists a mechanism $\langle A_0, \alpha_0, f \rangle$ such that condition 8 holds for every $\theta \in \Theta$.

To characterize the set of implementable value functions, we simply write the Nash equilibrium conditions for each state. In states $\theta$ and $\theta'$, the players will send message profiles $(\theta, \theta)$ and $(\theta', \theta')$, respectively, in equilibrium. It is essential that the outcome specified for message profile $(\theta', \theta)$ be sufficient to simultaneously (i) dissuade player 1 from declaring the state to be $\theta'$ when the state is actually $\theta$ and (ii) discourage player 2 from declaring "$\theta$" in state $\theta'$. Thus, letting $w$ and $w'$ denote the outcomes specified for message profiles $(\theta, \theta)$ and $(\theta', \theta')$, respectively, implementation relies on the existence of an outcome $\hat{w}$ satisfying

$$w_1(\theta) \geq \hat{w}_1(\theta) \quad \text{and} \quad w_2'(\theta') \geq \hat{w}_2(\theta'). \tag{9}$$

[10] See Watson (2005a) for more details on the material in this subsection.

With the following property, inequalities 9 hold if and only if $w_1(\theta) + w_2'(\theta') \geq \hat{w}_1(\theta) + \hat{w}_2(\theta')$.

**Definition 3:** *$W$ is called* **closed under constant transfers** *if [$w \in W$ and $m \in \mathbf{R}_0^2$] implies $w + m \in W$.*

The effect of ex ante renegotiation (that is, renegotiation prior to the message phase) is represented by constraining attention to mechanisms that yield efficient outcomes in equilibrium in every state.[11] If $W$ is closed under constant transfers, efficiency simply means maximizing the players' joint value.

**Definition 4:** *$W$ is said to* **possess state-contingent maximum joint values** *if $\gamma(\theta) \equiv \max\limits_{w \in W} w_1(\theta) + w_2(\theta)$ exists for each $\theta \in \Theta$.*

One can easily verify that, if $W$ possesses state-contingent maximum joint values and is closed under constant transfers, then randomization (a non-trivial specification of $\alpha_0$) is not needed to implement value functions with ex ante renegotiation.

Let $\tilde{V}$ denote the set of implementable value functions with ex ante renegotiation. To summarize the above analysis, if $W$ is closed under constant transfers and $W$ possesses state-contingent maximum joint values, then $\tilde{V}$ is characterized as follows.

$$\begin{aligned} \tilde{V} \;=\; & \{v \colon \Theta \to \mathbf{R}^2 \mid v_1(\theta) + v_2(\theta) = \gamma(\theta) \text{ for every } \theta \in \Theta, \\ & \text{and, for every } \theta, \theta' \in \Theta, \text{ there exists } \hat{w} \in W \\ & \text{such that } v_1(\theta) + v_2(\theta') \geq \hat{w}_1(\theta) + \hat{w}_2(\theta')\}. \end{aligned} \tag{10}$$

I refer to $\hat{w}_1(\theta) + \hat{w}_2(\theta')$ as the "punishment value" for states $\theta$ and $\theta'$. The set $\tilde{V}$ is closed under constant transfers.

The following result is a key component of the main result in this paper and also may be of independent interest.

**Lemma 1:** *Consider a two-player mechanism-design problem described by a set of states $\Theta$ and a set of outcomes $W$, such that $W$ possesses state-contingent maximum joint values and is closed under constant transfers. For any given two states $\theta, \theta' \in \Theta$ and for any given $\underline{w} \in W$, there is an implementable value function $v \colon \Theta \to \mathbf{R}^2$ such that*

*(i)* $v_1(\theta) + v_2(\theta') = \underline{w}_1(\theta) + \underline{w}_2(\theta')$*, and*

*(ii)* $v_1(\theta'') + v_2(\theta'') = \gamma(\theta'')$ *for every* $\theta'' \in \Theta$.

[11] For example, consider any given mechanism $f$ for which truthful reporting is a Nash equilibrium in every state and, ignoring randomization for simplicity, define mechanism $f'$ as follows. For every state $\theta$, define $f'(\theta, \theta)$ to be the outcome that the players would renegotiate to select, given the disagreement outcome $f(\theta, \theta)$. For every message profile $(\theta_1, \theta_2)$, where $\theta_1 \neq \theta_2$, let $f'(\theta_1, \theta_2) = f(\theta_1, \theta_2)$. It is not difficult to verify that, with mechanism $f'$, truthful reporting is a Nash equilibrium in every state; further, it yields the same state-contingent outcome as was achieved with $f$ renegotiated ex ante.

The proofs of this and the other lemmas are in the appendix.

## Recursive Mechanism-Design Representation

To analyze the contracting model of Section 2, it will be useful to think in terms of continuation value functions from the start of each period; recall that a value function gives the continuation value vector in a given period as a function of the state. I will represent sets of continuation-value functions at a particular period in terms of the productive actions taken earlier, rather than in terms of the outstanding contract $c$. Let $\tilde{V}^t(\lambda^{t-1})$ be the set of value functions that can be implemented from period $t$ contingent on productive actions $\lambda^{t-1}$ taken earlier. This set is defined across all contracts that are consistent with the history $\lambda^{t-1}$ of productive actions. To formalize, for each $t$ and every $\lambda^t$, let

$$L^t(\lambda^t) \equiv \{(c, h^t) \mid \lambda^t = \phi^t(c, h^t)\}.$$

Then

$$\begin{aligned}\tilde{V}^t(\lambda^{t-1}) \;\equiv\; & \{v\colon \Theta \to \mathbf{R}^2 \mid \text{there exist } c \in C \text{ and } h^{t-1}\text{, such that} \\ & (c, h^{t-1}) \in L^{t-1}(\lambda^{t-1}) \text{ and, for every } \theta \in \Theta,\, v(\theta) \in V^t(c, h^{t-1}, \theta)\}\end{aligned}$$

Observe that individual incentives in a given period depend only on the past productive actions and on the contractual terms for this and future periods. Also, the contractual terms $A^t, \alpha_0^t, \rho^t, \mu^t$ for a specific period $t$ are not constrained by the terms specified for other periods. This implies that the set of implementable value functions from period $t$ depends not on the contractual terms that were in force in previous periods, except as they affected the productive actions taken in previous periods. To be precise, for any $\lambda^{t-1}$ and any $(\hat{c}, \hat{h}^{t-1}) \in L^{t-1}(\lambda^{t-1})$,

$$\begin{aligned}&\{v\colon \Theta \to \mathbf{R}^2 \mid \text{there exists } c \in C^t(\hat{c}) \text{ such that,} \\ &\qquad \text{for every } \theta \in \Theta,\, v(\theta) \in V^t(c, h^{t-1}, \theta)\} \;=\; \tilde{V}^t(\lambda^{t-1})\end{aligned}$$

Note that $(\hat{c}, \hat{h}^{t-1}) \in L^{t-1}(\lambda^{t-1})$ and $c \in C^t(\hat{c})$ imply $(c, \hat{h}^{t-1}) \in L^{t-1}(\lambda^{t-1})$.

Given these facts, we can characterize the sets $\tilde{V}^t(\lambda^{t-1})$ recursively in terms of some standard, static mechanism-design problems. First, we define $\tilde{V}^{T+1}(\lambda^T) \equiv \{(0,0)\}$ for every $\lambda^T \in \Lambda^T$. That is, because the relationship ends at the end of period $T$, the continuation value after period $T$ is zero. Then, for each $t = 1, 2, \ldots, T$, we relate $\tilde{V}^t$ to $\tilde{V}^{t+1}$ by means of a standard, static mechanism-design problem. To this end, let $w^t : \Theta \to \mathbf{R}^2$ represent the continuation value vector following the message phase in a given period, as a function of the state. That is, $w^t$ is an outcome at period $t$ and is given by

$$w^t(\theta) = u^t((\lambda^{t-1}, p), \theta) + m + v^{t+1}(\theta), \tag{11}$$

where $(p, m)$ is the public action in period $t$ and $v^{t+1}$ gives the continuation payoffs from the start of the next period, as a function of the state. Given $\tilde{V}^{t+1}$ and a history of productive actions $\lambda^{t-1}$, the set of feasible outcomes in period $t$ is

$$\begin{aligned} W^t(\lambda^{t-1}) \equiv \{ & w^t : \Theta \to \mathbf{R}^2 \mid \text{there exist } p \in P^t, m \in \mathbf{R}^2_0, \text{ and a function} \\ & v^{t+1} \in \tilde{V}^{t+1}(\lambda^{t-1}, p), \text{ such that equation 11 holds for every } \theta \in \Theta \} \end{aligned}$$

The mechanism-design problem that relates $\tilde{V}^t(\lambda^{t-1})$ to the correspondence $\tilde{V}^{t+1}$ is described by the set of states $\Theta$ and the outcome set $W^t(\lambda^{t-1})$. The mechanism-design problem has ex ante renegotiation, because the players can renegotiate prior to the message phase. Applying the characterization of expression 10 on page 20, we have

$$\begin{aligned} \tilde{V}^t(\lambda^{t-1}) = \{ & v : \Theta \to \mathbf{R}^2 \mid v_1(\theta) + v_2(\theta) = \gamma^t(\lambda^{t-1}, \theta) \text{ for every } \theta \in \Theta, \\ & \text{and, for every } \theta, \theta' \in \Theta, \text{ there exists } \hat{w} \in W^t(\lambda^{t-1}) \\ & \text{such that } v_1(\theta) + v_2(\theta') \geq \hat{w}_1(\theta) + \hat{w}_2(\theta') \} \end{aligned} \tag{12}$$

## Proof of Theorem 2

To prove Theorem 2, I use an induction method. The key step is to demonstrate that, for any $T > 1$, implementation in any $T$-period contractual relationship is equivalent to implementation in a specific $(T-1)$-period relationship that is defined by consolidating into a single period the productive actions that were to be taken in periods $T-1$ and $T$. With this claim established, iterative application yields Theorem 2.

Take as given a $T$-period model—call it the *original model*—with the productive action set in period $t$ denoted by $P^t$ and the payoff vector in period $t$ given by $u^t(\lambda^t, \theta) + m^t$, for $t = 1, 2, \ldots, T$. I defined the *reduced model* with $T-1$ periods as follows. The reduced model is assumed to have exactly the same productive action sets and payoffs in periods 1 through $T-2$ as has the original model. In period $T-1$, the reduced model's productive action set is defined as $P^{T-1} \times P^T$ and its payoff vector is given by

$$u^{T-1}((\lambda^{T-2}, p^{T-1}), \theta) + u^T((\lambda^{T-2}, p^{T-1}, p^T), \theta) + m^{T-1},$$

where $u^{T-1}$ and $u^T$ are from the original model.

The original and reduced models are identical on periods 1 through $T-2$ and they share the set of $(T-2)$-period productive histories, $\Lambda^{T-2}$. Let $\tilde{V}^t$ denote the set of implementable continuation values (as a function of the history of productive actions) from period $t$ for the original model, where $t \in \{1, 2, \ldots, T\}$. Let $\tilde{V}^t_{\text{Red}}$ give the set of implementable continuation values from period $t$ for the reduced model, where here $t \in \{1, 2, \ldots, T-1\}$.

I claim that the original and reduced models' continuation-value sets from period $T-1$ are equal; that is, for each $\lambda^{T-2} \in \Lambda^{T-2}$,

$$\tilde{V}^{T-1}(\lambda^{T-2}) = \tilde{V}^{T-1}_{\text{Red}}(\lambda^{T-2}).$$

To see this, recall the characterization of $\tilde{V}^t(\lambda^{t-1})$ from expression 12. We can use expression 11 (on page 21) to write this directly in terms of productive actions. Doing so, we have for the case of the original model at period $T-1$:

$$\begin{aligned}\tilde{V}^{T-1}(\lambda^{T-2}) \;=\; \{v\colon \Theta \to \mathbf{R}^2 \mid\ & v_1(\theta)+v_2(\theta)=\gamma^{T-1}(\lambda^{T-2},\theta) \text{ for every } \theta\in\Theta,\\ & \text{and, for every } \theta,\theta'\in\Theta, \text{ there exist } p^{T-1}\in P^{T-1}\\ & \text{and } v^T\in\tilde{V}^T(\lambda^{T-2},p^{T-1}) \text{ such that } v_1(\theta)+v_2(\theta)\geq J(p^{T-1},v^T)\},\end{aligned}$$

where

$$J(p^{T-1},v^T)\equiv u_1^{T-1}((\lambda^{T-2},p^{T-1}),\theta)+u_2^{T-1}((\lambda^{T-2},p^{T-1}),\theta')+v_1^T(\theta)+v_2^T(\theta').$$

The corresponding expression for the case of the reduced model is:

$$\begin{aligned}\tilde{V}^{T-1}_{\text{Red}}(\lambda^{T-2}) \;=\; \{v\colon \Theta \to \mathbf{R}^2 \mid\ & v_1(\theta)+v_2(\theta)=\gamma^{T-1}(\lambda^{T-2},\theta) \text{ for every } \theta\in\Theta,\\ & \text{and, for every } \theta,\theta'\in\Theta, \text{ there exist } p^{T-1}\in P^{T-1} \text{ and } p^T\in P^T\\ & \text{such that } v_1(\theta)+v_2(\theta)\geq K(p^{T-1},p^T)\},\end{aligned}$$

where

$$\begin{aligned}K(p^{T-1},p^T) \;\equiv\; & u_1^{T-1}((\lambda^{T-2},p^{T-1}),\theta)+u_2^{T-1}((\lambda^{T-2},p^{T-1}),\theta')\\ & +u_1^T((\lambda^{T-2},p^{T-1},p^T),\theta)+u_2^T((\lambda^{T-2},p^{T-1},p^T),\theta').\end{aligned}$$

The proof continues with two lemmas that relate $\tilde{V}^{T-1}$ and $\tilde{V}^{T-1}_{\text{Red}}$.

**Lemma 2:** *Fix $\lambda^{T-2}$. Take any two states $\theta,\theta'\in\Theta$ and any productive actions $p^{T-1}\in P^{T-1}$ and $p^T\in P^T$. There exists $v^T\in\tilde{V}^T(\lambda^{T-2},p^{T-1})$ such that $K(p^{T-1},p^T)=J(p^{T-1},v^T)$.*

This result addresses an outcome from period $T-1$ that is simultaneously designed to (i) punish player 1 from claiming the state to be $\theta'$ when the state is in fact $\theta$, and (ii) punish player 2 from claiming the state to be $\theta$ when the state is in fact $\theta'$. In the reduced model, the punishment occurs by prescribing productive actions $p^{T-1}$ and $p^T$ for message profile $(\theta',\theta)$. In the original model, productive action $p^{T-1}$ can be prescribed for period $T-1$, but the specification for period $T$ is subject to renegotiation at the beginning of period $T$. Thus, to get the same punishment value, one has to obtain it from a continuation value function $v^T$ that is implementable from period $T$. The proof of this lemma (contained in the appendix) uses Lemma 1 to find the appropriate value function.

**Lemma 3:** *Fix $\lambda^{T-2}$. Take any two states $\theta,\theta'\in\Theta$, any productive action $p^{T-1}\in P^{T-1}$, and any value function $v^T\in\tilde{V}^T(\lambda^{T-2},p^{T-1})$. There exists a productive action $p^T\in P^T$ such that $K(p^{T-1},p^T)\leq J(p^{T-1},v^T)$.*

The proof of this lemma (in the appendix) locates the productive action $p^T$ by examining how $v^T$ is implemented from period $T$. In particular, $p^T$ is chosen to be the productive action associated with the punishment value for states $\theta$ and $\theta'$.

Note that Lemma 2 implies

$$\tilde{V}^{T-1}_{\text{Red}}(\lambda^{T-2}) \subset \tilde{V}^{T-1}(\lambda^{T-2}),$$

whereas Lemma 3 implies the reverse inclusion relation. Thus,

$$\tilde{V}^{T-1}_{\text{Red}}(\lambda^{T-2}) = \tilde{V}^{T-1}(\lambda^{T-2}).$$

Because the original and reduced models are identical on periods 1 through $T-2$, and because the continuation value sets are characterized by backward induction, we conclude that

$$\tilde{V}^{t}_{\text{Red}} \equiv \tilde{V}^{t}$$

for $t = 1, 2, \ldots, T-1$. In particular, the equivalence holds for $t = 1$, which means that the set of implementable value functions in the original and reduced models are the same.

# 4 Other Results

In this section, I provide a result on the form of optimal contracts and additional results for the special case of a durable trading opportunity.

## Details of Optimal Contractsin the Multiple-Trading-Periods Model

I have portrayed the multiple-trading-periods model as an accurate depiction of real contractual relationships, whereas the simplified model is an abstract construct that (with Theorem 2) facilitates calculating the set of implementable value functions. If we are interested in examining the details of real contracts, it is useful to go beyond the question of what is implementable and also work to understand the structure of optimal contracts in the multiple-trading-periods model. Theorem 2 does not provide the details of optimal contracts, because an optimal contract in the simplified model cannot be applied directly to the real world of multiple trading periods.

However, analysis of the relation between the two models gives some intuition about optimal contracts in the multiple-trading-periods model. First, as noted in the analysis of the previous section, successive application of the revelation principle justifies limiting attention to contracts in which messages are direct reports of the state. Second, contracts can be structured so that renegotiation does not actually occur in any period, regardless of whether players sent equilibrium message profiles in previous periods. Third, one can focus on contracts in which messages are used in a given period only if the players have consistently disagreed about the state in previous periods; that is, if the players send the same report in some period then the external enforcer can ignore all messages in future periods.

To formalize these properties, I shall use the following terms in describing a contract $c = \langle A^t, \alpha_0^t, \rho^t, \mu^t \rangle_{t=1}^T$.

**Direct reporting:** $A_1^t = A_2^t = \Theta$ for every $t = 1, 2, \dots, T$ and, in the contractual equilibrium, players report truthfully in every state and period, whatever the history;

**Renegotiation-proof:** $V^t(c, h^{t-1}, \theta) = \text{eff } Y^t(c, h^{t-1}, \theta)$ for every $t$, $h^{t-1} \in H^{t-1}(c)$, and $\theta \in \Theta$;

**Renegotiation-proof in equilibrium:** $V^t(c, h^{t-1}, \theta) = \text{eff } Y^t(c, h^{t-1}, \theta)$ for every $t$, $\theta \in \Theta$, and $h^{t-1} = ((\theta, \theta), (\theta, \theta), \dots, (\theta, \theta))$;

**Minimal message usage:** for every $t$, $h^t = (a^1, a^2, \dots, a^t)$, and $h'^t = (a'^1, a'^2, \dots, a'^t)$ with $h^t, h'^t \in H^t(c)$, if $h^t$ and $h'^t$ coincide through some period $\tau < t$ and if $a_1^\tau = a_2^\tau$, then $\rho^t(h^t) = \rho^t(h'^t)$ and $\mu^t(h^t) = \mu^t(h'^t)$.

Here "eff" stands for the Pareto frontier. Also, "$h^t$ and $h'^t$ coincide through $\tau$" means that $a^l = a'^l$ for $l = 1, 2, \dots, \tau$. On the renegotiation-proofness property, recall that $Y^t(c, h^{t-1}, \theta)$ is the set of continuation payoffs that can be supported from the beginning of period $t$ with contract $c$, whereas $V^t$ is the set implied when players renegotiate from points in $Y^t$ to the Pareto frontier of feasible continuation payoffs. The condition $V^t(c, h^{t-1}, \theta) = \text{eff } Y^t(c, h^{t-1}, \theta)$ means that there is no surplus of renegotiation; the players achieve efficient continuation values by staying with their current externally enforced contract $c$. Note that renegotiation-proofness in equilibrium means that the players do not actively renegotiate on the equilibrium path, where truthful reporting occurred in previous periods.

**Theorem 3:** *Consider any contractual relationship with multiple trading periods. Every implementable value function can be implemented by a contract that features direct reporting and is renegotiation-proof. Also, every implementable value function can be implemented by a contract that features direct reporting and minimal message usage, and is renegotiation-proof in equilibrium.*

**Proof:** On the first claim, the direct reporting and renegotiation-proofness properties follow from the analysis in the previous section. That is, in the recursive mechanism-design formulation, we have the revelation principle at each stage of the analysis. Ex ante renegotiation at each stage is represented by the Pareto criterion on the equilibrium path in every state (as shown in expression 10 on page 20); contracts so identified are renegotiation-proof in every state and period, whatever the history of messages.

On the second claim, consider a contract $c'$ that implements $v$, that features direct reporting, and is renegotiation-proof. We can define a related contract $c$ that is identical to $c'$ for all histories in which the players have sent different reports of the state in every period. However, at the first instance in which $a_1^t = a_2^t$, contract $c$ prescribes the sequence of productive actions and transfers from period $t$ that would result under contract $c'$ if the players were to send this same message profile in each of the remaining periods. The sequence of public actions may still depend on a random draw, but it no longer is conditioned on the players' messages in periods $t+1, t+2, \ldots, T$. It is easy to see that, with contract $c$, the players have the incentive to report truthfully in every state. Also, $c$ is renegotiation-proof in equilibrium, because (by $c'$ being renegotiation-proof) the sequence of productive actions prescribed when players consistently name the same state over time is efficient in this state.[12] *Q.E.D.*

Minimal message usage has practical appeal. Optimal contracts can be designed so that, over time, players need to continue communicating with the external enforcer only if they had always disputed in the past by sending different reports.

## Durable Trading Opportunities and Reversibility

Many contractual relationships can be described as having *durable* trading opportunities, meaning that (i) the traded good or service yields a flow payoff over time and (ii) the trade can be delayed or reversed. For example, a retail firm may contract with a computer software company to design and install specialized software for inventory control. The software will generate for the retailer a flow of value over time, starting as soon as the software is installed. Suppose the software can be installed as early as in January; furthermore, if the seller fails to install the software in January, it can still be installed in February, or March, or later. However, if it is installed in, say, March, then the buyer will not obtain the value of the software in January or February. Also, the software can be uninstalled, yielding a subsequent flow payoff as though the software were never installed.

I represent durability by assuming that the set of feasible productive actions is constant across periods—that is, there is a set $P$ such that $P^t = P$ for every period $t$. If the productive action $p$ is taken in the current period, then it can be reversed by selecting any other productive action in the next period. Selection of the same productive action in the next period means that the action today is not reversed. Assume that the players' utility in a given period can be written in the following separable way. For each $t \geq 2$,

$$u^t(\lambda^t, \theta) = \psi(p^t, \theta) - \kappa^t(p^{t-1}, p^t, \theta),$$

and, for $t = 1$,

$$u^1(\lambda^1, \theta) = \psi(p^1, \theta).$$

The function $\psi : P \times \Theta \to \mathbf{R}^2$ gives the per-period flow payoff of the current productive action. The function $\kappa^t : P \times P \times \Theta \to \mathbf{R}^2$ gives the cost of changing the productive action from period $t-1$ to period $t$. Thus, assume that $\kappa^t(p, p, \theta) = (0, 0)$ and $\kappa^t(p, p', \theta) \geq (0, 0)$, for every $t \in \{2, 3, \ldots, T\}$, all $p, p' \in P$, and every $\theta \in \Theta$.

A special case of a durable trading opportunity is a stationarity environment with no cost of reversing productive actions—that is, where $\kappa^t \equiv (0, 0)$ for all $t \in \{2, 3, \ldots, T\}$. The next result shows that, for stationary environments, it suffices to consider stationary

[12] Contract $c$ is not necessarily renegotiation-proof out of equilibrium.

contracts. Regarding the definition of a stationary contract, recall that the externally enforced component of the contract is

$$c = \langle A^t, \alpha_0^t, \rho^t, \mu^t \rangle_{t=1}^T.$$

A stationary contract has the property that (i) $A^t$ and $\alpha_0^t$ are the same across periods, (ii) $\rho^t : H^t \to P^t$ and $\mu^t : H^t \to \mathbf{R}_0^2$ depend only on announcements made in period $t$ and so can be written as functions $\rho^t : A^t \to P^t$ and $\mu^t : A^t \to \mathbf{R}_0^2$, and (iii) $\rho^t$ and $\mu^t$ are the same across periods.

**Theorem 4:** *In any stationary contracting environment, if a value function $v$ is implementable then it can be implemented using a stationary contract.*

On the applied side, this theorem establishes that optimal contracts can always take a simple, stationary form, whereby the players interact in the same way in each period. On the technical side, the theorem shows that the analysis of long-term contracting reduces to selecting a one-period mechanism that is repeated over time. That is, players choose a long-term contract that requires them to play the same short-term mechanism in each period.

**Proof of Theorem 4:** By Theorem 2, we can study the simplified ($\hat{T} = 1$) version of the model to calculate implementable value functions. In a stationary environment, payoffs in the simplified model are just a scaled version of the payoffs in any single period of the original model, in the following sense. First, in each state, the efficient sequence of productive actions is a constant sequence. That is, for state $\theta$, there is a productive action $p^*(\theta)$ such that $\lambda^*(\theta) = (p^*(\theta), p^*(\theta), \ldots, p^*(\theta))$ is the efficient sequence of productive actions. Further, in the original model, it is efficient in the continuation from a given period $t$ to select $p^*(\theta)$ thereafter, regardless of the productive actions taken earlier. Second, the best (lowest) punishment value for states $\theta$ and $\theta'$ is achieved with some constant sequence of productive actions $(\underline{p}(\theta, \theta'), \underline{p}(\theta, \theta'), \ldots, \underline{p}(\theta, \theta'))$.

Thus, if value function $v$ can be implemented in the simplified model, then $v/T$ is implementable in a given period of the original model when this period is considered in isolation of the others. A stationary contract treats each period in isolation of the others, and so a stationary contract can be used to implement $v/T$ on a per-period basis, implying that $v$ is implemented. *Q.E.D.*

On the effect of reversal costs, recall that the intuition from the example is that higher reversal costs imply a greater scope of implementability. In a sense, reversal costs work much like money burning, which has beneficial effects when applied in out-of-equilibrium contingencies. The next theorem establishes the claim generally.

**Theorem 5:** *Consider any contracting environment with a durable trading opportunity. Fix the function $\psi$ and consider two reversal-cost technologies given by $\kappa^1, \ldots, \kappa^T$ and $\hat{\kappa}^1, \ldots, \hat{\kappa}^T$. Let $V$ and $\hat{V}$ denote the sets of implementable value functions in the two cases. If $\hat{\kappa}^t \geq \kappa^t$ for every period $t$, then $V \subset \hat{V}$. Furthermore, the inclusion relation is strict for large enough reversal costs.*

Thus, increasing the reversal costs implies an increase in the set of implementable value functions. The technical meaning of "large enough reversal costs" in the theorem is that there exists a number $L$ (which depends on $\kappa^1, \ldots, \kappa^T$) such that, for some period $t$, some state $\theta$, and some productive actions $p, p' \in P$, we have $\hat{\kappa}^t(p, p', \theta) \geq \kappa^t(p, p', \theta) + L$. This theorem suggests that it may be fruitfull to examine reversal costs in applied settings. Reversability is perhaps straightforward to estimate empirically and reversal costs may differ substantially within industries.

**Proof of Theorem 5:** By Theorem 2, we can study the simplified ($\hat{T} = 1$) version of the model to calculate implementable value functions. Recall that without the reversal costs, in each state, the efficient sequence of productive actions is a constant sequence. Because $\kappa^t(p, p, \theta) = (0, 0)$—that is, there is no cost of keeping the same productive action from period to period—this continues to be true even in the setting of positive reversal costs. By the renegotiation proofness principle, we consider mechanisms that specify efficient sequences of productive actions in equilibrium in each state; further, the efficient sequences and implied values are unchanged as reversal costs increase.

However, increased reversal costs imply weakly lower punishment values (used for out-of-equilibrium message profiles) and so increases the set of implementable value functions. It is not difficult to see that, for a large enough $L$, the model with higher reversal costs allows for a lower punishment value for some states $\theta, \theta'$. Lemma 1 then implies the existence of an implementable value function in the model with higher reversal costs that cannot be implemented in the model with lower reversal costs. *Q.E.D.*

# 5 Conclusion

This paper takes a modest step in my research program of exploring how technological details affect the form and efficacy of contracts. The modeling exercise demonstrates the value of accounting for how trade involves a sequence of productive actions, while showing that one can easily analyze such settings with standard tools. By examining the sequence of productive actions in a given contractual relationship, one will generally see ways for the parties to be punished off the equilibrium path (for example, using reversals of productive actions) that may not have been apparent with more stylized accounts of trade. I believe that empirical work will benefit from looking closely at the technological details of real contractual relationships

I note that transferrable utility (monetary transfers and payoffs that are linear in money) is a key assumption for Theorem 2. In particular, Lemma 1 relies on being able to (i) shift utility away from player 1 when both players report the state to be $\theta$ and (ii) shift utility away from player 2 when both players report the state to be $\theta'$, just enough so that $v_1(\theta) + v_2(\theta') = \underline{w}_1(\theta) + \underline{w}_2(\theta')$. Further, the characterization of implementation in expression 10 on page 20—in particular, the inequality condition—relies on transferrable utility. If one were to dispense with the transferrable utility assumption, Lemma 1 would not longer hold and it would generally be possible to implement more in the simplified model than can be done in the original multiple-trading-periods model.

In the model studied here, I assumed that renegotiation occurs before messages are sent in each period. One might also be interested in the case in which renegotiation occurs between the time messages are sent and when productive actions are taken. I have two points to make on this setting. First, if one continues to model the productive actions as public then all of the results herein extend to the case of renegotiation between messages and productive actions. The statement and proof of Lemma 1, for instance, extend with the modification that $\underline{w}_1(\theta)$ and $\underline{w}_2(\theta')$ are replaced with values that the players obtain when renegotiating from $\underline{w}$ in states $\theta$ and $\theta'$.

Second, however, it may no longer be appropriate to treat the productive actions as public. In fact, Watson (2005a) shows that artificial constraints are created by modeling individual actions as public when they are immediately preceded by renegotiation. That is, if the contracting parties are the ones to physically take the productive actions, and if renegotiation can occur just before the productive actions, then modeling the actions as public generally creates a distortion. Further, if we model the productive actions as individual, and there is renegotiation after messages are sent, then it is not clear whether the results of this paper survive.[13]

Analyzing settings with individual productive actions and post-message renegotiation in each period seems like a reasonable item for a "research to do" list. Another direction worth exploring is to look at settings in which verifiable and unverifiable actions mingle over time. Further, I think it would be useful to examine situations with infinite horizons, which is something I have started to do in other work and I would encourage others to consider.

[13] In the related and recent literature, many papers feature models in which verifiable trade actions are treated as public and in which renegotiation takes place between messages and trade. One might be inclined to think that such a model approximates a real situation in which there is a durable trading opportunity and the parties can always reverse the trade outcome after the external enforcer establishes a ruling (after the mechanism). Considering the durable-trading-opportunity setting studied here, this sort of approximation seems correct if one assumes that the players cannot communicate with the external enforcer past some period $\tau$, where $\tau$ is close to 1, $T$ is large, and the players are assumed to be patient. The communication barrier may be interepreted as an incompleteness assumption.

# A Proofs of the Lemmas

In this appendix, the lemmas are restated and proved.

**Lemma 1:** Consider a two-player mechanism-design problem described by a set of states $\Theta$ and a set of state-contingent value functions (outcomes) $W$, such that $W$ possesses state-contingent maximum joint values and is closed under constant transfers. For any given two states $\theta, \theta' \in \Theta$ and for any given $\underline{w} \in W$, there is an implementable value function $v \colon \Theta \to \mathbf{R}^2$ such that

(i) $v_1(\theta) + v_2(\theta') = \underline{w}_1(\theta) + \underline{w}_2(\theta')$, and

(ii) $v_1(\theta'') + v_2(\theta'') = \gamma(\theta'')$ for every $\theta'' \in \Theta$.

**Proof of Lemma 1:** Take any $\underline{w}$ and fix $\theta$ and $\theta'$ for the duration of this proof. We can find an outcome $w$ satisfying

$$w_1(\theta) = \underline{w}_1(\theta) \quad \text{and} \quad w_1(\theta) + w_2(\theta) = \gamma(\theta) \tag{13}$$

and an outcome $w'$ satisfying

$$w'_2(\theta') = \underline{w}_2(\theta') \quad \text{and} \quad w'_1(\theta') + w'_2(\theta') = \gamma(\theta'). \tag{14}$$

We can then find an outcome $\overline{w} \in W$ satisfying

$$\overline{w}_2(\theta) = w_2(\theta) \quad \text{and} \quad \overline{w} = \underline{w} + (\beta, -\beta) \tag{15}$$

for some scalar $\beta$. These three outcomes exist because $W$ is closed under constant transfers and attains a maximum in each state.

Define mechanism $f \colon \Theta^2 \to W$ as follows. (There will be no need for randomization, so $A_0$ is left out.) Players send reports of the state, and the report vector is $(a_1, a_2)$. Specify that $f(\theta, \theta) = w$; that is, if both players report the state to be $\theta$ then the outcome is $w$. Further, for all $\theta'' \neq \theta$, specify $f(\theta'', \theta) = \underline{w}$ and $f(\theta, \theta'') = \overline{w}$. Finally, let $w'$ be the outcome specified for all other announcement profiles, including when both players report the state to be $\theta'$.

By construction of $f$, reporting truthfully is a Nash equilibrium in state $\theta$ and it yields an efficient outcome in this state, with payoff vector $w(\theta)$. Regarding incentives in state $\theta'$, note that, by construction, player 2's report of "$\theta'$" is a best response to truthfull reporting by player 1. Player 1 also prefers to report honestly in state $\theta'$ if and only if $w'_1(\theta') \geq \overline{w}_1(\theta')$. I next demonstrate that this inequality holds.

Define "inefficiency amounts" $\sigma(\theta)$ and $\sigma(\theta')$ by

$$\sigma(\theta) \equiv \gamma(\theta) - [\underline{w}_1(\theta) + \underline{w}_2(\theta)]$$

and

$$\sigma(\theta') \equiv \gamma(\theta') - [\underline{w}_1(\theta') + \underline{w}_2(\theta')].$$

Expression 13 yields the equation $w_2(\theta) = \gamma(\theta) - \underline{w}_1(\theta)$. Using expression 15 to substitute for $w_2(\theta)$, we obtain

$$\underline{w}_2(\theta) - \beta = \gamma(\theta) - \underline{w}_1(\theta).$$

We can then substitute for $\beta$ by again using expression 15 (in particular $\overline{w}_1(\theta') = \underline{w}_1(\theta') + \beta$), to get

$$\overline{w}_1(\theta') = \underline{w}_1(\theta') + \underline{w}_1(\theta) + \underline{w}_2(\theta) - \gamma(\theta).$$

Substituting for the last three terms using the definition of $\sigma(\theta)$ yields

$$\underline{w}_1(\theta') = \overline{w}_1(\theta') + \sigma(\theta). \tag{16}$$

Starting from scratch, we see that expression 14 yields the equation $w_1'(\theta') = \gamma(\theta') - \underline{w}_2(\theta')$. Substituting for $\underline{w}_2(\theta')$ using the definition of $\sigma(\theta')$, we obtain

$$w_1'(\theta') = \underline{w}_1(\theta') + \sigma(\theta'). \tag{17}$$

Combining equations 16 and 17 to substitute for $\underline{w}_1(\theta')$, we obtain

$$w_1'(\theta') = \overline{w}_1(\theta') + \sigma(\theta) + \sigma(\theta').$$

The amounts $\sigma(\theta)$ and $\sigma(\theta')$ are nonnegative, which implies that $w_1'(\theta') \geq \overline{w}_1(\theta')$. This means that player 1's report of "$\theta'$" is a best response in state $\theta'$ to honest reporting by player 2. Thus, truthful reporting is a Nash equilibrium in state $\theta'$ and yields an efficient outcome in this state, with payoff vector $w'(\theta')$. Because there are only three outcomes specified by $f$ over all report profiles, there exist Nash equilibria in all other states as well. Selecting Nash equilibria $(\theta, \theta)$ and $(\theta', \theta')$ in states $\theta$ and $\theta'$, respectively, and an arbitrary selection of equilibria in the other states, we conclude that mechanism $f$ implements some value function $\tilde{v}$ satisfying $\tilde{v}(\theta) = w(\theta)$ and $\tilde{v}(\theta') = w'(\theta')$. We can employ the revelation principle and the "renegotiation-proofness principle" used earlier to find another mechanism $f'$ that implements a value function $v$ such that $v(\theta) = w(\theta)$, $v(\theta') = w'(\theta')$, and $v_1(\theta'') + v_2(\theta'') = \gamma(\theta'')$ for every state $\theta''$. Finally, by construction, we have

$$v_1(\theta) + v_2(\theta') = w_1(\theta) + w_2'(\theta') = \underline{w}_1(\theta) + \underline{w}_2(\theta').$$

*Q.E.D.*

**Lemma 2:** Fix $\lambda^{T-2}$. Take any two states $\theta, \theta' \in \Theta$ and any productive actions $p^{T-1} \in P^{T-1}$ and $p^T \in P^T$. There exists $v^T \in \tilde{V}^T(\lambda^{T-2}, p^{T-1})$ such that $K(p^{T-1}, p^T) = J(p^{T-1}, v^T)$.

**Proof of Lemma 2:** Note that, for the case of the original model, the set $\tilde{V}^T(\lambda^{T-2}, p^{T-1})$ is the solution of the mechanism-design problem defined by outcome set

$$\begin{aligned} W^T(\lambda^{T-2}, p^{T-1}) \equiv \{ & w^T : \Theta \to \mathbf{R}^2 \mid \text{there exist } p^T \in P^T \text{ and } m \in \mathbf{R}_0^2 \\ & \text{such that } w^T(\theta'') = u^T((\lambda^{T-1}, p^T), \theta'') + m \text{ for every } \theta'' \in \Theta\}, \end{aligned}$$

which is clearly closed under constant transfers and attains a maximum in every state. Lemma 1 therefore applies. Take any two states $\theta, \theta' \in \Theta$ and any productive actions $p^{T-1} \in P^{T-1}$ and $p^T \in P^T$. Let $w^T$ be defined by

$$\underline{w}(\theta'') = u^T((\lambda^{T-2}, p^{T-1}, p^T), \theta''),$$

for every $\theta'' \in \Theta$. Think of $\underline{w}$ as the outcome of specifying productive action $p^T$ (following $(\lambda^{T-2}, p^{T-1})$) with no monetary transfer. By Lemma 1, there exists a value function $v^T \in \tilde{V}^T(\lambda^{T-2}, p^{T-1})$ such that

$$v_1^T(\theta)+v_2^T(\theta') = \underline{w}_1(\theta)+\underline{w}_2(\theta') = u_1^T((\lambda^{T-2}, p^{T-1}, p^T), \theta)+u_2^T((\lambda^{T-2}, p^{T-1}, p^T), \theta).$$

Comparing the definitions of $J$ and $K$ then shows that they are equal for the selected $p^{T-1}$, $p^T$, and $v^T$. *Q.E.D.*

**Lemma 3:** Fix $\lambda^{T-2}$. Take any two states $\theta, \theta' \in \Theta$, any productive action $p^{T-1} \in P^{T-1}$, and any value function $v^T \in \tilde{V}^T(\lambda^{T-2}, p^{T-1})$. There exists a productive action $p^T \in P^T$ such that $K(p^{T-1}, p^T) \leq J(p^{T-1}, v^T)$.

**Proof of Lemma 3:** Let $W^T(\lambda^{T-2}, p^{T-1})$ be defined as in the proof of Lemma 2 above. By implementability of $v^T$ from period $T$, there is an outcome $\hat{w} \in W^T(\lambda^{T-2}, p^{T-1})$ such that

$$v_1^T(\theta) + v_2^T(\theta') \geq \hat{w}_1(\theta) + \hat{w}_2(\theta').$$

We know that, in characterizing $\hat{w}$, there are a productive action $p^T \in P^T$ and a constant transfer $m \in \mathbf{R}^2$ such that

$$\hat{w}(\theta'') = u^T((\lambda^{T-1}, p^T), \theta'') + m$$

for every $\theta'' \in \Theta$. Thus, we have that

$$\hat{w}_1(\theta) + \hat{w}_2(\theta') = u_1^T((\lambda^{T-1}, p^T), \theta) + u_2^T((\lambda^{T-1}, p^T), \theta').$$

The transfer $m$ does not appear because the players' components cancel. Comparing the definitions of $J$ and $K$ yields the claimed inequality. *Q.E.D.*

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