Reduction-consistency and the Condorcet principle in collective choice problems*

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Abstract

We study the implications of reduction-consistency and the Condorcet principle in the context of choosing alternatives from a set of feasible alternatives over which each agent has a strict preference. We show that reduction-consistency is incompatible with a weaker version of the Condorcet principle. On the domain for which majority rule is always nonempty and agents' preferences are strict, we provide two characterizations of majority rule: (1) it is the only efficient rule satisfying reduction-consistency and (2) it is the only single-valued and efficient rule satisfying the converse of reduction-consistency. *Journal of Economic Literature* Classification Numbers: D63; D70; D71.

Keywords: consistency; converse consistency; the Condorcet principle; Arrovian abstract choice problems; majority rule.

1 Introduction

We consider the problem of selecting alternatives from a set of feasible alternatives over which each agent has a strict preference (no indifference between any two alternatives). Such a problem is called a "collective choice problem" and arises naturally in a committee election: members of a committee have to elect new members from a slate of candidates. How should the choice(s) be made? A rule is a correspondence that associates with each such problem a non-empty subset of the feasible set. The literature devoted to the mathematical analysis of the problem is traced to Borda (1781) and Condorcet (1785). Substantial contributions to the early development of the literature are due to Arrow (1950; 1951).

The goal of this paper is to explore the implications of an invariance property of rules proposed by Yeh (2004). The property is motivated by the following observation. In real life, we often observe that once a collective choice is made, then by law, it must be implemented and independent of certain changes in the number of agents and in the number of alternatives. Namely, the stability of a collective choice is guaranteed by law. Thus, searching for rules that generate such stability turns out to be very appealing. The invariance property serves for this purpose.

Consider a problem and an alternative x chosen by a rule for it. Imagine now that some agents leave the scene and reassess the situation from the viewpoint of the remaining agents. Since the departing agents left with the understanding that x would be chosen, a condition for an alternative to be acceptable as a choice by the remaining agents is that each of the departing agents be indeed guaranteed a certain welfare level that is at least as well off as what he was initially promised. The revised preferences of the remaining agents are then obtained by restricting their original preferences to those acceptable alternatives. The new problem so obtained is called the associated reduced problem. The invariance property, *reduction-consistency*, requires that in the reduced problem, x should still be chosen. In this paper, we also consider the converse of *reduction-consistency* proposed by Hwang and Yeh (2004a), *converse reduction-consistency*, which says that if x is chosen for all of its associated reduced problems, x should be chosen for the original problem.

We first investigate the "Copeland rule" (Copeland, 1951), the "Simpson rule" (Moulin, 1988b), and the "uncovered set" (Miller, 1980; Fishburn, 1977; McKelvey, 1986) in light of *reduction-consistency*. As we show, unfortunately, they are not *reduction-consistent* (Examples 1, 2, and 3). Note that these rules are members of a family of rules that satisfy the so-called *Condorcet principle* (Condorcet, 1785): if an alternative "majority-dominates" any other alternative, then it should be chosen. One may wonder whether no rule satisfies the *Condorcet principle* and *reduction-consistency*. The answer is yes. In fact, a more general impossibility result can be proved: *reduction-consistency* is incompatible with a weaker version of the *Condorcet principle*, the *q Condorcet principle* (Moulin, 1988a): given a real number q between $\frac{1}{2}$ and 1, if an alternative is preferred over any other alternative by more than the fraction q of the population, then it should be chosen (Theorem 1).

The *Condorcet principle* is an appealing property and has been received much attention from many authors. For instance, Moulin (1988a) shows that the principle implies an incentive compatibility property, *no-show paradox*: agents can be better off by abstaining from voting. Campbell and Kelly (1998) show that the *Condorcet principle* is incompatible with another incentive compatibility property, *strategy-proofness*: no agent can be better off by misrepresenting his preference. Theorem 1 offers another Condorcet incompatibility result from the aspect of a general notion of consistency rather than from incentive compatibility viewpoints.

Our Condorcet incompatibility result is valid under the assumption that rules are always non-empty on the domain for which agents have strict preferences. This restriction excludes a well-known rule that satisfies the *Condorcet principle* from consideration. It is majority rule (Campbell and Kelly, 2003), which chooses the alternative, if it exists, that majority-dominates any other alternative. We then switch attention to majority rule and check whether it is *reduction-consistent.* Since majority rule may be empty on the present domain, we focus on a smaller domain for which the rule is always non-empty and agents have strict preferences.

On this smaller domain, we find that majority rule satisfies *reduction-consistency* (Proposition 1). Clearly, the rule also satisfies a basic requirement, *efficiency*: if an alternative is chosen, there is no other alternative that all agents strictly prefer. Is there any *efficient* rule other than majority rule that satisfies *reduction-consistency*? Surprisingly, as we show, the answer is no (Theorem 2). We next check whether majority rule is *conversely reduction-consistent.* As we show, the answer is yes (Proposition 2). Is there any *efficient* rule other than majority rule that satisfies *converse reduction-consistency*? The answer is yes. The Pareto rule, which chooses all "Pareto-efficient" alternatives, is another example. However, it is not "single-valued:" only one alternative should be chosen. Of course, when majority rule is non-empty, it is *single-valued.* We ask whether majority rule is the only *single-valued* and *efficient* rule satisfying *converse reduction-consistency.* We show that the answer is yes (Theorem 3).

Several characterizations of majority rule has been established by many authors. For example, May (1952) characterizes majority rule on the basis of an invariance property, *positive responsiveness*: if an alternative is chosen and some agent changes his preference by ranking the alternative first and all other agents' preferences remain unchanged, then the alternative should still be chosen. Campbell and Kelly (2003) base a characterization of the rule on *strategy-proofness.* Our characterizations of majority rule provide axiomatic arguments in favor of majority rule on the basis of a general notion of consistency.

The rest of the paper is organized as follows. Section 2 introduces the model and the main properties. Section 3 introduces the central rules and presents the Condorcet incompatibility result. Section 4 introduces majority rule, provides reduction-consistency characterizations of majority rule, and shows the independence of the properties listed in each of the characterizations.

2 The model and the main properties

There is an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$. Let $\mathcal{N}$ denote the class of non-empty and finite subsets of $\mathbb{N}$. Let $\mathbb{X}$ be a countably infinite set of potential alternatives. Let $\mathcal{X}$ denote the class of non-empty and finite subsets of $\mathbb{X}$. We use $\subset$ for strict set inclusion and $\subseteq$ for weak set inclusion.

Given $N \in \mathcal{N}$, $X \in \mathcal{X}$, and $i \in N$, **agent $\boldsymbol{i}$'s preference relation on $\boldsymbol{X}$**, denoted by $\boldsymbol{R_i}$, is a binary relation on X. We assume that R_i satisfies the following two conditions. We say that R_i is **complete** if for each $\{x, y\} \subseteq X$, we have either $x \; R_i \; y$ or $y \; R_i \; x$. Thus, completeness implies that for each $x \in X$, $x \; R_i \; x$. Also, R_i is **transitive** if for each $\{x, y, z\} \subseteq X$, $x \; R_i \; y$ and $y \; R_i \; z$ together imply $x \; R_i \; z$. Throughout our presentation, we restrict attention to preference relations for which distinct alternatives are never indifferent. Namely, R_i is **strict** if for each $\{x, y\} \subseteq X$, $x \; R_i \; y$ and $y \; R_i \; x$ together imply $x = y$. Let P_i denote the strict preference relation derived from R_i. Let $\mathcal{R}_{st}(X)$ denote the class of strict preference relations on X. A **preference profile on $\boldsymbol{X}$** is a list $P \equiv (P_i)_{i \in N}$ such that for each $i \in N$, $P_i \in \mathcal{R}_{st}(X)$. A choice problem for N or simply a **problem for $\boldsymbol{N}$** is a pair (X, P) such that $X \in \mathcal{X}$ and $P \in \mathcal{R}^N_{st}(X)$.[1] Let $\mathcal{D}^N_{st}$ denote the class of problems for N with strict preference relations and $\mathcal{D}_{st} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N_{st}$. Given a class of problems $\mathcal{D} \subseteq \mathcal{D}_{st}$, a choice rule on D or simply a **rule on $\boldsymbol{\mathcal{D}}$** is a correspondence that associates with each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$ a non-empty subset of X. Our generic notation for rules is φ.

[1]By $\mathcal{R}^N_{st}(X)$, we mean the Cartesian product of $|N|$ copies of $\mathcal{R}_{st}(X)$, indexed by the elements of N. Similar expressions in the rest of the paper should be interpreted in the same manner.

We now introduce the central properties. Our first requirement is that if an alternative is chosen, there is no other alternative that all agents strictly prefer.

Efficiency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, and each $x \in \varphi(X, P)$, there is no $y \in X \backslash \{x\}$ such that for each $i \in N$, $y \; P_i \; x$.

Next is the invariance property proposed by Yeh (2004). Consider a problem and an alternative x chosen by a rule for it. Imagine now that some agents leave with the understanding that x would be chosen. The remaining agents face a new problem in which the options open to them are specified in such a way that the departing agents are made at least as good as they are at x. Thus, an alternative is acceptable as a choice in the new problem if each of the departing agents finds it at least as desirable as x. The preferences of the remaining agents in the new problem are obtained by restricting their original preferences to those acceptable alternatives. The new problem is called the reduced problem with respect to the remaining agents and x. We say that a rule is "reduction-consistent" if x is still chosen by the rule in this reduced problem.

Formally, let $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}^N$, $x \in X$, and $N' \in \mathcal{N}$ with $N' \subset N$. Let $X' \equiv \{y \in X \mid \text{for each } i \in N \backslash N', \; y \; P_i \; x \} \bigcup \{x\}$. For each $i \in N'$, let $P_i|_{X'}$ denote the restriction of P_i to X'. That is, $P_i|_{X'}$ is a preference relation on X' and the ranking of each $\{y, z\} \subseteq X'$ by $P_i|_{X'}$ is identical to the ranking of y and z by P_i. Then, the **reduced problem of $(\boldsymbol{X}, \boldsymbol{P})$ relative to $\boldsymbol{N'}$ and $\boldsymbol{x}$**, denoted $r^x_{N'}(X, P)$, is defined by

$$r^x_{N'}(X, P) \equiv \left(X', (P_i|_{X'})_{i \in N'}\right).$$

Reduction-consistency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, each $x \in \varphi(X, P)$, and each $N' \in \mathcal{N}$ with $N' \subset N$, we have $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$ and $x \in \varphi(r^x_{N'}(X, P))$.

The next property is the converse of *reduction-consistency* proposed by Hwang and Yeh (2004). It says that if an alternative is chosen for all of its associated reduced problems, then it should be chosen for the original problem.

Converse reduction-consistency: For each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N$, and each $x \in X$, if for each $N' \in \mathcal{N}$ with $N' \subset N$, $r^x_{N'}(X, P) \in \mathcal{D}^{N'}$ and $x \in \varphi(r^x_{N'}(X, P))$, then $x \in \varphi\,(X, P)$.

The last property is proposed by Condorcet (1785) and says that if an alternative is preferred over any other alternative by more than half of the population, then the alternative should be chosen.

Condorcet principle: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$, if there is $x \in X$ such that for each $y \in X \backslash \{x\}$, $|\{i \in N \mid x\ P_i\ y\}| > \frac{|N|}{2}$, then $\{x\} = \varphi\,(X, R)$.

3 Impossibility of reduction-consistency and the Condorcet pricinple

We first investigate whether on the domain, $\mathcal{D}_{st}$, there exist rules satisfying the *Condorcet principle* and *reduction-consistency*. We consider the following rules that obviously satisfy the *Condorcet principle*, and examine them in light of *reduction-consistency*. The first rule is the "Copeland rule" (Copeland, 1951; Moulin, 1988b). To define it, we introduce the following binary relation. Given $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}^N_{st}$, and $\{x, y\} \subseteq X$, we say that ***x* majority-dominates *y***, written $x\ P_{MD}\ y$, if the number of agents who strictly prefer x to y is more than half of the population, namely, $|\{i \in N\,|x\ P_i\ y\}| > \frac{|N|}{2}$.

For each $x \in X$, the Copeland rule counts the number of alternatives that x majority-dominates, and the number of alternatives that majority-dominate x. The Copeland score of x is defined as the difference between these numbers. The Copeland rule then chooses the alternative(s) with the highest Copeland score. Formally, for each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}^N_{st}$, and each $\{x, y\} \subseteq X$, the **Copeland score of x with respect to y** is defined by

$$c(x, y, P) \equiv \begin{cases} 1 & \text{if } x \; P_{MD} \; y, \\ -1 & \text{if } y \; P_{MD} \; x, \\ 0 & \text{otherwise.} \end{cases}$$

Copeland rule, ***Cope***: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}_{st}^N$,

$$Cope(X, P) \equiv \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } y \in X \setminus \{x\}, \\ \sum_{z \in X \setminus \{x\}} c(x, z, P) \geq \sum_{z \in X \setminus \{y\}} c(y, z, P) \end{array} \right\}.$$

The second rule first counts for each $x \in X$, the number of agents who prefer x to another alternative $y \in X$. Thus for each alternative, there are $|X| - 1$ corresponding numbers. The lowest number is interpreted as a measure of "minimal support for x." The Simpson rule (Moulin, 1988b) then chooses the alternative(s) with the highest minimal support. Formally, for each $N \in \mathcal{N}$, each $(X, P) \in \mathcal{D}_{st}^N$, and each $\{x, y\} \subseteq X$, the **Simpson score of x** is defined by

$$s(x, P) \equiv \min_{y \in X \setminus \{x\}} \big|\{i \in N \mid x \; P_i \; y\}\big|.$$

Simpson rule, ***Simp***: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}_{st}^N$,

$$Simp(X, P) \equiv \big\{x \in X \;\big|\; \text{for each } y \in X \setminus \{x\},\, s(x, P) \geq s(y, P)\big\}.$$

To define the next rule, we introduce another binary relation derived from the majority-domination relation. Given $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}_{st}^N$, and $\{x, y\} \subseteq X$, **x covers y in X with respect to P_{MD}** if (i) x majority-dominates y, and (ii) x majority-dominates each alternative that is majority-dominated by y. In other words, $x \; P_{MD} \; y$ and $\{z \in X \setminus \{x, y\} \mid y \; P_{MD} \; z\} \subseteq \{z \in X \setminus \{x, y\} \mid x \; P_{MD} \; z\}$.

The uncovered set (Miller, 1980; Fishburn, 1977) chooses the alternatives which are not covered by any other alternative.

Uncovered set, *UC*: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}_{st}^N$,

$$UC(X,P) \equiv \left\{ x \in X \;\middle|\; \begin{array}{c} \text{there is no } y \in X \text{ such that} \\ y \text{ covers } x \text{ in } X \text{ with respect to } P_{MD} \end{array} \right\}.$$

As the next three examples show, the Copeland rule, the Simpson rule, and the uncovered set violate *reduction-consistency*.

Example 1 On the domain, $\mathcal{D}_{st}$, the Copeland rule is not *reduction consistent*. Let $N \equiv \{1, 2, 3, 4\}$ and $X \equiv \{u, w, x, y, z\}$. Consider the following preference profile:

P_1	P_2	P_3	P_4
x	z	w	y
z	w	y	x
u	u	u	u
w	y	x	z
y	x	z	w

	u	w	x	y	z	Cope #
u		0	0	0	0	0
w	0		0	1	-1	0
x	0	0		-1	1	0
y	0	-1	1		0	0
z	0	1	-1	0		0

Clearly, $Cope(X, P) = \{u, w, x, y, z\}$. In particular, $x \in Cope(X, P)$. Let $N' \equiv \{1, 2, 3\}$.

$P_1\|_{X'}$	$P_2\|_{X'}$	$P_3\|_{X'}$
x	y	y
y	x	x

	x	y	Cope #
x		-1	-1
y	1		1

Then $r_{N'}^x(X, P) \in \mathcal{D}_{st}^{N'}$ and the feasible set of $r_{N'}^x(X, P)$ is $X' \equiv \{x, y\}$. Note that $Cope(r_{N'}^x(X, P)) = \{y\}$. Thus, $x \notin Cope(r_{N'}^x(X, P))$. *Q.E.D.*

Example 2 On the domain, $\mathcal{D}_{st}$, the Simpson rule is not *reduction consistent*. Let $N \equiv \{1, 2, \ldots, 8\}$ and $X \equiv \{x, y, z\}$. Consider the following preference profile:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
z	y	y	z	x	x	z	y
y	x	x	y	z	z	x	x
x	z	z	x	y	y	y	z

	x	y	z	Simp #
x		3	5	3
y	5		3	3
z	3	5		3

Clearly, $Simp(X, P) = \{x, y, z\}$. In particular, $x \in Simp(X, P)$. Let $N' \equiv \{1, 2, \ldots, 7\}$.

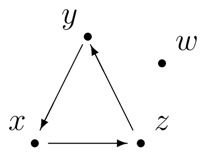
$P_{\{1,2,3,4\}}\|_{X'}$	$P_{\{5,6,7\}}\|_{X'}$
y	x
x	y

	x	y	Simp #
x		3	3
y	4		4

Then $r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st}$ and the feasible set of $r^x_{N'}(X, P)$ is $X' \equiv \{x, y\}$. Note that $Simp(r^x_{N'}(X, P)) = \{y\}$. Thus, $x \notin Simp(r^x_{N'}(X, P))$. *Q.E.D.*

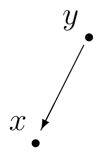
Example 3 On the domain, $\mathcal{D}_{st}$, the uncovered set is not *reduction-consistent*. Let $N \equiv \{1, 2, \ldots, 10\}$ and $X \equiv \{w, x, y, z\}$. Consider the following preference profile:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
y	z	z	y	x	w	w	w	w	w
x	y	y	x	y	z	x	y	z	x
z	x	x	z	z	x	z	x	y	z
w	w	w	w	w	y	y	z	x	y



Each arrow in the graph indicates the majority-domination relation. For instance, y majority-dominates x with respect to P_{MD}. It can be checked that $UC(X, P) = \{w, x, y, z\}$. In particular, $x \in UC(X, P)$. Let $N' \equiv \{2, 3, \ldots, 10\}$.

$P_{\{2,3,4\}}\|_{X'}$	$P_{\{5,6,7\}}\|_{X'}$	$P_{\{8,9\}}\|_{X'}$	$P_{10}\|_{X'}$
y	x	y	x
x	y	x	y



Then $r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st}$ and the feasible set of $r^x_{N'}(X, P)$ is $X' \equiv \{x,\ y\}$. Note that

$$|\{i \in N' \,|y \; P_i \; x\}| = 5 > \frac{|N'|}{2}.$$

It follows that $UC(r^x_{N'}(X,P)) = \{y\}$. Thus, $x \notin UC(r^x_{N'}(X,P))$. *Q.E.D.*

Since the Copeland rule, the Simpson rule, and the uncovered set violate *reduction-consistency* and they are members of the family of rules that satisfy the *Condorcet priniciple*, one may wonder whether on the domain, $\mathcal{D}_{st}$, no rule satisfies the two properties. The answer is yes. In fact, a more general impossibility result can be proved: *reduction-consistency* is incompatible with a weaker version of the *Condorcet principle*, the *q Condorcet principle* (Moulin, 1988a): given $q \in \mathbb{R}_+$ be such that $\frac{1}{2} \leq q < 1$, if an alternative is preferred over any other alternative by more than the fraction q of the population, then it should be chosen.[2]

***q* Condorcet principle:** For each $N \in \mathcal{N}$ and each $(X,P) \in \mathcal{D}^N$, if there is $x \in X$ such that for each $y \in X \backslash \{x\}$, $|\{i \in N \mid x\ P_i\ y\}| > q\,|N|$, then $\{x\} = \varphi\,(X,R)$.

Theorem 1 On the domain, $\mathcal{D}_{st}$, no rule satisfies the *q Condorcet principle* and *reduction-consistency*.

Proof. Let $q \in \mathbb{R}_+$ be such that $\frac{1}{2} \leq q < 1$. Let φ be a rule on $\mathcal{D}_{st}$ that satisfies the *q Condorcet principle*. We show that φ violates *reduction-consistency*.

Since $q \in \mathbb{R}_+$ and $q \leq 1$, there is a rational number q' such that $q < q' < 1$. Let $q' \equiv \frac{t}{v}$ where $v \in \mathbb{N}$, $t \in \mathbb{N}$, and $t \leq v-1$. Let $N \equiv \{1,2,\ldots,v+1\}$ and $X \equiv \{x_1, x_2, \ldots, x_{v+1}\}$. Consider the following preference profile.

P_1	P_2	$\cdots$	P_{v+1}
x_1	x_2	$\cdots$	x_{v+1}
x_2	x_3	$\cdots$	x_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$
x_{v+1}	x_1	$\cdots$	x_v

Suppose that $x_1 \in \varphi(X,P)$. Let $N' \equiv \{1,2,\ldots,v\}$.

[2] By $\mathbb{R}_+$, we denote the set of positive real numbers, $\mathbb{R}_+ \equiv \{x \in \mathbb{R} \mid x \geq 0\}$.

$P_1\|_{X'}$	$P_2\|_{X'}$	$P_3\|_{X'}$	$\cdots$	$P_v\|_{X'}$
x_1	x_{v+1}	x_{v+1}	$\cdots$	x_{v+1}
x_{v+1}	x_1	x_1	$\cdots$	x_1

Then $r_{N'}^{x_1}(X,P) \in \mathcal{D}_{st}^{N'}$ and the feasible set of $r_{N'}^{x_1}(X,P)$ is $X' \equiv \{x_1, x_{v+1}\}$. Note that

$$\frac{|\{i \in N' \mid x_{v+1}\ P_i\ x_1\}|}{|N'|} = \frac{v-1}{v} \geq q' > q.$$

By the *q Condorcet principle*, $\{x_{v+1}\} = \varphi(r_{N'}^{x_1}(X,P))$. It follows that $x_1 \notin \varphi(r_{N'}^{x_1}(X,P))$. Thus, φ violates *reduction-consistency*. By a similar argument, it can also be shown that φ violates *reduction-consistency* if $x_1 \in \varphi(X,P)$, $x_2 \in \varphi(X,P)$, $\cdots$, or $x_{v+1} \in \varphi(X,P)$. *Q.E.D.*

4 Reduction-consistency characterizations of majority rule

Theorem 1 is valid under the assumption that rules are always non-empty on $\mathcal{D}_{st}$. This restriction excludes a well-known rule that satisfy the *Condorcet principle* from consideration. It is the so-called majority rule (Campbell and Kelly, 2003), which chooses the alternative, if it exists, that majority-dominates any other alternative.[3] Since the rule may be empty on $\mathcal{D}_{st}$, we then restrict attention to a smaller domain, denoted by $\mathcal{D}_{st\cap C_m\neq\emptyset}$, for which agents have strict preferences and majority rule is always non-empty.

Majority rule, $\boldsymbol{C_m}$: For each $N \in \mathcal{N}$ and each $(X,P) \in \mathcal{D}_{st\cap C_m\neq\emptyset}^N$,

$$C_m(X,P) \equiv \{x \in X \mid \text{for each } y \in X\backslash\{x\},\ x\ P_{MD}\ y\}.$$

[3]We can define a weaker version of majority rule, weak majority rule, which choose the alternatives, if they exist, that are not majority-dominated by any other alternative (McKelvey, 1986; Austen-Smith and Banks, 1999). Hwang and Yeh (2004b) provide a characterization of weak majority rule on the basis of a stronger version of *reduction-consistency* and a weaker version of *converse reduction-consistency*.

We now check whether on $\mathcal{D}_{st \cap C_m \neq \emptyset}$, majority rule satisfies *reduction-consistency.* As the next result shows, the answer is yes.

Proposition 1 On the domain, $\mathcal{D}_{st \cap C_m \neq \emptyset}$, majority rule is *reduction-consistent.*

Proof. Let $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$, $x \in C_m(X, P)$, and $N' \in \mathcal{N}$ with $N' \subset N$. We show that $r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st \cap C_m \neq \emptyset}$ and $x \in \varphi(r^x_{N'}(X, P))$. The proof is in two steps.

Step 1: $\boldsymbol{r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st \cap C_m \neq \emptyset}}$. Since agents' preferences are strict, it follows that $r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st}$. We now show that $C_m(r^x_{N'}(X, P)) \neq \emptyset$. Suppose, by contradiction, that $C_m(r^x_{N'}(X, P)) = \emptyset$. Note that x is a feasible alternative in $r^x_{N'}(X, P)$. It follows that there exists $y \in X \setminus \{x\}$ such that (i) $\left|\left\{i \in N' \mid y\ P_i\ x\right\}\right| > \frac{|N'|}{2}$ and (ii) for each $j \in N \backslash N'$, $y\ P_j\ x$. Thus,

$$\begin{aligned} \left|\left\{i \in N \mid y\ P_i\ x\right\}\right| &> |N| - |N'| + \frac{|N'|}{2} \\ &= \frac{|N| + |N| - |N'|}{2} \\ &> \frac{|N|}{2}. \end{aligned}$$

It follows that $x \notin C_m(X, P)$, in violation of $x \in C_m(X, P)$.

Step 2: $\boldsymbol{x \in C_m(r^x_{N'}(X, P))}$. Suppose that $x \notin C_m(r^x_{N'}(X, P))$. By Step 1, there exists $y \in X \backslash \{x\}$ such that $y \in C_m(r^x_{N'}(X, P))$. Thus, $\left|\left\{i \in N' \mid y\ P_i\ x\right\}\right| > \frac{|N'|}{2}$. Note that for each $i \in N \backslash N'$, $y\ P_i\ x$. It follows that

$$\begin{aligned} \left|\left\{i \in N \mid y\ P_i\ x\right\}\right| &> |N| - |N'| + \frac{|N'|}{2} \\ &= \frac{|N| + |N| - |N'|}{2} \\ &> \frac{|N|}{2}. \end{aligned}$$

Thus, $x \notin C_m(X, P)$, in violation of $x \in C_m(X, P)$. *Q.E.D.*

Clearly, majority rule satisfies *efficiency*. Is there any *efficient* rule other than majority rule that satisfies *reduction-consistency*? Surprisingly, we show that the answer is no. To prove this assertion, we make use of the next two facts. The first one indicates that majority rule satisfies the following property, which says that only one alternative should be chosen.

Single-valuedness: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N$, we have $|\varphi(X, P)| = 1$.

The second fact indicates that majority rule satisfies the converse of *reduction-consistency*.

Proposition 2 On the domain, $\mathcal{D}_{st \cap C_m \neq \emptyset}$, majority rule satisfies *converse reduction-consistency*.

Proof. Let $N \in \mathcal{N}$, $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$, and $x \in X$. For each $N' \in \mathcal{N}$ with $N' \subset N$, let $r^x_{N'}(X, P) \in \mathcal{D}^{N'}_{st \cap C_m \neq \emptyset}$ and $x \in C_m\left(r^x_{N'}(X, P)\right)$. We show that $x \in C_m(X, P)$. Suppose, by contradiction, that $x \notin C_m(X, P)$. Since $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$, there exists $y \in X \setminus \{x\}$ such that $y \in C_m(X, P)$. It follows that $\left|\left\{i \in N \mid y \; P_i \; x\right\}\right| > \frac{|N|}{2}$. Let $j \in \left\{i \in N \mid y \; P_i \; x\right\}$. Note that y is a feasible alternative in $r^x_{N \setminus \{j\}}(X, P)$., and that agents' preferences are strict. Thus, we have either

$$\text{(i)} \; \left|\left\{i \in N \setminus \{j\} \mid y \; P_i \; x\right\}\right| > \frac{|N \setminus \{j\}|}{2}$$

or

$$\text{(ii)} \; \left|\left\{i \in N \setminus \{j\} \mid y \; P_i \; x\right\}\right| = \frac{|N \setminus \{j\}|}{2} = \left|\left\{i \in N \setminus \{j\} \mid x \; P_i \; y\right\}\right| .$$

In both cases, we have $x \notin C_m\left(r^x_{N \setminus \{j\}}(X, P)\right)$, which contradicts $x \in C_m\left(r^x_{N \setminus \{j\}}(X, P)\right)$. *Q.E.D.*

Thanks to Propositions 1 and 2, we are now ready to prove the announced characterization of majority rule.

Theorem 2 On the domain, $\mathcal{D}_{st \cap C_m \neq \emptyset}$, majority rule is the only rule satisfying *efficiency* and *reduction-consistency*.

Proof. Clearly, majority rule is *efficient*. As shown in Proposition 1, it is *reduction-consistent*. Conversely, let φ be a rule satisfying the properties. Let $N \in \mathcal{N}$ and $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$. We show that $\varphi(X, P) = C_m(X, P)$. The proof is by induction on $|N|$.

Case 1: $|N| = 1$. Since agents' preferences are strict, by *efficiency*, $\varphi(X, P) = C_m(X, P)$.

Case 2: $|N| > 1$. Let $k \in \mathbb{N}$. The induction hypothesis is that for each $N \in \mathcal{N}$ with $1 \leq |N| \leq k$ and $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$, $\varphi(X, P) = C_m(X, P)$. We show that for each $N \in \mathcal{N}$ with $|N| = k + 1$ and each $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$, $\varphi(X, P) = C_m(X, P)$. Since majority rule is *single-valued*, it suffices to show that $\varphi(X, P) \subseteq C_m(X, P)$. Let $x \in \varphi(X, P)$. By *reduction-consistency*, $x \in \varphi(r^x_{N'}(X, P))$. Note that $|N'| < k + 1$. By the induction hypothesis, $\varphi(r^x_{N'}(X, P)) = C_m(r^x_{N'}(X, P))$. Thus, $x \in C_m(r^x_{N'}(X, P))$. As shown in Proposition 2, majority rule is *conversely reduction-consistent*. Thus, $x \in C_m(X, P)$. *Q.E.D.*

In the proof of Theorem 2, we make use of the fact that majority rule is *converse reduction-consistency*. One may wonder whether another characterization of majority rule can be obtained by replacing *reduction-consistency* with *converse reduction-consistency* in Theorem 2. The answer is no. The Pareto rule, which chooses all "Pareto-efficient" alternatives, also satisfies *efficiency* and *converse reduction-consistency*. However, the Pareto rule is not *single-valued*. Is there any *single-valued* rule other than majority rule satisfying *efficiency* and *converse reduction-consistency*? The answer is no. To prove this assertion, we introduce a lemma that indicates a logical relation between *single-valuedness*, *converse reduction-consistency*, and *reduction-consistency*.

Lemma 1 On the domain, $\mathcal{D}_{st \cap C_m \neq \emptyset}$, if a rule satisfies *single-valuedness* and *converse reduction-consistency*, then it satisfies *reduction-consistency*.

Proof. Let φ be a rule satisfying *single-valuedness* and *converse reduction-consistency*. We show that φ satisfies *reduction-consistency*. The proof is in two steps. Step 1 shows that for each $N \in \mathcal{N}$ with $|N| > 1$ and $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$, majority rule is the only *single-valued* rule satisfying *converse reduction-consistency*. Step 2 concludes by invoking that majority rule is *reduction-consistent*.

Step 1: For each $N \subset \mathcal{N}$ with $|N| > 1$ and each $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$, $\varphi(X,P) = C_m(X,P)$. Let $N \in \mathcal{N}$ with $|N| \geq 2$ and $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$. Let $\{x\} = C_m(X,P)$. We show that $\{x\} = \varphi(X,P)$. The proof is by induction on $|N|$.

Substep 1.1: $|N| = 2$. Note that $\{x\} = C_m(X,P)$ and agents' preferences are strict. If $|X| = 1$, then we are done. Suppose that $|X| > 1$. It follows that for each $i \in N$ and $y \in X \setminus \{x\}$, $x \; P_i \; y$. Thus, for each $N' \subset N$, x is the only feasible alternative in $r^x_{N'}(X,P)$. It follows that $\{x\} = \varphi(r^x_{N'}(X,P))$. By *converse reduction-consistency*, $\{x\} = \varphi(X,P)$.

Substep 1.2: $|N| > 2$. Let $k \in \mathbb{N}$. The induction hypothesis is that for each $N \in \mathcal{N}$ with $2 \leq |N| \leq k$ and $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$, $\varphi(X,P) = C_m(X,P)$. We show that for each $N \in \mathcal{N}$ with $|N| = k+1$ and each $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$, $\varphi(X,P) = C_m(X,P)$. As shown in Proposition 1, majority rule is *reduction-consistent*. It follows that for each $N' \subset N$ with $|N'| \geq 2$, $\{x\} = C_m(r^x_{N'}(X,P))$. Note that $|N'| < k+1$. By the induction hypothesis, $C_m(r^x_{N'}(X,P)) = \varphi(r^x_{N'}(X,P))$. It follows that $\{x\} = \varphi(r^x_{N'}(X,P))$. By *converse reduction-consistency* and *single-valuedness*, $\{x\} = \varphi(X,P)$.

Step 2: Completion of the proof. Let $N \in \mathcal{N}$ and $(X,P) \in \mathcal{D}^N_{st\cap C_m \neq \emptyset}$. By *single-valuedness*, let $\{x\} = \varphi(X,P)$. Let $N' \subset N$. We show that $r^x_{N'}(X,P) \in \mathcal{D}^{N'}_{st\cap C_m \neq \emptyset}$ and $\{x\} = \varphi(r^x_{N'}(X,P))$. Suppose that $|N'| \geq 2$. By Step 1, $\varphi(r^x_{N'}(X,P)) = C_m(r^x_{N'}(X,P))$. Note that $\{x\} = \varphi(X,P) = C_m(X,P)$ and majority rule is *reduction-consistent*. It follows that $\{x\} =$

$\varphi\left(r^x_{N'}(X,P)\right)$ and $r^x_{N'}(X,P) \in \mathcal{D}^{N'}_{st\cap C_m\neq\emptyset}$. We now consider $|N'|=1$. We first show that if $\{x\}=C_m(X,P)$, then for each $N' \subset N$ with $|N'|=1$, x is the only feasible alternative in $r^x_{N'}(X,P)$. Suppose, by contradiction, that there is another feasible alternative y. Since for each $i \in N\backslash N'$, $y \; P_i \; x$, it follows that $\{x\} \neq C_m(X,P)$, in violation of $\{x\}=C_m(X,P)$. Note that $|N| \geq 2$. By Step 1, $\{x\}=C_m(X,P)$. Thus, x is the only feasible alternative in $r^x_{N'}(X,P)$. It follows that $\{x\}=\varphi\left(r^x_{N'}(X,P)\right)$ and $r^x_{N'}(X,P) \in \mathcal{D}^{N'}_{st\cap C_m\neq\emptyset}$. *Q.E.D.*

Thanks to Theorem 2 and Lemma 1, we are now ready to prove the announced characterization of majority rule.

Theorem 3 On the domain, $\mathcal{D}_{st\cap C_m\neq\emptyset}$, majority rule is the only *single-valued* rule satisfying *efficiency* and *converse reduction-consistency*.

Proof. Clearly, majority rule is *single-valued* and *efficient*. As shown in Proposition 2, it is *conversely reduction-consistent*. Conversely, let φ be a rule satisfying the properties. Let $N \in \mathcal{N}$ and $(X,P) \in \mathcal{D}^{N}_{st\cap C_m\neq\emptyset}$. We show that $\varphi(X,P)=C_m(X,P)$. Suppose that $|N|=1$. Since agents' preferences are strict, by *efficiency*, $\varphi(X,P)=C_m(X,P)$. By Lemma 1 and Theorem 2, we conclude that $\varphi(X,P)=C_m(X,P)$ for $|N|>1$. *Q.E.D.*

As shown in Lemma 1, *converse reduction-consistency* together with *single-valuedness* implies *reduction-consistency*. One may wonder whether the converse is true. The answer is no. The following *single-valued* rule satisfies *reduction-consistency* but violates *converse reduction-consistency*. When there is an alternative Pareto-dominated by any other alternative, the rule chooses the alternative; otherwise, the rule chooses the alternative selected by majority rule.

We now show that the properties listed in Theorems 2 and 3 are logically independent. For this purpose, we introduce additional rules. The first rule chooses all "Pareto-efficient" alternatives.

Pareto rule, $\boldsymbol{PE}$: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$,

$$PE\,(X, P) \equiv \left\{ x \in X \;\middle|\; \text{there is no } y \in X \text{ such that for each } i \in N,\ y\ P_i\ x \right\}.$$

Next is a family of fixed-order rules proposed by Yeh (2004). This family of rules is inspired by the family of "target rules" studied by Ching and Thomson (1992) in the context of choosing a point in an interval over which each agent is endowed with a single-peaked preference. Given a point or a target in an interval, the associated target rule is described as follows: if the target is "Pareto-efficient," then the rule chooses this point; otherwise, it chooses the point in the set of Pareto-efficient points that is closest to the target.

The family of fixed-order rules is defined in a similar way as the target rules, but in the context of collective choice problems. Formally, let $P_0 \in \mathcal{R}_{st}(\mathbb{X})$ be a strict preference relation on $\mathbb{X}$. We interpret P_0 to be the preference relation of an "arbitrator." Then, the fixed-order rule chooses the most preferred alternative according to P_0 from those alternatives chosen by the Pareto rule .

Fixed-order rule relative to $\boldsymbol{P_0}$, $\boldsymbol{F^{P_0}}$: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$,

$$F^{P_0}(X, P) \equiv \{x \in PE(X, P) \mid \text{for each } y \in PE\,(X, P) \setminus \{x\}\,,\ x\ P_0\ y\}.$$

The last rule is defined as follows. Let $\{x, y, z\} \subset \mathbb{X}$. If there is only one agent, the feasible set is $\{x, y, z\}$, and x is the "middle alternative" according to the preference of that agent, then the rule chooses x; otherwise, the rule chooses the alternative selected by majority rule.

Constant-Majority rule, $\boldsymbol{CM}$: For each $N \in \mathcal{N}$ and each $(X, P) \in \mathcal{D}^N_{st \cap C_m \neq \emptyset}$,

$$CM\,(X, P) \equiv \begin{cases} \{x\} & \text{if } N \equiv \{i\}\,,\ X \equiv \{x, y, z\}\,, \text{ and } y\ P_i\ x\ P_i\ z; \\ C_m\,(X, P) & \text{otherwise} \end{cases}$$

Property / Rule	PE	F^{P_0}	CM
single-valuedness	No	Yes	Yes
efficiency	Yes	Yes	No
reduction-consistency	No	No	Yes
converse reduction-consistency	Yes	No	Yes

Table 1: **Independence of the properties in Theorems 2 and 3.** The notation "Yes" ("No") means that a certain rule satisfies (violates) a certain property.

Table 1 shows that the properties listed in Theorems 2 and 3 are logically independent. For example, the Pareto rule satisfies *efficiency* but violates *reduction-consistency.* The constant-majority rule satisfies *reduction-consistency* but violates *efficiency.* Thus, the properties listed in Theorem 2 are independent.

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