Robust Monopoly Pricing: The Case of Regret*

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First Version: May 2003

This Version: January 2005

Preliminary and Incomplete

Abstract

We consider a robust version of the classic problem of optimal monopoly pricing with incomplete information. The robust version of the problem is distinct in two aspects: (i) the seller minimizes regret rather than maximizes revenue, and (ii) the seller only knows that the true distribution of the valuations is in a neighborhood of a given model distribution.

The robust pricing policy is characterized as the solution to a minimax problem for the case of small and large uncertainty faced by the seller. In the case of small uncertainty, the robust pricing policy prices closer to the median at a rate determined by the curvature of the static profit function. Regret increases linearly in the uncertainty.

Keywords: Monopoly, Optimal Pricing, Regret, Robustness

Jel Classification: C79, D82

*The first author gratefully acknowledges support by NSF Grants #SES-0095321, #CNS-0428422 and a DFG Mercator Research Professorship at the Center of Economic Studies at the University of Munich. We would like to thank Peter Klibanoff and David Pollard for helpful suggestions. We are grateful to seminar participants at the University of Wisconsin and the Cowles Foundation Conference "Uncertainty in Economic Theory" at Yale University for many helpful comments. Sahar Malik provided excellent research assistance.

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1 Introduction

1.1 Motivation

In this paper we investigate a robust version of the classic problem of a monopolist selling a product under incomplete information. We introduce robustness by enriching the canonical model in two key aspects.

First, instead of a given true distribution regarding the valuations of the buyers, in our set-up the seller only knows that the true distribution will be in a neighborhood of a given model distribution. The enlargement of the set of possible priors is meant to represent the uncertainty about the nature of the true distribution. As such, it represents the possibility that the model of the seller is misspecified.

Second, the objective function of the seller is formulated as a regret minimization problem rather than a revenue maximization problem. For a given neighborhood of possible distributions around a model distribution, we characterize the pricing policy which minimizes maximal regret. We refer to the resulting pricing policy as the robust policy.

The main objective of the paper is to describe how the robust policies depend on the model distribution and the size of the uncertainty as represented by the size of the neighborhood. As part of the analysis, we also determine how the regret varies with the amount of uncertainty faced by the seller. Naturally, the specification of the neighborhood matters in the determination of the robust pricing policy. For the most part, we shall adopt the notion of a contaminated neighborhood in which the true distribution is represented as a convex combination of the model distribution and an arbitrary distribution. The size of the neighborhoods is then represented by the weight permitted on the arbitrary distribution. A common variant to the contamination model are neighborhoods described the Prohorov norm, which we shall discuss in future versions of the paper as well.

We choose to pursue the robust analysis with the notion of regret rather than revenue as the regret objective delivers interesting insights into the decision problem by the seller even in the presence of complete uncertainty. In contrast, with revenue as an objective function and nature unconstrained in its choice of distributions over the valuations, the trivial solution is that adversary nature will place probability one on the lowest possible realization. Presumably for this reason, a common evaluation criterion in theoretical computer science for the performance of an auction is the ratio between realized revenue and maximal feasible revenue (which is equal to the realized value). In this literature, a good mechanism is a mechanism which maximizes the ratio over all realizations and the resulting mechanism is called $\alpha-$competitive if the largest attainable ratio is α (see Neeman (1999) for a competitive analysis of the second price auction). Yet the notion of competitive ratio lacks in decision theoretic foundations and it is difficult to link the competitive ration to a related Bayesian decision problem. For these reasons, we suggest regret as a criterion which delivers interesting insights in the presence of complete uncertainty and has some desirable decision theoretic foundations. The idea of a minimax regret rule was advocated by Savage (1954) and appears to have originated in Wald (1950), for a more critical stance regarding minimax regret see Chernoff (1954) and Savage (1970). Apart from the emphasis on regret rather than revenue, we can therefore think of the seller as ambiguity averse in the sense of Gilboa & Schmeidler (1989). As regret compares the realized revenue with maximal revenue for every possible realization it is in general impossible to find a zero regret policy. In a sense it represent a too difficult benchmark for the monopolist as even in the best possible situation, he will only know the true distribution but never the realized valuation. For this reason, we consider in the second half of the paper a weaker notion of regret, referred to as interim regret. Under interim regret the decision maker only compares the expected realized revenue with maximal expected revenue the seller could obtain if he were to know the true distribution of valuations. This can also be interpreted as the relevant notion of regret when facing a continuum of buyers and not able to price discriminate even if the true distribution were known.

With large uncertainty, the solution to the regret and the interim regret problem coincide exactly. When we consider local uncertainty, this coincidence ceases to exists. The robust pricing policy under regret is then determined by the location of the maximal exposure risk relative to the optimal price before robustness considerations. As the seller acts to minimize regret, the optimal price will therefore locally move in the direction of the exposure. This logic will attach a prominent role to the midpoint of the interval as expected prices will move towards this midpoint. In contrast, when the value of the buyer is known almost certainly, the robust pricing policy under interim regret has only downward pressure on prices. Misperceptions of the buyer value distribution at high value levels are now ignored as they do not enter the optimal profits even if the true distribution were known. Both regret and interim regret increase linearly in the size of the neighborhood. We compare robust policies to the performance of the optimal price to the monopolist's beliefs.

The reminder of the paper is organized as follows. In Section 2 we present the model, the notion of regret and the neighborhoods. In Section 3 we characterize the robust pricing policy with large uncertainty. In this case, the neighborhood is unrestricted. In Section 4, we characterize the robust policy in the presence of small uncertainty represented by a small neighborhood. Here, an important benchmark will be the case of near certainty in which the model distribution is a Dirac function. In Section 5 we shall consider the robust pricing policy when we relax the notion of regret to interim regret. Section 6 concludes with final remarks and a discussion of open problems.

1.2 Related Literature

The basic framework for robust analysis in statistic in presented in Huber (1981) and Hampel, Ronchetti, Rousseeuw & Stahel (1986). Linhart & Radner (1989) analyze minimax regret strategies in a bilateral bargaining framework. In their framework, the valuation of the buyer and the cost function of the seller depend on a choice variable q which may represent quantity or quality. In contrast to the incomplete information environment here, the bulk of the analysis in Linhart & Radner (1989) is concerned with bilateral trade under complete information. In addition, they largely restrict their analysis to deterministic strategies, even though mixed strategies will typically lead to lower regrets.

Bose, Ozdenoren & Pape (2004) investigate the nature of the optimal auction in the presence of an ambiguity averse seller as well as ambiguity averse bidders. In contrast, we consider a monopoly pricing problem with either a single buyer or a continuum of buyers. In either case, as the product is produced at a constant marginal cost, there is no competition across the buyers. It follows that the ambiguity aversion (or robustness concern) is of no consequence for the behavior of the buyers. The optimal selling mechanism is a fixed price mechanism and each buyer's belief about the other buyers is immaterial. A recent article by Prasad (2003) presents negative result, an in particular shows that the standard optimal pricing policy of the monopolist is not robust. In the case of an ambiguity averse seller and (two) ambiguity neutral buyers, their main result, Proposition 9, asserts the *existence* of a selling mechanism which maximizes the minimum expected revenue. (For a related result with a risk averse seller, see Eso & Futo (1999).) A recent manuscript, Hansen & Sargent (2004), surveys and establishes new results for robust policies in dynamic and recursive problem within a macroeconomic context.

In Bergemann & Schlag (2004) we pursue the robust analysis in a more general mechanism design context while focusing on revenue rather than regret. In Selten (1989) first and second price auction are considered under a modified form of regret.

2 Model

2.1 Regret

We consider the optimal pricing problem of a monopolist. In contrast to classical problem where the monopolist seeks to maximize expected revenue for a given prior distribution over valuations, we consider the problem where the monopolist seeks minimize the regret against an adversary who chooses a distribution F over valuation so as to maximize the regret of the seller.

We begin with a classic environment in which the seller faces a single potential buyer who has unit demand for this good. The marginal costs of production are constant and normalized to zero. The buyer values the good at a value v which is private information to the buyer. The buyer purchases the good if and only if his net surplus from the transaction is positive, or $v-p \geq 0$. The valuation v of the buyer is drawn from a probability distribution on $[\underline{v}, 1]$ with $0 \leq \underline{v} < 1$. Let $\tilde{v}$ be the associated random variable and let $F_v(v) = \Pr(\tilde{v} \leq v)$ be the c.d.f. and let f_v denote the corresponding density whenever needed. The expected revenue, or profit, of the monopolist is for a given distribution F_v is denoted by:

$$\pi(p, F_v) = p \Pr(\tilde{v} \geq p) = p \int_p^1 dF_v(v).$$

The objective of the monopolist is to minimize his regret from an optimal selling policy. The *regret* of the monopolist represents the opportunity loss attributed to the fact that he does not first learn the value of the buyer and then sets his price. More specifically the regret of the monopolist charging price p facing a buyer with value v is defined as the difference between (a) the profit the monopolist could make if he would learn the value v of the buyer before setting his price and (b) the profit he makes without this information. Formally, regret is defined as:

$$r(p, v) \triangleq v - p\mathbb{I}_{\{v \geq p\}}.$$

We observe that regret is non-negative and can only vanish if $p = v$. The regret of the monopolist is strictly positive in either of two cases: (i) the value v exceeds the price p, the indicator function is then $\mathbb{I}_{\{v \geq p\}} = 1$, and the regret is the difference between possible revenue and actual price or (ii) the value v is below the price p, the indicator function is then $\mathbb{I}_{\{v \geq p\}} = 0$, and the regret is foregone surplus due to a too high price. The regret from adopting a mixed strategy pricing policy according to F_p when facing a buyer with value drawn according to F_v is derived by taking expectations:

$$r(F_p, F_v) \triangleq \mathbb{E}[r(\tilde{p}, \tilde{v})] = \int \int r(p, v)\, dF_p(p)\, dF_v(v) = \int v dF_v(v) - \int \pi(p, F_v)\, dF_p(p).$$

The strategy space of the monopolist is the set of all random pricing policy with support on the positive real line. The random variable associated with the mixed strategy is denoted by $\tilde{p}$, the c.d.f. by $F_p \in \Delta\mathbb{R}_+$ and the density by f_p. In contrast to the classic monopoly problem in which the monopolist wishes to maximizes the expected return

$$\max_{F_p} \left\{ \int \pi(p, F_v)\, dF_p(p) \right\},$$

we seek to find the policy which minimizes over all policies the maximum regret over all distributions over valuations F_v:

$$\inf_{F_p} \sup_{F_v} r(F_p, F_v) = \inf_{F_p} \sup_{F_v} \left\{ \int v dF_v(v) - \int \pi(p, F_v)\, dF_p(p) \right\}$$

where formally we use sup instead of max.

We will relate the above problem to the dual problem in which the monopolist knows the value distribution and nature chooses this to maximize regret, i.e. nature solves

$$\sup_{F_v} \inf_{F_p} \left\{ \int v dF_v(v) - \int \pi(p, F_v)\, dF_p(p) \right\}. \qquad (r)$$

and to the solution of the zero sum game in which the monopolist plays a simultaneous move zero sum game against nature where the monopolist aims to minimize regret while nature aims to maximize regret of the monopolist. We refer here to distributions F_p^* and F_v^* that solve

$$r\left(F_p^*, F_v\right) \leq r\left(F_p^*, F_v^*\right) \leq r\left(F_p, F_v^*\right)$$

for all F_p and F_v.

The above notion of regret is an ex-post criterion as we compare the realized revenue of the monopolist with the revenue he could have realized for every realization of the random variable $\widetilde{v}$. This suggests a weaker version of regret (denoted by $R = R(F_p, F_v)$) in form of an interim criterion where the regret is the difference between the expected revenue the monopolist could obtain if he knew the true distribution and the expected revenue he actually obtains.

$$R(F_p, F_v) = \sup_p \pi(p, F_v) - \int \pi(p, F_v)\, dF_p(p).$$

We shall refer to this interim notion of regret as *interim regret*.[1] The resulting min max problem is given by:

$$\inf_{F_p} \sup_{F_v} R(F_p, F_v). \tag{R}$$

There is also an alternative interpretation of interim regret when there is a continuum of buyers. Regret investigates the impact of not knowing the value of a single buyer. Consider now a continuum of buyers. There are two possible benchmarks with which expected payoffs are compared. One is where the monopolist is able to price discriminate among buyers if she knew their values. Here the notion of regret applies and the monopolist prices as if facing a single buyer. However, if the monopolist is not able to price discriminate, but instead has to set a single price then regret no longer makes sense. Without the ability to price discriminate, a monopolist who knows F_v will charge $p^* \in \arg\max_o \pi(p, F_v)$. Thus, the natural concept of regret when facing many buyers and unable to price discriminate is interim regret.

2.2 Neighborhoods

In contrast to the standard model of incomplete information, in our robust version the seller is uncertain about the true distribution over the buyer's valuations. The uncertainty (or ambiguity in the language of Ellsberg (1961)) is represented by a model distribution $F_0(v)$ and the requirement that the true distribution $F_v(v)$ is in a neighborhood of the model distribution $F_0(v)$. The size of the uncertainty can then be quantified by the size of neighborhood around the model distribution. For most of the paper, we shall consider the contamination "neighborhood" $\mathcal{N}_\varepsilon(F_0)$ identified by model distribution and the bandwidth ε:

$$\mathcal{N}_\varepsilon(F_0) = \{F_v \,|F_v = (1-\varepsilon) F_0 + \varepsilon G,\ G \in \Delta[\underline{v}, 1]\}.$$

[1] The notion of interim regret is called "regret risk" in Chamberlain (2000).

The contamination neighborhood therefore permits that with a *small* probability there is *large* change in the valuations. We shall refer to the case of $\varepsilon = 1$ as the case of *large uncertainty.* With $\varepsilon = 1$, the neighborhood is not anchored by any model distribution F_0 at all. Similarly, we refer to the case of small ε as the case of *small uncertainty.* If the model distribution F_0 is given by a Dirac function δ_v (and ε is small), then we refer to it as the case of *near certainty*, and for arbitrary model distributions F_0 with a small ε, we refer to it as local uncertainty.

A related concept of neighborhoods is generated by the Prohorov distance, $\mathcal{P}_\varepsilon(F_0)$:

$$\mathcal{P}_\varepsilon(F_0) = \{F_v \,|F_v(A) \leq F_0(A^\varepsilon) + \varepsilon, \ \forall A\},$$

where the set A^ε denotes the closed ε neighborhood of A, or

$$A^\varepsilon = \left\{ v \in [\underline{v}, 1] \;\middle|\; \inf_{y \in A} d(x, y) \leq \varepsilon \right\}.$$

In contrast with the contamination neighborhood, the Prohorov metric allows for a *small* probability of *large* changes in the valuations as well as *large* probabilities of *small* changes in the valuations. The difference between the two notions of neighborhoods is illustrated in the graphics below for a Dirac function $\delta_{\hat{v}}$ at $\hat{v}$. The contamination through the distribution $F_0 = \delta_{\hat{v}}$ allow F to be any distribution function between the two dotted lines, but it has to have a mass point at $\hat{v}$ as the distribution F is formed by convex combination involving the Dirac function $\delta_{\hat{v}}$.

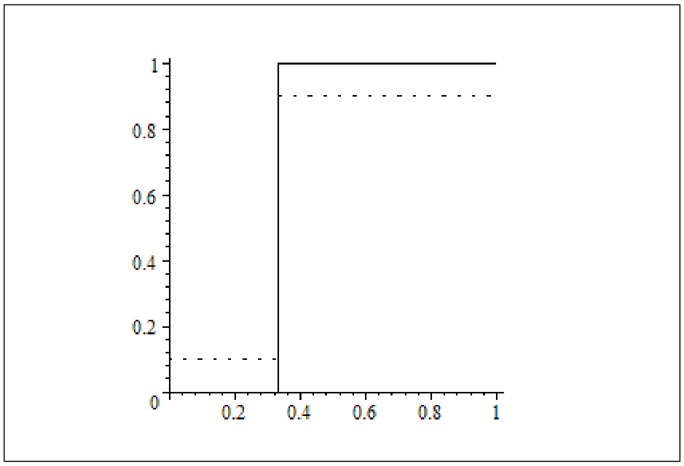


Figure 1: Illustration of the contamination neighborhood of size $\varepsilon = 0.1$ around $\delta_{1/3}$.

In contrast, the Prohorov neighborhood permits that the probability mass at $\hat{v}$ is accumulated over the interval $[\hat{v} - \varepsilon, \hat{v} + \varepsilon]$ and in consequence a distribution F close to $F_0 = \delta_{\hat{v}}$ can be between the two dotted lines which forms a bandwidth around the entire graph of the distribution function F_0 rather than only a bandwidth along its horizontal components.

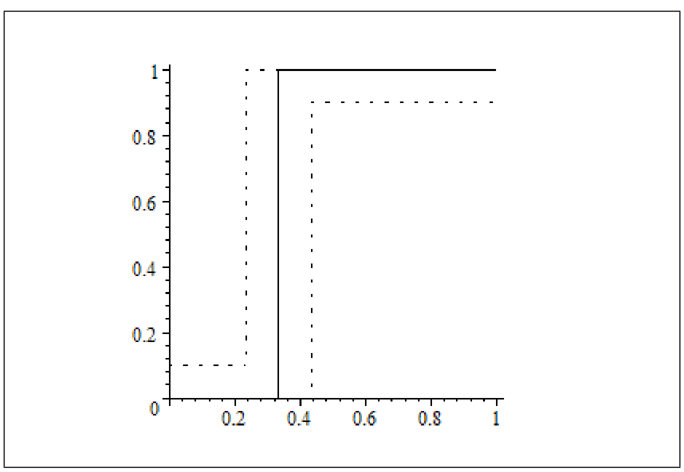


Figure 2: Illustration of the Prohorov neighborhood of size $\varepsilon = 0.1$ around $\delta_{1/3}$.

Conceptually therefore, the Prohorov metric is the most attractive metric to work with.[2] However, it is not very manageable for actual calculations. In the following we shall employ the Prohorov metric only for very simple models of uncertainty and then mostly to contrast the robust policy with a Prohorov neighborhood to a robust policy with respect to the contamination neighborhood.

With the notion of regret we consider here, there is another reason why the Prohorov metric might not be the natural metric. Consider for the moment pure strategies by the seller and nature. Since the regret displays a discontinuity at $v = p$, two pure strategies by nature which are nearby according to the Prohorov metric, say $v = p - \varepsilon$ and $v = p$ yield very different payoffs to the seller and nature. Yet, with the contamination neighborhood no pure strategy is close to another pure strategy, which also demonstrates that in a specific sense the contaminations neighborhoods are small.[3,4]

3 Large Uncertainty

We search for a mixed pricing strategy that minimizes among all mixed pricing strategies the maximum regret among all distributions of buyer valuation. Accordingly, $F_p^* \in \Delta\mathbb{R}_+$ *attains minimax regret* if

$$F_p^* \in \arg \min_{F_p \in \Delta\mathbb{R}} \sup_{F_v \in \Delta[\underline{v},1]} r(F_p, F_v) = \arg \min_{F_p \in \Delta\mathbb{R}} \sup_{v \in [\underline{v},1]} r(F_p, v).$$

Notice that although we use the term minimax regret the expressions themselves use supremum instead of maximum as $\int_p r(p,v)\, dF_p(p)$ need not be continuous in v. The lowest upper bound that regret can attain is denoted by:

$$\bar{r} \triangleq \inf_{F_p \in \Delta\mathbb{R}} \sup_{v \in [\underline{v},1]} r(F_p, v),$$

where $\bar{r}$ will be referred to as the value of *minimax regret*. The characterization of the minimax regret will be achieved by finding an equilibrium strategy of the monopolist for

[2] The Prohorov distance is actually a metric for weak convergence of probability measures (for more on additional properties of the Prohorov distance see (Dudley (2002)) and (Pollard (2002)))

[3] There is a close relationship between the contamination neighborhood model and the variational norm, see Shiryaev (1995), p.360.

[4] The issue of an appropriate neighbohood also appears in stability analysis of evolutionary games, see Oechssler & Riedel (2001) and Oechssler & Riedel (2002).

a specific zero sum game. The game is between the monopolist and nature where the monopolist aims to minimize regret r and nature aims to maximize regret r. The solution to the dual problem of the adversary is given by F_v^* which solves:

$$F_v^* \in \arg \max_{F_v \in \Delta[\underline{v},1]} \inf_{F_p \in \Delta\mathbb{R}} r\left(F_p, F_v\right).$$

F_v^* can be interpreted as the demand that maximizes the regret of a monopolist who knows the demand and maximizes expected payoffs. Accordingly, we sometimes refer to F_v^* as a *worst case demand.*

In the case of large uncertainty, we can obtain the optimal strategies by solving an associated saddle point problem. In general, the saddlepoint condition is only a sufficient, but not necessary condition for F_p^*, F_v^* to yield minimax regret and worst case demand.

Lemma 1 (Saddlepoint)
If $F_p^* \in \Delta\mathbb{R}_+$ *and* $F_v^* \in \Delta\left[\underline{v}, 1\right]$ *satisfy*

$$r\left(F_p^*, F_v\right) \leq r\left(F_p^*, F_v^*\right) \leq r\left(F_p, F_v^*\right), \qquad \forall F_p \in \Delta\mathbb{R}_+, \, \forall F_v \in \Delta\left[\underline{v}, 1\right] \tag{1}$$

then:

1. F_p^* *attains minimax regret,*

2. F_v^* *is worst case demand, and*

3. *minimax regret* $\bar{r} = r\left(F_p^*, F_v^*\right)$, *and* $\operatorname{supp}\left(F_p^*\right) \subseteq \arg\max_p \pi\left(p, F_v^*\right).$

Proof. F_p^* attains minimax regret and $F_v^* \in \arg\max_{F_v \in \Delta[\underline{v},1]} \inf_{F_p \in \Delta\mathbb{R}_+} r\left(F_p, F_v\right)$ as

$$\begin{aligned} r\left(F_p^*, F_v^*\right) &= \max_{F_v} r\left(F_p^*, F_v\right) \geq \inf_{F_p} \sup_{F_v} r\left(F_p, F_v\right) = \bar{r} \\ &\geq \sup_{F_v} \inf_{F_p} r\left(F_p, F_v\right) \geq \min_{F_p} r\left(F_p, F_v^*\right) = r\left(F_p^*, F_v^*\right) \end{aligned}$$

holds given the general fact that $\inf\sup \geq \sup\inf$.

From the above with linearity of r we obtain $\operatorname{supp}\left(F_p^*\right) \subset \arg\min_p r\left(p, F_v^*\right)$. Since $r\left(p, F_v^*\right) = \int v dF_v^*\left(v\right) - \pi\left(p, F_v^*\right)$ it follows that $\operatorname{supp}\left(F_p^*\right) \subset \arg\max_p \pi\left(p, F_v^*\right)$. ■

We observe that the saddlepoint condition (1) is equivalent to

$$r\left(F_p^*, v\right) \leq r\left(F_p^*, F_v^*\right) \leq r\left(p, F_v^*\right), \forall p \in \mathbb{R}_+, \quad \forall v \in \left[\underline{v}, 1\right], \tag{2}$$

which we use in the proof. The saddlepoint result allows us to connect minimax regret behavior to payoff maximizing behavior under a prior as follows. When minimax regret is derived from the equilibrium characterization in (1) then any price chosen by a monopolist who minimizes maximal regret, is at the same time a price which maximizes *expected revenues* against a particular demand, namely any arbitrary worst case demand. We now use the above lemma to establish optimal strategies for seller and adversary.

The optimal pricing strategy of the monopolist minimizes his regret. The regret arises qualitatively from two, very different exposures. If the valuation of the buyer is very high, then the regret may arise from having offered a price too low relative to the valuation. On the other hand, by having offered a price too high, the buyer risks to have a valuation below the price and the regret of the seller arises from not selling at all. Of course, for low valuations of the buyer, the possible regret is small and the first source of regret is more of a concern. For this reason, the value of the lowest valuation, $\underline{v}$, plays a particular role in the pricing strategy. A critical value for $\underline{v}$ is given by $\frac{1}{e}$ ($\approx 0.367\,88$) and we denote by κ:

$$\kappa \triangleq \max\left\{\frac{1}{e}, \underline{v}\right\}.$$

Proposition 1 (Large Uncertainty)

1. *The minimax regret strategy F_p^* is given by a continuous density:*

$$f_p^*(p) = \frac{1}{p}, \quad \text{for} \quad \kappa \leq p \leq 1$$

   and a lower mass point $\Pr(\tilde{p}^* = \underline{v}) = 1 + \ln \underline{v}$ *if* $\kappa = \underline{v} > \frac{1}{e}$.

2. *The worst case demand F_v^* is given by a continuous density*

$$f_v^*(v) = \frac{\kappa}{v^2}, \text{ for} \quad \kappa \leq v \leq 1$$

   and an upper mass point $\Pr(\tilde{v}^* = 1) = \kappa$.

3. *The minimax regret is*

$$\bar{r} = 1 - \mathbb{E}[\tilde{p}^*] = \begin{cases} \frac{1}{e} & \text{if} \quad \underline{v} \leq \frac{1}{e} \\ -\underline{v}\ln\underline{v} & \text{if} \quad \underline{v} > \frac{1}{e} \end{cases}.$$

Proof. Let $\tilde{v}^*$ have c.d.f. $F_v^* \in \Delta[\kappa, 1]$ with density $\frac{\kappa}{v^2}$ and where $\Pr(\tilde{v}^* = 1) = \kappa$. Hence $\Pr(\tilde{v}^* \geq v) = \frac{\kappa}{v}$ for $v \in [\kappa, 1]$ and

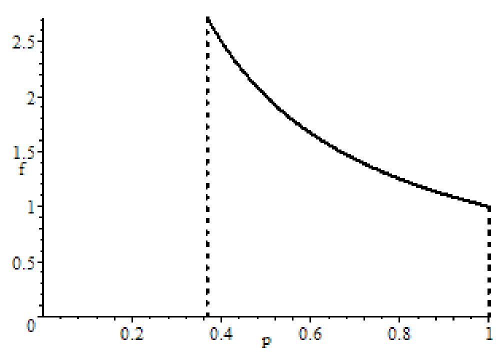
$$\begin{aligned}
r(p, F_v^*) &= \kappa + \int_\kappa^1 v\frac{\kappa}{v^2}dv - p\frac{\kappa}{p} = -\kappa \ln \kappa \text{ for } p \in [\kappa, 1] \\
r(p, F_v^*) &= \kappa + \int_\kappa^1 v\frac{\kappa}{v^2}dv - p > -\kappa \ln \kappa \text{ for } 0 \leq p < \kappa \\
r\left(F_p^*, v\right) &= v - \int_\kappa^v p\frac{1}{p}dp - \kappa(1 + \ln \kappa) = -\kappa \ln \kappa \text{ for } v \in [\kappa, 1] \\
r\left(F_p^*, v\right) &= v < -\kappa \ln \kappa = \frac{1}{e} \text{ for } \underline{v} < v < \frac{1}{e} = \kappa \ .
\end{aligned}$$

Hence, $\left(F_p^*, F_v^*\right)$ satisfies the conditions of Lemma 1. Note that $\bar{r} = r\left(F_p^*, 1\right)$ so $\bar{r} = 1 - \mathbb{E}[\tilde{p}]$ which completes the proof. ■

The proof of Proposition 1 is based on constructing the worst case demand F_v^*. Under the worst case demand the buyer does not purchase the object with a probability equal to

$$\int_\kappa^1 Pr(\tilde{v}^* < p)\frac{1}{p}dp = \kappa - 1 - \ln \kappa$$

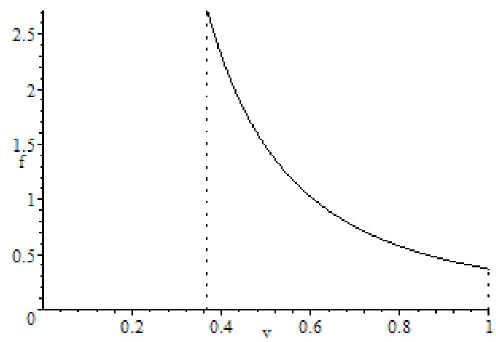
which is decreasing, convex and ranges from $\frac{1}{e}$ when $\kappa = \frac{1}{e}$ to 0 when $\kappa = 1$. The density function f_p^* of the mixed pricing strategy is illustrated below for $\underline{v} \leq 1/e$.



Graph of density of the minimax pricing strategy when $\underline{v} \leq \frac{1}{e}$.

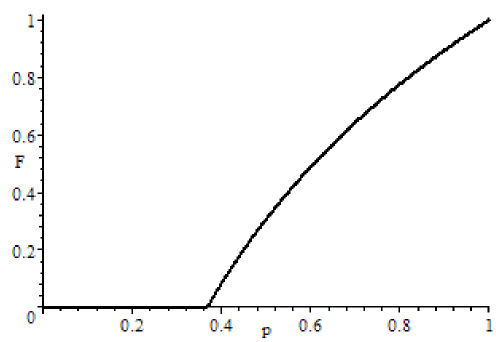
The density function f_v^* of the worst case demand is decreasing more rapidly, by the factor

$\frac{\kappa}{v}$, than the density f_p^* of the pricing rule.



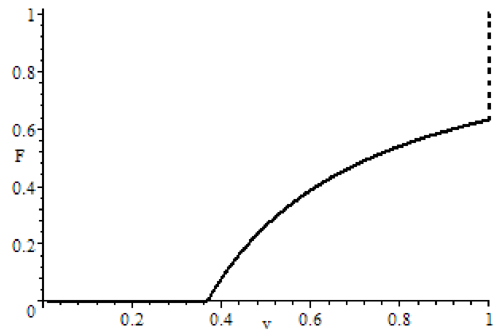
Graph of the density of the worst case demand F^* when $\underline{v} \leq \frac{1}{e}$.

The associated distribution function F_p^* is given by:



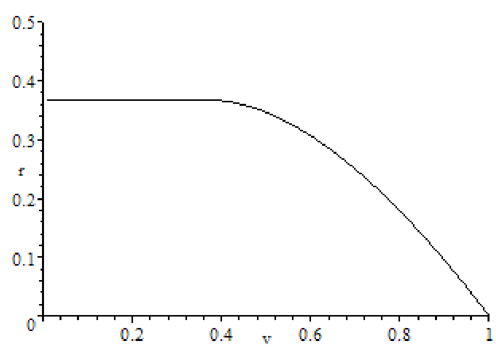
Graph of cumulative distribution of mixed pricing strategy when $\underline{v} \leq \frac{1}{e}$.

and for the worst case demand, it is given by $F_{\underline{v}}^{*}$, which includes the mass point at $v = 1$.



Graph of the cumulative distribution of the worst case demand F^{*} when $\underline{v} \leq \frac{1}{e}$.

Finally we represent the regret of the seller (which equals $1 -$ expected price) as a function of the lowest possible value $\underline{v}$:



Value of minimax regret as a function of $\underline{v}$.

Naturally, the regret is decreasing in the lowest possible value $\underline{v}$ as a higher lower bound $\underline{v}$ reduces the basic uncertainty for the seller. It is however constant for very low $\underline{v}$ as the seller optimal disregards small regrets caused by low realizations of v.

Remark 1 (Deterministic Strategies)

The above analysis allowed seller and adversary to select a random strategy. If the monop-

olist can only set deterministic prices then it is easily verified that the monopolist will set $p^* = \frac{1}{2}$ *if* $\underline{v} < 1/2$ *and set* $p^* = \underline{v}$ *if* $\underline{v} \geq 1/2$. *Maximal regret then equals* $\frac{1}{2}$ *if* $\underline{v} < 1/2$ *and equals* $1 - \underline{v}$ *if* $\underline{v} \geq 1/2$.

Remark 2 *Consider instead the case of pricing to maximize minimum payoff. The worst case is that the buyer has value* $\underline{v}$ *and hence the unique solution to the maximin problem when* $\underline{v} > 0$ *is to set* $p = \underline{v}$.

4 Small Uncertainty

In this section we describe the robust policy and some of its properties when the uncertainty is small and ε is close to zero. We start with the case of near certainty, where the model distribution F_0 is a Dirac function at v_0. We then extend the analysis to an arbitrary model distribution F_0.

In this variation, F_p^* attains minimax regret if $F_p^* \in \arg\min_{F_p \in \Delta\mathbb{R}_+} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(F_p, F_v)$. The saddle point condition (1) is adjusted to yield

$$r\left(F_p^*, F_v\right) \leq r\left(F_p^*, F_v^*\right) \leq r\left(F_p, F_v^*\right) \text{ for all } F_p \in \Delta\mathbb{R}_+ \text{ and } F_v \in \mathcal{N}_\varepsilon\left(F_0\right) \tag{3}$$

which is equivalent to

$$r\left(F_p^*, v\right) \leq r\left(F_p^*, G^*\right) \text{ for all } v \in [0,1] \text{ and } \pi\left(F_p^*, F_v^*\right) = \max_{p \in [0,1]} \pi\left(p, F_v^*\right)$$

We observe that this equivalence builds on the fact that $r(F_p, F_v) = (1-\varepsilon) r(F_p, F_0) + \varepsilon r(F_p, G)$.

4.1 Near Certainty

In the following assume that buyer values v are drawn from $(1-\varepsilon) v_0 + \varepsilon\tilde{v}$ where $\varepsilon \in (0,1)$ and $v_0 \in [0,1]$ are known while the distribution of $\tilde{v}$ is unknown. Let

$$F_v(v) \triangleq (1-\varepsilon)\mathbb{I}_{\{v \geq v_0\}} + \varepsilon G.$$

The regret of choosing price p is then given by

$$\begin{aligned} r(p, F_v) &= (1-\varepsilon) r(p, v_0) + \varepsilon r(p, F_v) \\ &= (1-\varepsilon) v_0 + \varepsilon \int_{\underline{v}}^1 v dF_v(v) - (1-\varepsilon)\pi(p, v_0) - \varepsilon\mathbb{E}\left[\pi(p, \tilde{v})\right]. \end{aligned}$$

The following proposition characterizes minimax regret behavior for $v_0 > 0$.

Proposition 2 (Near Certainty)

The following pair $\left(F_p^*, F_v^*\right)$ attains minimax regret and represents worst case demand.

1. *If $\varepsilon < v_0 \leq \frac{1}{2}$, then $\Pr\left(\tilde{p}^* = v_0\right) = \Pr\left(\tilde{v}^* = 1\right) = 1,$ and $\bar{r} = \varepsilon\left(1 - v_0\right).$*

2. *If $\varepsilon < 1 - e^{-2+\frac{1}{v_0}}$ and $\frac{1}{2} < v_0 \leq 1$, then*

   (a) *the minimax regret strategy is given by a continuous density*

   $$f_p^*(p) = \frac{1}{p}, \qquad \text{for } p \in [(1-\varepsilon)\, v_0, v_0),$$

   and an upper mass point $\Pr\left(\tilde{p}^* = v_0\right) = 1 + \ln(1-\varepsilon)$;

   (b) *the worst case demand $F_v^*(v)$ is given by a continuous density*

   $$f_v^*(v) = \frac{(1-\varepsilon)\, v_0}{\varepsilon v^2}, \qquad \text{for } v \in [(1-\varepsilon)\, v_0, v_0),$$

   and an upper mass point $\Pr\left(\tilde{v}^* = v_0\right) = 1 - \varepsilon$;

   (c) *$\mathbb{E}[\tilde{p}] = (1 + \varepsilon + \ln(1-\varepsilon))\, v_0$ and $\bar{r} = -(1-\varepsilon)\, v_0 \ln(1-\varepsilon)$;*

   (d) *$\mathbb{E}[\tilde{p}]$ is strictly decreasing in ε with $\frac{d}{d\varepsilon}\mathbb{E}[\tilde{p}]\,|_{\varepsilon=0} = 0$ and $\frac{d}{d\varepsilon}\bar{r}|_{\varepsilon=0} = v_0$.*

Proof. (1.) $r\left(F_p^*, F_v^*\right) = \varepsilon(1 - v_0)$ follows immediately. We next show that: $F_p^* \in \arg\min r\left(F_p, F_v^*\right)$. The only alternative candidate is to set $p = 1$ which yields $r\left(1, F_{v^*}\right) = (1-\varepsilon)\, v_0$. Note that $\varepsilon(1 - v_0) < (1-\varepsilon)\, v_0$ if $\varepsilon < v_0$ so claim is proven. We then show that: $1 \in \arg\max_v r(v_0, v)$. As $v_0 \leq \frac{1}{2}$ it follows that F_v^* is a maximizer.

(2.) Note that

$$F_v^*(v) = \begin{cases} 0, & \text{if} \quad v \in [0, (1-\varepsilon)\, v_0], \\ 1 - \frac{(1-\varepsilon) v_0}{v}, & \text{if} \quad v \in ((1-\varepsilon)\, v_0, v_0), \\ 1, & \text{if} \quad v \in [v_0, 1]. \end{cases}$$

We first show that: $F_p^* \in \arg\max_{F_p \in \Delta\mathbb{R}_+} \pi\left(F_p, F_{v^*}\right)$. If $p \in [(1-\varepsilon)\, v_0, v_0]$ then $\pi\left(p, F_{v^*}\right) = p\left(1 - F_v^*(p)\right) = (1-\varepsilon)\, v_0$. If $p < (1-\varepsilon)\, v_0$ or $p > v_0$ then clearly $\pi\left(p, F_{v^*}\right) < (1-\varepsilon)\, v_0$.Next we show that: $G^* \in \arg\max_{G \in \Delta[0,1]} r\left(F_p^*, G\right)$. If $(1-\varepsilon)\, v_0 < v < v_0$ then

$$r\left(F_p^*, v\right) = v - \int_{(1-\varepsilon)v_0}^{v} p \frac{1}{p} dp = (1-\varepsilon)\, v_0$$

If $v \geq v_0$ then

$$r\left(F_p^*, v\right) = v - \mathbb{E}[\tilde{p}] = v - (1 + \varepsilon + \ln(1-\varepsilon))\, v_0$$

as

$$\mathbb{E}[\tilde{p}] = \int_{(1-\varepsilon)v_0}^{v_0} p\frac{1}{p}dp + (1+\ln(1-\varepsilon))v_0 = (1+\varepsilon+\ln(1-\varepsilon))v_0.$$

If $v < (1-\varepsilon)v_0$ then $r\left(F_p^*, v\right) = v$. Comparing we find that $(1-\varepsilon)v_0 < 1-\varepsilon v_0 - (1+\ln(1-\varepsilon))v_0$ iff $\varepsilon < 1-e^{-2+\frac{1}{v_0}}$. Finally,

$$\begin{aligned} r\left(F_p^*, F_v^*\right) &= (1-\varepsilon)r\left(F_p^*, v_0\right) + \varepsilon r\left(F_p^*, (1-\varepsilon)v_0\right) \\ &= (1-\varepsilon)(v_0 - \mathbb{E}[\tilde{p}]) + \varepsilon(1-\varepsilon)v_0 \\ &= -(1-\varepsilon)v_0\ln(1-\varepsilon), \end{aligned}$$

which concludes the proof. ■

For small v_0 (case 1) we find $\frac{d}{d\varepsilon}\bar{r} = 1 - v_0$ and for large v_0 (case 2) we find $\frac{d}{d\varepsilon}\bar{r}|_{\varepsilon=0} = v_0$. Thus uncertainty has the largest impact on maximal regret for extreme values of v_0.

Remark 3 *In the special case of $v_0 = 0$, not covered by Proposition 2, the equilibrium solution is exactly identical to the solution under large uncertainty. More precisely, if $v_0 = 0$ then*

$$r\left(p, (1-\varepsilon)\mathbb{I}_{\{v\geq 0\}} + \varepsilon\mathbb{I}_{\{v\geq v'\}}\right) = \varepsilon\left(v' - p\mathbb{I}_{\{p\leq v'\}}\right)$$

and it follows immediately that the solution from the case of large uncertainty attains minimax regret. So F_p^* is given by a continuous density $f_p^*(p) = \frac{1}{p}$ for $\frac{1}{e} \leq p \leq 1$. The worst case demand in turn is given by the convex combination of the Dirac function at 0 and the solution F_v^* from Proposition 1, or $(1-\varepsilon)\delta_0 + \varepsilon F_v^*$ (in slight abuse of notation).

Remark 4 (Deterministic Strategies)
If the monopolist is confined to choose among deterministic prices it is easily verified that $p^* = v_0$ if $\varepsilon < \frac{1}{2}$. In this case, $\bar{r}_D := \min_p \sup_{F_v\in\mathcal{N}_\varepsilon(F_0)} r(p, F_v) = \varepsilon\max\{v_0, 1-v_0\}$ so locally near $\varepsilon = 0$ we find a similar performance as under mixed pricing.

4.2 Near Smooth Profits

Consider now the setting where values are distributed according to $(1-\varepsilon)F_0 + \varepsilon G$ where $\varepsilon \in (0,1)$ and F_0 are known but $G \in \Delta[0,1]$ is unknown. Let $\tilde{v}_0$ denote the random variable associated to F_0.

Proposition 3 (Small Uncertainty (I))

Assume that $\{p_0\} = \arg\max \pi(p, F_0)$ *with* $p_0 < \frac{1}{2}$, f_0 *continuously differentiable in a neighborhood of* p_0 *and* $\pi(p, F_0)$ *as a function of* p *is continuous on* $[0,1]$ *and strictly concave in a neighborhood of* p_0. *Then*

1. *there exists* $\bar{\varepsilon} > 0$ *such that for each* $\varepsilon \in (0, \bar{\varepsilon})$ *there exists* $p^* \in \left(0, \frac{1}{2}\right)$ *such that* p^* *attains minimax regret and* $(1-\varepsilon)\mathbb{I}_{\{v \geq v_0\}} + \varepsilon \mathbb{I}_{\{v \geq 1\}}$ *is a worst case demand;*

2. $\frac{d}{d\varepsilon} p^*(0) = -\frac{1}{\pi''(p_0, F_0)} > 0$ *and* $\frac{\partial}{\partial \varepsilon} \bar{r}|_{\varepsilon=0} = 1 - p_0 - r(p_0, F_0) > 0$.

Proof. Assume that $F_v^* = \mathbb{I}_{\{v \geq 1\}}$. Let $\delta \in \left(0, \frac{1}{2} - p_0\right)$ be such that f is differentiable and $\pi(p, F_v^*) = (1-\varepsilon)p(1 - F_0(p)) + \varepsilon p$ is strictly concave for all $p \in B = (p_0 - \delta, p_0 + \delta)$. Note that such a δ exists.

We first construct $\bar{\varepsilon} > 0$ and $p^*(\varepsilon) < \frac{1}{2}$ for $\varepsilon < \bar{\varepsilon}$ such that $p^* \in \arg\max_p \pi(p, F_v^*) = \arg\min_p r(p, F_v^*)$ and derive some properties of p^*. Since $\pi(p, F_0)$ is continuous we obtain that $\arg\max_p \pi(p, F_v^*)$ is upper hemi-continuous in ε. So there exists $\bar{\varepsilon} > 0$ such $\arg\max_p \pi(p, F_v^*) \cap B \neq \emptyset$ for all $\varepsilon < \bar{\varepsilon}$. For $\varepsilon < \bar{\varepsilon}$ choose $p^* \in \arg\max_p \pi(p, F_v^*) \cap B$. With this we obtain $p^* < \frac{1}{2}$ and p^* solves the first order conditions

$$(1-\varepsilon)(1 - F_0(p^*) - p^* f_0(p^*)) + \varepsilon = 0.$$

Differentiating with respect to ε we obtain:

$$-(1 - F_0(p^*) - p^* f_0(p^*)) + 1 + (1-\varepsilon)\left(-2f_0(p^*) - p^* f_0'(p^*)\right)\frac{d}{d\varepsilon}p^* = 0$$

so

$$\frac{d}{d\varepsilon}p^* = \frac{F_0(p^*) + p^* f_0(p^*)}{(1-\varepsilon)(2f_0(p^*) + p^* f_0'(p^*))} > 0$$

with

$$\frac{d}{d\varepsilon}p^*(0) = \frac{1}{2f_0(p^*) + p^* f_0'(p^*)} > 0.$$

Given that $p^* < \frac{1}{2}$ for $\varepsilon < \bar{\varepsilon}$ it follows that $\{1\} = \arg\max r(p^*, v)$. Applying the saddle point condition it follows that p^* attains minimax regret.

We obtain $\bar{r} = (1-\varepsilon)\left(\int v f_0(v)\,dv - p^*(1 - F_0(p^*))\right) + \varepsilon(1 - p^*)$, so

$$\frac{\partial}{\partial \varepsilon}\bar{r}|_{\varepsilon=0} = -\left(\int v f_0(v)\,dv - p_0(1 - F_0(p_0))\right) + 1 - p_0 = 1 - p_0 - r(p_0, F_0),$$

where $r(p_0, F_0) < 1 - p_0$ follows directly from $p_0 < \frac{1}{2}$. ■

Notice that $\sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p_0, F_v) = (1-\varepsilon) r(p_0, F_0) + \varepsilon(1-p_0)$ and hence $\frac{d}{d\varepsilon} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p_0, F_v) = \frac{\partial}{\partial \varepsilon} \bar{r}|_{\varepsilon=0}$.

Next we consider the case where the best response price under certainty is above $\frac{1}{2}$.

Proposition 4 (Small Uncertainty (II))

Assume that $\pi(p, F_0)$ is strictly concave in p, F_0 is without mass points, $p_0 > \frac{1}{2}$ given $\{p_0\} = \arg\max_p \pi(p, F_0)$ and that ε is sufficiently small. The following pair $\left(F_p^*, F_v^*\right)$ attains minimax regret and represents worst case demand:

1. *(a) the minimax regret strategy is given by a continuous density*

$$f_p^*(p) = \frac{1}{p}, \qquad \textit{for } p \in [a, p_0),$$

and an upper mass point $\Pr(\tilde{p}^* = v_0) = 1 + \ln \frac{a}{p_0}$*;*

(b) the worst case demand $F_v^*(v)$ is given by

$$F_v^*(v) = \begin{cases} (1-\varepsilon) F_0(v), & \textit{if} \quad v \in [0, a], \\ 1 - \frac{(1-\varepsilon)\pi(p_0, F_0)}{v}, & \textit{if} \quad v \in (a, p_0), \\ (1-\varepsilon) F_0(v) + \varepsilon, & \textit{if} \quad v \in [p_0, 1]. \end{cases}$$

where $a < p_0$ is the unique solution to $(1-\varepsilon)\pi(a, F_0) + a\varepsilon = (1-\varepsilon)\pi(p_0, F_0)$.

Proof. We first observe that strict concavity of $\pi(p, F_0)$ makes a well defined. Note that

$$G^*(v) = \begin{cases} 0 & \text{if } v \in [0, a] \\ \frac{1}{\varepsilon}\left(1 - (1-\varepsilon) F_0(v) - \frac{(1-\varepsilon)\pi(p_0, F_0)}{v}\right) & \text{if } v \in (a, p_0) \\ 1 & \text{if } v \in [p_0, 1] \end{cases}$$

for $v \in [a, p_0]$ so G^* has no mass points. Then it is easily verified that both F_v^* and G are well defined c.d.f.

First we show that $G^* \in \arg\max r\left(F_p^*, G\right)$ if ε is sufficiently small. We find

$$\mathbb{E}[\tilde{p}] = \int_a^{p_0} p \frac{1}{p} dp + \left(1 + \ln \frac{a}{p_0}\right) p_0 = 2p_0 - a + p_0 \ln \frac{a}{p_0},$$

so if $v \in (a, p_0)$ then

$$r\left(F_p^*, v\right) = v - \int_a^v p \frac{1}{p} dp = a.$$

If $v \geq p_0$, then

$$r\left(F_p^*, v\right) = v - \mathbb{E}\left[\tilde{p}\right] = v - \left(2p_0 - a + p_0 \ln \frac{a}{p_0}\right);$$

and if $v \leq a$ then $r\left(F_p^*, v\right) = v$. So $[a, p_0) = \arg\max_v r\left(F_p^*, v\right)$ if

$$1 - \left(2p_0 - a + p_0 \ln \frac{a}{p_0}\right) < a,$$

where the latter holds if and only $a > p_0 e^{-2+\frac{1}{p_0}}$. Notice that $a > p_0 e^{-2+\frac{1}{p_0}}$ holds when $\frac{1}{2} < a < p_0$ which is true if ε is sufficiently small as $a \to p_0$ as $\varepsilon \to 0$. Hence our first claim is proven.

Now we show that $F_p^* \in \arg\max_{F_p} \pi\left(F_p, F_v^*\right)$ if ε is sufficiently small. It follows immediately from our definition of F_v^* that

$$\pi\left(p, F_v^*\right) = p\left(1 - F_v^*\left(p\right)\right) = \left(1 - \varepsilon\right) \pi\left(p_0, F_0\right) \quad \text{for} \quad p \in [a, p_0].$$

For $p > p_0$ we obtain $\pi\left(p, F_v^*\right) = \left(1 - \varepsilon\right) \pi\left(p, F_0\right) < \left(1 - \varepsilon\right) \pi\left(p_0, F_0\right) = \pi\left(p_0, F_v^*\right)$. Finally for $p < a$ we obtain $\pi\left(p, F_v^*\right) = \left(1 - \varepsilon\right) \pi\left(p, F_0\right) + \varepsilon p < \pi\left(a, F_v^*\right)$. This proves our second claim. ■

Proposition 5 (Increasing Uncertainty with High Price)
With the maintained assumptions of the Proposition 4, the expected price is strictly decreasing and regret is strictly increasing in ε:

$$\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \mathbb{E}\left[\tilde{p}\right] = \frac{1}{\pi''\left(p_0, F_0\right)} < 0 \text{ and } \frac{d}{d\varepsilon} \bar{r}|_{\varepsilon = 0} = p_0 - r\left(p_0, F_0\right) > 0.$$

Proof. From the proof of Proposition 4, we find $\mathbb{E}\left[\tilde{p}\right] = 2p_0 - a + p_0 \ln \frac{a}{p_0}$. Since

$$\left(1 - \varepsilon\right) \frac{1}{a} \left(\pi\left(a, F_0\right) - \pi\left(p_0, F_0\right)\right) + \varepsilon = 0,$$

we have

$$\varepsilon = 1 - \frac{1}{1 + \frac{1}{a}\left(\pi\left(p_0, F_0\right) - \pi\left(a, F_0\right)\right)}.$$

We note that:

$$\frac{d}{da} \varepsilon\left(a\right) = \frac{1}{\left(1 + \frac{1}{a}\left(\pi\left(p_0, F_0\right) - \pi\left(a, F_0\right)\right)\right)^2} \frac{1}{a^2} \left(a^2 f_0\left(a\right) - p_0\left(1 - F_0\left(p_0\right)\right)\right).$$

Since $\pi\left(p, F_0\right)$ is strictly concave we have that $1 - F_0\left(a\right) - a f_0\left(a\right) > 0$ (as $a < p_0$) so

$$a^2 f_0\left(a\right) - p_0\left(1 - F_0\left(p_0\right)\right) < a\left(1 - F_0\left(a\right)\right) - p_0\left(1 - F_0\left(p_0\right)\right) < 0$$

and hence $\frac{d}{da}\varepsilon(a) < 0$. In particular,

$$\frac{d}{da}\varepsilon(p_0) = \frac{1}{p_0^2}\left(p_0^2 f_0(p_0) - p_0(1 - F_0(p_0))\right) = -\frac{1}{p_0}(1 - F_0(p_0) - p_0 f_0(p_0)) = 0 \ .$$

Using the chain rule we find

$$\begin{aligned} \frac{d}{d\varepsilon}\mathbb{E}[\tilde{p}] &= \frac{\partial}{\partial a}\mathbb{E}[\tilde{p}] / \frac{d}{da}\varepsilon(a) = \left(-1 + \frac{p_0}{a}\right)\frac{a^2\left(1 + \frac{1}{a}(p_0(1 - F_0(p_0)) - \pi(a, F_0))\right)^2}{\left(a^2 f_0(a) - p_0(1 - F_0(p_0))\right)} \\ &= -\frac{a - p_0}{a^2 f_0(a) - p_0^2 f_0(p_0)} a\left(1 + \frac{1}{a}\left(p_0^2 f_0(p_0) - \pi(a, F_0)\right)\right)^2 \end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\mathbb{E}[\tilde{p}] = -p_0\frac{1}{\frac{d}{da}a^2 f_0(a)\left|_{a=p_0}\right.} = -\frac{1}{2f_0(p_0) + p_0 f_0'(p_0)} < 0 \ .$$

The proof of Proposition 4 shows that

$$\begin{aligned} \bar{r} &= (1-\varepsilon)\int_{[0,a]\cup[p_0,1]} v f_0(v)\,dv + \int_a^{p_0} v\frac{(1-\varepsilon)\pi(p_0, F_0)}{v^2}dv - a\left((1-\varepsilon)(1 - F_0(a)) + \varepsilon\right) \\ &= (1-\varepsilon)\int_{[0,a]\cup[p_0,1]} v f_0(v)\,dv + (1-\varepsilon)\pi(p_0, F_0)\ln\frac{p_0}{a} - a\left(1 - (1-\varepsilon)F_0(a)\right) \end{aligned}$$

so we obtain

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\bar{r} &= -\int_{[0,a]\cup[p_0,1]} v f_0(v)\,dv - \pi(p_0, F_0)\ln\frac{p_0}{a} - aF_0(a) \\ &\to -\int_0^1 v f_0(v)\,dv - p_0 F_0(p_0) \ \text{ as } \varepsilon\to 0 \\ \frac{\partial}{\partial a}\bar{r} &= (1-\varepsilon)af_0(a) - (1-\varepsilon)\pi(p_0, F_0)\frac{1}{a} - (1 - (1-\varepsilon)F_0(a)) + (1-\varepsilon)af_0(a) \\ &= 2(1-\varepsilon)af_0(a) + (1-\varepsilon)F_0(a) - (1-\varepsilon)\pi(p_0, F_0)\frac{1}{a} - 1 \\ &= \frac{1}{a}\left(2(1-\varepsilon)a^2 f_0(a) - (1-\varepsilon)(\pi(p_0, F_0) + \pi(a, F_0)) - \varepsilon\right) \\ &\to 2p_0 f_0(p_0) + 2F_0(p_0) - 2 = 0 \text{ as } \varepsilon\to 0 \end{aligned}$$

where

$$\begin{aligned} \frac{\frac{\partial}{\partial a}\bar{r}}{\frac{d}{da}\varepsilon(a)} &= \frac{1}{a}\left(2(1-\varepsilon)a^2 f_0(a) - (1-\varepsilon)(\pi(p_0, F_0) + \pi(a, F_0)) - \varepsilon\right)\frac{a^2\left(1 + \frac{1}{a}(\pi(p_0, F_0) - \pi(a, F_0))\right)^2}{a^2 f_0(a) - \pi(p_0, F_0)} \\ &= a\left(1 + \frac{1}{a}(\pi(p_0, F_0) - \pi(a, F_0))\right)^2\frac{(1-\varepsilon)\left(2a^2 f_0(a) - (\pi(p_0, F_0) + \pi(a, F_0))\right) - \varepsilon}{a^2 f_0(a) - \pi(p_0, F_0)} \\ &\to 2p_0 \end{aligned}$$

as

$$\frac{(1-\varepsilon)\left(2a^2 f_0(a) - (\pi(p_0,F_0) + \pi(a,F_0))\right) - \varepsilon}{a^2 f_0(a) - \pi(p_0,F_0)} \approx 2 + \frac{\pi(p_0,F_0) - \pi(a,F_0)}{a^2 f_0(a) - \pi(p_0,F_0)} - \frac{\varepsilon}{a^2 f_0(a) - \pi(p_0,F_0)}$$

and

$$\frac{\pi(p_0,F_0) - \pi(a,F_0)}{a^2 f_0(a) - \pi(p_0,F_0)} \approx \frac{-\pi'(p_0,F_0)}{2p_0 f_0(p_0) + p_0^2 f_0'(p_0)} = 0$$

given $\frac{d}{dp}(1 - F_0 - pf_0) = -2f_0 - pf_0' < 0$ and as

$$\frac{\varepsilon}{a^2 f_0(a) - \pi(p_0,F_0)} \approx \frac{\varepsilon'(p_0)}{2p_0 f_0(p_0) + p_0^2 f_0'(p_0)} = 0$$

Summarizing we find

$$\frac{d}{d\varepsilon}\bar{r}|_{\varepsilon=0} = -\int_0^1 v f_0(v)\, dv - p_0 F_0(p_0) + 2p_0 = p_0 - r(p_0,F_0),$$

where $r(p_0,F_0)|_{\varepsilon=0} < p_0$ follows directly from $p_0 > \frac{1}{2}$. ∎

The analysis of the regret strategies in the presence of small uncertainty assigns a prominent role to optimal price without uncertainty being equal to $\frac{1}{2}$. The special role of $p^0 = \frac{1}{2}$ can be easily accounted for. Clearly, with small uncertainty the optimal regret policy will not divert too much from the policy without uncertainty and in consequence the expected price $\mathbb{E}[p^*]$ will stay close to p^0. For the adversary, the question is then for which realizations the small regret can be maximized. If the price p^0 is large than $\frac{1}{2}$, then the regret is maximized by realizing values just below p^0 as the seller would then fail to make any sales. As $p^0 \to \frac{1}{2}$, the maximal regret which can be achieved with valuations v below p^0 is converging to $\frac{1}{2}$. On the other hand, if $p^0 < \frac{1}{2}$, then the maximal regret can be achieved by realization of $v = 1$ as the regret which emerges from here is the foregone revenue which is equal to $v - p^0$, and for $v = 1$ and $p^0 \to \frac{1}{2}$, the maximal regret converges to $\frac{1}{2}$, and hence $p^0 = \frac{1}{2}$, is the critical value in the determination of the optimal strategy of the adversary.

Corollary 1 (Deterministic Pricing with High Price)

Under the assumptions of Proposition 4, if the monopolist can only set a deterministic price then p^* *solves* $\pi'(p^*,F_0) = \frac{\varepsilon}{1-\varepsilon}$ *where* $p^* \in (a, p_0)$,

$$\frac{d}{d\varepsilon}p^*|_{\varepsilon=0} = \frac{1}{\pi''(p_0,F_0)} < 0 \text{ and } \frac{d}{d\varepsilon}\bar{r}_D|_{\varepsilon=0} = p_0 - r(p_0,F_0) > 0.$$

Proof. If ε is sufficiently small then p^* will be close to p_0, in particular $p^* > \frac{1}{2}$. For $p > \frac{1}{2}$ we obtain

$$\sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p, F_v) = (1-\varepsilon)\left(\int_0^1 v f_0(v)\, dv - p(1 - F_0(p))\right) + \varepsilon p,$$

where $\frac{d}{dp} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p, F_v) = -(1-\varepsilon)\pi'(p, F_0) + \varepsilon$ and where

$$\frac{d^2}{(dp)^2} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p, F_v) = -(1-\varepsilon)\pi''(p, F_0).$$

So if p is sufficiently close to p_0 then $\frac{d^2}{(dp)^2} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p, F_v) < 0$. So $\pi'(p^*, F_0) = \frac{\varepsilon}{1-\varepsilon}$ which implies $p^* < p_0$ and $\lim_{\varepsilon \to 0} p^* = p_0$. We then recall that

$$(1-\varepsilon)\frac{1}{a}(\pi(a, F_0) - \pi(p_0, F_0)) + \varepsilon = 0,$$

and so

$$\frac{\pi(p_0, F_0) - \pi(a, F_0)}{a} = \frac{\varepsilon}{1-\varepsilon} = \pi'(p^*, F_0).$$

If $p^* \leq a$ then

$$\frac{\varepsilon}{1-\varepsilon} = \pi'(p^*, F_0) \geq \pi'(a, F_0) > \frac{\pi(p_0, F_0) - \pi(a, F_0)}{p_0 - a} > \frac{\pi(p_0, F_0) - \pi(a, F_0)}{a} = \frac{\varepsilon}{1-\varepsilon},$$

if $p_0 < 2a$ which is true if ε is sufficiently small.

Given $\bar{r}_D := \min_p \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p, F_v) = \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p^*, F_v)$ it follows that

$$\frac{d}{d\varepsilon}\bar{r}_D|_{\varepsilon=0} = -\left(\int_0^1 v f_0(v)\, dv - p_0(1 - F_0(p_0))\right) + p_0 = p_0 - r(p_0, F_0)|_{\varepsilon=0},$$

which completes the proof. ■

We finally observe that the choice of p_0 yields

$$\sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p_0, F_v) = (1-\varepsilon)\int_0^1 v f_0(v)\, dv + \varepsilon p_0 - (1-\varepsilon) p_0 (1 - F_0(p_0))$$

and hence $\frac{d}{d\varepsilon} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} r(p_0, F_v)|_{\varepsilon=0} = \frac{d}{d\varepsilon}\bar{r}_D|_{\varepsilon=0} = \frac{d}{d\varepsilon}\bar{r}|_{\varepsilon=0}$.

Remark 5 *In all cases of almost smooth payoffs investigated above we have observed that mixed strategy pricing and optimal pricing perform similarly in terms of maximal regret when ε is sufficiently small.*

5 Interim Regret

Earlier we defined interim regret by:

$$R(F_p, F_v) = \sup_p \pi(p, F_v) - \int \pi(p, F_v)\, dF_p(p)$$

where F_p^* attains minimax interim regret if

$$F_{p^*} \in \arg \min_{F_p \in \Delta\mathbb{R}^+} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} R(F_p, F_v)$$

where $\bar{R} = \min_{F_p \in \Delta\mathbb{R}^+} \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} R(F_p, F_v)$ is the value of minimax interim regret. Comparing this to regret $r(F_p, F_v)$ we obtain immediately $R(F_p, F_v) \le r(F_p, F_v)$.

5.1 Saddle Point

Notice that a saddle point characterization in terms of R as obtained under regret is not available. To see this, assume that $\left(F_p^*, F_v^*\right)$ solves $R\left(F_p^*, F_v\right) \le R\left(F_p^*, F_v^*\right) \le R\left(F_p, F_v^*\right)$ for all $F_p \in \Delta[0,1]$ and all $F_v \in U_\varepsilon(F_0)$. Note that $\arg\min_{F_p} R(F_p, F_v^*) = \arg\max_{F_p} \pi(F_p, F_v^*) \supseteq \arg\max_p \pi(p, F_v^*)$ which means that $R\left(F_p^*, F_v^*\right) = 0$ and consequently $R\left(F_p^*, F_v\right) = 0$ for all $F_v \in U_\varepsilon(F_0)$ should the saddle point condition hold. However it is immediate that this is not true and hence we obtain a contradiction.

The way to obtain a saddle point characterization is to use the following trick of using the linearization of $R(F_p, F_v) = R(F_p, (1-\varepsilon) F_0 + \varepsilon G)$ in terms of G instead. Let $R_L(F_p, F_v)$ be defined by taking expected values using $R\left(p, (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \ge v'\}}\right)$, i.e. $R_L(F_p, F_v) = \int\int R\left(p, (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \ge v'\}}\right) dG(v')\, dF_p(p)$ so

$$\begin{aligned} R_L(p, F_v) &= \int \sup_{p' \in [0,1]} \pi\left(p', (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \ge v'\}}\right) dG\left(v'\right) - \pi(p, F_v) \\ R_L(F_p, F_v) &= \int \sup_{p' \in [0,1]} \pi\left(p', (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \ge v'\}}\right) dG\left(v'\right) - \int \pi(p, F_v)\, dF_p(p) \end{aligned}$$

To simplify notation we will write $\delta_{\varepsilon, v'} = (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \ge v'\}}$. Note that $R_L\left(F_p, \delta_{\varepsilon, v'}\right) = R\left(F_p, \delta_{\varepsilon, v'}\right)$ but that $R_L(p, F_v)$ and $R(p, F_v)$ need not coincide.

The reason why we can work with R_L instead of R is as follows. We will show that R is quasi-convex in G which means that it $R(F_p, \cdot)$ attains its maxima on the set $\left\{\delta_{\varepsilon, v'} : v' \in [0,1]\right\}$. $R_L(F_p, \cdot)$ is linear so it also attains its maxima on the set of Dirac measures. So in order to calculate min sup we obtain that R and R_L yield the same solution. It however turns out that R_L often does have a saddle point.

The following lemma shows that we can calculate minimax interim regret by establishing the saddle point for R_L instead of for R.

Lemma 2 (Auxiliary Saddlepoint)
If $\left(F_p^*, F_v^*\right) \in \Delta[0,1] \times U_\varepsilon(F_0)$ *solves*

$$R_L\left(F_p^*, F_v\right) \leq R_L\left(F_p^*, F_v^*\right) \leq R_L\left(F_p, F_v^*\right), \quad \forall F_p \in \Delta[0,1], \ \forall F_v \in U_\varepsilon(F_0)$$

which is equivalent to

$$R_L\left(F_p^*, \delta_{\varepsilon,v}\right) \leq R_L\left(F_p^*, F_v^*\right) \text{ for all } v \in [0,1],$$

and

$$\pi\left(p, F_v^*\right) \leq \pi\left(F_p^*, F_v^*\right) \text{ for all } p \in [0,1],$$

and then F_p^* *attains minimax interim regret and* $R_L\left(F_p^*, F_v^*\right) = \min_{F_p \in \Delta[0,1]} \sup_{F_v \in U_\varepsilon(F_0)} R\left(F_p, F_v\right)$.

Proof. For any given F_p we claim that $\sup_{F_v \in U_\varepsilon(F_0)} R\left(F_p, F_v\right) = \sup_{v' \in [0,1]} R\left(F_p, \delta_{\varepsilon,v'}\right)$. So we have to show that

$$\begin{aligned} & \sup_{F_v \in U_\varepsilon(F_0)} \left(\sup_{p' \geq 0} \left(p' \Pr\left(\tilde{v} \geq p'\right)\right) - \int p \Pr\left(\tilde{v} \geq p\right) dF_p(p) \right) \\ = & \sup_{F_v \in \left\{\delta_{\varepsilon,v'} : v' \in [0,1]\right\}} \left(\sup_{p' \geq 0} \left(p' \Pr\left(\tilde{v} \geq p'\right)\right) - \int p \Pr\left(\tilde{v} \geq p\right) dF_p(p) \right) \end{aligned}$$

The above equality however follows from the fact that

$$h\left(F_v\right) = \left(\sup_{p' \geq 0} \left(p' \Pr\left(\tilde{v} \geq p'\right)\right) - \int p \Pr\left(\tilde{v} \geq p\right) dF_p(p) \right)$$

is quasi-convex in F_v for any given F_p and hence h attains its suprema in the set $\left\{\delta_{\varepsilon,v'} : v' \in [0,1]\right\}$.

Using the fact that $R_L\left(F_p, \delta_{\varepsilon,v'}\right) = R\left(F_p, \delta_{\varepsilon,v'}\right)$ we obtain that

$$\arg \min_{F_p \in \Delta\mathbb{R}^+} \sup_{F_v \in U_\varepsilon(F_0)} R\left(F_p, F_v\right) = \arg \min_{F_p \in \Delta\mathbb{R}^+} \sup_{v' \in [0,1]} R_L\left(F_p, \delta_{\varepsilon,v'}\right) .$$

Given that $\left(F_p^*, F_v^*\right)$ is a saddle point with respect to R_L we know that

$$F_p^* \in \arg \min_{F_p \in \Delta\mathbb{R}^+} \sup_{v' \in [0,1]} R_L\left(F_p, \delta_{\varepsilon,v'}\right)$$

and by the above identity it follows that F_p^* attains minimax interim regret. ■

Corollary 2

If a saddle point $\left(F_p^*, F_v^*\right)$ *as described in Lemma 2 exists then neither* F_p^* *nor* G^* *are Dirac measures.*

Proof. Assume that $F_p^* = \mathbb{I}_{\{p \geq \bar{p}\}}$ for some $\bar{p}$. Since $\bar{p}$ is a best response to F_v^* we obtain that if then $R_L\left(\bar{p}, F_v^*\right) = 0$ and hence $\bar{R} = 0$. However it is easy to show that $\bar{R} > 0$.

Assume that $G^* = \mathbb{I}_{\{v \geq \bar{v}\}}$ for some $\bar{v}$. As $R_L\left(F_p^*, \delta_{\varepsilon,\bar{v}}\right) = R\left(F_p^*, \delta_{\varepsilon,\bar{v}}\right)$ this would mean that $\left(F_p^*, F_v^*\right)$ is a saddle point when replacing R_L by R. However, as we explained at the beginning of this section this is not possible. ■

In order to derive a saddle point under R_L we still have to deal with

$$\sup_{p'} \pi\left(p', (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \geq v'\}}\right).$$

Here is a trick. Look for a saddle point where $v' \in C\left(G^*\right)$ implies:

$$v' \in \arg\sup_{p'} \pi\left(p', (1-\varepsilon) F_0 + \varepsilon \mathbb{I}_{\{v \geq v'\}}\right).$$

This means that the profit maximizing price is precisely equal to the value set by nature with probability ε. If this is possible then use the fact that

$$R_L\left(p, \delta_{\varepsilon,v'}\right) = \pi\left(v', \delta_{\varepsilon,v'}\right) - \pi\left(p, \delta_{\varepsilon,v'}\right)$$

which is a much simpler expression. This trick is used in the results below.

5.2 Large Uncertainty

If $\varepsilon = 1$ then $R\left(F_p, v'\right) = r\left(F_p, v'\right)$ and hence $R_L\left(F_p, F_v\right) = r\left(F_p, F_v\right)$ which means that minimax regret and minimax interim regret coincide.

5.3 Small Uncertainty

5.3.1 Near Certainty

Proposition 6 (Interim Regret with Near Certainty)

Let F_p^* *have density* $f_p^*(p) = \frac{1}{\varepsilon p}$ *on* $\left[e^{-\varepsilon} v_0, v_0\right]$. *If* $\varepsilon \leq v_0$ *then* F_p^* *attains minimax interim regret where* $\mathbb{E}\left[\tilde{p}\right] = v_0 \frac{1-e^{-\varepsilon}}{\varepsilon}$ *is strictly decreasing in* ε, $\frac{d}{d\varepsilon}\mathbb{E}\left[\tilde{p}\right]|_{\varepsilon=0} = -\frac{1}{2}v_0$ *and* $\bar{R} = \left(\varepsilon - 1 + e^{-\varepsilon}\right)\frac{v_0}{\varepsilon}$ *with* $\frac{d}{d\varepsilon}\bar{R}|_{\varepsilon=0} = \frac{v_0}{2}$.

Proof. We will show that $\left(F_p^*, F_v^*\right)$ is a saddle point where F_v^* is such that $G^*(v) = \frac{1}{\varepsilon}\left(1 - \frac{1}{v}e^{-\varepsilon}v_0\right)$ (and hence $g^*(v) = \frac{1}{\varepsilon v^2}e^{-\varepsilon}v_0$) for $v \in [e^{-\varepsilon}v_0, v_0)$ and $G^*(v_0) = 1$ so G^* has point mass $1 - \frac{1}{\varepsilon}\left(1 - e^{-\varepsilon}\right)$ at $v = v_0$. So

$$F_v^*(v) = \begin{cases} 0 & \text{if } v \in [0, e^{-\varepsilon}v_0] \\ 1 - \frac{1}{v}e^{-\varepsilon}v_0 & \text{if } v \in (e^{-\varepsilon}v_0, v_0) \\ 1 & \text{if } v \in [v_0, 1] \end{cases}$$

and hence

$$f_v^*(v) = \begin{cases} 0 & \text{if } v \in [0, e^{-\varepsilon}v_0] \\ \frac{1}{v^2}e^{-\varepsilon}v_0 & \text{if } v \in (e^{-\varepsilon}v_0, v_0) \\ 0 & \text{if } v \in [v_0, 1] \end{cases} \quad \text{and } \Pr(\tilde{v} = v_0) = e^{-\varepsilon}$$

Note that $(1-\varepsilon)\pi(v_0, F_0) < (1-\varepsilon)\pi(p, F_0) + \varepsilon p$ for all $p \in (e^{-\varepsilon}v_0, v_0)$ as $(1-\varepsilon)v_0 < p$ is implied by the fact that $1 - \varepsilon \leq e^{-\varepsilon}$. So for $p \in [e^{-\varepsilon}v_0, v_0]$ we obtain

$$\pi(p, F_v^*) = p\left(e^{-\varepsilon} + \int_p^{v_0} \frac{1}{v^2}e^{-\varepsilon}v_0 dv\right) = e^{-\varepsilon}v_0$$

and for $v \in [e^{-\varepsilon}v_0, v_0]$ we find that

$$R_L\left(F_p^*, \delta_{\varepsilon,v}\right) = v - (1-\varepsilon)\int_{e^{-\varepsilon}v_0}^{v_0} p\frac{1}{\varepsilon p}dp - \varepsilon\int_{e^{-\varepsilon}v_0}^{v} p\frac{1}{\varepsilon p}dp = \left(\varepsilon - 1 + e^{-\varepsilon}\right)\frac{v_0}{\varepsilon}\ .$$

We also need that $R_L\left(F_p^*, \delta_{\varepsilon,1}\right) \leq R_L\left(F_p^*, \delta_{\varepsilon,v_0}\right)$ and hence

$$\sup_{p\in[0,1]}\left(p\left((1-\varepsilon)\mathbb{I}_{\{p\leq v_0\}} + \varepsilon\mathbb{I}_{\{p\leq 1\}}\right)\right) = v_0.$$

which holds if $v_0 \geq \varepsilon$. The rest of the conditions are easily verified. ■

Let us briefly compare the above to our results under regret. The interesting case is where $v_0 > \frac{1}{2}$, where for small uncertainty the expected price is lower under interim regret than under regret. Only if v_0 and ε are sufficiently large then the opposite can hold, more specifically, if $v_0 > 0.81$ and $0.53 < \varepsilon < 1 - e^{-2+\frac{1}{v_0}}$ then the expected price under interim regret is larger than it is under regret.

We consider now selection among deterministic pricing policies and hence investigate $\bar{R}_D := \inf_p \sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} R(p, F_v)\,.$

Corollary 3 (Deterministic Strategies)

Assume $\varepsilon < v_0$.

(i) If the monopolist is confined to choosing among deterministic prices then $p^* = \frac{1}{1+\varepsilon}v_0$ *attains minimax interim regret where* $p^*(\varepsilon)$ *is strictly decreasing in* ε, $\frac{d}{d\varepsilon}p^*|_{\varepsilon=0} = -v_0$ *and* $\bar{R}_D = \frac{\varepsilon}{1+\varepsilon}v_0$ *so* $\frac{d}{d\varepsilon}\bar{R}_D|_{\varepsilon=0} = v_0$.

(ii) If the monopolist chooses the optimal price v_0 *without noise then* $\sup_{F_v \in \mathcal{N}_\varepsilon(F_0)} \bar{R}(v_0, F_v) = \varepsilon v_0$.

Proof. If $p > v_0$ then $\sup_{v' \in [0,1]} R\left(p, \delta_{\varepsilon,v'}\right) \geq v_0$. If $(1-\varepsilon)v_0 < p \leq v_0$ and $\varepsilon < v_0$ then $\sup_{v' \in [0,1]} R\left(p, \delta_{\varepsilon,v'}\right) = \max\{v_0 - p, p - (1-\varepsilon)p\} = \max\{v_0 - p, \varepsilon p\}$. If $p \leq (1-\varepsilon)v_0$ then $\sup_{v' \in [0,1]} R\left(p, \delta_{\varepsilon,v'}\right) \geq v_0 - p \geq \varepsilon v_0$. ■

Note that $\frac{1-e^{-\varepsilon}}{\varepsilon} > \frac{1}{1+\varepsilon} > e^{-\varepsilon}$ which means that price under deterministic solution is lower than expected price under mixed strategy solution while still within the support of the mixed strategy solution.

Remark 6 *We obtain* $\frac{d}{d\varepsilon}\bar{R}|_{\varepsilon=0} = \frac{v_0}{2} < v_0 = \frac{d}{d\varepsilon}\bar{R}_D|_{\varepsilon=0}$. *So unlike our previous analysis of regret we find that mixed pricing outperforms any pure pricing policy even when* ε *is small.*

5.3.2 Near Smooth Profits

An important condition for our proof to work is that $v \in \arg\sup_p \pi(p, \delta_{\varepsilon,v})$ for all v in the support of G^*. Assuming this we set up the saddle point conditions. Let $[a,c]$ be the support of G^*. For $f_p^*(p)$ makes nature indifferent so for $v \in [a,c]$ we obtain

$$\frac{d}{dv}\left(\pi(v, \delta_{\varepsilon,v}) - \int \pi(p, \delta_{\varepsilon,v}) f_p^*(p)\,dp\right) = \frac{d}{dv}\left((1-\varepsilon)\pi(v, F_0) + \varepsilon v - \varepsilon \int_a^v p f_p^*(p)\,dp\right) = 0$$

so

$$\varepsilon v f_p^*(v) = (1-\varepsilon)\pi'(v, F_0) + \varepsilon$$

which implies that

$$f_p^*(p) = \frac{1-\varepsilon}{\varepsilon p}\pi'(p, F_0) + \frac{1}{p} \text{ for } p \in [a,c].$$

The mixed strategy of nature with c.d.f. F_v^* makes the individual indifferent so $p \in (a,c)$ implies

$$\frac{d}{dp}\pi(p, F_v^*) = 0$$

which implies $F_v^*(v) = 1 - \frac{\gamma}{v}$ for $v \in [a,c]$ where $\gamma > 0$ is an appropriate constant. Since $G^*(a) = 0$ we obtain $F_v^*(a) = (1-\varepsilon)F_0(a)$ so

$$F_v^*(v) = 1 - \frac{(1-(1-\varepsilon)F_0(a))\,a}{v} = 1 - \frac{(1-\varepsilon)\pi(a, F_0) + \varepsilon a}{v} \text{ for } v \in [a,c].$$

So

$$\begin{aligned} G^*(v) &= \frac{1}{\varepsilon}\left(1 - \frac{(1-(1-\varepsilon)F_0(a))a}{v} - (1-\varepsilon)F_0(v)\right) \\ &= 1 + \frac{(1-\varepsilon)(\pi(v,F_0) - \pi(a,F_0))}{\varepsilon v} - \frac{a}{v} \end{aligned}$$

and

$$g^*(v) = \frac{(1-\varepsilon)(v\pi'(v,F_0) - \pi(v,F_0) + \pi(a,F_0)) + \varepsilon a}{\varepsilon v^2} .$$

F_p^* will have support $[a,c]$ with a point mass at c while G^* will have support $[a,c]$ with a point mass at $v = 1$. We impose two conditions on a and c. First we choose $g^*(c) = 0$. This ensures that the monopolist will set a point mass at $p = c$ and hence does not have an incentive to deviate to lower prices and thus to capture the density of G^* that is close to c. The second condition is given by the indifference of nature between choosing $v = 1$ and v slightly below c. Denote the maximizer of $(1-\varepsilon)\pi(p,F_0) + \varepsilon p$ by z so $\pi'(z,F_0) = -\frac{\varepsilon}{1-\varepsilon}$ where $z > p_0$ if ε is sufficiently small. Then $\max_p \pi(p,\delta_{\varepsilon,1}) = (1-\varepsilon)\pi(z,F_0) + \varepsilon z$ is larger than $\max_p \pi(p,\delta_{\varepsilon,c}) = (1-\varepsilon)\pi(c,F_0) + \varepsilon c$. But when $v = 1$ then the monopolist obtains as compared to when v is slightly below c an additional profit of $\varepsilon c \Pr(\tilde{p}^* = c)$. So

$$\lim_{v \to < c} R_L(F_p^*, \delta_{\varepsilon,v}) = R_L(F_p^*, \delta_{\varepsilon,1}) \text{ if } (1-\varepsilon)(\pi(z,F_0) - \pi(c,F_0)) + \varepsilon(z-c) = \varepsilon c\left(1 - \int_a^c f_p^*(p)\,dp\right) .$$

It actually turns out that $c < z$ so $R_L(F_p^*, \delta_{\varepsilon,1}) = R_L(F_p^*, \delta_{\varepsilon,z})$ and nature can either put a point mass at $v = 1$ or the same mass anywhere on $[z,1]$.

Proposition 7 (Interim Regret with Small Uncertainty)
If π is twice continuously differentiable and if ε is sufficiently small then there exists a, c with $a < p_0 < c$ such that

1. (a) *the minimax regret strategy is given by a continuous density*

$$f_p^*(p) = \frac{1-\varepsilon}{\varepsilon p}\pi'(p,F_0) + \frac{1}{p}, \qquad \textit{for } p \in [a,c),$$

and an upper mass point $\Pr(\tilde{p}^* = c) = \frac{1-\varepsilon}{\varepsilon c}(\pi(z,F_0) - \pi(c,F_0)) + \frac{z-c}{c}$;

(b) *the worst case demand $F_v^*(v)$ is given by a continuous density*

$$f_v^*(v) = \frac{(1-\varepsilon)(v\pi'(v,F_0) - \pi(v,F_0) + \pi(a,F_0)) + \varepsilon a}{\varepsilon v^2}, \qquad \textit{for } v \in [a,c),$$

and an upper mass point $\Pr(\tilde{v}^* = 1) = -\frac{(1-\varepsilon)\pi'(c,F_0)}{\varepsilon}$.

Proof. The proof is built up of six major steps.

(1) Solve for a, c such that $a < p_0 < c < z$ and:

$$(1-\varepsilon)\left(c\pi'(c,F_0) - \pi(c,F_0) + \pi(a,F_0)\right) + \varepsilon a = 0 \quad (4)$$

$$(1-\varepsilon)(\pi(z,F_0) - \pi(c,F_0)) + \varepsilon(z-c) - \varepsilon c\left(1 - \int_a^c \left(\frac{1-\varepsilon}{\varepsilon p}\pi'(p,F_0) + \frac{1}{p}\right)dp\right) = 0 \quad (5)$$

(1a) Notice first that if $a = p_0$ then $c = p_0$ and $\varepsilon = 0$ solve (4) and (5). It would be nice to invoke the implicit function theorem to show that there are solutions $a(\varepsilon)$ and $c(\varepsilon)$ when ε is close to 0. However, it turns out that $a'(0) \to -\infty$ which prevents us of following this path. Instead we invoke the implicit function theorem to show that there exist solutions c and ε as functions of a if a is in a sufficiently small neighborhood of p_0. Later we show that $\varepsilon(a)$ is invertible.

Let $h_1 = h_1(a,c,\varepsilon)$ and $h_2 = h_2(a,c,\varepsilon)$ denote the left hand sides of (4) and (5) respectively. Then

$$\begin{aligned}
\frac{d}{dc}h_1 &= (1-\varepsilon)c\pi''(c,F_0) \\
\frac{d}{d\varepsilon}h_1 &= -\left(c\pi'(c,F_0) - \pi(c,F_0) + \pi(a,F_0)\right) + a \\
\frac{d}{dc}h_2 &= -(1-\varepsilon)\pi'(c,F_0) - \varepsilon - \varepsilon\left(1 - \int_a^c \left(\frac{1-\varepsilon}{\varepsilon p}\pi'(p,F_0) + \frac{1}{p}\right)dp\right) + \varepsilon c\left(\frac{1-\varepsilon}{\varepsilon c}\pi'(c,F_0) + \frac{1}{c}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\varepsilon}h_2 &= -(\pi(z,F_0) - \pi(c,F_0)) + (z-c) - c\left(1 - \int_a^c \left(\frac{1-\varepsilon}{\varepsilon p}\pi'(p,F_0) + \frac{1}{p}\right)dp\right) - c\int_a^c \left(\frac{1}{\varepsilon p}\pi'(p,F_0)\right)dp \\
&= -(\pi(z,F_0) - \pi(c,F_0)) + z - 2c + c\int_a^c \left(-\frac{1}{p}\pi'(p,F_0) + \frac{1}{p}\right)dp
\end{aligned}$$

The relevant matrix is $M = M_{a,c,\varepsilon} := D_{c,\varepsilon}h$ where

$$M_{p_0,p_0,0} = \begin{bmatrix} p_0\pi''(p_0,F_0) & p_0 \\ 0 & -p_0 \end{bmatrix}.$$

Since $\det M_{p_0,p_0,0} = -p_0^2\pi''(p_0,F_0) > 0$ the implicit function theorem states for a in a neighborhood of p_0 that there exist differentiable $c(a)$ and $\varepsilon(a)$ that solve the equations (4) and (5).

(1b) We now have to show for $a < p_0$ that $\varepsilon > 0$ and $p_0 < c < z$. We know that

$M_{p_0,p_0,0}\left(c'(p_0), \varepsilon'(p_0)\right)^T = -D_a h(p_0, p_0, 0)$ and calculate

$$\begin{aligned} \frac{d}{da}h_1 &= (1-\varepsilon)\pi'(a, F_0) + \varepsilon \\ \frac{d}{da}h_2 &= -\varepsilon c\left(\frac{1-\varepsilon}{\varepsilon a}\pi'(a, F_0) + \frac{1}{a}\right) \end{aligned}$$

to find that $-D_a h(p_0, p_0, 0) = 0$ and hence $c'(p_0) = \varepsilon'(p_0) = 0$. More generally

$$M_{a,c,\varepsilon} = \begin{bmatrix} (1-\varepsilon)c\pi''(c, F_0) & -\left(c\pi'(c, F_0) - \pi(c, F_0) + \pi(a, F_0)\right) + a \\ -\varepsilon + \varepsilon\int_a^c \left(\frac{1-\varepsilon}{\varepsilon p}\pi'(p, F_0) + \frac{1}{p}\right)dp & -\left(\pi(z, F_0) - \pi(c, F_0)\right) + z - 2c + c\int_a^c \left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp \end{bmatrix}$$

and

$$-D_a h = \begin{pmatrix} -(1-\varepsilon)\pi'(a, F_0) - \varepsilon \\ \frac{c}{a}\left((1-\varepsilon)\pi'(a, F_0) + \varepsilon\right) \end{pmatrix}$$

with $M_{a,c(a),\varepsilon(a)}\left(c'(a), \varepsilon'(a)\right)^T = -D_a h|_{a,c(a),\varepsilon(a)}$ if a is in a neighborhood of p_0. Since $c'(p_0) = \varepsilon'(p_0) = 0$ we use first order approximations and replace c by p_0 and ε by 0 in the expressions for M and $-D_a h$ to obtain:

$$\begin{bmatrix} p_0\pi''(p_0, F_0) & \pi(p_0, F_0) - \pi(a, F_0) + a \\ 0 & -p_0 + p_0\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp \end{bmatrix} \begin{pmatrix} c'(a) \\ \varepsilon'(a) \end{pmatrix} \approx \begin{pmatrix} -\pi'(a, F_0) \\ \frac{p_0}{a}\pi'(a, F_0) \end{pmatrix}$$

Thus,

$$\varepsilon'(a) \approx -\frac{\pi'(a, F_0)}{a - a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp} < 0$$

and

$$\begin{aligned} p_0\pi''(p_0, F_0)c'(a) &\approx -\pi'(a, F_0) + \left(\pi(p_0, F_0) - \pi(a, F_0) + a\right)\frac{\pi'(a, F_0)}{a - a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp} \\ &= \pi'(a, F_0)\left(-1 + \frac{\pi(p_0, F_0) - \pi(a, F_0) + a}{a - a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp}\right) \\ &= \pi'(a, F_0)\left(\frac{a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp + \pi(p_0, F_0) - \pi(a, F_0)}{a - a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp}\right) > 0 \end{aligned}$$

as

$$\begin{aligned} &\frac{d}{da} \left(a\int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp + \pi(p_0, F_0) - \pi(a, F_0)\right) \\ &= \int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp - a\left(-\frac{1}{a}\pi'(a, F_0) + \frac{1}{a}\right) - \pi'(a, F_0) \\ &= \int_a^{p_0}\left(-\frac{1}{p}\pi'(p, F_0) + \frac{1}{p}\right)dp - 1 \to -1 \text{ as } a \to p_0. \end{aligned}$$

Consequently, $a < p_0$ implies $\varepsilon > 0$ and $c > p_0$. Moreover, $\varepsilon(a)$ is invertible. Since $h_1(a,z,\varepsilon) = (1-\varepsilon)(\pi(a,F_0) - \pi(z,F_0)) + \varepsilon(a-z) < 0$ and since $\frac{d}{dc}h_1 < 0$ we obtain that $c < z$.

(2) F_p^* is well defined:

(2a) Since $c < z$ we have $(1-\varepsilon)\pi'(p,F_0) + \varepsilon > 0$ for $p \in [a,z]$ and hence $f_p^*(p) > 0$ for $p \in [a,z]$

(2b) $\Pr\left(F_p^* = c\right) > 0$ by definition of z.

(3) G is well defined:

(3a) Since $g^*(c) = 0$ by definition of a and c and since

$$\frac{d}{dv}\left((1-\varepsilon)\left(v\pi'(v,F_0) - \pi(v,F_0) + \pi(a,F_0)\right) + \varepsilon a\right) = (1-\varepsilon)v\pi''(v,F_0) < 0$$

it follows that $g^*(v) \geq 0$ for $v \in [a,c]$.

(3b) $1 - G^*(c) = \frac{a}{c} - \frac{(1-\varepsilon)(\pi(c,F_0)-\pi(a,F_0))}{\varepsilon c} = -\frac{(1-\varepsilon)\pi'(c,F_0)}{\varepsilon} > 0$ as $c > p_0$.

(4) To show $v' \in \arg\max_{p'} \pi\left(p', (1-\varepsilon)F_0 + \varepsilon\mathbb{I}_{\{v \geq v'\}}\right)$ for all $v' \in [a,c]$. For $p \in [a,c]$ we have that $((1-\varepsilon)\pi(p,F_0) + \varepsilon p)$ is concave and is maximized at $p = z$. Since $c < z$ we need to verify that $((1-\varepsilon)\pi(a,F_0) + \varepsilon a) > (1-\varepsilon)\pi(p_0,F_0)$ which is true for small ε as

$$\frac{d}{d\varepsilon}((1-\varepsilon)\pi(a,F_0) + \varepsilon a - (1-\varepsilon)\pi(p_0,F_0)) = a - \pi(a,F_0) + \left((1-\varepsilon)\pi'(a,F_0) + \varepsilon\right)a'(\varepsilon) + \pi(p_0,F_0) \to p_0$$

as $\varepsilon \to 0$.

(5) Incentives of monopolist:

(5a) to show that $\pi(c,F_v^*) \geq \pi(p,F_v^*)$ for all $p > c$.By construction, $\pi(c,F_v^*) = (1-\varepsilon)\pi(a,F_0) + \varepsilon a$ and:

$$\pi(p,F_v^*) = (1-\varepsilon)\pi(p,F_0) - \varepsilon\frac{(1-\varepsilon)\pi'(c,F_0)}{\varepsilon}p = (1-\varepsilon)\left(\pi(p,F_0) - p\pi'(c,F_0)\right)$$

so

$$\pi(c,F_v^*) - \pi(p,F_v^*) = \varepsilon a + (1-\varepsilon)\left(\pi(a,F_0) - \pi(p,F_0) + p\pi'(p,F_0)\right),$$

which is decreasing in p in a neighborhood U of p_0 that is independent of ε.

Note that if ε is sufficiently small then $\pi(c,F_v^*) > \pi(p,F_v^*)$ if $p \notin U$.

(5b) by construction of f_v^* we have that $\pi(p,F_v^*)$ is constant for all $p \in [a,c]$.

(5c) clearly $\pi(c,F_v^*) > \pi(p,F_v^*)$ for $p < a$.

(6) Incentives of nature:

(6a) by construction $\lim_{v \to_< c} R_L\left(F_p^*, \delta_{\varepsilon,v}\right) = R_L\left(F_p^*, \delta_{\varepsilon,1}\right)$

(6b) by construction of f_p^* we have $R_L\left(F_p^*, \delta_{\varepsilon,v}\right)$ is constant on (a, c)

(6c) clearly $R_L\left(F_p^*, \delta_{\varepsilon,v}\right) < R_L\left(F_p^*, \delta_{\varepsilon,a}\right)$ if $v < a$. This completes the proof. ■

In future research will solve for comparative statics of the expected price and of $\bar{R}$ in terms of ε.

Proposition 8 (Determinstic Strategies)
If the seller is restricted to setting a deterministic price and if ε is sufficiently small then there exists a unique $\hat{p} = \hat{p}(\varepsilon)$ that solves

$$\varepsilon\hat{p} = (1-\varepsilon)\left(\pi(z, F_0) - \pi(\hat{p}, F_0)\right) + \varepsilon(z - \hat{p})$$

and where $\hat{p}$ attains minimax interims regret, $\hat{p}(0) = p_0$, $\hat{p}$ is decreasing in ε, $\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\hat{p}(\varepsilon) = -\infty$, $\bar{R}_D = \varepsilon\hat{p}$ and $\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\bar{R}_D = p_0$.

Proof. Consider only p in a neighborhood U of p_0 such that $\pi(p, F_0)$ is concave. Let

$$\varepsilon\hat{p} = (1-\varepsilon)\left(\pi(z, F_0) - \pi(\hat{p}, F_0)\right) + \varepsilon(z - \hat{p})$$

and

$$(1-\varepsilon)\pi(\bar{p}, F_0) + \varepsilon\bar{p} = (1-\varepsilon)\pi(p_0, F_0),$$

so $\bar{p} = \frac{1}{2(1-\varepsilon)}\left(1 - \sqrt{\varepsilon(2-\varepsilon)}\right)$ and $\hat{p} = \frac{1}{2}\frac{1+\varepsilon-\sqrt{2\varepsilon+\varepsilon^2}}{1-\varepsilon}$ in the uniform case. We show that $\bar{p} < \hat{p} < p_0$. Let

$$h(\varepsilon, p) = (1-\varepsilon)\left(\pi(z, F_0) - \pi(p, F_0)\right) + \varepsilon(z-p) - \varepsilon p.$$

Then $\frac{\partial}{\partial p}h = -(1-\varepsilon)\pi'(p, F_0) - 2\varepsilon < 0$ for $p < z$ and

$$h(\varepsilon, \bar{p}) = (1-\varepsilon)\left(\pi(z, F_0) - \pi(p_0, F_0)\right) + \varepsilon(z - \bar{p}) > (1-\varepsilon)\left(\pi(z, F_0) - \pi(p_0, F_0)\right) + \varepsilon(z - p_0) > 0,$$

as $\bar{p} < p_0$ clearly holds, and

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}h &= (1-\varepsilon)\pi'(z, F_0)z'(\varepsilon) - \left(\pi(z, F_0) - \pi(p, F_0)\right) + z - 2p \\ &= \frac{\varepsilon}{(1-\varepsilon)^2\pi''(z, F_0)} - \left(\pi(z, F_0) - \pi(p, F_0)\right) + z - 2p\end{aligned}$$

as $\pi'(z, F_0) = -\frac{\varepsilon}{1-\varepsilon}$ implies $\pi''(z, F_0)z'(\varepsilon) = -\frac{1}{(1-\varepsilon)^2}$. Hence we have

$$\hat{p}'(\varepsilon) = -\frac{-\frac{\varepsilon}{(1-\varepsilon)^2\pi''(z, F_0)} + \pi(z, F_0) - \pi(\hat{p}, F_0) - z + 2\hat{p}}{(1-\varepsilon)\pi'(\hat{p}, F_0) + 2\varepsilon} < 0$$

and $\hat{p}(0) = p_0$.

The only candidates for v' that solve $\sup_{v\in[0,1]} R(p, \delta_{\varepsilon,v})$ are $v' = 1$ and v' slightly below p. Note that

$$\begin{aligned} R(p, \delta_{\varepsilon,1}) &= \max_{p'} \left\{ \left((1-\varepsilon)\pi\left(p', F_0\right) + \varepsilon p' \right) \right\} - \left((1-\varepsilon)\pi(p, F_0) + \varepsilon p \right) \\ &= (1-\varepsilon)\left(\pi(z, F_0) - \pi(p, F_0)\right) + \varepsilon(z-p). \end{aligned}$$

If $p > z$ then

$$\sup_{v'} R\left(p, \delta_{\varepsilon,v'}\right) = (1-\varepsilon)\left(\pi(z, F_0) - \pi(p, F_0)\right) + \varepsilon z$$

which is increasing for $p \in U$. Assume $p < z$. Then

$$\lim_{v'\to_< p} R\left(p, \delta_{\varepsilon,v'}\right) = \max\left\{(1-\varepsilon)\pi(p_0, F_0), (1-\varepsilon)\pi(p, F_0) + \varepsilon p\right\} - (1-\varepsilon)\pi(p, F_0).$$

where $(1-\varepsilon)\pi(p_0, F_0) > (1-\varepsilon)\pi(p, F_0) + \varepsilon p$ holds if and only if $p < \bar{p}$.Assume that $p \le \bar{p}$. So $\lim_{v'\to_< p} R\left(p, \delta_{\varepsilon,v'}\right) = (1-\varepsilon)\left(\pi(p_0, F_0) - \pi(p, F_0)\right)$. Since both $\lim_{v'\to_< p} R\left(p, \delta_{\varepsilon,v'}\right)$ and $R(p, \delta_{\varepsilon,1})$ are decreasing in p we find that minimax regret price will not be strictly less than $\bar{p}$.

Now assume $p > \bar{p}$. Then

$$\lim_{v'\to_< p} R\left(p, \delta_{\varepsilon,v'}\right) = (1-\varepsilon)\pi(p, F_0) + \varepsilon p - (1-\varepsilon)\pi(p, F_0) = \varepsilon p$$

where $\varepsilon p < (1-\varepsilon)\left(\pi(z, F_0) - \pi(p, F_0)\right) + \varepsilon(z-p)$ and hence $\lim_{v'\to_< p} R\left(p, \delta_{\varepsilon,v'}\right) < R(p, \delta_{\varepsilon,1})$ if and only if $p < \hat{p}$. So if $\bar{p} < p < \hat{p}$ then $\sup_{v'} R\left(p, \delta_{\varepsilon,v'}\right) = R(p, \delta_{\varepsilon,1})$ which is decreasing in p.

Finally, if $\hat{p} \le p < z$ then $\sup_{v'} R\left(p, \delta_{\varepsilon,v'}\right) = \varepsilon p$ which is increasing in p. From the above it follows that $\hat{p}$ attains minimax interims regret among the deterministic pricing strategies where $\hat{p} = \hat{p}(\varepsilon)$ is strictly decreasing and $\lim_{\varepsilon\to 0} \frac{d}{d\varepsilon}\hat{p}(\varepsilon) = -\infty$. Note that $\bar{R}_D = \varepsilon\hat{p}$.

Finally,

$$\frac{d}{d\varepsilon}\bar{R}_D = \hat{p} - \varepsilon \frac{-\frac{\varepsilon}{(1-\varepsilon)^2 \pi''(z, F_0)} + \pi(z, F_0) - \pi(\hat{p}, F_0) - z + 2\hat{p}}{(1-\varepsilon)\pi'(\hat{p}, F_0) + 2\varepsilon}.$$

Since $\lim_{\varepsilon\to 0} \frac{\pi'(\hat{p}, F_0)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon}\pi'(\hat{p}, F_0) = \infty$ we obtain $\frac{d}{d\varepsilon}\bar{R}_D \to p_0$ as $\varepsilon \to 0$. ∎

Notice that if the monopolist chooses the optimal price p_0 then $\sup_{F_v\in\mathcal{N}_\varepsilon(F_0)} R(p_0, F_v) = \varepsilon p_0$ so $\frac{d}{d\varepsilon}\sup_{F_v\in\mathcal{N}_\varepsilon(F_0)} R(p_0, F_v) = p_0 = \frac{d}{d\varepsilon}\bar{R}_D|_{\varepsilon=0}$.

6 Conclusion

In this paper we analyzed robust pricing policies by a monopolist.

We began by considering globally robust policies as solution to minimax regret strategies with an unconstrained adversary. The resulting policy is random and generates an expected price that is always above $1 - \frac{1}{e} \approx 0.63$. Even the deterministic globally robust policy will never price below $\frac{1}{2}$. Notice that $\frac{1}{2}$ is the price where regret after purchase of a buyer with maximal value 1 equals regret after no sale due to setting a price slightly above the value of the buyer. As comparison, notice that the maximin policy requires to price at the lowest possible value of the buyer.

We then considered locally robust policies by restricting the strategy space by the adversary to contain only nearby distributions. The monopolist anticipates that his prior is only a noisy forecast of the true distribution. The value $\frac{1}{2}$ remains focal. Adding noise pushes pricing towards the median. The price increases when the optimal price without noise is below $\frac{1}{2}$ while expected price decreases in the level of the noise when the optimal price lies above $\frac{1}{2}$. The magnitude of the change in expected price is approximately indirectly proportional to the curvature of the profit function at the optimal price. Regret increases linearly under the locally robust policy approximately at the same degree as it would if the monopolist would stick to the optimal price under no noise. For small noise, the robust pricing policy does not substantially improve regret as compared to the optimal price under no noise.

Finally we considered a monopolist facing many buyers. This can also be interpreted as a monopolist facing a single buyer but applying regret to not knowing the true distribution instead of not knowing the buyer's value. There is no change in globally robust policies. Our results on locally robust policies under almost certainty are most complete. Here we find expected price to be strictly decreasing even when expecting the buyer's value to be below $\frac{1}{2}$. We also show that (interim) regret is strictly increasing at approximately 50% the rate that both the optimal price and the locally robust pure pricing policy generate. In other words, while (for very small noise levels) there is no advantage in terms of regret to moving away from the optimal price to an alternative pure price but there is when moving to a mixed pricing policy.

The problem of optimal monopoly price is in many respects the most elementary mechanism design problem. In a sequel to this paper, Bergemann & Schlag (2004), we consider optimal policies for a wide class of design problems, including the discriminating monopolist (as in Mussa & Rosen (1978) and Maskin & Riley (1984)) who can offer different qualities or quantities and optimal auctions for a single unit. In contrast to the approach pursued here, there we focus local robustness with revenue maximization rather than regret minimization. In addition, as we consider multi-agent design problems, it becomes desirable to "robustify" both the decisions of the buyers and the seller. [5] The complete solution of these problems poses a rich field for future research.

[5] See however Segal (2003) and Chung & Ely (2003) for sufficient conditions for the existence of dominant strategies for the bidders in optimal auctions.

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