Large Newsvendor Games

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Abstract

We consider a game, called *newsvendor game*, where several retailers, who face a random demand, can pool their resources and build a centralized inventory that stocks a single item on their behalf. The inventory costs have to be allocated in a way that is advantageous to all the retailers. A game in characteristic form is obtained by assigning to each coalition its optimal expected cost. Müller et al. (2002) proved that the anticore of this game is always nonempty for every possible joint distribution of the random demands.

In this paper we consider newsvendor games with possibly an infinite number of newsvendors. We generalize some results contained in Müller et al. (2002) and we show that in a game with a continuum of players, under a nonatomic condition on the demand, the core is a singleton. For a particular class of demands we show how the core shrinks to a singleton when the number of players increases.

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1 Introduction

The newsvendor problem is a classic textbook example in optimization. A newsvendor sells a product (newspapers) during a short selling period (a morning) with stochastic demand. The newsvendor can order inventory before the selling period and has no additional replenishment opportunity. If the order quantity is greater than the realized demand, the newsvendor must dispose of the remaining stock at a loss. If the order quantity is lower than realized demand, the newsvendor forgoes some profit. Therefore, in choosing an order quantity the newsvendor must balance the costs of ordering too little against the costs of ordering too much.

In recent years strategic versions of the newsvendor problem have been considered by several authors. The reader is referred to Cachon and Netessine (2004) for a nice survey of both cooperative and noncooperative models in supply chain.

Noncooperative versions of the newsvendor game appear in different variations. For instance Parlar (1988), Wang and Parlar (1994), Ernst and Kouvelis (1999), and Netessine and Rudi (2003) study the role of inventory in the competition among retailers and determine uniqueness of the Nash equilibrium. Lippman and McCardle (1997) study competition between firms in a single-period setting, where a consumer may switch among firms to find available inventory. In Cachon and Lariviere (1999a,b) retailers behave strategically when ordering from a supplier with limited capacity.

Cooperative versions have been considered for instance by Eppen (1979), Gerchak and Gupta (1991), Robinson (1993), Hartman and Dror (1996), Hartman et al. (2000), Müller et al. (2002), and Slikker et al. (2005). In a cooperative newsvendor game several retailers can pool their resources and build a centralized inventory that stocks a single item on their behalf. The inventory costs have to be allocated in a way that is advantageous to all the retailers. Otherwise some of them will prefer not to join the centralized inventory.

If we assign to each coalition the expected cost that it incurs if it stocks the optimal number of newspapers (as in the single-newsvendor model), then we have a cooperative game in characteristic form. Every newsvendor will find convenient to build the centralized inventory if the anticore of the game is nonempty (i.e. if the game is balanced). The anticore, rather than the core, is the relevant concept here, because the characteristic function represents costs. Müller et al. (2002) prove that if the costs are linear and homogeneous across newsvendors, then the newsvendor game is balanced for every possible joint distribution of the random demands. Slikker et al. (2005) prove a similar result for games with transsphipment costs.

In this paper we consider large newsvendor games having a structure as in Müller et al. (2002). In order to prove our results it is useful to see the newsvendor game as an infinite-dimensional measure game as in Milchtaich (1998). This allows to treat both finite and infinite games. The anticore of the game is now a set of charges (finitely additive measures) that are dominated by the game.

Several results from Müller et al. (2002) are proved in greater generality with different techniques. For instance, in order to show that the game is balanced, a charge, which is always in the anticore, is explicitely computed. Conditions for exactness, supermodularity, and monotonicity of the game are established.

The main result of the paper is that a for nonatomic newsvendor game the anticore is a singleton, whenever the aggregate demand has a continuous distribution.

For a particular class of games it is shown that the anticore shrinks to a singleton when the number of newsvendors increases.

The paper is organized as follows. In Section 2 the newsvendor game is presented and some general results are stated. Section 3 deals with nonatomic newsvendor games and provides the main result of the paper. In Section 4 a class of newsvendor games with many players is considered and some asymptotic results are stated. Section 5 studies some useful properties of an operator that is used in the newsvendor problem. Section 6 contains the proofs of the results.

2 Newsvendor games

2.1 The newsvendor problem

We introduce the newsvendor problem in an abstract setting that will prove suitable for the analysis of the newsvendor game.

A newsvendor has to decide how many newspapers to stock in order to face an unknown demand. No replenishment is allowed. If a newsvendor faces a demand x and orders a quantity y, then she incurs a cost $\varphi(y-x)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$\varphi(t) = \begin{cases} h \cdot t & \text{if} \quad t \geq 0, \\ -\pi \cdot t & \text{if} \quad t < 0, \end{cases} \tag{2.1}$$

for some positive constants h, π. The constant h represents the holding cost of stocking more newspapers than are actually sold, and the constant π is the penalty cost of not ordering enough newspapers to meet the demand. Formula (2.1) amounts to linearity in costs.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random quantities will be defined, and the space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ of all integrable random variables. We are interested in the operator $\Gamma : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, defined as

$$\Gamma(X) = \min_{y \in \mathbb{R}} \mathbb{E}\left[\varphi(y - X)\right]. \tag{2.2}$$

The operator Γ represents the expected cost for a newsvendor who orders the optimal amount of newspapers. Its properties will be studied extensively in Section 5.

2.2 The game

A general newsvendor game is defined on a measurable space of agents $(I, \mathcal{C})$, along the lines of the model described by Müller et al. (2002), where the analysis is restricted to games with a finite number of participants.

The set I is a set of players (newsvendors), and $\mathcal{C}$ is a σ-algebra of subsets of I. Elements of $\mathcal{C}$ are then feasible coalitions of newsvendors. Any coalition that forms orders a fixed number of newspapers to face a random demand. As in the newsvendor problem, if a coalition faces a demand x and orders a quantity y, then it incurs a cost $\varphi(y-x)$, where φ is defined as in (2.1).

The random demand will be represented by a function $X : \Omega \times \mathcal{C} \to \mathbb{R}$ that satisfies the following conditions

- for all $A \in \mathcal{C}$, the map $\omega \to X(\omega, A)$ is an integrable random variable,
- for all $\omega \in \Omega$, the map $A \to X(\omega, A)$ is an additive measure on $(I, \mathcal{C})$.

With a slight abuse of notation, throughout the paper we identify an integrable random variable Z with its class of equivalence $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

By this identification, we can define an *additive vector measure* $D : \mathcal{C} \to L^1(\Omega, \mathcal{F}, \mathbb{P})$. For any coalition A, $D(A) = X(\cdot, A) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is interpreted as the joint demand faced by coalition A. Details about vector measures can be found in the next subsection.

The optimal amount ordered by coalition A is

$$y_A^* = \arg\min_{y \in \mathbb{R}} \mathbb{E}\varphi(y - D(A)),$$

that is the amount that minimizes the expected cost for the coalition. The minimizer y_A^* exists and is a $\pi/(h+\pi)$-quantile of the distribution of $D(A)$. Hence the optimal expected cost for coalition A is

$$\Gamma(D(A)) = \min_{y \in \mathbb{R}} \mathbb{E}[\varphi(y - D(A))] = \mathbb{E}[\varphi(y_A^* - D(A))].$$

The newsvendor game is then defined as

$$\nu(A) = \Gamma(D(A)) \tag{2.3}$$

for all $A \in \mathcal{C}$. The amount $\nu(A)$ is the cost that members of the coalition A jointly incur.

Definition (2.3) of newsvendor game through the vector-valued measure $A \to D(A)$ presents some analytical advantage, since the newsvendor game can be viewed as an *infinite dimensional measure game* (see Milchtaich (1998)).

Example 2.1. If we set $I = \{1, 2, \ldots, d\}$, $\mathcal{C} = 2^I$, and $D(\{i\}) = X_i \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, we get the finite newsvendor game studied by Müller et al. (2002). Here, $\nu(A) = \Gamma\left(\sum_{i \in A} X_i\right)$.

2.3 Notation

Here we introduce some notation and definitions that will be used throughout the paper.

Two random variables $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ are called *comonotone* if for all $\omega' \in \Omega$ we have $\mathbb{P}(\omega : (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0) = 1$.

Given a measurable space $(I, \mathcal{C})$, a coalitional game is a set-function $\nu : \mathcal{C} \to \mathbb{R}$ such that $\nu(\varnothing) = 0$. In this paper the games under study are *cost games*, therefore the characteristic function $\nu(A)$ has to be understood as the cost faced by the members of the coalition $A \in \mathcal{C}$. We list some standard terminology utilized in cooperative games literature.

A game ν is

- *bounded* if $\sup_{A\in\mathcal{C}} |\nu(A)| < +\infty$;
- *monotone* if $\nu(A) \leq \nu(B)$ when $A \subseteq B$;
- *subadditive* if $\nu(A\cup B) \leq \nu(A) + \nu(B)$ for all pairwise disjoint A and B;
- *submodular* if $\nu(A\cup B) + \nu(A\cap B) \leq \nu(A) + \nu(B)$ for all A and B;
- *additive* (a *charge*) if $\nu(A\cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint A and B;
- σ-*additive* (a *measure*) if $\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ whenever $A_i \cap A_j = \varnothing$ for $i \neq j$.
- *continuous at* A, if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $A_n \downarrow A$ and $A_n \uparrow A$;
- *continuous* if ν is continuous at A for all $A \in \mathcal{C}$.

The set of bounded charges (i. e., additive measures) is denoted by $\mathrm{ba}(\mathcal{C})$, the set of bounded measures (i. e., countably additive measures) is denoted by $\mathrm{ca}(\mathcal{C})$, and the set of positive bounded measures is denoted by $\mathrm{ca}^{+}(\mathcal{C})$. Given a positive measure λ, the set of all measures which are absolutely continuous with respect to λ is denoted by $\mathrm{ca}(\mathcal{C}, \lambda)$.

An outcome of the game ν is an element of $\mathrm{ba}(\mathcal{C})$. Since we deal with cost games, we use of the notion of anticore, rather than the more usual notion of core. The *anticore* of a cost game ν is the set

$$\mathrm{acore}(\nu) = \{\mu \in \mathrm{ba}(\mathcal{C}) : \mu(I) = \nu(I) \text{ and } \mu(A) \leq \nu(A) \text{ for all } A \in \mathcal{C}\}.$$

The anticore is always weak*-compact subset of $\mathrm{ba}(\mathcal{C})$.

A cost game is said to be *balanced* if $\mathrm{acore}(\nu) \neq \varnothing$. Given a game ν and a coalition $A \in \mathcal{C}$, we may consider the game $\nu_A : \mathcal{C}_A \to \mathbb{R}$ which is the restriction of ν to the coalitions in A. A cost game is *totally balanced* if $\mathrm{acore}(\nu_A) \neq \varnothing$ for all $A \in \mathcal{C}$. Any totally balanced cost game is subadditive.

A cost game ν is *exact* if $\mathrm{acore}(\nu) \neq \varnothing$ and

$$\nu(A) = \max_{\mu\in\mathrm{acore}(\nu)} \mu(A), \quad \forall A \in \mathcal{C}.$$

Clearly, any exact game is totally balanced. Furthermore any submodular and bounded game is exact (see Marinacci and Montrucchio (2004a, Proposition 1)).

2.4 General results

The result that follows is rather general and almost free of assumptions. It shows that the anticore of all newsvendor games is nonempty. More importantly, it offers a specific solution in the anticore. We will see in the sequel that in the atomic case the anticore may be quite large. On the other hand, under some mild conditions, in the nonatomic setting the anticore turns out be a singleton, therefore agreeing with the solution (2.4) below.

Theorem 2.2. *Any newsvendor game is totally balanced. Moreover, if* $\sup\{\|D(A)\| : A \in \mathcal{C}\} < \infty$*, and the aggregate demand* $D(I)$ *has a continuous distribution, then* $\mu \in \text{acore}(\nu)$*, where* μ *is a bounded charge defined as*

$$\mu(A) = -h \int_{D(I) \le y^*} D(A)\ \mathrm{d}\mathbb{P} + \pi \int_{D(I) \ge y^*} D(A)\ \mathrm{d}\mathbb{P} \tag{2.4}$$

for all $A \in \mathcal{C}$*, where* y^* *is a* $\pi/(h+\pi)$*-quantile of* $D(I)$*.*

The element of the anticore defined by (2.4) is particularly appealing for some classes of games, as the next proposition shows.

Proposition 2.3. *Consider a finite newsvendor game. If all the marginals distributions of the random demands* $X_1, \ldots, X_d$ *are equal, then the measure* μ *defined by* (2.4) *is the unique element in* $\text{acore}(\nu)$ *such that* $\mu_i = \mu(\{i\}) = \Gamma(D(I))/d$ *for all* $i \in I$*. Furthermore,* μ *is the barycenter of* $\text{acore}(\nu)$*, provided* $(X_i)_{i \in I}$ *are exchangeable.*

Though all newsvendor games are totally balanced, they are not necessarily exact. Next proposition states a sufficient condition that ensures this property.

Proposition 2.4. *If* $D(I)$ *and* $D(A)$ *are comonotone for all coalitions* $A \in \mathcal{C}$*, then the newsvendor game is exact.*

At least an important example is contemplated by this proposition. If there is no aggregate risk, i.e., $D(I)$ is nonrandom, then $D(I)$ and $D(A)$ are comonotone for every $A \in \mathcal{C}$.

The next result is a straightforward extension of Müller et al. (2002, Theorem 3.3).

Proposition 2.5. *If* $\mathbb{E}[D(A) \mid D(B)]$ *and* $D(B)$ *are comonotone for all coalitions* A *and* B *such that* $A \cap B = \varnothing$*, then the game is monotone. Therefore, all members of the anticore are nonnegative.*

The following result provides a strong property for newsvendor games with a particular structure of the demand D. Two random variables X and Y are of the same type, provided $F_X(x) = F_Y(ax+b)$, for some $a > 0$ and b.

Proposition 2.6. *If the following conditions hold*

(i) *all* $D(A) \neq 0$ *are of the same type,*

(ii) $D(A)$ *have finite variance,*

then the newsvendor game is

$$\nu(A) = k\sqrt{\text{Var}[D(A)]} \tag{2.5}$$

for some $0 < k \le \max\{h, \pi\}$*.*

In addition, if the random variables $D(A)$ *and* $D(B)$ *are uncorrelated for all* A *and* B *such that* $A \cap B = \varnothing$*, then the newsvendor game is submodular.*

It is interesting that the nature of the game does not depend on π and h.

Specializing Proposition 2.6 to finite games yields a remarkable result as long as the demands X_i have Gaussian distributions. We obtain another explicit solution in $\operatorname{acore}(\nu)$. Let $e_i \in \mathbb{R}^d$ be the vector whose i-th element is 1 and the others are 0. The vector e_A is defined as $e_A = \sum_{i \in A} e_i$.

Proposition 2.7. *Let ν be a finite newsvendor game with multinormal demands $(X_i)_{i \in I}$. Denoting by $\Sigma = [\sigma_{ij}]$ its covariance matrix, we have*

$$\nu(A) = k \left(e_A' \Sigma e_A\right)^{1/2} = k \left(\sum_{(i,j) \in A \times A} \sigma_{ij} \right)^{1/2}, \tag{2.6}$$

for all coalitions $A \in 2^I$. The measure $\mu \in \operatorname{acore}(\nu)$ if

$$\mu(\{i\}) = k' \sum_{j=1}^{n} \sigma_{ij} \tag{2.7}$$

for a suitable normalization factor k'. In addition, if the $(X_i)_{i \in I}$ are exchangeable, with variance σ^2 and correlation coefficient $-1/(n-1) \leq \rho \leq 1$, then we get the symmetric game

$$\nu(A) = k\sigma \sqrt{(1-\rho)|A| + \rho |A|^2}, \tag{2.8}$$

which is submodular.

As a by-product, Proposition 2.7 shows that the sufficient condition for submodularity used in Proposition 2.6 is not necessary.

In general solutions (2.7) and (2.4) do not coincide. They do when the demands are exchangeable. In this case, solution (2.7) turns out be the barycenter of $\operatorname{acore}(\nu)$. In submodular games the barycenter is necessarily the Shapley value (see Shapley (1971/72).

3 Nonatomic newsvendor games

In this section we examine a nonatomic version of the newsvendor game, namely, a version where there is a continuum of players and each on of them has a negligible weight.

First we state a known result about nonatomic vector measures. We recall that a vector measure D is said to be *nonatomic* if $D(A) \neq 0$ implies the existence of some $B \in \mathcal{C}$, with $B \subseteq A$, such that $D(B) \neq 0$ and $D(A \setminus B) \neq 0$.

Proposition 3.1. *Assume that the demand vector measure D has Radom-Nikodym derivative, i.e,*

$$D(A) = \int_A \delta \, \mathrm{d}\lambda, \text{ with } \lambda \in \operatorname{ca}^+(\mathcal{C}) \text{ and } \lambda \text{ nonatomic.} \tag{3.1}$$

Then $A \mapsto D(A)$ is nonatomic.

The main result of this section establishes that in the nonatomic setting, when the aggregate demand has a continuous distribution, the anticore of newsvendor games is a singleton.

Theorem 3.2. *Assume that in the newsvendor game* $\nu(A) = \Gamma(D(A))$ *the demand vector measure satisfies* (3.1)*. Furthermore let the aggregate demand* $D(I)$ *have a continuous distribution. Then*

(i) $\operatorname{acore}(\nu) \subset L^1(I, \mathcal{C}, \lambda)$ *is a singleton, given by* (2.4)*.*

(ii) *the Radom-Nikodym derivative* $\mathrm{d}\mu/\mathrm{d}\lambda$ *of the unique element* $\mu \in \operatorname{acore}(\nu) \subset L^1(I, \mathcal{C}, \lambda)$ *is*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = -h \int_{D(I) \leq y^*} \overline{\delta}(i, \omega) \ \mathrm{d}\mathbb{P} + \pi \int_{D(I) \geq y^*} \overline{\delta}(i, \omega) \ \mathrm{d}\mathbb{P}. \tag{3.2}$$

Nonatomicity poses restrictions on the distribution of the demands. For instance, the following proposition shows that the last claim of Proposition 2.6 is not effective in the nonatomic framework.

Proposition 3.3. *Assume that* $A \mapsto D(A)$ *is nonatomic and bounded-variation, and that the distributions of all* $D(A) \neq 0$ *are continous and of the same type, with finite variance. For each coalition A, for which* $D(A) \neq 0$*, there exist two disjoint subcoalitions* $A_1, A_2 \subseteq A$*, such that* $\operatorname{Cov}(D(A_1), D(A_2)) \neq 0$*.*

4 Large newsvendor games

Here we study the shrinking of the anticore of newsvendor games as the number of players increases. We restrict our analysis to Gaussian case studied in Proposition 2.7. First we deal with the case of exchangeable demands.

Let the demand $X = (X_1, \dots, X_d)$ have a multinormal distribution with expectation μ and covariance matrix Σ. Suppose further that the demands are exchangeable, namely, $\sigma_{ii} = \sigma^2$, $\sigma_{ij} = \sigma^2 \rho$, for $i \neq j$, where $-1/(d-1) \leq \rho \leq 1$.

Consider a game with d players and a multinormal exchangeable demands such that $\nu(I) = 1$. In view of (2.8), setting $|A| = a$, we have the symmetric game

$$\nu(A) = \left(\frac{a^2 \rho + a(1-\rho)}{d^2 \rho + d(1-\rho)} \right)^{1/2}. \tag{4.1}$$

By Proposition 2.7 this game is submodular. Therefore it is easy to compute the extreme points of the anticore by using Shapley's theorem Shapley (1971/72). We recall that for finite submodular games the extreme points of the anticore are one-to-one with the so-called marginal worth associated with the maximal chains. More specifically, if $\varnothing = C_0 \subset C_1 \subset \cdots \subset C_{d-1} \subset C_d = I$ is a maximal chain, then there exists one and only one measure μ such that $\mu(C_i) = \nu(C_i)$. Clearly, for all $j \in I$, there is an index i such that $\{j\} = C_i \setminus C_{i-1}$. Hence $\mu_j = \mu(\{j\}) = \nu(C_i) - \nu(C_{i-1})$. By taking the chain $\overline{C}_i = \{1, 2, \dots, i\}$ we get the extreme point in the anticore $\overline{\mu}_i = \nu(\overline{C}_i) - \nu(\overline{C}_{i-1})$. Clearly $\overline{\mu}_i$ is decreasing in i. Further, as the game is symmetric, all the extreme points are obtained by permuting the sequence $(\overline{\mu}_i)$.

Denote by $\overline{\mu}_\pi$ any such measure, where $\pi \in \Pi$, the set of all permutations of $\{1, 2, \dots, d\}$. Indicating by $\|\mu\| = |\mu|(I) = \sum_{i=1}^d |\mu_i|$ the total variation norm of μ, the diameter Φ of the anticore is

$$\Phi = \max_{\mu,\mu' \in \text{acore}(\nu)} \|\mu - \mu'\| = \max_{\pi \in \Pi} \|\overline{\mu} - \overline{\mu}_\pi\|,$$

where the last step is a consequence of Bauer maximum principle. Assume first that the number d of players is even. It is easy to see that the maximum is achieved by taking $\overline{\mu}_\pi = \left(\overline{\mu}_d, \overline{\mu}_{d-1}, \dots, \overline{\mu}_1\right)$. Hence,

$$\begin{aligned}\Phi(d,\rho) &= 2\sum_{i=1}^{d/2}\left(\overline{\mu}_i - \overline{\mu}_{d-i}\right)\\ &= 2\overline{\mu}\left(\overline{C}_{d/2}\right) - 2\overline{\mu}\left(\overline{C}^c_{d/2}\right)\\ &= 2\left[2\overline{\mu}\left(\overline{C}_{d/2}\right) - \overline{\mu}(I)\right]\\ &= 2\left[2\nu\left(\overline{C}_{d/2}\right) - 1\right].\end{aligned} \tag{4.2}$$

Using (4.1), we obtain

$$\frac{1}{2}\Phi(d,\rho) = \left(\frac{d^2\rho + 2d(1-\rho)}{d^2\rho + d(1-\rho)}\right)^{1/2} - 1.$$

The diameter is decreasing in ρ. Its value is zero when $\rho = 1$, which corresponds to the case of comonotone demands. It diverges to infinity as $\rho \to -1/(d-1)$ (which corresponds to the case where the aggregate demand is nonrandom).

If ρ is fixed and $\rho > 0$, then $\Phi(d,\rho) \to 0$, as $d \to \infty$. Actually, we have

$$\Phi(d,\rho) = \frac{1-\rho}{\rho}\left[d^{-1} - \frac{5}{4}\left(\frac{1-\rho}{\rho}\right)d^{-2}\right] + o\left(d^{-2}\right)$$

which shows that the diameter of the anticore shrinks with rate $1/d$.

If the number of players in odd, we get a similar result, where (4.2) is replaced by

$$\Phi(d,\rho) = 2\left[\nu\left(\overline{C}_{(d-1)/2}\right) + \nu\left(\overline{C}_{(d+1)/2}\right) - 1\right].$$

Notice that if $\rho = 0$ the diameter does not vanish asymptotically. In fact, (4.1) becomes $\nu(A) = \sqrt{|A|/d}$ which "approaches" the game $\sqrt{\ell(A)}$ where ℓ is the Lebesgue measure of the interval $[0,1]$. An indirect way to understand why for $\rho = 0$ the shrinking does not occur is to invoke Proposition 3.3.

If ρ is allowed to vary with d and to assume negative values, then it is possible that the anticore does not shrink to a singleton.

The shrinkage of the anticore holds for a larger class of games with multinormal demands, even without requiring exchangeability. Let $\{\nu_d\}_{d\in\mathbb{N}}$ be a sequence of normalized Gaussian games where the set of players is $I_d = \{1, 2, \dots, d\}$ and $\nu_d(I_d) = 1$. Let $\Sigma^d = [\sigma_{ij}(d)]$ be the covariance matrix for the game ν_d, where $\sigma_{ii}(d) \equiv \sigma^2(d)$, and $\sigma_{ij}(d) = \sigma^2(d)\rho_{ij}(d)$ with $i \neq j$. Set

$$\rho(d) = \max_{i \neq j \in I_d} \rho_{ij}(d).$$

Proposition 4.1. *Under the condition $\rho(d) \geq \eta$ for some $\eta > 0$, the diameters of* $\operatorname{acore}(\nu_d)$ *shrink to zero as $d \to \infty$.*

5 The operator Γ

Most of the proofs of the results stated in the previous sections rely on properties of the operator Γ defined in (2.2). In this section we study such properties.

The following facts are proved in Müller et al. (2002).

- Γ is a well-defined and finite map on $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, Γ is convex and positively homogeneous.
- The operator Γ is comonotonically additive, namely, $\Gamma(X+Y) = \Gamma(X) + \Gamma(Y)$, whenever X and Y are comonotone.
- The minimizers $\arg\min_{y \in \mathbb{R}} \mathbb{E}[\varphi(y - X)]$ exist for all X, but not necessarily unique. They are the $\pi/(h+\pi)$-quantiles of the distribution of X.
- Γ is convex-order monotone, namely, if $\mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)]$ for all convex functions ψ, then $\Gamma(X) \leq \Gamma(Y)$.

The following useful result establishes the Lipschitz continuity of Γ.

Proposition 5.1. *For all $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ we have*

$$|\Gamma(X) - \Gamma(Y)| \leq \gamma \|X - Y\|, \tag{5.1}$$

with $\gamma = \max\{h, \pi\}$.

By (5.1), the operator Γ is continuous. Hence, Γ is a *support function* (see for instance Hörmander (1955)). For $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ we write $\langle X, Y \rangle = \int XY \, \mathrm{d}\mathbb{P} = \mathbb{E}[XY]$. By Hormander's theorem, there exists a unique weak*-compact and convex set $\Gamma^* \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Gamma(X) = \max_{Y \in \Gamma^*} \langle Y, X \rangle. \tag{5.2}$$

The next proposition, which plays an important role in the proof of many of our results, provides a complete characterization of the set Γ^*.

Proposition 5.2. $Y \in \Gamma^*$ *if and only if*

(i) $\int Y \, \mathrm{d}\mathbb{P} = 0$,

(ii) $-h \leq Y \leq \pi$ $\mathbb{P}$-*a.s.*

In addition, if X *has continuous distribution, then there exists a unique* $\overline{Y} \in \Gamma^*$ *such that*

$$\Gamma(X) = \langle \overline{Y}, X \rangle, \tag{5.3}$$

whose expression is

$$\overline{Y} = -h 1_{\{X \leq y^*\}} + \pi 1_{\{X \geq y^*\}}. \tag{5.4}$$

where y^* *is a* $\pi / (h + \pi)$*-quantile of the distribution of* X*.*

Formula (5.3) is substantially one of several forms of Hartman et al. (2000, equation (2)). Next statement is easily obtained by straightforward algebraic manipulation of (5.4).

Proposition 5.3. *Let* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *have continuous distribution* F_X*. Then*

$$\begin{aligned}\Gamma(X) &= h(y^* - \mathbb{E}(X)) + (h + \pi) \int_{y^*}^{\infty} (x - y^*) \, \mathrm{d}F_X(x) \\ &= h\mathbb{E}(|X - y^*|) + (\pi - h) \int_{y^*}^{\infty} (x - y^*) \, \mathrm{d}F_X(x).\end{aligned}$$

The first equation generalizes the one given by Eppen (1979) for normal variables.

Remark 5.4. The set of $Y \in \Gamma^*$ such that $\Gamma(X) = \langle Y, X \rangle$ can be studied also when the distribution of X is not continuous. As long as $\mathbb{P}(X = y^*) = 0$ it is clear from the proof of Proposition 5.2 that Γ^* remains a singleton.

Remark 5.5. By convex analysis, it turns out that $\Gamma^* = \partial\Gamma(0)$, where $\partial\Gamma(0)$ is the subdifferential of the convex function Γ. Likewise, the set of elements $Y \in \Gamma^*$ such that $\Gamma(X) = \langle X, Y \rangle$ is nothing but $\partial\Gamma(X)$. Therefore saying that $\partial\Gamma(X)$ is a singleton, when X has continuous distribution, amounts to affirming that Γ is Gateaux differentiable at X (some more details are discussed in the proof of Theorem 3.2).

The following proposition is not used to prove properties of newsvendor games, but it has its own mathematical interest. It shows that the functional Γ is a Choquet integral.

Proposition 5.6. *For all* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$*, we have*

$$\Gamma(X) = \int X \, \mathrm{d}\sigma = \int_{-\infty}^{+\infty} \sigma(X \geq t) \, \mathrm{d}t,$$

where σ *is the submodular set-function* $\sigma(E) = \min\{\pi\mathbb{P}(E), h\mathbb{P}(E^c)\}$*,* $E \in \mathcal{F}$*.*

The differentiability properties of Choquet integrals have been extensively studied by Marinacci and Montrucchio (2004a). The differentiability of Γ at a point X having continuous distribution is perfectly consistent with their general results.

6 Proofs

First we introduce some more notation and known results that will be used throughout the proofs.

Given a game ν, a coalition $N \in \mathcal{C}$ is ν-null, whenever $\nu(A \cup N) = \nu(A)$ for all $A \in \mathcal{C}$. For $\lambda \in \mathrm{ca}^+(\mathcal{C})$, a game ν is called λ-continuous if $\lambda(A) = 0$ implies that A is ν-null.

As well known, $\mathrm{ba}(\mathcal{C})$ is (isometrically isomorphic to) the norm dual of the space $B(\mathcal{C})$ of all bounded and measurable functions (endowed with the supnorm), the duality being $\langle f, \mu \rangle = \int f \, \mathrm{d}\mu$, with $f \in B(\mathcal{C})$ and $\mu \in \mathrm{ba}(\mathcal{C})$. We consider the relevant subset $B_1(\mathcal{C}) = \{f \in B(\mathcal{C}) : 0 \leq f \leq 1\}$, whose members are often called *ideal coalitions* (see Aumann and Shapley (1974)).

The set of ideal coalitions can be endowed with the na-*topology* due to Aumann and Shapley (1974), which is the coarsest topology for which all the functionals $f \mapsto \int f \, \mathrm{d}\mu$, with μ nonatomic, are continuous. By Lyapunov's theorem the indicator functions are na-dense in $B_1(\mathcal{C})$. Therefore, any game ν, when viewed as the function $1_A \mapsto \nu(A)$ defined on a space of indicator functions, has at most one na-continuous extension to $B_1(\mathcal{C})$. We use na-extensions of newsvendor games in our main Theorem 3.2.

Given a game ν, its *variation norm* $\|\nu\|$ is given by

$$\sup \sum_{i=1}^{n} |\nu(A_i) - \nu(A_{i-1})|,$$

where the supremum is taken over all finite chains $\varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = I$. If ν is a charge, the variation norm $\|\nu\|$ reduces to the total variation norm. We denote by $\mathrm{bv}(\Sigma)$ the vector space of all games ν having finite variation norm. This is the games setting adopted by Aumann and Shapley (1974). The newsvendor games is defined through vector-valued measures $F : \mathcal{C} \to X$ where X is a Banach space (specifically, $X = L^1(\Omega, \mathcal{F}, \mathbb{P})$, the space of integrable random variables defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Diestel and Uhl (1977) is the standard reference for them. We recall just some definitions.

An additive measure $F : \mathcal{C} \to X$ is bounded, if $\sup\{\|F(A)\| : A \in \mathcal{C}\} < \infty$. If F is countably additive then F is necessarily bounded (see Diestel and Uhl (1977, Cor. 19, p. 9)). We recall that we can associate with any $F : \mathcal{C} \to X$, its *semivariation* $\|F\|$ which is a scalar subadditive set-function (see Diestel and Uhl (1977, p. 2)). The measure F is said to be of bounded semivariation if $\|F\|(I) < +\infty$. Any countably additive vector measure F is of bounded semivariation (see Diestel and Uhl (1977, Proposition 11, p. 4)).

Given a vector measure $F : \mathcal{C} \to X$, the variation of F is the extended nonnegative measure $|F|$ defined as $|F|(A) = \sup_\pi \sum_{B \in \pi} \|F(B)\|$, where the supremum is taken over all partitions of A into a finite number of pairwise disjoint members of $\mathcal{C}$. If $|F|(I) < +\infty$, F is then called of *bounded variation,* a more stringent condition than bounded semivariation.

We recall that if $\mu \in \mathrm{ca}(\mathcal{C})$, a μ-measurable function $f : I \to X$, where X is a Banach space, is *Bochner integrable* if $\int \|f\| \, \mathrm{d}\mu < \infty$, where $\|f\|$ is the norm function: $\|f\|(i) = \|f(i)\|$ (see Diestel and Uhl (1977, p. 45)). Given a μ-Bochner integrable function $f : I \to X$, we can define the X-valued measure $F(A) = \int_A f \, \mathrm{d}\mu$, for $A \in \mathcal{C}$.

Section 5

We prove results of Section 5 first because they are used in the proofs of the other results.

Proof of Proposition 5.1. Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Since $\varphi(t) \leq \gamma |t|$, we have

$$\Gamma(X) \leq \mathbb{E}[\varphi(-X)] \leq \gamma \mathbb{E}[|X|] = \gamma \|X\|.$$

By subadditivity

$$\begin{aligned}\Gamma(X) &= \Gamma(X - Y + Y) \\ &\leq \Gamma(X - Y) + \Gamma(Y) \\ &\leq \gamma \|X - Y\| + \Gamma(Y),\end{aligned}$$

that is, $\Gamma(X) - \Gamma(Y) \leq \gamma \|X - Y\|$. Interchanging the role of X and Y we obtain (5.1). $\square$

Proof of Proposition 5.2. First we prove that conditions (i) and (ii) are satisfied by any $Y \in \Gamma^*$. Fix $Y \in \Gamma^*$. We have

$$\Gamma(X) = \min_y \mathbb{E}[\varphi(y - X)] \geq \langle X, Y \rangle, \quad \text{for all } X \in L^1.$$

Hence, $\mathbb{E}[\varphi(y - X)] \geq \langle X, Y \rangle$ for all $X \in L^1$ and all $y \in \mathbb{R}$. Fix an element $X \in L^1$ and a scalar $y \neq 0$, and consider the parametrized family of random variables $X_\lambda = (\lambda - 1) y + X$ with $\lambda \in \mathbb{R}$. We obtain

$$\begin{aligned}\mathbb{E}[\varphi(\lambda y - X_\lambda)] &\geq \langle X_\lambda, Y \rangle \\ \mathbb{E}[\varphi(y - X)] &\geq (\lambda - 1) y \langle 1, Y \rangle + \langle X, Y \rangle\end{aligned}$$

which holds for all $\lambda \in \mathbb{R}$. Clearly, this implies that $\langle 1, Y \rangle = \int Y \, \mathrm{d}\mathbb{P} = 0$.

From $\mathbb{E}[\varphi(y - X)] \geq \langle X, Y \rangle$, by setting $y = 0$ and replacing X with $-X$, we get $\mathbb{E}[\varphi(X)] + \langle X, Y \rangle \geq 0$. In view of (2.1), we have

$$\int_{X \geq 0} X(h + Y) \, \mathrm{d}\mathbb{P} + \int_{X < 0} X(Y - \pi) \, \mathrm{d}\mathbb{P} \geq 0$$

which must hold for all X. In particular, if X is nonnegative, we have $\int X(h + Y) \, \mathrm{d}\mathbb{P} \geq 0$ for all $X \geq 0$. Clearly, this implies that $Y \geq -h$ almost surely. By using nonpositive random variables, we get $Y \leq \pi$.

Conversely, we prove that any Y satisfying (i) and (ii) lies in Γ^*. Consider the difference

$$\int_\Omega \varphi(y - X) \, \mathrm{d}\mathbb{P} - \langle X, Y \rangle$$

where $X \in L^1$ and $y \in \mathbb{R}$. In view of condition (i), we have

$$\begin{aligned}\int_\Omega \varphi(y - X) \, \mathrm{d}\mathbb{P} - \langle X, Y \rangle &= \int_\Omega \varphi(y - X) \, \mathrm{d}\mathbb{P} - \langle X - y, Y \rangle \\ &= \int_\Omega \varphi(Z) \, \mathrm{d}\mathbb{P} + \langle Z, Y \rangle,\end{aligned}$$

where $Z = y - X$. On the other hand,

$$\int_{\Omega} \varphi(Z) \ \mathrm{d}\mathbb{P} + \langle Z, Y \rangle = \int_{Z \geq 0} Z(h + Y) \ \mathrm{d}\mathbb{P} + \int_{Z<0} Z(Y - \pi) \ \mathrm{d}\mathbb{P} \geq 0,$$

where the two addenda are nonnegative by condition (ii). This proves that $\Gamma(X) \geq \langle X, Y \rangle$ and, in turn, that $Y \in \Gamma^*$.

To prove the last statement, it suffices to calculate $\Gamma(X) - \langle Y, X \rangle$. Since X has a continuous distribution, it follows that $\mathbb{P}(X = y^*) = 0$. Therefore,

$$\Gamma(X) - \langle Y, X \rangle = \int_{X \leq y^*} (h + Y)(y^* - X) \ \mathrm{d}\mathbb{P} + \int_{X \geq y^*} (\pi - Y)(X - y^*) \ \mathrm{d}\mathbb{P}.$$

Hence, $\Gamma(X) - \langle Y, X \rangle = 0$ if and only if $Y = \overline{Y}$ as defined in (5.4). □

Proof of Proposition 5.6. The representation of functionals by Choquet integrals goes back to Schmeidler (1986). His result cannot be used here as our set-function is not monotone. We use instead a generalization to nonmonotonic case due to Marinacci and Montrucchio (2004b, Theorem 4.5). It is easy to check that $\Gamma(1_E) = \sigma(E)$ for all $E \in \mathcal{F}$. Consider the space $B(\mathcal{F})$ of all bounded and $\mathcal{F}$-measurable function on Ω, endowed with the sup-norm, denoted by $\|.\|_\infty$. As $\|X\|_{L_1} \leq \|X\|_\infty$, by (5.1) the functional Γ is Lipschitz continuous over $B(\mathcal{F})$. In view of Thoerem 4.5 of Marinacci and Montrucchio (2004b), comonotonic additivity and sup-norm continuity imply that $\Gamma(X) = \int X \ \mathrm{d}\sigma$ holds for all $X \in B(\mathcal{F})$. If $X_1 = X_2$, P-a.e, then $\int X_1 \ \mathrm{d}\sigma = \int X_2 \ \mathrm{d}\sigma$. We can hence assert that $\Gamma(X) = \int X \ \mathrm{d}\sigma$ holds for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. As $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is dense in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and Γ is continuous on this space, our statement will be true if $\int X \ \mathrm{d}\sigma$ is well-defined and continuous over $L^1(\Omega, \mathcal{F}, \mathbb{P})$. On the other hand, as $\sigma(\Omega) = 0$,

$$\int X \ \mathrm{d}\sigma = \int_0^{+\infty} \sigma(X \geq t) \ \mathrm{d}t + \int_{-\infty}^0 \sigma(X \geq t) \ \mathrm{d}t.$$

Since σ is manifestly of bounded variation, the integrands are of bounded variations as well and thus the integrals are generalized Riemann integrals. Moreover,

$$\begin{aligned}
\int X \ \mathrm{d}\sigma &\leq \pi \int_0^{+\infty} \mathbb{P}(X \geq t) \ \mathrm{d}t + h \int_{-\infty}^0 \mathbb{P}(X < t) \ \mathrm{d}t \\
&\leq \gamma \left[\int_0^{+\infty} \mathbb{P}(X \geq t) \ \mathrm{d}t + \int_0^{+\infty} \mathbb{P}(X \leq -t) \ \mathrm{d}t \right] \\
&= \gamma \int_0^{+\infty} \mathbb{P}(|X| \geq t) \ \mathrm{d}t \\
&= \gamma \|X\|_{L_1},
\end{aligned}$$

and the generalized integral is well-defined and finite. If we prove that the Choquet integral is subadditive, we are done, as

$$\left| \int X \ \mathrm{d}\sigma - \int Y \ \mathrm{d}\sigma \right| \leq \gamma \|X - Y\|_{L_1}$$

that proves the claim. As σ is submodular, the subadditivity property $\int (X+Y) \ \mathrm{d}\sigma \leq \int X \ \mathrm{d}\sigma + \int Y \ \mathrm{d}\sigma$ is known to be true for bounded functions X and Y. Consider then the n-truncations of X and Y. Namely, $X_n = X \wedge (n1_\Omega) \vee (-n1_\Omega)$. We have $\int (X_n + Y_n) \ \mathrm{d}\sigma \leq \int X_n \ \mathrm{d}\sigma + \int Y_n \ \mathrm{d}\sigma$. □

Section 2

Proof of Theorem 2.2. To prove that the game is totally balanced it suffices to check that, for any coalition $A \in \mathcal{C}$, if $\sum_i \lambda_i 1_{B_i} = 1_A$, where $\{B_i\}_i$ are finitely many coalitions, then $\sum_i \lambda_i \nu(B_i) \geq \nu(A)$. This is classical result, due to Bondareva (1963) and Shapley (1967), holds also in an infinite setting (see Marinacci and Montrucchio (2004b, Theorem 4.1) for a proof).

Given a simple function $\varphi = \sum_i \mu_i 1_{A_i}$, $D(\varphi)$ denotes $\sum_i \mu_i D(A_i)$. It is well-known that the map $\varphi \to D(\varphi)$ is a linear operator on the space of simple functions. Hence, from $\sum_i \lambda_i 1_{B_i} = 1_A$, it follows

$$\begin{aligned}\nu(A) &= \Gamma(D(A)) \\ &= \Gamma\left(\sum_i \lambda_i D(B_i)\right) \\ &\leq \sum_i \lambda_i \Gamma(D(B_i)) \\ &= \sum_i \lambda_i \nu(B_i).\end{aligned}$$

Consequently the game is totally balanced.

Define now the additive measure $\mu(A) = \langle \overline{Y}, D(A) \rangle$, $A \in \mathcal{C}$, where $\overline{Y} \in \Gamma^*$ is given by $\overline{Y} = -h1_{\{D \leq y^*\}} + \pi 1_{\{D \geq y^*\}}$.

In view of Proposition 5.2, we have $\nu(I) = \Gamma(D(I)) = \langle \overline{Y}, D(I) \rangle = \mu(I)$. Moreover, for all $A \in \mathcal{C}$, $\nu(A) = \Gamma(D(A)) \geq \langle \overline{Y}, D(A) \rangle = \mu(A)$. If D is bounded, we have $|\mu(A)| = |\langle \overline{Y}, D(A) \rangle| \leq \|\overline{Y}\| \|D(A)\| \leq M$. Hence, $\mu \in \mathrm{ba}(\mathcal{C})$ and $\mu \in \mathrm{acore}(\nu)$. Clearly μ is nothing but (2.4). □

Proof of Proposition 2.3. If all the random variables X_i have the same distributions, then

$$\mu_i = -h \int_{D(I) \leq y^*} X_i \ \mathrm{d}\mathbb{P} + \pi \int_{D(I) \geq y^*} X_i \ \mathrm{d}\mathbb{P}$$

is independent of i. Suppose now that the X_i are exchangeable. Let $\pi : I \to I$ be any permutation. If $\lambda \in \mathrm{acore}(\nu)$, then $\lambda_\pi \in \mathrm{acore}(\nu)$, where $\lambda_\pi(A) = \lambda(\pi A)$. If λ is an extremal point of $\mathrm{acore}(\nu)$, then λ_π is, too. Hence μ agrees with $\sum_\pi \frac{1}{n!} \lambda_\pi$, since $\sum_\pi \frac{1}{n!} \lambda_\pi$ is uniform over I. Note that the elements λ_π are not necessarily all different but they can be regrouped into distinct classes of the same cardinality. Therefore $\mu = \sum_\pi \frac{1}{n!} \lambda_\pi$ is the barycenter of $\mathrm{acore}(\nu)$. □

Proof of Proposition 2.4. Fix $A \in \mathcal{C}$. Set $X_1 = D(A)$ and $X_2 = D(I)$. By assumption, X_1 and X_2 are comonotone. Hence, $\Gamma(X_1 + X_2) = \Gamma(X_1) + \Gamma(X_2)$. Set

$$\begin{aligned}
\Gamma_1^* &= \{Y \in \Gamma^* : \langle Y, X_1 \rangle = \Gamma(X_1)\} \\
\Gamma_2^* &= \{Y \in \Gamma^* : \langle Y, X_2 \rangle = \Gamma(X_2)\} \\
\Gamma_3^* &= \{Y \in \Gamma^* : \langle Y, X_1 + X_2 \rangle = \Gamma(X_1 + X_2)\}.
\end{aligned}$$

Clearly $\Gamma_1^* \cap \Gamma_2^* = \Gamma_3^*$. For, if $Y \in \Gamma_1^* \cap \Gamma_2^*$,

$$\begin{aligned}
\Gamma(X_1 + X_2) &= \Gamma(X_1) + \Gamma(X_2) = \langle Y, X_1 \rangle + \langle Y, X_2 \rangle \\
&= \langle Y, X_1 + X_2 \rangle
\end{aligned}$$

and $Y \in \Gamma_3^*$. The converse can be proved in a similar way.

As Γ_3^* is nonempty, there exist some $\overline{Y} \in \Gamma_1^* \cap \Gamma_2^*$. The measure $\mu(E) = \left\langle \overline{Y}, D(E) \right\rangle$ lies in the anticore by construction. Further, $\mu(A) = \left\langle \overline{Y}, D(A) \right\rangle = \left\langle \overline{Y}, X_1 \right\rangle = \Gamma(X_1) = \Gamma(D(A)) = \nu(A)$. This proves that the game is exact. □

Proof of Proposition 2.5. The proof rests on this simple fact. If $\mathcal{F}_1$ is a σ-subalgebra of $\mathcal{F}$, then $\Gamma(\mathbb{E}[X \mid \mathcal{F}_1]) \leq \Gamma(X)$. Actually, by Jensen's inequality

$$\mathbb{E}[\varphi(y - X)] = \mathbb{E}[\mathbb{E}[\varphi(y - X) \mid \mathcal{F}_1]] \geq \mathbb{E}[\varphi(y - \mathbb{E}[X \mid \mathcal{F}_1])],$$

which implies $\Gamma(X) \geq \Gamma(\mathbb{E}[X \mid \mathcal{F}_1])$.

Let A and B be disjoint. Then,

$$\begin{aligned}
\nu(A \cup B) &= \Gamma(D(A) + D(B)) \\
&\geq \Gamma(D(B) + \mathbb{E}[D(A) \mid D(B)]) \\
&= \Gamma(D(B)) + \Gamma(\mathbb{E}[D(A) \mid D(B)]) \\
&\geq \Gamma(D(B)) \\
&= \nu(B),
\end{aligned}$$

where we are using the fact that Γ is comonotonically additive. Hence, ν is monotone. Clearly, the elements in the anticore are nonnegative. For, if $\mu \in \operatorname{acore}(\nu)$, from $\mu(A^c) \leq \nu(A^c)$ it follows $\mu(A) \geq \nu(I) - \nu(A^c) \geq 0$. □

Lemma 6.1. *We have*

$$|\Gamma(X) - \Gamma(Y)| \leq k(\pi, h, p) \|X - Y\|_p$$

with $X, Y \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, and $k(\pi, h, p) \leq \gamma$. If $\mathbb{P}$ is nonatomic then

$$k(\pi, h, p) = \left(\frac{h\pi \left(h^{q-1} + \pi^{q-1}\right)}{h + \pi} \right)^{1/q}, \tag{6.1}$$

where $q^{-1} + p^{-1} = 1$.

Proof. By representation (5.2) and Holder's inequality we have

$$\Gamma(X) \leq \max_{Y \in \Gamma^*} |\langle Y, X \rangle| \leq \|X\|_p \max_{Y \in \Gamma^*} \|Y\|_q .$$

In view of Proposition 5.2, $|Y| \leq \gamma$ if $Y \in \Gamma^*$. Therefore, $\|Y\|_q \leq \gamma$. Hence, $\Gamma(X) \leq \gamma \|X\|_p$. By the subadditivity of Γ, we deduce that $|\Gamma(X) - \Gamma(Y)| \leq \gamma \|X - Y\|_p$.

If $\mathbb{P}$ is nonatomic, the identification of the extreme points of Γ^* is easy. Specifically, $Y \in \operatorname{ext} \Gamma^*$, if there is a measurable set A such that $Y = \pi$ on A and $Y = -h$ on A^c. Moreover, by (i) of Proposition 5.2, it follows that $\pi \mathbb{P}(A) - h \mathbb{P}(A^c) = 0$. Hence, $\mathbb{P}(A) = h(h+\pi)^{-1}$. Note that if $\mathbb{P}$ fails to be nonatomic, this last argument does not work, since the equation $\mathbb{P}(A) = h(h+\pi)^{-1}$ may not have solutions. By Bauer theorem (see for instance Aliprantis and Border (1994, Theorem 5.118)), we have

$$\Gamma(X) = \max_{Y \in \operatorname{ext} \Gamma^*} \langle Y, X \rangle \leq \max_{Y \in \operatorname{ext} \Gamma^*} |\langle Y, X \rangle| \leq \|X\|_p \max_{Y \in \operatorname{ext} \Gamma^*} \|Y\|_q .$$

But $\|Y\|_q$ equals (6.1) for all $Y \in \operatorname{ext} \Gamma^*$ and this proves the claim. □

Proof of Proposition 2.6. Since by assumption the variance of $D(A)$ is finite, we have $D(A) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\|D(A)\|_2$ its L^2-norm. Let Z be a random variable with mean 0 and variance 1, whose distribution is of the same type as $D(A)$. Set $\Gamma(Z) = k$. By Lemma 6.1, $k \leq \gamma$, hence

$$\begin{aligned} \nu(A) &= \Gamma(D(A)) \\ &= \Gamma[D(A) - \mathbb{E}D(A)] \\ &= k \|D(A) - \mathbb{E}D(A)\|_2 \\ &= k (\operatorname{Var}[D(A)])^{1/2}, \end{aligned}$$

which is also valid if $D(A) = 0$.

As far as the last statement is concerned, it suffices to observe that in this case the set function $A \to \operatorname{Var}[D(A)]$ is additive. Actually, $A \cap B = \varnothing$ implies

$$\begin{aligned} \operatorname{Var}[D(A \cup B)] &= \operatorname{Var}[D(A) + D(B)] \\ &= \operatorname{Var}[D(A)] + \operatorname{Var}[D(B)] + 2 \operatorname{Cov}[D(A), D(B)] \\ &= \operatorname{Var}[D(A)] + \operatorname{Var}[D(B)]. \end{aligned}$$

Since $t \to t^{1/2}$ is concave, it is well-known that $\nu(A) = k (\operatorname{Var}[D(A)])^{1/2}$ is submodular. □

Proof of Proposition 2.7. Representation (2.6) follows easily from From (2.5). If the demands are exchangeable, then $\sigma_{ii} = \sigma^2$ and $\sigma_{ij} = \sigma^2 \rho$ for $i \neq j$. This leads to (2.8). The function $x \to ((1-\rho)x + \rho x^2)^{1/2}$ is concave over $\mathbb{R}_+$, hence these games are submodular, provided the X_i are exchangeable. We need to prove that (2.7) gives an element in the anti-core. Observe that the game $\nu(A) = k(e_A' \Sigma e_A)^{1/2}$ has a natural extension to $[0,1]^n$, given by the function $\widetilde{\nu}(x) = k(x' \Sigma x)^{1/2}$, with $x \in [0,1]^n$ and where a coalitions A is identified with the extremal points e_A of $[0,1]^n$. The function $\widetilde{\nu}(x)$ is convex and linearly homogeneous. $\widetilde{\nu}(x)$ is differentiable, consequently the derivative D of $\widetilde{\nu}(x)$ at $2^{-1}e$, with $e = e_I$, is a subdifferential. By a standard argument (see the proof of Theorem 3.2), the derivative belongs to acore(ν). Straightforward computation leads to $\mathrm{D}\,\widetilde{\nu}(2^{-1}e) = k(e'\Sigma e)^{-1/2}\Sigma e = k'\Sigma e$, which is the desired result. $\square$

Section 3

Proof of Proposition 3.1. This result is well-known, but we provide a proof for sake of completeness. Assume that A is an atom of D and let $D(E) = 0$ for some $E \subseteq A$. As D assumes just the two values 0 and $D(A)$ on the subsets of A, it follows that $|D|(E) = 0$. Hence, $|D|(A \setminus E) = |D|(A)$. Namely, $\int_{A\setminus E} \|\delta\| \ \mathrm{d}\lambda = \int_A \|\delta\| \ \mathrm{d}\lambda \Longrightarrow \int_E \|\delta\| \ \mathrm{d}\lambda = 0 \Longrightarrow \lambda(E) = 0$. Since A is an atom of D, we have $D(A_1) = 0$ or $D(A \setminus A_1) = 0$ for all $A_1 \subseteq A$. By what has been proved, we infer that either $\lambda(A_1) = 0$ or $\lambda(A \setminus A_1) = 0$. As $\lambda(A) > 0$, A would be an atom for λ, which is a contradiction. $\square$

The following technical lemmata are crucial to prove our main theorem. Notice that the functional Γ is clearly weakly lower semicontinuous, since Γ is convex, but it may fail to be weakly continuous over $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 6.2. *The function Γ is weakly continuous when restricted to any relatively norm compact set.*

Proof. Let $K \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ be relatively norm compact and B^* be the unit ball of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

First we prove that the bilinear map $(X,Y) \to \langle X, Y\rangle$ is jointly continuous over $K \times B^*$ where K is endowed with the weak topology and B^* with the weak* topology.

With each $X \in K$, we associate the continuous function $\widehat{X} \in C(B^*)$, defined by $\widehat{X}(Y) = \langle X, Y\rangle$. Observe that the linear map $X \to \widehat{X}$ is an isometry. Actually, by Hanh-Banach

$$\|X\|_{L_1} = \max_{Y\in B^*} |\langle X, Y\rangle| = \max_{Y\in B^*} \left|\widehat{X}(Y)\right| = \left\|\widehat{X}\right\|_\infty,$$

where $\|\cdot\|_\infty$ is the supnorm of $C(B^*)$. Since K is relatively norm compact, so is the image set $\widehat{K}$. By Ascoli-Arzelà's theorem, the family of functions $\widehat{X}$ in $\widehat{K}$ is equicontinuous. Fix $(X_0, Y_0) \in K \times B^*$. We have

$$\begin{aligned} |\langle X, Y\rangle - \langle X_0, Y_0\rangle| &= \left|\widehat{X}(Y) - \widehat{X}_0(Y_0)\right| \\ &\le \left|\widehat{X}(Y) - \widehat{X}(Y_0)\right| + \left|\widehat{X}(Y_0) - \widehat{X}_0(Y_0)\right|. \end{aligned}$$

Given an $\varepsilon > 0$, there exists a weak* neighborhood $U(Y_0)$ of Y_0 such that $\left|\widehat{X}(Y) - \widehat{X}_0(Y_0)\right| \le \varepsilon/2$ for all $X \in K$, due to the equicontinuity of $\widehat{K}$. Further, there is a weak neighborhood $U\left(\widehat{X}_0\right)$ of $\widehat{X}_0$ such that $X \in U\left(\widehat{X}_0\right) \cap K$ implies $\left|\widehat{X}(Y_0) - \widehat{X}_0(Y_0)\right|$. Therefore,

$|\langle X,Y\rangle - \langle X_0,Y_0\rangle| \leq \varepsilon$, for all $(X,Y) \in U\left(\widehat{X}_0\right) \times U(Y_0)$. This proves the continuity of the bilinear function $\langle \cdot,\cdot \rangle$.

Clearly the same properties holds if one replaces $K \times B^*$ by $K \times B^*_\rho$, where $B^*_\rho = \rho B^*$ is the ball with radius ρ. Chose B^*_ρ such that $\Gamma^* \subseteq B^*_\rho$. As the function Γ is a support function, we have

$$\Gamma(X) = \max_{Y \in \Gamma^*} \langle X,Y \rangle, \quad X \in K.$$

Γ turns out to be weakly continuous over K by Berge's maximum theorem (see Aliprantis and Border (1994, Theorem 16.31)). □

Lemma 6.3. *Assume that $A \to D(A)$ is σ-additive. Then:*

(i) *ν is bounded,*

(ii) *ν is continuous,*

(iii) $\mathrm{acore}(\nu) \subset \mathrm{ca}(\mathcal{C})$,

(iv) *there exists a nonnegative real-valued countably additive measure $\overline{\lambda}$ on $(I,\mathcal{C})$ such that* $\mathrm{acore}(\nu) \subset \mathrm{ca}\left(\mathcal{C},\overline{\lambda}\right) \equiv L^1\left(I,\mathcal{C},\overline{\lambda}\right)$

(v) *ν is $\overline{\lambda}$-continuous.*

Proof. (i) As D is σ-additive, D is bounded (see Diestel and Uhl (1977, Cor. 19, p. 9)). Namely, $\|D(A)\| \leq N$ for all $A \in \mathcal{C}$ and for some scalar N. In view of (5.1), we have $0 \leq \nu(A) \leq \gamma N$ and ν is bounded.

(ii) If, for instance, $A_n \uparrow A$, then $\|D(A_n) - D(A)\| \to 0$. Proposition 5.1 implies that $\nu(A_n) \to \nu(A)$.

(iii) It is well known that the anticore of games, continuous at $\varnothing$ and at the grand coalition I, consists of countably additive measures (see Aumann and Shapley (1974, p. 173) or Marinacci and Montrucchio (2004b, Proposition 4.4)).

(iv) By Bartle-Dunford-Schwartz Theorem (see Diestel and Uhl (1977, Cor. 6, p. 14)) there is a positive σ-additive measure $\overline{\lambda}$ such that $\overline{\lambda}(E) \to 0$ iff $\|D\|(E) \to 0$ where $\|D\|$ denotes the semivariation. In particular, we have the implications $\overline{\lambda}(E) = 0 \Longrightarrow \|D\|(E) = 0 \Longrightarrow \|D(E)\| = 0$. If $\mu \in \mathrm{acore}(\nu)$ and $\overline{\lambda}(E) = 0$, then $\mu(E) \leq \Gamma(D(E)) = 0$. Moreover,

$$\mu(E) \geq \Gamma(D(I) - D(E)) - \Gamma(D(I)) = 0.$$

Hence, μ is absolutely continuous with respect $\overline{\lambda}$.

(v) We must prove that for each coalitions N, for which $\overline{\lambda}(N) = 0$, N is ν, i.e., $\nu(F \cup N) = \nu(F)$ for all $F \in \mathcal{C}$. By (iv), $\overline{\lambda}(N) = 0 \Longrightarrow \|D\|(N) = 0 \Longrightarrow D(N_1) = 0$ for all $N_1 \subseteq N$. Hence,

$$\begin{aligned}\nu(F \cup N) &= \Gamma(D(F \cup N)) = \Gamma(D(F \cup N \setminus F)) \\ &= \Gamma(D(F) + D(N \setminus F))) = \Gamma(D(F)) = \nu(F).\end{aligned}$$

□

Lemma 6.4. *Assume that in the newsvendor game $\nu(A) = \Gamma(D(A))$ the demand vector measure satisfies* (3.1). *Then the game ν admits an* na*-continuous extension to the set of the ideal coalitions $B_1(\mathcal{C})$, which is convex and positively homogeneous,*

Proof. Consider the map $T : L^\infty(I, \mathcal{C}, \lambda) \to \mathbb{R}$, given by $f \to \int f \, \mathrm{d}D$. It is well defined, as $\lambda(A) = 0$ implies $D(A) = 0$. By a consequence of Bartle-Dunford-Schwartz's theorem, the map T is a weak*-to-weak continuous linear operator (see Diestel and Uhl (1977, Cor. 7, p. 14)). Restrict this operator to the subset

$$\mathcal{I}^\infty(\lambda) = \{f \in L^\infty(I, \mathcal{C}, \lambda) : 0 \leq f \leq 1, \lambda\text{ -a.e.}\}. \tag{6.2}$$

Clearly, $T(\mathcal{I}^\infty(\lambda))$ is the extended range of the vector measure D. By Uhl's theorem (see Diestel and Uhl (1977, Theorem 10 p. 206)), the extended range in norm compact. By invoking Lemma 6.2, we deduce that the functional $f \to \Gamma\left(\int f \, \mathrm{d}D\right)$ is weak* continuous over $\mathcal{I}^\infty(\lambda)$. Consider the space $B_1(\mathcal{C})$ of the ideal coalitions. As λ is nonatomic, the map $f \to [f]$ from $B_1(\mathcal{C})$ to $\mathcal{I}^\infty(\lambda)$ is na-to-weak* continuous. As a consequence, the functional $\nu^*(f) = \Gamma\left(\int f \, \mathrm{d}D\right)$ is the na-continuous extension of the game ν to the ideal coalitions, is convex and linearly homogeneous. □

Proof of Theorem 3.2. (i) The proof is somewhat related to Einy et al. (1999, Theorem A), although they use dna-continuous extensions and here we exploit the na-extension $\nu^*(f)$ defined over $\mathcal{I}^\infty(\lambda)$, as defined in (6.2) of Lemma 6.4. We think of $\mathcal{I}^\infty(\lambda) \subset L^\infty(I, \mathcal{C}, \lambda)$, endowed with two topologies. The first one is the strong topology of the uniform convergence. The second one is the weak* topology. Consider the subdifferential $\partial\nu^*\left(\frac{1}{2}1_I\right)$ of the convex function $\nu^* : \mathcal{I}^\infty(\lambda) \to \mathbb{R}$ at the point $(1/2)\,1_I$. The elements of $\partial\nu^*\left(\frac{1}{2}1_I\right)$ lie in $(L^\infty(I, \mathcal{C}, \lambda))' = \mathrm{ba}(I, \mathcal{C}, \lambda)$.

If $p \in \partial\nu^*((1/2)\,1_I)$, we have

$$\nu^*(f) \geq (1/2)\,\nu(I) + \langle p, f\rangle - (1/2)\,p(I)$$

for all $f \in \mathcal{I}^\infty(\lambda)$. Setting $f = 0$ and $f = 1_I$, we deduce that $p(I) = \nu(I)$. Setting $f = 1_A$, for any coalition A, we obtain $p(A) \leq \nu(A)$. Consequently, $p \in \mathrm{acore}(\nu)$. By Lemma 6.3, $p \in L^1(I, \mathcal{C}, \lambda)$. Hence, $\partial\nu^*((1/2)\,1_I) \subseteq \mathrm{acore}(\nu) \subset L^1(I, \mathcal{C}, \lambda)$.

We now prove that $\mathrm{acore}(\nu) = \partial\nu^*((1/2)\,1_I)$. Let $m \in \mathrm{acore}(\nu)$. We know that $m \in L^1(I, \mathcal{C}, \lambda)$ and $m(A) \leq \nu(A)$ for all $A \in \mathcal{C}$. Namely, $\langle m, 1_A\rangle \leq \nu^*(1_A)$. Both ν^* and $\langle m, .\rangle$ are w^*-continuous. By Lyapunov theorem (see Kingman and Robertson

(1968)), the indicator functions are weak* dense. Hence $\langle m, f\rangle \leq \nu^*(f)$ holds for all $f \in \mathcal{I}^\infty(\lambda)$. Therefore

$$\nu^*(f) \geq \nu^*\left(\frac{1}{2}1_I\right) + \left\langle m, f - \frac{1}{2}1_I\right\rangle$$

and $m \in \partial\nu^*((1/2)\,1_I)$.

As a last step, we prove that $\partial\nu^*((1/2)\,1_I)$ is a singleton, namely that ν^* is differentiable at $(1/2)\,1_I$. First observe that the functional $\Gamma : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is differentiable at $D(I)$, provided $D(I)$ has continuous distribution. Using a hyperplane separation theorem it is easy to see that $\partial\Gamma(0) = \Gamma^*$ (see Proposition 5.2). This in turn implies that the subdifferential at any point $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is $\partial\Gamma(X) = \{Y \in \Gamma^* : \langle Y, X\rangle = \Gamma(X)\}$. In view of Proposition 5.2, $\partial\Gamma(X)$ is a singleton when X has continuous distribution. Therefore, Γ is Gateaux differentiable at X, with derivative $\mathrm{D}\,\Gamma(X) = \overline{Y}$ given by (5.4).

Now compute the directional derivative of ν^* at $(1/2)\,1_I$, that is

$$\mathrm{D}\,\nu^*((1/2)\,1_I; h) = \lim_{t\to 0^+} \frac{\nu^*((1/2)\,1_I + th) - \nu^*((1/2)\,1_I)}{t}$$

with $h \in L^\infty(I, \mathcal{C}, \lambda)$. Denoting $Tf = \int f \,\mathrm{d}D$ and T^* its transpose, we obtain

$$\begin{aligned}\mathrm{D}\,\nu^*((1/2)\,1_I; h) &= \lim_{t\to 0^+} \frac{\Gamma(D(I) + 2tTh) - \Gamma(D(I))}{2t} \\ &= \langle \overline{Y}, Th\rangle = \langle T^*\overline{Y}, h\rangle,\end{aligned}$$

where $\overline{Y} = \mathrm{D}\,\Gamma(D(I))$. Since the directional derivative $\mathrm{D}\,\nu^*((1/2)\,1_I; h)$ is linear, ν^* is differentiable and $\mathrm{D}\,\nu^*((1/2)\,1_I) = T^*\overline{Y}$. As a consequence, $\partial\nu^*((1/2)\,1_I) = \mathrm{acore}(\nu)$ is a singleton. In view of Theorem 2.2 the element in the anticore is given by (2.4).

(ii) It is well known (see Dunford and Schwartz (1988, Theorem 17, p. 198)) that a perfectly equivalent way of giving a λ-Bochner integrable function $\delta : I \to L^1(\Omega, \mathcal{F}, \mathbb{P})$ is to assign a $\lambda \otimes P$-integrable function $\overline{\delta} : I \times \Omega \to \mathbb{R}$ such that $\delta(i) = \overline{\delta}(i, .)$, λ-a.e., and $\left(\int \delta \,\mathrm{d}\lambda\right)(\omega) = \int \overline{\delta}(i, \omega)\ \mathrm{d}\lambda$, P-a.e. This allows to explicitly write the density of the unique element of the anticore as in (3.2).

□

Lemma 6.5. *The results of Theorem 3.2 are true for any σ-additive, nonatomic, and bounded-variation vector measure $D(A)$ with values in $L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 < p < \infty$.*

Proof. The proof of Theorem 3.2 uses only the fact that the vector measure $D(A)$ takes values in a Banach space X having the Radon-Nikodym property. By Phillips's theorem (see Diestel and Uhl (1977, p. 76)), any space $L^p(\Omega, \mathcal{F}, \mathbb{P})$, with $1 < p < \infty$ has the Radon-Nikodym property. This proves the result. □

Proof of Proposition 3.3. Assume by contradiction the existence of a coalition A, such that $D(A) \neq 0$ and for all $A_1, A_2 \subseteq A$, with $A_1 \cap A_2 = \varnothing$, $D(A_1)$ and $D(A_2)$ are uncorrelated. Consider the restriction ν_A to the coalition A of the game ν. ν_A turns out to be nonatomic and of bounded variations. As D takes values on L^2, we can invoke Lemma 6.5 and so $\text{acore}(\nu_A)$ is a singleton. On the other hand, by Proposition 2.6, ν_A is submodular. If a submodular game has a singleton anticore, it is additive. In view of Proposition 2.6, $\nu_A(B) = k\sqrt{\text{Var}[D(B)]}$, for all $B \subseteq A$. Taking any two coalitions $B \subseteq A$ and $A \setminus B$, we have $\sqrt{\text{Var}[D(A)]} = \sqrt{\text{Var}[D(B)]} + \sqrt{\text{Var}[D(A \setminus B)]}$, which implies either $\text{Var}[D(B)] = 0$ or $\text{Var}[D(A \setminus B)] = 0$. Namely, either $D(B) = 0$ or $D(A \setminus B) = 0$. The coalition A would be an atom, a contradiction. □

Section 4

Proof of Proposition 4.1. In view of (2.6) the games ν_d are defined as

$$\nu_d(A) = \left(\frac{e'_A \Sigma e_A}{e'_{I_d} \Sigma e_{I_d}}\right)^{1/2}$$

for $A \subseteq I_d$. We construct a new sequence of games $\widetilde{\nu}_d$ which is a cover of ν_d. That means a sequence such that $\nu_d \leq \widetilde{\nu}_d$ and $\nu_d(I_d) = \widetilde{\nu}_d(I_d) = 1$. Clearly, $\text{acore}(\nu_d) \subseteq \text{acore}(\widetilde{\nu}_d)$. For each $a \in I_d$, define

$$\rho(a; d) = \max_{i \neq j \in A,\ |A| = a} \rho_{ij}(d).$$

Clearly $\rho(a; d)$ is increasing in a, and $\rho(d; d) \equiv \rho(d)$. The games

$$\begin{aligned} \widetilde{\nu}_d &= \left[\frac{\rho(d)|A|^2 + (1 - \rho(|A|; d)|A|)}{\rho(d) d^2 + (1 - \rho(d) d)}\right]^{1/2} \\ &= \left[\frac{(|A|/d)^2 + \varphi(|A|, d)|A|/d}{1 + \varphi(d, d)}\right]^{1/2}, \end{aligned}$$

where

$$\varphi(|A|, d) = \frac{1 - \rho(|A|; d)}{d\rho(d)},$$

are the desired cover sequence. Notice that under our assumption on $\rho(d)$, $\varphi(|A|, d) \to 0$ as $d \to \infty$, uniformly in $|A|$. To evaluate the asymptotic behavior, it is useful to consider a new sequence of games defined over $[0, 1]$, endowed with the its Borel σ-algebra $\mathcal{B}$. Consider the injective map $i_d : I_d \to [0, 1]$ given by $i_d(a) = a/d$ for $a \in I_d$. These maps induce naturally a family of games in $[0, 1]$ by setting

$$\overline{\nu}_d(E) = i_d^*(\widetilde{\nu}_d)(E) = \widetilde{\nu}_d\left(i_d^{-1}(E)\right)$$

for all Borel set $E \in [0, 1]$. All games $\overline{\nu}_d$ have a finite carrier given by $i_d(I_d)$. Further, it is easy to check that $\text{acore}(\overline{\nu}_d) = i_d^*(\text{acore}(\widetilde{\nu}_d))$, where i_d^* is the usual forward images of measures. Since i_d^* is an isometry, the diameters are preserved.

Denote by λ the Lebsgue measure on the unit interval. Denote by λ_d the measure having mass $1/d$ at each point of I_d. It is easy to write games $\overline{\nu}_d$ by using measures λ_d. In fact,

$$\overline{\nu}_d(E) = \left[\frac{(\lambda_d(E))^2 + \varphi(d\lambda_d(E), d)\lambda_d(E)}{1 + \varphi(d,d)}\right]^{1/2}.$$

The measures λ_d converge weakly to λ. But, more important, we have $\lambda_d(J) \to \lambda(J)$ uniformly over all the intervals of $[0,1]$ (see for instance, Billingsley (1999, Ex. 25.3)). We infer that $\overline{\nu}_d(J) \to \lambda(J)$ uniformly over the intervals as well as over the finite union of intervals. That means that $|\overline{\nu}_d(J) - \lambda(J)| \le \varepsilon$ for all J and $d \ge d(\varepsilon)$.

If $\mu, \mu' \in \text{acore}(\overline{\nu}_d)$, we have $\mu(J) \le \overline{\nu}_d(J)$ and $\mu'(J) \ge 1 - \overline{\nu}_d(J^c)$. Therefore, if $d \ge d(\varepsilon)$, then

$$\begin{aligned}\mu(J) - \mu'(J) &\le \overline{\nu}_d(J) - 1 + \overline{\nu}_d(J^c)\\ &\le \lambda(J) + \varepsilon - 1 + \lambda(J^c) + \varepsilon\\ &= 2\varepsilon.\end{aligned}$$

This implies $|\mu(J) - \mu'(J)| \le 2\varepsilon$ that, in turn, implies that the diameter is less than 2ε. This is the desired result. □

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