# Some Results on *Adjusted Winner*

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June 27, 2005

## 1 Introduction

**Abstract:** *We study the* Adjusted Winner *procedure of Brams and Taylor for dividing goods fairly between two individuals, and prove several results. In particular we show rigorously that as the differences between the two individuals become more acute they both benefit. We study some rather odd knowledge-theoretic properties of strategizing. We introduce a geometic approach which allows us to give alternate proofs of some of the Brams-Taylor results and which gives some hope for understanding the many-agent case also. We also point out that while honesty may not always be the best policy, it* is *as Parikh and Pacuit [PPsv] point out in the context of voting, the only* safe *one. Finally, we also show that provided that the assignments of valuation points are allowed to be real numbers, the final result is a continuous function of the valuations given by the two agents.*

In this paper we study one particular algorithm, or procedure, for settling a dispute between two players over a finite set of goods. The algorithm we are interested in is called *Adjusted Winner* (*AW*) and due to Steven Brams and Alan Taylor [BT1]. Suppose there are two players, called Ann ($A$) and Bob ($B$), and $n$ (divisible[1]) goods ($G_1, \dots, G_n$) which must be distributed to Ann and Bob. The goal of the Adjusted Winner algorithm is to *fairly* distribute the $n$ goods between Ann and Bob. We begin by discussing an example which illustrates the Adjusted Winner algorithm.

Suppose Ann and Bob are dividing three goods: $G_1, G_2$, and $G_3$. *Adjusted Winner* begins

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[1] Actually all we need to assume is that *one* good is divisible. However, since we do not know before the algorithm begins *which* good will be divided, we assume all goods are divisible. See [BT1, BT2] for a discussion of this fact.

by giving both Ann and Bob 100 points to divide among the three goods. Suppose that Ann and Bob assign these points according to the following table.

| Item | Ann | Bob |
|---|---|---|
| $G_1$ | $\underline{10}$ | 7 |
| $G_2$ | $\underline{65}$ | 43 |
| $G_3$ | 25 | $\underline{50}$ |
| **Total** | 100 | 100 |

The first step of the procedure is to give $G_1$ and $G_2$ to Ann since she assigned more points to those items, and item $G_3$ to Bob. However this is not an equitable outcome since Ann has received 75 points while Bob only received 50 points (each according to their personal valuation). We must now transfer some of Ann's goods to Bob. In order to determine which goods should be transfered from Ann to Bob, we look at the ratios of Ann's valuations to Bob's valuations. For $G_1$ the ratio is $10/7 \approx 1.43$ and for $G_2$ the ratio is $65/43 \approx 1.51$. Since 1.43 is less than 1.51, we transfer as much of $G_1$ as needed from Ann to Bob[2] to achieve equitability.

However, even giving all of item $G_1$ to Bob will not create an equitable division since Ann still has 65 points, while Bob has only 57 points. In order to create equitability, we must transfer part of item $G_2$ from Ann to Bob. Let $p$ be the proportion of item $G_2$ that Ann will keep. $p$ should then satisfy

$$65p = 100 - 43p$$

yielding $p = 100/108 = 0.9259$, so Ann will keep 92.59% of item $G_2$ and Bob will get 7.41% of item $G_2$. Thus both Ann and Bob receive 60.185 points. It turns out that this allocation (Ann receives 92.59% of item $G_2$ and Bob receives all of item $G_1$ and item $G_3$ plus 7.41% of item $G_2$) is *envy-free*, *equitable* and *efficient*, or *Pareto optimal*. In fact, Brams and Taylor show that Adjusted Winner *always* produces such an allocation [BT1]. We will discuss these properties in more detail below.

# 2 The Adjusted Winner Procedure

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods, or items. A **valuation** of these goods is a vector of natural numbers $\langle a_1, \ldots, a_n \rangle$ whose sum is 100. Let $\alpha, \alpha', \alpha'', \ldots$ denote possible

[2] When the ratio is closer to 1, a unit gain for Bob costs a smaller loss for Ann.

valuations for Ann and $\beta, \beta', \beta'', \ldots$ denote possible valuations for Bob. An **allocation** is a vector of $n$ real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows. For each $i = 1, \ldots, n$, $s_i$ is the proportion of $G_i$ given to Ann. Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, "Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob." Thus $AW$ can be viewed as a function that accepts Ann's valuation $\alpha$ and Bob's valuation $\beta$ and returns an allocation $\sigma$. It is not hard to see that every allocation produced by $AW$ will have a special form: all components except one will be either 1 or 0.

We now give the details of the procedure. Suppose that Ann and Bob are each given 100 points to distribute among $n$ goods as he/she sees fit. In other words, Ann and Bob each select a valuation, $\alpha = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, \ldots, b_n \rangle$ respectively. For convenience rename the goods so that

$$a_1/b_1 \geq a_2/b_2 \geq \cdots a_r/b_r \geq 1 > a_{r+1}/b_{r+1} \geq \cdots a_n/b_n$$

Let $\alpha/\beta$ be the above vector of real numbers (after renaming of the goods). Notice that this renaming of the goods ensures that Ann, based on her valuation $\alpha$, values the goods $G_1, \ldots, G_r$ at least as much as Bob; and Bob, based on his valuation $\beta$, values the goods $G_{r+1}, \ldots, G_n$ more than Ann does. Then the $AW$ algorithm proceeds as follows:

1. Give all the goods $G_1, \ldots, G_r$ to Ann and $G_{r+1}, \ldots, G_n$ to Bob. Let $X, Y$ be the number of points received by Ann and Bob respectively. Assume for simplicity that $X \geq Y$.
2. If $X = Y$, then stop. Otherwise, transfer a portion of $G_r$ from Ann to Bob which makes $X = Y$. If equitability is not achieved even with all of $G_r$ going to Bob, transfer $G_{r-1}, G_{r-2}, \ldots, G_1$ in that order to Bob until equitability is achieved.

Thus the $AW$ procedure is a function from pairs of valuations to allocations. Let $\mathsf{AW}(\alpha, \beta) = \sigma$ mean that $\sigma$ is the allocation given by the procedure $AW$ when Ann announces valuation $\alpha$ and Bob announces valuation $\underline{e}ta$. In [BT1, BT2], it is argued that $AW$ is a "fair" procedure, where fairness is judged according to the following properties.

Let $\alpha = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, \ldots b_n \rangle$ be valuations for Ann and Bob respectively. An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is

- **Proportional** if both Ann and Bob receive at least 50% of their valuation. That is, $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n}(1-s_i)b_i \geq 50$

- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation. That is, $\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1-s_i)a_i$ and $\sum_{i=1}^{n}(1-s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$.

- **Equitable** if both players receive the same total number of points. That is $\sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n}(1-s_i)b_i$

- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party. That is for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1-s'_i)b_i < \sum_{i=1}^{n}(1-s_i)b_i$. (Similarly for Bob).

In order to simplify notation, let $V_A(\alpha, \sigma)$ be the total number of points Ann receives according to valuation $\alpha$ and allocation $\sigma$ and $V_B(\beta, \sigma)$ the total number of points Bob receives according to valuation $\beta$ and allocation $\sigma$.

It is not hard to see that for two-party disputes, proportionality and envy-freeness are equivalent. For a proof, notice that

$$\sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i(1-s_i) = \sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i s_i = 100$$

Then if $\sigma$ is envy free for Ann, then $\sum_{i=1}^{n} a_i s_i \geq \sum_{i=1}^{n} a_i(1-s_i)$. Hence, $2\sum_{i=1}^{n} a_i s_i \geq \sum_{i=1}^{n} a_i = 100$. And so, $\sum_{i=1}^{n} a_i s_i \geq 50$. The argument is similar for Bob.
Conversely, suppose that $\sigma$ is proportional. Then since $\sum_{i=1}^{n} a_i s_i \geq 50$, $\sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i s_i \geq 100 = \sum_{i=1}^{n} a_i$. Then $\sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} a_i s_i - \sum_{i=1}^{n} a_i \geq 0$. Hence, $\sum_{i=1}^{n} a_i s_i - \sum_{i=1}^{n} a_i(1-s_i) \geq 0$. And so, $\sum_{i=1}^{n} a_i s_i \geq \sum_{i=1}^{n} a_i(1-s_i)$. The proof is similar for Bob.

Returning to $AW$, it is easy to see the $AW$ only produces equitable allocations (equitability is essentially built in to the procedure). Brams and Taylor go on to show that $AW$, in fact, satisfies all of the above properties.

**Theorem 1 (Brams and Taylor [BT1])** *AW produces an allocation of the goods based on the announced valuations that is efficient, equitable and envy-free.*

A formal proof of this Theorem is provided in [BT1].

## 2.1 The Proportional-Allocation Procedure

In this section we briefly discuss a procedure related to AW called Proportional-Allocation ($PA$). $PA$, as the name implies, allocates goods proportionally. As before, assume there are $n$ goods $G_1, \ldots, G_n$ and assume that Ann announces a valuation $\alpha = \langle a_1, \ldots, a_n \rangle$ and Bob announces a valuation $\beta = \langle b_1, \ldots, b_n \rangle$. For simplicity suppose that for each $i$, either $a_i \neq 0$ or $b_i \neq 0$. Then under $PA$ Ann is allocated the fraction $a_i/(a_i + b_i)$ of $G_i$, and Bob the fraction $b_i/(a_i + b_i)$.

For an example, consider the distribution of points from the introduction:

| Item | Ann | Bob |
|---|---|---|
| $G_1$ | $\underline{10}$ | 7 |
| $G_2$ | $\underline{65}$ | 43 |
| $G_3$ | 25 | $\underline{50}$ |
| **Total** | 100 | 100 |

Under $PA$, Ann is awarded 10/17 of $G_1$, 65/108 of $G_2$, and 25/75 of $G_3$ giving her a total of 53.33 points. Bob also receives 53.33 points. Recall that under $AW$ both Ann and Bob would be awarded 60.185 points. Thus, $PA$ is not efficient but it is equitable and envy-free.

**Theorem 2 (Brams and Taylor [BT1])** *$PA$ produces an allocation of the goods based on the announced values that is equitable and envy-free.*

The principal advantage of $PA$ over $AW$ is that it discourages (unilateral) departures from truthfulness. Refer to [BT1] for an extended discussion. Note that the effect of AW *could* be achieved by PA if Ann and Bob co-operate. Suppose Bob 'cedes' $r$ items to Ann, allocating 0 points to these, Ann 'cedes' $n - r - 1$ items to Bob, again allocating 0 points to these, and they both lay claim to the one remaining item in appropriate proportions. In that case PA will give the same result as AW does.

# 3 A Geometrical Interpretation of $AW$

In this section and the one on continuity, it will be useful to think of both valuations and allocations as vectors in $n$-space, and to use vector notation where such notation will assist our geometric intuition.

Notice that the $AW$ procedure only produces allocations in which all components, except possibly one, are either 1 or 0. In this section, we show that this is not an accident. We will be working in $\mathbb{R}^k$ for $k \geq 1$. An allocation is a vector $\vec{x} \in \mathbb{R}^k$ where each component is a non-negative real less than or equal to 1. Thus the set of all possible allocations is a hypercube in $\mathbb{R}^k$. Let $\mathcal{C}_k = \{\vec{x} \mid \forall i \;\; 0 \leq x_i \leq 1\}$ be this hypercube of dimension $k$ (we will leave out the $k$ when possible).

A **valuation** is a vector $\vec{P} \in \mathbb{R}^k$ where $\sum_{i=1}^k P_i = 100$. Let $\cdot$ denote the dot product, that is $\vec{x} \cdot \vec{P} = \sum_{i=1}^k x_i P_i$. Now, let $\vec{P}$ and $\vec{Q}$ be two fixed vectors (Ann's valuation and Bob's valuation). As we want to ensure that Ann and Bob both receive the same valuation, we are interested in the hyperplane $\mathcal{H}_{\vec{P},\vec{Q}}$ generated by the following equation

$$\vec{x} \cdot \vec{P} = (\vec{1} - \vec{x}) \cdot \vec{Q}$$

Since $\vec{1} \cdot \vec{Q} = 100$, we have

$$\vec{x} \cdot (\vec{P} + \vec{Q}) = \vec{x} \cdot (\vec{Q} + \vec{P}) = \vec{x} \cdot \vec{Q} + (\vec{1} - \vec{x}) \cdot \vec{Q} = \vec{1} \cdot \vec{Q} = 100$$

Thus $\mathcal{H}_{\vec{P},\vec{Q}} = \{\vec{x} \mid \vec{x} \cdot (\vec{P} + \vec{Q}) = 100\}$. Again we will leave out the subscripts when possible.

For a fixed $\vec{P}$ and $\vec{Q}$, wanting efficency, we can ask for the allocations $\vec{x}$ that maximize $\vec{x} \cdot \vec{P}$ (subject to the above constraints): Let $\mathcal{I} = \mathcal{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$. Define the function $f : \mathcal{I} \to \mathbb{R}$ by $f(\vec{x}) = \vec{x} \cdot \vec{P}$. Then, since $\mathcal{I}$ is a closed and bounded subset of $\mathbb{R}^k$ (hence compact by the Heine-Borel Theorem), $f$ has a maximum value on $\mathcal{I} = \mathcal{C}_k \cap \mathcal{H}_{\vec{P},\vec{Q}}$. Let $m$ be this maximum value, so that for each $\vec{x} \in \mathcal{I}$, $f(\vec{x}) \leq m$ and the set $\mathcal{M} = \{\vec{x} \mid f(\vec{x}) = m\} \neq \emptyset$.

We claim that there is a point of $\mathcal{M}$ which lies on an edge of the hypercube $\mathcal{C}_k$. More formally,

**Theorem 3** *There is a point $\vec{x} \in \mathcal{M}$ with all components either 1 or 0 except possibly one. I.e., $\exists j$ such that $\forall i$, if $i \neq j$ then $x_i = 1$ or $x_i = 0$.*

**Proof** We will show that
$(*)$ if $\vec{x} \in \mathcal{M}$ with $0 < x_i < 1$ and $0 < x_j < 1$ for $i \neq j$, then there is a point $\vec{x}' \in \mathcal{M}$ with $x_l = x'_l$ for all $l \neq i, j$ and either $x'_i = 1$ or $x'_j = 1$.

To see that this statement implies the theorem, take an arbitrary element $\vec{x} \in \mathcal{M}$ (such an element exists since $\mathcal{M}$ is nonempty). Now, each time that $(*)$ is used, the number of strictly fractional components (not 0 or 1) decreases by one. Thus when we are finished there will be at most one fractional component left.

To prove $(*)$ WLOG we may assume that $i = 1$ and $j = 2$. Thus we have

$$x_1P_1 + x_2P_2 + \sum_{i=3}^{k} x_iP_i = m$$

where $m$ is the maximum of the function $f$. Now we must show that either there is $0 \leq x_1' \leq 1$

$$x_1'P_1 + P_2 + \sum_{i=3}^{k} x_iP_i = m$$

or there is $0 \leq x_2' \leq 1$ such that

$$P_1 + x_2'P_2 + \sum_{i=3}^{k} x_iP_i = m$$

Now if we set $x_1' = \frac{x_1P_1+x_2P_2-P_2}{P_1}$, and $x_2' = 1$ then it is not hard to see that $x_1'P_1 + P_2 + \sum_{i=3}^{k} x_iP_i = m$. Similarly, if we set $x_2'' = \frac{x_1P_1+x_2P_2-P_1}{P_2}$ and $x_1'' = 1$. But to show that one of the other of these assignments work, we still need to show that either $0 \leq x_1' \leq 1$ or $0 \leq x_2'' \leq 1$.

Since $x_1$ and $x_2$ are both between 0 and 1, $x_1P_1 + x_2P_2 < P_1 + P_2$. Thus using basic algebra, $x_1' < 1$ and $x_2'' < 1$.

Suppose that $x_1' < 0$ and $x_2'' < 0$. Then since $P_1$ and $P_2$ are both positive real numbers, $x_1P_1 + x_2P_2 - P_2 < 0$ and $x_1P_1 + x_2P_2 - P_1 < 0$. Therefore, $x_1P_1 + x_2P_2 < P_2$ and $x_1P_1 + x_2P_x < P_1$ and so $x_1P_1 + x_2P_2 < \frac{1}{2}P_1 + \frac{1}{2}P_2$ . Thus

$$\frac{1}{2}P_1 + \frac{1}{2}P_2 + \sum_{i=3}^{k} x_iP_i > x_1P_1 + x_2P_2 + \sum_{i=3}^{k} x_iP_i = m$$

which is a contradiction since we could clearly have used $\frac{1}{2}, \frac{1}{2}$ as our values, and $m$ is the maximum.

$\square$

# 4 Continuity

In this section we will think of the $\mathsf{AW}$ as a function that takes two vectors of *real* numbers and returns a real number. Our goal is to show that $\mathsf{AW}$ is continuous in both vectors.

Assume there are $k$ goods. Suppose that $\alpha$ is Ann's valuation, $\beta$ is Bob's valuation and $\sigma$ is the allocation produced by $AW$ (that is $\mathsf{AW}(\alpha, \beta) = \sigma$). Let $r$ be the ratio $a_i/b_i$ where $G_i$ is the item that is divided by the procedure. Define $I = \{l \mid a_l/b_l = r\}$, i.e., $I$ is the set of indices of the goods that have the same ratio as the item which is divided by the procedure.

**Lemma 4** *Suppose that $\alpha, \beta, \sigma$ and $I$ are defined as above. Suppose that $y_1, y_2, y_3$ where $y_2$ is Ann's value of the item being split and $y_1, y_3$ are the totals of Ann's values of all other items in $I$. Suppose that we choose another item from $I$ to split, call this allocation $\sigma'$. Say $z_1, z_2, z_3$ are integers where $z_2$ is Ann's value of the (new) item being split and $z_1, z_3$ are the totals of Ann's values for all other items in $I$. Then $V_A(\alpha, \sigma) = V_A(\alpha, \sigma')$, i.e., Ann (and hence Bob) receives the same number of points.*

**Proof** Let $X$ be the value of allocation out side $I$ that will be allocated to Bob by his valuation. Let $Y$ be the value of allocation out side $I$ that will be allocated to Ann by her valuation. Then

$$V_A(\alpha, \sigma) = X + ry_1 + pry_2 = Y + y_3 + (1-p)y_2$$

where $p$ is the percentage that Bob will get from the item that correspond to $y_2$. On the other hand

$$V_A(\alpha, \sigma') = X + rz_1 + qrz_2 = Y + z_3 + (1-q)z_2$$

where $q$ is the percentage that Bob will get from the item that correspond to $z_2$. Also note that $y_1 + y_2 + y_3 = z_1 + z_2 + z_3$. Let $S = y_1 + y_2 + y_3$.

Let $A = ry_1 + pry_2$ and let $B = y_3 + (1-p)y_2$ then $A/r + B = S$ and that gives us $A = r(S - B)$. Substitute in the above equation we get $V_A(\alpha, \sigma) = X + r(S - B) = Y + B$ then $(Y + B)(1 + r) = X + rS + rY$ and that give us $V_A(\alpha, \sigma) = Y + B = (X + rS + rY)/(1 + r)$.

In a similar argument, Let $A' = ry_1 + pry_2$ and let $B' = y_3 + (1-p)y_2$ then $A'/r + B' = S$ and that gives us $A' = r(S - B')$. Substitute in the above equation we get $V_A(\alpha, \sigma') = X + r(S - B') = Y + B'$ then $(Y + B')(1 + r) = X + rS + rY$ and that give us $V_A(\alpha, \sigma) = Y + B' = (X + rS + rY)/(1 + r)$. Thus we $V_A(\alpha, \sigma) = V_A(\alpha, \sigma')$.

$\square$

# 5 The Distance Between Announced Allocations

In this section we formalize the intuition that the more the valuations differ, the more points each agent will receive. Since $AW$ only produces equitable allocations, we can think of the function $\mathsf{AW}$ as a function from pairs of valuations to real numbers. Let $V_{AW}(\alpha, \beta)$ denote the total points that $AW$ allocates to each agent – say Ann, (according to the announced valuations $\alpha$ and $\beta$). Formally, $V_{AW}(\alpha, \beta)$ is defined to be $V_A(\alpha, \mathsf{AW}(\alpha, \beta))$. Of course, we coild define it in terms of Bob's valuation, but they are equal so it does not matter which definition is used.

Given an allocation $\alpha$ for Ann, if Ann increases any component then she must decrease another component as the sum of the components must be 100. Now if Ann wants to accentuate the difference between her allocation and Bob's allocation, then she will only increase points on goods that she values more than Bob. Let $\alpha, \alpha'$ and $\beta, \beta'$ be two valuations for Ann and Bob, respectively. We say that $(\alpha, \beta) \prec^A_{ij} (\alpha', \beta')$ if

1. $\beta = \beta'$
2. $\alpha_i > \beta_i$, $\alpha_j < \beta_j$, $\alpha'_i = \alpha_i + 1$ and $\alpha'_j = \alpha_j - 1$.
3. for all $k \neq i, j$, $\alpha'_k = \alpha_k$

Similarly, we define $\prec^B_{ij}$ with respect to Bob's valuation. The intuition is that if $(\alpha, \beta) \prec^A_{ij} (\alpha', \beta')$, then the pair $(\alpha', \beta')$ represents a situation in which Ann has "increased" by 1 unit the difference between $\alpha$ and $\beta$. We say $(\alpha, \beta) \prec (\alpha', \beta')$ if there is a sequence of pairs of valuations linearly ordered by the $\prec^A_{ij}, \prec^B_{ij}$ relations (with varying $i, j$) that begins with $(\alpha, \beta)$ and ends with $(\alpha', \beta')$. Thus $\prec$ is the transitive closure of the **union** of the relations $\prec^A_{ij}$ and $\prec^B_{ij}$. It is not hard to see that $\prec$ is a (non-reflexive) partial order. The main theorem of this section is

**Theorem 5** *If $(\alpha, \beta) \prec (\alpha', \beta')$, then $V_{AW}(\alpha, \beta) < V_{AW}(\alpha', \beta')$.*

Before proving this theorem we will prove a number of facts that will turn out to be useful throughout the paper.

**Lemma 6** *If* $\alpha = \beta$ *then* $V_{AW}(\alpha, \beta) = 50$

**Proof** Suppose that $\alpha = \beta$. Let $G_1, G_2, \ldots$ be the order of goods induced by the *AW* procedure. Now the *AW* procedure will distribute the goods so that

$$a_1 + a_2 + \cdots + pa_r = (1-p)b_r + b_{r+1} + \cdots b_n$$

Since $\alpha = \beta$, for each $j = r, \ldots, n$, $b_j = a_j$. Hence, we have

$$a_1 + a_2 + \cdots + pa_r = (1-p)a_r + a_{r+1} + \cdots a_n$$

Now, since $\sum_{i=1}^{n} a_i = 100$,

$$a_1 + a_2 + \cdots + pa_r = (1-p)a_r + 100 - (a_1 + \cdots + a_r)$$

Thus $2(a_1 + a_2 + \cdots + pa_r) = 100$ and so $a_1 + \cdots + pa_r = 50$. Hence, $V_{AW}(\alpha, \beta) = 50$.

$\square$

**Lemma 7** *If* $V_{AW}(\alpha, \beta) = 50$*, then* $\alpha = \beta$*.*

**Proof** Suppose that $V_{AW}(\alpha, \beta) = 50$. Suppose that $\alpha \neq \beta$. Then there exist $i$ and $j$ such that $a_i > b_i$ and $a_j < b_j$. The *AW* procedure produces an allocation where (after renaming the goods)

$$a_1 + \cdots + pa_r = (1-p)b_r + \cdots + b_n = 50$$

Furthermore, the procedure ensures that $i \leq r$. WLOG we can assume $i = 1$ by simply choosing the $i$ that maximizes the ratio $a_i/b_i$. Using basic algebra, we have

$$a_1 + a_2 + \cdots + a_{r-1} + b_{r+1} + b_{r+2} + \cdots b_n = 100 - pa_r - (1-p)b_r$$

Since $a_1 > b_1$ and for each $k = 2, \ldots, r-1$, $a_k \geq b_k$, we have

$$100 - pa_r - (1-p)b_r = a_1 + a_2 + \cdots + a_{r-1} + b_{r+1} + b_{r+2} + \cdots b_n > b_1 + b_2 + \cdots + b_{r-1} + b_{r+1} + \cdots + b_n$$

Hence,

$$100 - pa_r - pb_r > b_1 + b_2 + \cdots b_n = 100$$

This is a contradiction since $p, a_r, b_r > 0$.

□

**Lemma 8** *For all $\alpha, \beta$, $V_{AW}(\alpha, \beta) \geq 50$.*

**Proof** Suppose not. That is suppose that $V_{AW}(\alpha, \beta) < 50$. Then the goods can be reordered so that

$$a_1 + \cdots + pa_r = (1-p)b_r + \cdots + b_n < 50$$

Hence $a_1 + \cdots + pa_r + (1-p)b_r + \cdots + b_n < 100$. Now since for each $j = 1, \ldots, r$, $a_j \geq b_j$, we have

$$100 > a_1 + \cdots + pa_r + (1-p)b_r + \cdots + b_n + pa_r + (1-p)b_r + \cdots + b_n \geq b_1 + \cdots + pb_r + (1-p)b_r + \cdots + b_n$$

This is a contradiction since $b_1 + \cdots + pb_r + (1-p)b_r + \cdots + b_n = 100$.

□

These lemmas each show that the $AW$ procedure produces proportional and hence envy-free allocations.

We return to the proof of the main theorem of this section (Theorem 5). The proof of the theorem is an easy consequence of the following fact.

**Lemma 9** *Suppose that $(\alpha, \beta) \prec^A_{ij} (\alpha', \beta')$, then $V_A(\alpha, \mathsf{AW}(\alpha, \beta)) < V_A(\alpha', \mathsf{AW}(\alpha', \beta'))$.*

**Proof** To see this, note that when Ann increased some valuations by 1, where it already exceeded Bob's valuation for that item, then she gets that item in the initial allocation both before this change and after the change. Hence Ann receives more points in her first allocation, and Bob must be compensated for this fact in the final allocation. Thus Bob's final score will increase. But since both Ann and Bob receive the same final score, they will both benefit. We postpone the details and the arithmetic to the final version of the paper.

□

# 6 Strategizing

In this section we consider the question of whether Ann can improve her total allocation by misrepresenting her preferences. It turns out that she *can* improve her allocation. The following example from [BT1] illustrates how Ann can deceive Bob. Suppose that Ann and Bob are dividing two paintings: one by Matisse and one by Picasso. Suppose that Ann and Bob's actual valuations are given by the following table.

| Item | Ann | Bob |
|---|---|---|
| Matisse | 75 | 25 |
| Picasso | 25 | 75 |

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

But now suppose Ann knows Bob's preferences, but Bob does not know Ann's. Can Ann benefit from being insincere? Suppose that Ann announces the following allocation:

| Item | Ann | Bob |
|---|---|---|
| Matisse | 26 | 25 |
| Picasso | 74 | 75 |

So Ann will get the Matisse, receiving 26 of her announced (and insincere) points and Bob gets 75 of his announced points. Let $x$ be the fraction of the Picasso that Ann will get, then we want

$$26 + 74p = 75 - 75p$$

Solving for $p$ gives us $p = 0.33$ and each gets 50 of his or her announced preference. In terms of Ann's *true* preference, however, the situation is very different. She is getting from her true preference $75 + 0.33 * 25 = 83.33$.

Suppose *both* players know each other's preferences but neither knows that the other knows their own. Their announced point allocations might then be as follows:

| Item | Ann | Bob |
|---|---|---|
| Matisse | 26 | 74 |
| Picasso | 74 | 26 |

Each will get 74 of his or her announced points, but each one is really getting only 25 of his or her *true* points. The following theorem of Brams and Taylor describes the situation when agents divide two goods.

**Theorem 10 (Brams and Taylor [BT1])** *Assume there are two goods, $G_1$ and $G_2$, all true and announced values are restricted to integers, and suppose Bob's announced valuation of $G_1$ is $x$, where $x \geq 50$. Assume Ann true valuation of $G_1$ is $b$. Then her optimal announced valuation of $G_1$ is:*

$$\begin{cases} x+1 & \text{if } b > x \\ x & \text{if } b = x \\ x-1 & \text{if } b < x \end{cases}$$

**Corollary 11** *Assume all true and announced valuations are restricted to the integers and suppose Bob's true valuation of $G_1$ is $b$ and Ann true valuation of $G_1$ is $a$ and $a > b$. Then a Nash equilibrium is the following ordered pairs of announced valuation for $G_1$ by Bob and Ann:*
$(x+1, x)$ *if* $b < x < a-1$
$(a, a)$ *if* $a = b$

Suppose *both* players know each other's preferences. Moreover, Ann knows that Bob knows her preference and Bob doesn't know that Ann knows, then the announced allocation will be as follows:

| Item | Ann | Bob |
|---|---|---|
| Matisse | 73 | 74 |
| Picasso | 27 | 26 |

Now suppose they both know each other's preference and each know that the other person knows his or her preference. Then the announced valuations will be:

| Item | Ann | Bob |
|---|---|---|
| Matisse | 73 | 27 |
| Picasso | 27 | 73 |

What happens as the level of knowledge increases?

# 7 Honesty is Safety

In this section we point out that while honesty may not always be the best policy, it is the only safe one; i.e., it is the only one which will guarantee 50%.

For suppose that Ann's actual valuation is $(a_1, ..., a_n)$ but she reports $(c_1, ..., c_n)$. We show how she can end up with less than 50%. Suppose that Bob also reported $(c_1, ..., c_n)$. We know that in that case both Ann and Bob would get *exactly* 50% of their declared valuations. So Ann would receive 50% according to her *declared* valuation and this might be different from her actual valuation. To see how it might be *less* consider the eventuality that Bob reports slightly more than $c_i$ when $c_i < a_i$ and slightly less than $c_i$ when $c_i > a_i$. In the initial allocation then Bob will get all the pieces where Ann's declared valuation is less than her actual valuation, and Ann will get those where it is more. There will be adjustments of course, but Ann will still tend to get pieces where her declared valuation is *more* than her actual valuation. If she gets (approximately) 50% by her declared valuation, then it will be *less than* 50% by her actual valuation.

Thus she can lose out by being dishonest (unless of course she *knows* something about Bob's declared values).

# 8 More Than Two Players

In this section we discuss the situation when there are more than two players.

This example was given by two Dutch mathematicians J. H. Reijnierse and J. A. M. Potters.

| Items | Ann | Bob | Nan |
|---|---|---|---|
| X | 40 | 30 | 30 |
| Y | 50 | 40 | 30 |
| Z | 10 | 30 | 40 |

The only efficient and equitable allocation turns out to be give $X$ to Ann, $Y$ to Bob, and $Z$ to Nan. Obviously, this 40-40-40 allocation is equitable; it can be shown to be efficient. But it is not envy-free. Obviously Ann prefers $Y$, which went to Bob, to $X$, which she herself got.

# References


[BT1] Steven Brams and Alan Taylor, *Fair Division*, Cambridge University Press, 1996.

[BT2] Steven Brams and Alan Taylor, *The Win-Win Solution: Guaranteeing Fair Shares to Everybody*, W. W. Norton & Company, New York, 1999.

[PPsv] Rohit Parikh and Eric Pacuit, "Safe votes, sincere votes, and strategizing", presented at the *Workshop on Uncertainty in Economics*, Singapore 2005.