Smooth Ex-Post Implementation with Multi-Dimensional Information

Claudio Mezzetti

Sérgio O. Parreiras

Department of Economics, UNC at Chapel Hill

sergiop@unc.edu

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Abstract

This paper provides sufficient conditions for ex-post implementation of social choice rules. The main feature of our approach is that the set of outcomes of the social choice function include randomization of alternatives and attention is restrict attention to smooth, regular social choice functions.

1 Introduction

We provide sufficient conditions for ex-post implementation of social choice rules. The set of outcomes of the social choice function include randomization of alternatives and we restrict attention to smooth, regular social choice functions. This allows us to use the differential approach of Laffont and Maskin [3, 4], which, for instance, allows an elementary proof of [2]'s impossibility result for the class of social choice rules we consider.

The work closest to our is McAfee and McMillan, [7] who study one-agent, multi-dimensional, incentive-compatible mechanisms. For one-agent mechanisms, interim-incentive compatibility is equivalent to ex-post incentive compatibility. Their work also characterizes transfers as a solution of a system of partial differential equations PDEs, however conditions for the existence of the solution of the system are not provided.conditions.

Another related paper is Manelli and Vicent, [5]. They focus on mechanisms where, given the report of the agents, the probability that any alternative is chosen is either zero or one. In contrast, we restrict attention to social choice functions where the probability of an alternative being chosen is positive for almost all reports.

2 The Model

There are N agents and each agent i receives a private signal $s_i \in S_i$ where,

Assumption 1 *The set $S_i \subset \mathbb{R}^{d_i}$ is a compact, convex, d_i-dimensional manifold with boundary. The interior of S_i is denoted by S_i°.*

There are $K+1$ social alternatives. A social choice rule, $\psi : \prod_{k=0}^N S_i \longrightarrow \triangle^K$, maps the agents' signals into outcomes, which are probability distributions over alternatives, $\triangle^K = \left\{\psi \in \mathbb{R}^K : \psi^k \geq 0, \sum_{k=1}^K \psi^k \leq 1\right\}$.

Agents preferences, $U^i(s, \tau, \xi)$, over outcomes $\xi \in \triangle^K$, and transfers $\tau \in \mathbb{R}$, may depend on the profile of signals, s.

Assumption 2 *Quasi-linear preferences: $U^i : \prod_{i=1}^N S_i \times \mathbb{R} \times \triangle^K \longrightarrow \mathbb{R}$ is given by $U^i(s, \tau, \xi) = u^i(s, \xi) - \tau$.*

Assumption 3 *Von-Neuman–Morgenstern preferences:*
$u^i(s, \xi) = \xi^\top V^i(s) = \sum_{k=1}^K \xi_k V_k^i(s)$,

where $V^i(s) = \left(V_1^i(s), ..., V_k^i(s)\right)$, $V_k^i(s)$ is the utility agent i gets when alternative k is chosen, and ξ_k is the probability of k. The utility of alternative zero, the *status-quo,*, is normalized to zero without any loss of generality, $V_0^i(s)$ for all i and s.

In a single-object auction without externalities: If $i \neq k$, $V_k^i(s)$ is zero since the object is not allocated to agent i, while $V_i^i(s)$ is the agent's valuation for the object.

Taxation Principle. Transfers in a ex-post incentive compatible direct revelation mechanism do not depend directly on the agent's report. They only depend on the agent's report throughout the decision rule; that is ψ: $\psi(s_i, s_{-i}) = \psi(\widehat{s}_i, s_{-i})$ implies $t^i(s_i, s_{-i}) = t^i(\widehat{s}_i, s_{-i})$. Therefore, we write to transfer t^i as a function of the others' agents reports s_{-i}, and the outcome induced by truthful reporting, ψ: $t^i(\psi, s_{-i})$.

Definition 4 *The choice rule ψ is* ex-post incentive compatible *(*EPIC*) if for any agent i there is a transfer function $t^i : \triangle^K \times \prod_{j \neq i} S^j \longrightarrow \mathbb{R}$ such that $\forall \widehat{s}_i \in S_i$:*

$$\begin{aligned} & \psi(s_i, s_{-i})^\top V^i(s_i, s_{-i}) - t^i\left(\psi(s_i, s_{-i}), s_{-i}\right) \geq \\ \geq \; & \psi(\widehat{s}_i, s_{-i})^\top V^i(s_i, s_{-i}) - t^i(\psi(\widehat{s}_i, s_{-i}), s_{-i}) \end{aligned} \tag{1}$$

Remark 5 *Without any loss of generality, alternative zero is the* status-quo*: all agents utility for at status-quo is zero regardless of their signals.*

This paper restricts attention to smooth environments.

Assumption 6 *The social choice rule, utilities and transfers are infinitely differentiable: $\psi(\cdot, s_{-i}) \in C^\infty(S_i)$, $V^i(\cdot, s_{-i}) \in C^\infty(S_i)$ and $t^i(\cdot, s_{-i}) \in C^\infty(\triangle^K)$ for all i and s_{-i}.*

Any piecewise constant function, for instance the efficient rule, can be approximated by smooth functions. Consequently, the restriction to smooth rules still allows for virtually efficient rules.

The first-order condition for truth-telling of an interior type $s_i \in S_i^\circ$ is,

$$\nabla_{s_i}\psi(s)\left[V^i(s) - \nabla_\psi t^i(\psi, s_{-i})\right] = 0. \tag{2}$$

Later, we analyze

Definition 7 *We say that ψ is full-rank at s if, for every agent i, the matrix $\nabla_{s_i}\psi(s)$ has full rank. The rule ψ is full-rank if it is full-rank in an open and dense subset of S.*

In this paper, we restrict our attention to social choice functions which are full-ranked. Unless stated, any social choice function is full-rank.

When ψ is regular (full-rank and the dimension of the agent's signal space is greater than or equal to the dimension of the set of alternatives), $d_i \geq K$, (2) is equivalent to:

$$V^i(s) = \nabla_\psi\, t^i(\psi, s_{-i}). \tag{3}$$

In the case where ψ is not full-rank or the dimension of the agent's signal space is smaller than the dimension of the set of alternatives, $d_i < K$, (3) is *sufficient* for the first-order conditions (2) to hold.

Lemma 8 *The sufficient condition (3) for the first-order conditions (2) implies that*

$$\psi(s_i, s_{-i}) = \psi(\widehat{s}_i, s_{-i}) \Rightarrow V^i(s_i, s_{-i}) = V^i(\widehat{s}_i, s_{-i}). \tag{4}$$

Proof. By the Taxation Principle, two signals that induce the same choice over alternatives must be associated to the same transfer, $t^i(\psi(s_i, s_{-i}), s_{-i}) = t^i(\psi(\widehat{s}_i, s_{-i}), s_{-i})$. Therefore $\nabla_\psi\, t^i(\psi(s_i, s_{-i}), s_{-i}) = \nabla_\psi\, t^i(\psi(s_i, \widehat{s}_{-i}), s_{-i})$ and the result follows immediately from (3) ■

3 Ex-Post Implementation

3.1 Transfers

Lemma 9 *If, for all i and s_{-i}, the diagram (5) commutes, that is, there exists H^i, continuous on S_i and smooth on S_i°, such that $H^i(\psi(s), s_{-i}) = V^i(s)$ and the integrability condition, $\nabla_\psi H^i$ is symmetric, is satisfied, then ψ satisfies the first-order condition for* EPIC*.*

$$\begin{tikzcd} & V^i(S_i, s_{-i}) \subset \mathbb{R}^K \\ S_i \subset \mathbb{R}^{d_i} \arrow[ur, "V^i(\cdot, s_{-i})"] \arrow[r, "\psi(\cdot, s_{-i})"'] & \psi(S_i, s_{-i}) \subset \triangle^K \arrow[u, "H^i(\cdot, s_{-i})"'] \end{tikzcd} \tag{5}$$

Proof. Since ψ is continuous and S_i is compact, $\psi(S_i, s_{-i})$ is compact. Moreover, since H^i is continuous and $\psi(S_i, s_{-i})$ is compact and the integrability restrictions are satisfied, by Frobënius Theorem [1], the system of partial differential equations $\nabla_\psi t^i = H^i(\psi)$ has a unique local solution for any arbitrary initial condition. Finally, since $\psi(S_i, s_{-i})$ is compact, the solution can be extended to a global solution. ■

In order to prove that the mapping H^I is well-defined and smooth, we consider two cases: The first case, when $d_i \leq K$ is covered by Lemma 10. The second case, when $d_i \geq K$ is covered by Lemma 11.

Lemma 10 *If $\psi(\cdot, s_{-i})$ is one-to-one then there exists smooth H^i satisfying (5).*

Proof. When $\psi(\cdot, s_{-i})$ is one-to-one, it is possible to define $H^i : \psi(S_i, s_{-i}) \longrightarrow V^i(S_i, s_{-i})$ such that $H^i(\psi(s)) = V^i(s)$. Moreover, if $\psi(\cdot, s_{-i})$ is one-to-one then $d_i \leq K$ and in this case, $\psi(\cdot, s_{-i})$ is an immersion. Moreover, since S_i is compact, $\psi(\cdot, s_{-i})$ is an embedding and so its image, $\psi(S_i, s_{-i})$, is a manifold.

By the local form of immersions, for any $s_i \in S_i^\circ$ there are open neighborhoods: $X \ni s_i$; $Y \ni \psi(s)$; $\mathbb{R}^{d_i} \supset W \ni 0$; $\mathbb{R}^K \supset Z \ni 0$; and parameterizations α and β such that $\alpha(0) = s_i$, $\beta(0) = \psi(s)$ and the diagram below, where $\pi(w_1, ..., w_{d_i}) = (w_1, ..., w_{d_i}, 0, ..., 0)$, commutes:

$$
\begin{tikzcd}
 & V^i(S_i, s_{-i}) \subset \mathbb{R}^K \\
X \subset S_i \arrow[ur, "V^i(\cdot, s_{-i})"] \arrow[r, "\psi(\cdot, s_{-i})"'] & Y \subset \psi(S_i, s_{-i}) \arrow[u, "H^i(\cdot, s_{-i})"'] \\
W \arrow[u, "\alpha"] \arrow[r, "\pi"] & Z \arrow[u, "\beta"']
\end{tikzcd}
\tag{6}
$$

Notice that $H^i|_Y = V^i(\cdot, s_{-i}) \circ \alpha \circ \pi^{-1} \circ \beta^{-1}$. The function V^i is smooth by assumption; β^{-1} and α are smooth by definition; π^{-1} is a projection and hence also smooth. Consequently, H^i is smooth as composition of smooth functions. ■

The next lemma takes care of the case when $d_i > K$ which was not covered by the previous Lemma.

Lemma 11 *If ψ satisfies (4) and $d_i > K$ then there exists smooth H^i satisfying (5).*

Proof. Clearly, since ψ satisfies (4), it is possible to define $H^i : \psi(S_i, s_{-i}) \longrightarrow V^i(S_i, s_{-i})$ such that $H^i(\psi(s)) = V^i(s)$. Moreover, since $d_i > K$, ψ is a submersion. Any submersion is an open mapping and so $\psi(S_i, s_{-i})$ is an open set in $\triangle^K$, which implies that $\psi(S_i, s_{-i})$ is a manifold.

By the local form of submersions, for any $s_i \in S_i^\circ$ there are open neighborhoods: $X \ni s_i$; $Y \ni \psi(s)$; $\mathbb{R}^{d_i} \supset W \ni 0$; $\mathbb{R}^K \supset Z \ni 0$; and parameterizations α and β such that $\alpha(0) = s_i$, $\beta(0) = \psi(s)$ and the diagram below, where $\pi(w_1, ..., w_{d_i}) = (w_1, ..., w_K)$, commutes:

$$\begin{tikzcd}
& V^i(S_i, s_{-i}) \subset \mathbb{R}^K \\
X \subset S_i \arrow[ur, "V^i(\cdot, s_{-i})"] \arrow[r, "\psi(\cdot, s_{-i})"'] & Y \subset \psi(S_i, s_{-i}) \arrow[u, "H^i(\cdot, s_{-i})"'] \\
W \arrow[u, "\alpha"] \arrow[r, "\pi"] & Z \arrow[u, "\beta"]
\end{tikzcd} \tag{7}$$

Notice that $H^i|_Y = V^i(\cdot, s_{-i}) \circ \alpha \circ (\beta^{-1}(\cdot), 0, ..., 0)$. The function V^i is smooth by assumption; β^{-1} and α are also smooth by definition; π^{-1} is a projection and hence also smooth. Consequently, H^i is smooth as composition of smooth functions. ■

3.2 Boundary and Critical Points

Observe that all the above arguments used in defining H^i are local. Therefore, since for a given s_{-i}, $\psi(\cdot, s_{-i})$ is full-rank for almost all s_i, the $H^i(\cdot, s_{-i})$ function is well-defined and smooth for almost all interior points.

If s_i is a boundary point or if $\psi(\cdot, s_{-i})$ is not full-rank at s_i, the function $H^i(\cdot, s_{-i})$ can be defined at $\psi(s_i, s_{-i})$ by continuity. Clearly, the extension of H^i also satisfies $H^i(\psi(s), s_{-i}) = V^i(s)$.

3.3 Second-Order Conditions

Lemma 12 *Let $V^i = H^i \circ \psi$ and t^i as in lemma (9), the second-order condition for truth telling is satisfied, if the matrix $\nabla_\psi H^i$ is positive semi-definite for every ψ.*

Proof. The second-order condition for truth telling is that the matrix,

$$\nabla^2_{s_i} U^i(s_i|s) = \sum_{k=1}^{K} \nabla^2_{s_i} \psi^k \left[V^i_k - \frac{\partial}{\partial \psi^k} t^i \right] - \nabla_{s_i} \psi \, \nabla^2_{\psi} t^i \, \nabla_{s_i} \psi^\top = \tag{8}$$

$$= -\nabla_{s_i} \psi \, \nabla^2_{\psi} t^i \, \nabla_{s_i} \psi^\top, \tag{9}$$

be negative semi-definite. Moreover, from the first-order condition it follows that,

$$\nabla_{\psi} t^i = V^i \Rightarrow \nabla_{s_i} \psi \, \nabla^2_{\psi} t^i = \nabla_{s_i} V^i, \tag{10}$$

and from Lemma 9,

$$\nabla_{s_i} V^i = \nabla_{s_i} \psi \, \nabla_{\psi} H^i \Rightarrow \nabla_{s_i} \psi = \nabla_{s_i} V^i \, \left(\nabla_{\psi} H^i \right)^{-1}.$$

Combining these results,

$$\nabla_{s_i} \psi \, \nabla^2_{\psi} t^i \, \nabla_{s_i} \psi^\top = \nabla_{s_i} V^i \, \left[\left(\nabla_{\psi} H^i \right)^{-1} \right]^\top \, \nabla_{s_i} V^{i\,\top} \tag{11}$$

■

3.4 The Integrability Constraints

The integrability constraints required that $D_\psi H^i$ is symmetric.

Lemma 13 *If $n_i \geq K$ and $D_{s_i} V^i(s) D_{s_i} \psi(s)^\top$ is a symmetric matrix then $D_\psi H^i$ is symmetric.*

$$H^i(\psi(s), s_{-i}) = V^i(s)$$

$$D_\psi H^i(\psi(s), s_{-i}) D_{s_i} \psi(s) = D_{s_i} V^i(s)$$

Since $n_i \geq K$,

$$D_\psi H^i(\psi(s), s_{-i}) D_{s_i} \psi(s) D_{s_i} \psi(s)^\top = D_{s_i} V^i(s) D_{s_i} \psi(s)^\top$$

$$D_\psi H^i(\psi(s), s_{-i}) = D_{s_i} V^i(s) D_{s_i} \psi(s)^\top \left(D_{s_i} \psi(s) D_{s_i} \psi(s)^\top \right)^{-1}$$

Moreover since $D_{s_i} \psi(s) D_{s_i} \psi(s)^\top$ is symmetric, $D_\psi H^i(\psi(s), s_{-i})$ is symmetric if $D_{s_i} V^i(s) D_{s_i} \psi(s)^\top$ is symmetric.

Theorem 14 *If, for every agent, ψ satisfies (4) and $\nabla_{s_i} V^i \nabla_{s_i} \psi^\top$ is everywhere positive semi-definite then ψ is EPIC.*

Proof. First observe that $\nabla_\psi H^i$ is positive semi-definite, if and only if, $\nabla_{s_i} V^i \nabla_{s_i} \psi^\top$ is positive semi-definite. Since $\nabla_{s_i} V^i \nabla_{s_i} \psi^\top = \nabla_{s_i} \psi \, \nabla_\psi H^i \, \nabla_{s_i} \psi^\top$ and $\nabla_{s_i} \psi^\top$ has full-rank. From the previous lemmata, the first and second order conditions for EPIC are satisfied.

■

4 Auctions

This section considers the model *without externalities.*

Lemma 15 *Any* EPIC ψ *satisfies:*

$$\left[V^i(s_i, s_{-i}) - \frac{\partial}{\partial \psi^i} t^i\left(\psi^i(s_i, s_{-i}), s_{-i}\right)\right] \nabla_{s_i} \psi^i(s_i, s_{-i}) = 0 \implies \tag{12}$$

$$\nabla_{s_i} \psi^i(s_i, s_{-i}) \neq 0 \implies V^i(s_i, s_{-i}) = \frac{\partial}{\partial \psi^i} t^i\left(\psi^i(s_i, s_{-i}), s_{-i}\right), \tag{13}$$

Proposition 16 *If* $\forall i$, $\nabla_{s_i} \psi^i(s_i, s_{-i}) = 0$ *then* ψ *is implementable with transfers* $t^i(s_{-i}) \leq \inf_{\{s_i \,:\, \psi^i(s_i, s_{-i}) > 0\}} \psi^i(s_i, s_{-i}) \, V^i(s_i, s_{-i})$.

Example 1 *[2]: signals are given by* $s^i = \left(p^i, c^i\right) \in [0,1]^2$ *and utilities are* $V^i\left(p^i, c^i, c^{-i}\right) = p^i + c^i c^{-i}$.

The allocations implementable accordingly to Proposition 16 are uninteresting in the sense that the probability an agent gets the object is not affect by his information. Nevertheless, the allocations are not trivial (constant) since the winning probability of an agent may depend on the other agents' information. For instance, in example 3.2, the allocation,

$$\psi = \left(\psi^1, \psi^2\right) = \left(\frac{c^2}{1+c^2} \frac{1+p^2}{1+2p^2}, \frac{c^1}{1+c^1} \frac{1+p^1}{1+2p^1}\right), \tag{14}$$

is implementable with transfers $t = (0,0)$.

Proposition 17 *Consider $g_i : \mathbb{R} \longrightarrow [0,1]$ such that $g_i' > 0$ and $\psi^i : S \longrightarrow [0,1]$ defined by $\psi^i(s) = g_i(V^i(s))$ satisfies $\sum_{i=1}^N \psi^i < 1$. Let $t^i : [0,1] \longrightarrow \mathbb{R}$ be a solution of the differential equation $\frac{\partial}{\partial \psi^i} t^i(\psi^i) = g_i^{-1}(\psi^i)$ with initial condition $t^i(0) = 0$. The allocation ψ is implementable with transfers $t = (t^1, ..., t^N)$.*

Proof Agent i's utility of misreporting type $\widehat{s}_i$ when s_i is the true and the other agents report truthfully s_{-i}, as well as, the corresponding Jacobian and Hessian are:

$$\Pi^i(\widehat{s}_i|s) = g_i(V^i(\widehat{s}_i, s_{-i}))V^i(s) - t^i(g_i(V^i(\widehat{s}_i, s_{-i}))), \tag{15}$$

$$\nabla_{\widehat{s}_i}\Pi^i(\widehat{s}_i|s) = g_i'(V^i(\widehat{s}_i, s_{-i}))\nabla_{\widehat{s}_i}V^i(\widehat{s}_i, s_{-i}) \left[V^i(s) - \frac{\partial}{\partial \psi^i} t^i(g_i(V^i(\widehat{s}_i, s_{-i})))\right], \tag{16}$$

$$\nabla^2_{\widehat{s}_i}\Pi^i = \left[\nabla_{\widehat{s}_i}V^i \nabla_{\widehat{s}_i}V^{i\top} g_i'' + g_i' \nabla^2_{\widehat{s}_i}V^i\right]\left[V^i(s) - \frac{\partial}{\partial \psi^i} t^i\right] - g_i' \nabla_{\widehat{s}_i}V^i \nabla_{\widehat{s}_i}V^{i\top}, \tag{17}$$

therefore, $\nabla_{\widehat{s}_i}\Pi^i(s_i|s) = 0$ and $\forall h$, $h^\top \nabla^2_{\widehat{s}_i}\Pi^i(s_i|s)\, h \leq 0$.

To illustrate Proposition 6, consider Example 1 again and let

$$\psi^i = \frac{3}{4}\frac{p^i + c^i c^{-i}}{1 + p^i + c^i c^{-i}} \tag{18}$$

then the required transfers are:

$$t^i(\psi^i) = -\frac{4}{3}\psi^i + \ln(1 - \frac{4}{3}\psi^i). \tag{19}$$

Remark 18 *Proposition 6 can easily be extended to cover the more general case where $\psi^i(s) = g_i(V^i(s), s_{-i})$. Let $g_i : \mathbb{R} \times S_{-i} \longrightarrow [0,1]$ be such that $\sum \psi^i \leq 1$ and $D_1 g_i > 0$. Let the transfer t^i satisfy the following differential equation $\frac{\partial}{\partial \psi^i} t^i(\psi^i) = g_i^{-1}(\cdot, s_{-i})(\psi^i)$ initial condition $t^i(\underline{V}^i(s_{-i}), s_{-i}) = \psi^i(\underline{V}^i(s_{-i}), s_{-i}))\, \underline{V}^i(s_{-i})$ where $\underline{V}^i(s_{-i}) =$* $\min s_i\, V^i(s_i, s_{-i})$.

To illustrated Remark 7, once more consider Example 1 and let

$$\psi^i = \frac{3}{4}\frac{p^i + c^i c^{-i}}{1 + p^i + c^i c^{-i}} + \frac{1}{2} - \frac{3}{4}\frac{p^{-i} + c^{-i}}{1 + p^{-i} + c^{-i}} \tag{20}$$

$$t^i(\psi^i, s_{-i}) = \frac{2}{3} - \frac{4}{3}\psi^i - \frac{p^{-i} + c^{-i}}{1 + p^{-i} + c^{-i}} + \ln\left(\frac{5}{3} - \frac{4}{3}\psi^i - \frac{p^{-i} + c^{-i}}{1 + p^{-i} + c^{-i}}\right) \tag{21}$$

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