Participation Incentives in Rank Order Tournaments with Endogenous Entry*

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Abstract

Rank order tournaments, in which the payment made to an agent is based upon relative observed performance, are a commonly used compensation scheme. Such tournaments induce agents to exert effort when the exact level of effort is not easily observable.

In many situations, agents will potentially compete in a series of such tournaments and must decide which events to enter. The decision of which events to enter will depend upon the prizes offered by each event. The primary focus of the present study is the entry decision of agents in a series of rank order tournaments. Of particular interest is the possibility that a field of higher quality may be attracted to an event with smaller prizes. Conditions are determined under which this counterintuitive outcome could possibly occur.

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1 Introduction

In many situations the absolute performance of an agent depends not only upon the level of effort exerted, but also upon factors beyond the control of the agent. Further, it is often the case that a principal can observe performance, but cannot decompose the observed performance into that resulting from effort versus that resulting from other factors. When the principal cannot accurately observe effort, agents may have an incentive to exert less than the optimal amount of effort. Prendergast (1999) presents an extensive overview of the broad literature on providing incentives within organizations. "Rank order tournaments," in which the level of compensation awarded to an agent is based upon relative observed performance, are one of the many compensation schemes discussed.

Rank order compensation schemes often provide agents with incentives to exert effort when the exact level of effort is not easily observable. The use of rank order compensation schemes has been examined extensively (see for example Lazear and Rosen (1981), Nalebuff and Stiglitz (1983), Green and Stokey (1983), and Mookherjee (1984)). The primary focus has been an environment in which agents compete in a single rank order tournament by choosing a level of effort to exert. Of particular interest are the incentives for agents to exert effort, and subsequently the efficiency (or inefficiency) of the chosen level of effort.

In many situations, agents will potentially compete in a series of tournaments and must decide which events to enter. Numerous examples of this nature can be found in professional sports (for example: golf, auto racing, bowling, and tennis). In each of these instances, there is a "series" of tournaments, and participants must choose which events to enter and which events not to enter.[1] The decision of which tournaments to enter is likely to depend

[1] Another economic example would be a situation in which multiple firms compete to hire agents from a common pool of applicants, by way of offering different schedules of rank order compensation (for example, compensation in the form of a "base salary" plus a "bonus based upon relative performance").

upon the compensation packages offered by each event. That is, “prizes” can be used not only to induce agents to exert effort in a particular tournament, but also to induce agents to enter a particular event.

In a series of tournaments, the objectives of the promoter (or promoters) of the events are not immediately clear. Two possible (and potentially competing) objectives are: to attract a high quality field of entrants to a particular tournament, or to have “parity” or “competitive balance” across events. In any case, the prize levels which best achieve the objectives of tournament organizers will depend upon the entry incentives for individual participants. Intuitively, one would expect that “larger prizes” (for a particular tournament) would attract entrants of “higher quality” (to that particular event). If true, there would be a positive correlation between “prize money” and “strength of field” (at a particular event).[2]

The existing theoretical literature primarily focuses on providing incentives for agents to exert effort in a single rank order tournament, while neglecting such a participation decision by agents in a series of tournaments. One exception is Friebel and Matros (2003), who consider a perfectly competitive environment with homogenous workers and heterogenous firms (differing in their probability of bankruptcy). They analyze a series of elimination tournaments in which each firm promotes a single worker to CEO. The “winner” receives a pre-announced prize (so long as the firm does not go bankrupt), while all other workers receive nothing. Based upon the known bankruptcy rates and announced prizes of each firm, workers decide which firm to enter and compete by choosing effort levels. In equilibrium workers exert less effort in firms that are more likely to go bankrupt. However, firms that are more likely to go bankrupt may offer a larger prize (that is, greater CEO compensation).

The primary focus of the present study is the entry decision of hetero-

[2]Empirically, Ehrenberg and Bognanno (1990) and Bronars and Oettinger (2001) analyze the “tournament participation decision” of professional golfers on the PGA Tour. Both studies suggest that, in some sense, “higher prizes” attract entrants of “higher ability.”

geneous agents (differing in their innate ability) in a series of rank order tournaments. A model is presented in Section 2 of a series of two tournaments, each with a field limited to two entrants. The competition between agents in a particular tournament is analyzed in Section 3. The incentives to enter each event for strategic agents of differing abilities are examined in Section 4. Of particular interest is the possibility that the agent of higher ability may be attracted to the event with the smaller prizes, an outcome which is investigated in greater detail in Sections 5 and 6. Section 7 concludes and discusses directions for future research.

2 A Series of Two Tournaments

A simplified theoretical model is developed, which is intended to capture many of the features of such an entry decision by agents across a series of tournaments. Consider a series of two tournaments, denoted Event 1 and Event 2, each of which has a field limited to two entrants. Suppose there are two "strategic entrants" of differing abilities, one of "high ability" (agent H) and one of "low ability" (agent L). Assume each agent wishes to participate in one and only one of these two tournaments. The field of any event that is not filled by these strategic agents will be filled by "non-strategic" agents (denoted N) of "very low ability."[3]

Suppose the prize structure of each event k is such that: p_k is awarded to the "winner" and αp_k is awarded to the "runner-up" (where $0 \leq \alpha < 1$).[4] If considering a compensation scheme in the form of a "base salary" along

[3]This structure is intended to model that of the PGA Tour in which "Tour Members" can essentially enter any open events without any constraint, with the remaining slots in the field filled by "Non-Tour Members." The assumption that agents do not wish to participate in every event is consistent with observed behavior of PGA Tour members who each typically compete in only 20 to 30 weekly events per year.

[4]A common value of α across events is observed in the prize structure offered on the PGA Tour, since of the total purse at a typical event: 18% is awarded to the first place finisher, 10% is awarded to the second place finisher, 6.8% is awarded to the third place finisher, on down to .2% awarded to the 70th place finisher.

with a "bonus," then the base salary is given by αp_k and the bonus is given by $(1-\alpha)\,p_k$. Without loss of generality, assume that $p_1 = \tau p_2$ with $\tau \geq 1$, so that Event 1 has "larger prizes."

The entry decision of the two strategic agents H and L is analyzed. It is assumed that the entry decision of the agents is sequential, with the agent of higher ability making his choice first. That is agent H first chooses to enter Event 1 or enter Event 2. After observing the choice of agent H, agent L then chooses to enter Event 1 or enter Event 2.

As a result of the decisions of these agents, each tournament realizes one of four possible fields of entrants: $\{H, L\}$, $\{H, N\}$, $\{L, N\}$, or $\{N, N\}$. So long as both strategic entrants choose to enter one of the events, there are four possible assignments of participants across the two events.[5] For example, if agent H enters Event 1 and agent L enters Event 2, then the resulting fields are $F_1 = \{H, N\}$ for Event 1 and $F_2 = \{L, N\}$ for Event 2. Once the field of each tournament is determined, the agents in each event compete by choosing levels of effort.

3 Tournament Level Competition

Consider a tournament with a first place prize of p and a second place prize of αp. Two agents, A and B, compete by simultaneously choosing effort levels, $e_A \geq 0$ and $e_B \geq 0$ respectively. Let $\delta\,(e_A, e_B)$ denote the probability with which A "outperforms" or "beats" B. Assume that $\delta\,(e_A, e_B)$ is homogeneous of degree zero with $\delta\,(e, e) = \delta \geq \frac{1}{2}$.[6] Further assume that for all $(e_A, e_B) \in$

[5] In practice, an agent i will only wish to enter an event if his expected payoff from doing so exceeds a potentially positive reservation wage r_i. In order to ease the presentation, it is implicitly assumed throughout most of the discussion that the prizes are sufficiently large so that this condition holds for all entrants. One exception is the discussion in subsection 5.3, in which an event organizer must choose prize levels subject to "participation constraints" of this nature.

[6] δ can be restricted to $\delta \geq \frac{1}{2}$ without loss of generality by simply assuming that "Agent A" is of "higher relative ability."

$(0,\infty) \times (0,\infty)$: $\frac{\partial \delta}{\partial e_A} \geq 0$; $\frac{\partial \delta}{\partial e_B} \leq 0$; $\frac{\partial^2 \delta}{\partial e_A^2} \leq 0$; and $\frac{\partial^2 \delta}{\partial e_B^2} \geq 0$.[7] Suppose that for each agent the cost of exerting effort of e is given by a continuous function $c(e)$ for which: $c(0) = 0$, and further $c'(e) > 0$ and $c''(e) \geq 0$ for all $e \geq 0$. The competition between two agents A and B in such a tournament is analyzed in Appendix A.

Without any further restrictions on the functional forms of either $\delta\left(e_A, e_B\right)$ or $c(e)$, there exists a unique equilibrium for which $e_A^* = e_B^* = e^*$ in the subgame defined by such a tournament. When agents choose these effort levels, agent A outperforms agent B with probability $\delta = \delta\left(e^*, e^*\right)$.

Further assuming $\delta\left(e_A, e_B\right) = \frac{\delta e_A^z}{(1-\delta) e_B^z + \delta e_A^z}$ (with $z \in [0,1]$) and $c(e) = e$: $e^* = zp(1-\alpha)\delta(1-\delta)$. This choice of effort leads to payoffs of

$$\Pi_{A,\{A,B\}} = p\left\{\alpha + (1-\alpha)\delta - z(1-\alpha)\delta(1-\delta)\right\}$$

and

$$\Pi_{B,\{A,B\}} = p\left\{1 - (1-\alpha)\delta - z(1-\alpha)\delta(1-\delta)\right\}.$$

From here, it is straightforward to apply these general insights to each of the four possible fields which a tournament could realize.

For instance, consider an event with field $F = \{H, L\}$. Letting $\lambda(e_H, e_L) = \frac{\lambda e_H^z}{(1-\lambda) e_L^z + \lambda e_H^z}$ denote the probability with which H outperforms L, and supposing each agent has effort costs of $c(e) = e$, we have: $e_H^* = e_L^* = e_{\{H,L\}}^* = zp(1-\alpha)\lambda(1-\lambda)$. These effort choices result in payoffs of

$$\Pi_{H,\{H,L\}} = p\left\{\alpha + (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\right\}$$

and

$$\Pi_{L,\{H,L\}} = p\left\{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\right\}$$

for agents H and L respectively.

In an event with field $F = \{H, N\}$, let $\omega(e_H, e_N) = \frac{\omega e_H^z}{(1-\omega) e_N^z + \omega e_H^z}$ denote the probability with which H beats N. With effort costs of $c(e) = e$, we have

[7] Attention is restricted to $\delta\left(e_A, e_B\right)$ for which either: $\delta\left(e_A, e_B\right) = \delta$ for all $\left(e_A, e_B\right) \in [0,\infty) \times [0,\infty)$; or $\delta(e_A, 0) = 1$ for all $e_A > 0$ and $\delta(0, e_B) = 0$ for all $e_B > 0$.

$e_H^* = e_N^* = e_{\{H,N\}}^* = zp(1-\alpha)\omega(1-\omega)$, which leads to a payoff of

$$\Pi_{H,\{H,N\}} = p\left\{\alpha + (1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)\right\}$$

for agent H.

For an event with field $F = \{L, N\}$, let $\rho(e_L, e_N) = \frac{\rho e_L^z}{(1-\rho)e_N^z + \rho e_L^z}$ specify the probability with which L outperforms N. With $c(e) = e$, the agents will exert $e_L^* = e_N^* = e_{\{L,N\}}^* = zp(1-\alpha)\rho(1-\rho)$, leading to a payoff for L of

$$\Pi_{L,\{L,N\}} = p\left\{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)\right\}.$$

If H is of higher ability than L, then it should be that: $\lambda > \frac{1}{2}$ (so that when competing against one another H is more likely to outperform L than vice versa) and $\omega > \rho$ (so that H is more likely than L to outperform N). Similarly, if L is of higher ability than N, we would expect: $\rho > \frac{1}{2}$ (so that when competing against one another L is more likely to outperform N than vice versa) and $1-\lambda > 1-\omega$ (so that L is more likely than N to outperform H). Finally, if H is of higher ability than L and L is of higher ability than N, it would be reasonable to presume that H is of higher ability than N. From here it should follow that: $\omega > \frac{1}{2}$ (so that when competing against one another H is more likely to outperform N than vice versa) and $\lambda > 1-\rho$ (so that H is more likely than N to outperform L). Together these restrictions can be summarized as: $\omega > \lambda > \frac{1}{2}$ and $\omega > \rho > \frac{1}{2}$. These restrictions on the base probabilities will be referred to as the "intuitive restrictions."

4 Allocation of Fields and Entry Decision

Based upon the entry decisions of H and L, one of four allocations of fields across the events will result. Denoting these by a, b, c, and d, we have:

Allocation a: $F_1^a = \{H, L\}$ and $F_2^a = \{N, N\}$. Both strategic agents enter the event with the larger prizes;

Allocation b**:** $F_1^b = \{H, N\}$ and $F_2^b = \{L, N\}$. The strategic agent of higher ability enters the event with the larger prizes, while the strategic agent of lower ability enters the event with the smaller prizes;

Allocation c**:** $F_1^c = \{L, N\}$ and $F_2^c = \{H, N\}$. The strategic agent of lower ability enters the event with the larger prizes, while the strategic agent of higher ability enters the event with the smaller prizes;

Allocation d**:** $F_1^d = \{N, N\}$ and $F_2^d = \{H, L\}$. Both strategic agents enter the event with the smaller prizes.

Allocations a and d will be referred to as "pooling allocations"; allocations b and c will be referred to as "separating allocations."

Let Π_i^m denote the expected payoff for agent i under allocation m. Let $p_2 = p$ (and therefore $p_1 = \tau p$). The expected payoff of each strategic agent for each allocation of fields is summarized in Table 1.

Table 1:

m	Π_H^m	Π_L^m
a	$\tau p\{\alpha + (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\}$	$\tau p\{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\}$
b	$\tau p\{\alpha + (1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)\}$	$p\{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)\}$
c	$p\{\alpha + (1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)\}$	$\tau p\{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)\}$
d	$p\{\alpha + (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\}$	$p\{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\}$

Consider the entry decisions of the strategic agents H and L, with no restrictions on the base probabilities other than the intuitive restrictions of $\omega > \lambda > \frac{1}{2}$ and $\omega > \rho > \frac{1}{2}$. Suppose H and L sequentially choose which event to enter (with agent H choosing first). Following a decision by H to enter Event 2, L will want to enter Event 1 if $\Pi_L^c \geq \Pi_L^d$, which is true so long as

$$\tau p\{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)\} \geq p\{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\}.$$

Since $\tau \geq 1$, this condition will hold if

$$\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho) > 1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda).$$

or equivalently

$$(\rho + \lambda - 1) + z\left[\lambda(1-\lambda) - \rho(1-\rho)\right] > 0. \tag{1}$$

If $\lambda \leq \rho$, then $\lambda(1-\lambda) - \rho(1-\rho) \geq 0$, and Condition 1 clearly holds.

Instead suppose $\lambda > \rho$, in which case $\lambda(1-\lambda) - \rho(1-\rho) < 0$. The expression on the left side of the inequality in Condition 1 is now decreasing in z. With $z = 1$, this expression becomes

$$(\rho + \lambda - 1) + \left[\lambda(1-\lambda) - \rho(1-\rho)\right] = \lambda(1-\lambda) - \left[1 - \rho^2\right].$$

Note that $\lambda(1-\lambda) - [1-\rho^2]$ is increasing in both ρ and λ (for $\frac{1}{2} < \rho \leq 1$ and $\frac{1}{2} < \lambda \leq 1$). With $\rho = \lambda = \frac{1}{2}$, $\lambda(1-\lambda) - [1-\rho^2] = 0$, implying that Condition 1 holds for any $\frac{1}{2} < \rho \leq 1$, $\frac{1}{2} < \lambda \leq 1$, and $0 \leq z \leq 1$. Since Condition 1 is always satisfied, we have that L will always want to enter Event 1 following a decision by H to enter Event 2.[8]

If instead H chooses to enter Event 1, L will choose to enter Event 1 so long as $\Pi_L^a \geq \Pi_L^b$. This condition holds when

$$\tau p\left\{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)\right\} \geq p\left\{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)\right\},$$

or, defining

$$\Psi(\alpha) = \tau\frac{\Pi_L^b}{\Pi_L^a} = \frac{\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)}{1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)},$$

this condition holds when $\tau \geq \Psi(\alpha)$.

It has already been argued that

$$\alpha + (1-\alpha)\rho - z(1-\alpha)\rho(1-\rho) \geq 1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda),$$

[8]An additional implication of Condition 1 being satisfied is that $\Pi_L^c > \Pi_L^a$.

implying $\Psi(\alpha) > 1$.[9] Fixing α, ρ, λ, and z: $\tau \geq \Psi(\alpha)$ for τ sufficiently large, in which case L will choose to enter Event 1 as well; $\tau < \Psi(\alpha)$ for τ sufficiently close to one, in which case L will instead choose to enter Event 2. Thus, when H chooses to enter Event 1, L might potentially choose to enter either Event 1 or Event 2.

Comparing the payoffs of H, it is clear that $\Pi_H^b > \Pi_H^c$. It follows that if L will choose to enter Event 2 following a choice by H to enter Event 1, then H will want to enter Event 1.

Additionally, $\Pi_H^b > \Pi_H^a$. However, a general comparison of Π_H^a to Π_H^c is not possible. Defining

$$\Theta(\alpha) = \tau\frac{\Pi_H^c}{\Pi_H^a} = \frac{\alpha + (1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)}{\alpha + (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)},$$

it follows that $\Pi_H^a \geq \Pi_H^c$ if and only if $\tau \geq \Theta(\alpha)$.[10] As such, if L will choose to enter Event 1 following a choice by H to enter Event 1, then: H may choose to enter Event 1 (which is the case if $\tau \geq \Theta(\alpha)$) or H may choose to enter Event 2 (which is the case if $\tau < \Theta(\alpha)$).

With no further restrictions on the base probabilities, it may be that in equilibrium: H and L both enter Event 1; H enters Event 1 and L enters Event 2; or H enters Event 2 and L enters Event 1.

H and L both enter Event 1 when $\Pi_H^a \geq \Pi_H^c$ and $\Pi_L^a \geq \Pi_L^b$, which occurs when $\tau \geq \max\{\Psi(\alpha), \Theta(\alpha)\}$. This is the only instance in which a pooling allocation is realized. Intuitively, it is clear that both agents will choose to enter Event 1 for τ "sufficiently large." To see this, suppose that the prizes in Event 1 are sufficiently large so that the second place prize in Event 1 exceeds the first place prize in Event 2 (that is, $\alpha p_1 > p_2$). In this case, entering Event 1 and exerting zero effort will lead to a payoff of at least αp_1, while entering Event 2 and exerting any amount of effort can never lead to

[9]More precisely, for values of ω, λ, and ρ satisfying the intuitive restrictions, along with $z \in [0,1]$: $\Psi(0) > 1$, $\Psi(1) = 1$, and further $\Psi'(\alpha) < 0$ and $\Psi''(\alpha) > 0$ for all $\alpha \in [0,1)$.

[10]It can be shown that for values of ω, λ, and ρ satisfying the intuitive restrictions, along with $z \in [0,1]$: $\Theta(0) > 1$, $\Theta(1) = 1$, and further $\Theta'(\alpha) < 0$ and $\Theta''(\alpha) > 0$ for all $\alpha \in [0,1)$.

a payoff above p_2. Thus, both agents will clearly choose to enter Event 1. Since $\tau = \frac{p_1}{p_2}$, it follows that $\alpha p_1 > p_2$ can equivalently be expressed as $\tau > \frac{1}{\alpha}$. Thus, for any $0 < \alpha < 0$, the pooling allocation (in which both agents choose to enter Event 1) will result for τ "sufficiently large."

H will enter Event 1 and L will enter Event 2 when $\Pi_L^a < \Pi_L^b$, which arises if $\tau < \Psi(\alpha)$. It is again intuitively clear that this must be the outcome for τ "sufficiently small." Consider $\tau \approx 1$, in which case $p_1 \approx p_2 = p$ (that is, the prizes offered by the two events are approximately equal to each other). When the prizes in the two events are almost equal, L will choose to "avoid H" and enter Event 2 following a choice by H to enter Event 1. To see this, first note that L is more likely to obtain the first place prize when competing against N than when competing against H. The expected prizes for L from competing in Event 2 against N are greater than the expected prizes for L from competing in Event 1 against H by approximately

$$[p\rho + \alpha p(1-\rho)] - [p(1-\lambda)\alpha p\lambda] = p(1-\alpha)\left[\lambda + \rho - 1\right] > 0.$$

From here, if $\rho \geq \lambda$, then L would exert less effort when competing against N than against H. In this case, L clearly has a higher payoff from competing against N in Event 2 than from competing against H in Event 1. If instead $\rho < \lambda$, then L would exert more effort when competing against N than against H. With $c(e) = e$, the cost of this additional effort is approximately $zp(1-\alpha)\left[\rho(1-\rho) - \lambda(1-\lambda)\right]$. This additional cost is greatest when $z = 1$, in which case it is equal to $p(1-\alpha)\left[\rho(1-\rho) - \lambda(1-\lambda)\right]$. However, even if $z = 1$ the additional cost of exerting more effort when competing against N instead of H is less than the increase in expected prizes from competing in Event 2 against N instead of competing in Event 1 against H. That is,

$$p(1-\alpha)\left[\lambda + \rho - 1\right] > p(1-\alpha)\left[\rho(1-\rho) - \lambda(1-\lambda)\right],$$

since $\rho^2 > (1-\lambda)^2$. Thus, even when L will have to exert more effort when competing against N than when competing against H, he will still wish to "avoid H" and enter Event 2 following a choice by H to enter Event 1 when the prizes in the two events are equal to each other. Recalling that L will always choose to enter Event 1 following a decision by H to enter Event 2, it follows that for either initial choice by H a separating allocation will result. Since prizes are larger in Event 1, H will enter Event 1 and L will subsequently enter Event 2 for τ "sufficiently small."

Finally, H will choose to enter Event 2 and L will enter Event 1 so long as $\Pi_L^a \geq \Pi_L^b$ and $\Pi_H^a < \Pi_H^c$, or equivalently if $\Psi(\alpha) \leq \tau < \Theta(\alpha)$. This is the case when: (relative to the prizes in Event 2) the prizes in Event 1 are sufficiently large so that L would rather compete against H in Event 1 than compete against N in Event 2, but at the same time (relative to the prizes in Event 2) the prizes in Event 1 are sufficiently small so that H would rather compete against N in Event 2 than compete against L in Event 1.

As a result of the sequential entry decisions of the strategic agents, we may observe either separating allocation or the pooling allocation in which both agents enter Event 1. To see that each of these outcomes is in fact possible, fix all parameter values other than τ, supposing: $\alpha = \frac{1}{2}$, $\rho = \lambda = \frac{251}{400} = .6275$, $\omega = 1$, and $z = 1$. These values lead to $\Psi(\alpha) = \frac{223{,}001}{182{,}201} \approx 1.2239$ and $\Theta(\alpha) = \frac{320{,}000}{223{,}001} \approx 1.4350$. It follows that $\tau > \frac{320{,}000}{223{,}001} \approx 1.4350$ will induce both strategic agents to enter Event 1. For $\tau < \frac{223{,}001}{182{,}201} \approx 1.2239$ H will enter Event 1 but L will enter Event 2. Finally, $1.2239 \approx \frac{223{,}001}{182{,}201} \leq \tau < \frac{320{,}000}{223{,}001} \approx 1.4350$ will result in H entering Event 2 and L entering Event 1.

The insights thus far can be summarized as:

Observation 1. *For outcome probabilities satisfying the intuitive restrictions $\omega > \lambda > \frac{1}{2}$ and $\omega > \rho > \frac{1}{2}$:*

1. A "pooling allocation" will be realized for τ sufficiently large; a "separating allocation" will be realized for τ sufficiently small.

2. Allocations a, b, or c may possibly be realized, but allocation d will never be realized.

The only counterintuitive insight of these observations is the possibility of allocation c resulting. Note that when allocation c is realized, H enters Event 2 and L enters Event 1, in which case the agent of higher ability is attracted to the event offering the smaller prizes. This can possibly occur whenever $\Psi(\alpha) < \Theta(\alpha)$.

5 H Enters Event 2?

If allocation c is realized, then agent H (the high ability agent) chooses to enter Event 2 (the event with the smaller prizes). If this outcome is possible, then changes in the prize structure may lead to unanticipated changes in the resulting fields of tournament entrants.

Consider ω, λ, ρ, and z such that for some $(\tau, \alpha) \in [1, \infty) \times [0, 1)$, c is the resulting allocation. Consider such a pair $(\bar{\tau}, \bar{\alpha})$. Since c is the realized allocation under $(\bar{\tau}, \bar{\alpha})$, it follows that $\Psi(\bar{\alpha}) \leq \bar{\tau} < \Theta(\bar{\alpha})$. Since $\Psi(\bar{\alpha}) > 1$ by the "intuitive restrictions," it follows that there exists $\hat{\tau} < \bar{\tau}$ for which $\hat{\tau} < \Psi(\bar{\alpha}) < \Theta(\bar{\alpha})$. Under $(\hat{\tau}, \bar{\alpha})$, the resulting allocation of fields is b, in which case H enters Event 1 and L enters Event 2. From here it follows that whenever c is the realized allocation for some $(\bar{\tau}, \bar{\alpha})$, there exists $1 < \hat{\tau} < \bar{\tau}$ such that b is realized under $(\hat{\tau}, \bar{\alpha})$.

Consider the two separating fields $\{H, N\}$ and $\{L, N\}$. Clearly $\{H, N\}$ would be viewed as the stronger of these two. Suppose that initially the prize structures are such that $(\tau, \alpha) = (\hat{\tau}, \bar{\alpha})$, in which case Event 1 attracts $\{H, N\}$. If Event 2 appropriately decreases the amount of their prizes (leading to an increase in τ) allocation c will result, in which case Event 2 attracts $\{H, N\}$ instead of $\{L, N\}$. That is, Event 2 is able to attract a field of *higher quality* by offering *smaller prizes.*[11]

As an example, consider $\omega = 1$ and $\lambda = \rho = \frac{251}{400} = .6275$, along with

[11] We could also envision having $(\bar{\tau}, \bar{\alpha})$ initially, and then realizing $(\hat{\tau}, \bar{\alpha})$ as a result of Event 1 offering smaller prizes. After such a change, Event 1 would attract a field of *higher quality* by offering *smaller prizes.*

$z = 1$ and $\alpha = \frac{1}{2}$. Recall that these values lead to $\Psi(\alpha) \approx 1.2239$ and $\Theta(\alpha) \approx 1.4350$. Suppose Event 1 offers $p_1 = 30,000$. If Event 2 offers $\hat{p}_2 = 25,000$, then $\tau = \hat{\tau} = \frac{30{,}000}{25{,}000} = 1.2$. In this case, allocation b results (and Event 2 attracts the field $\{L, N\}$) since $\hat{\tau} < \Psi(\alpha)$. If Event 2 instead sets a *lower* first place prize of $\bar{p}_2 = 24,000$, then $\tau = \bar{\tau} = \frac{30{,}000}{24{,}000} = 1.25$. Since $\Psi(\alpha) \leq \bar{\tau} < \Theta(\alpha)$, the resulting allocation is c (and Event 2 attracts the field $\{H, N\}$). Thus, Event 2 attracts a *stronger field* by offering *smaller prizes.*

To gain further insight into how this change in the prize structure leads to the resulting change in the allocation of entrants, recall that for τ sufficiently small H will enter Event 1 and L will enter Event 2 (since $\pi_H^c > \pi_H^a$ and $\pi_L^b > \pi_L^a$). Starting at such a value of τ, increase τ by decreasing p_2 (as was done in the preceding example). As τ is increased in this manner, the prizes offered by Event 2 will eventually become small enough so that $\pi_L^b < \pi_L^a$, in which case L would choose to enter Event 1 instead of Event 2 following a choice by H to enter Event 1. However (when $\Psi(\alpha) < \Theta(\alpha)$), for this decreased value of p_2 the prizes in Event 2 are still large enough so that $\pi_H^c > \pi_H^a$ (so that H will choose to enter Event 2). Therefore, as a result of this decrease in the level of their prizes, Event 2 attracts the stronger separating field of $\{H, N\}$ instead of the weaker separating field of $\{L, N\}$.

5.1 Conditions under which H Enters Event 2

Since allocation c results when $\Psi(\alpha) \leq \tau < \Theta(\alpha)$, it follows that this counterintuitive outcome can never occur if $\Psi(\alpha) \geq \Theta(\alpha)$ for all $\alpha \in [0, 1)$, or equivalently if (for fixed values of ω, λ, ρ, and z)

$$\eta(\alpha) = [\Psi(\alpha) - \Theta(\alpha)] [\alpha + (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)] [1 - (1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)]$$

is positive for all $\alpha \in [0, 1)$. Defining

$$\begin{aligned}
\eta_a &= (1-\rho)[1+z\rho](1-\lambda)[1+z\lambda] - (1-\omega)[1+z\omega]\lambda[1+z(1-\lambda)], \\
\eta_b &= (1-\omega)[1+z\omega](1-\lambda)[1-z\lambda] + \omega[1-z(1-\omega)]\lambda[1+z(1-\lambda)] \\
&\quad -(1-\rho)[1+z\rho]\lambda[1-z(1-\lambda)] - \rho[1-z(1-\rho)](1-\lambda)[1+z\lambda], \\
\eta_c &= \rho[1-z(1-\rho)]\lambda[1-z(1-\lambda)] - \omega[1-z(1-\omega)](1-\lambda)[1-z\lambda],
\end{aligned}$$

we have that $\eta(\alpha)$ can be expressed as

$$\eta(\alpha) = \alpha^2\eta_a - \alpha\eta_b + \eta_c.$$

Begin by noting that $\eta(1) = 0$ (since $\Psi(1) = \Theta(1) = 1$) and $\eta(0) = \eta_c$. Further, $\eta'(\alpha) = 2\alpha\eta_a - \eta_b$ and $\eta''(\alpha) = 2\eta_a$, the latter of which does not depend upon α. Since $\eta''(\alpha)$ does not depend upon α it follows that $\eta(\alpha) \geq 0$ for all $\alpha \in [0, 1)$ if and only if $\eta(0) = \eta_c \geq 0$ and $\eta'(1) = 2\eta_a - \eta_b \leq 0$.

Observe that

$$\begin{aligned}
\eta'(1)|_{z=1} &= 2\left\{\left(1-\rho^2\right)\left(1-\lambda^2\right) - \left(1-\omega^2\right)\lambda\left(2-\lambda\right)\right\} \\
&\quad - \left\{\left(1-\omega^2\right)\left(1-2\lambda+\lambda^2\right) + \omega^2\lambda\left(2-\lambda\right) - \left(1-\rho^2\right)\lambda^2 - \rho^2\left(1-\lambda^2\right)\right\} \\
&= \omega^2 - \rho^2 + 1 - 2\lambda.
\end{aligned}$$

Similarly

$$\begin{aligned}
\eta'(1)|_{z=0} &= 2\left\{(1-\rho)(1-\lambda) - (1-\omega)\lambda\right\} - \left\{(1-\omega)(1-\lambda) + \omega\lambda - (1-\rho)\lambda - \rho(1-\lambda)\right\} \\
&= \omega - \rho + 1 - 2\lambda.
\end{aligned}$$

Further note that

$$\frac{\partial\eta'(1)}{\partial z} = 2\frac{\partial\eta_a}{\partial z} - \frac{\partial\eta_b}{\partial z} = \rho(1-\rho) - \omega(1-\omega),$$

which is strictly positive for $\frac{1}{2} < \rho < \omega \leq 1$. From here it follows that in order for $\eta'(1) \leq 0$ for every $z \in [0, 1]$, it must be that $\eta'(1)|_{z=1} \leq 0$. This is the case so long as

$$\lambda \geq \frac{\omega^2 - \rho^2 + 1}{2}. \tag{2}$$

Additionally, if $\eta'(1) \leq 0$, then $\eta(0) \geq 0$. In order to see this, begin by recognizing that if $\eta'(1) \leq 0$ (for some $z \in [0, 1]$) it must be that $\eta'(1)|_{z=0} \leq 0$, or equivalently

$$\lambda \geq \frac{\omega - \rho + 1}{2}. \tag{3}$$

Observe that

$$\frac{\partial\eta(0)}{\partial\lambda} = \rho\left[1 - z(1-\rho)\right]\left\{\left[1 - z(1-\lambda)\right] + \lambda z\right\} + \omega\left[1 - z(1-\omega)\right]\left\{(1 - z\lambda) + z(1-\lambda)\right\} > 0.$$

From here it follows that if $\eta'(1) \leq 0$, then $\eta(0) \geq \eta(0)|_{\lambda=\frac{\omega-\rho+1}{2}}$. Note that $\eta(0)|_{\lambda=\frac{\omega-\rho+1}{2}} \geq 0$ if and only if

$$\left(\frac{\omega-\rho+1}{\omega}\right)\left(\frac{2-z(\rho-\omega+1)}{1-z(1-\omega)}\right) \geq \left(\frac{\rho-\omega+1}{\rho}\right)\left(\frac{2-z(\omega-\rho+1)}{1-z(1-\rho)}\right).$$

This condition is satisfied since, for any $\frac{1}{2} < \rho < \omega \leq 1$ and $z \in [0,1]$, $\frac{\omega-\rho+1}{\omega} > \frac{\rho-\omega+1}{\rho}$ and $\frac{2-z(\rho-\omega+1)}{1-z(1-\omega)} \geq \frac{2-z(\omega-\rho+1)}{1-z(1-\rho)}$. Therefore, $\eta(0) \geq 0$ whenever $\eta'(1) \leq 0$, implying that $\eta'(1) \leq 0$ is necessary and sufficient to ensure $\eta(\alpha) \geq 0$ for all $\alpha \in [0,1)$.

In summary, allocation c can never result if $\Psi(\alpha) \geq \Theta(\alpha)$ for all $\alpha \in [0,1)$. For this condition to hold for all $z \in [0,1]$, Condition 2 must be satisfied; for this condition to hold for any $z \in [0,1]$, Condition 3 must be satisfied.

If Condition 3 holds but Condition 2 does not, then allocation c can never result for relatively small values of z but can result for relatively large values of z. Thus, for fixed values of ω, λ, and ρ, the possibility of allocation c ever arising can more easily be ruled out when performance is less dependent upon effort (that is, for smaller values of z). Equivalently, the possible realization of allocation c can more easily occur in competitions in which performance is more dependent upon effort (that is, for larger values of z).

Further, Condition 2 is violated whenever the outcome probabilities are such that $\lambda < \frac{\omega^2-\rho^2+1}{2}$, in which case allocation c may result for some pairs of α and τ. For fixed values of ω and ρ, Condition 2 will not hold if λ is sufficiently small (that is, sufficiently close to $\frac{1}{2}$). Thus, fixing the probability with which each type of strategic agent outperforms N, Condition 2 will be violated if the probability with which H outperforms L is relatively low.

For a final interpretation of these conditions, begin by noting that Condition 2 can be stated as $\rho^2 - (1-\lambda) \geq \omega^2 - \lambda$ and Condition 3 can be stated as $\rho - (1-\lambda) \geq \omega - \lambda$. Recall that: λ is the probability with which H outperforms L (and therefore, $1-\lambda$ is the probability with which L outperforms H); ω is the probability with which H outperforms N; and ρ is the probability with which L outperforms N. Letting ibj denote the outcome that agent i outperforms (or "beats") agent j, Condition 2 can be expressed as

$$\{\Pr(LbN)\}^2 - \Pr(LbH) \geq \{\Pr(HbN)\}^2 - \Pr(HbL)$$

and Condition 3 can be expressed as

$$\Pr(LbN) - \Pr(LbH) \geq \Pr(HbN) - \Pr(HbL).$$

5.2 Resulting Fields for $(\tau, \alpha) \in [1, \infty) \times [0, 1)$

It is possible to gain further insight into the resulting allocation of fields for general pairs $(\tau, \alpha) \in [1, \infty) \times [0, 1)$, by focusing on $\eta(\alpha)$ in even greater detail. Recall that: $\eta(1) = 0$, and $\eta''(\alpha)$ does not depend upon α. Further, it has been argued that if $\eta'(1) \leq 0$, then $\eta(0) \geq 0$. With these restrictions on $\eta(\alpha)$, it must be that either:

i. $\eta(\alpha) \geq 0$ for all $\alpha \in [0, 1)$,

ii. there exists a unique $\hat{\alpha} \in [0, 1)$ such that $\eta(\alpha) \geq 0$ if and only if $\alpha \in [0, \hat{\alpha}]$, or

iii. $\eta(\alpha) < 0$ for all $\alpha \in [0, 1)$.

For ω, λ, and ρ satisfying the intuitive restrictions, each of these three outcomes is possible.

Recall that $\eta(0) = \eta_c$ and $\eta'(1) = 2\eta_a - \eta_b$. Supposing $z = 1$: $\eta(0) = \rho^2\lambda^2 - \omega^2(1-\lambda)^2$ and $\eta'(1) = 1 - 2\lambda + \omega^2 - \rho^2$. For outcome (i) to result it must be that $\eta(0) \geq 0$ and $\eta'(1) \leq 0$, conditions which are satisfied for $\omega = .8$ and $\lambda = \rho = \frac{2}{3}$. Outcome (ii) will occur so long as $\eta(0) \geq 0$ and $\eta'(1) > 0$, both of which hold for $\omega = 1$ and $\lambda = \rho = .6275$. Finally, for $\omega = 1$ and $\lambda = \rho = .55$, $\eta(0) < 0$ and $\eta'(1) > 0$ leading to outcome (iii).

The resulting relation of $\Psi(\alpha)$ to $\Theta(\alpha)$ in each case is illustrated in Figure 1. Remember that allocation a results for (τ, α) such that $\tau \geq \max\{\Psi(\alpha), \Theta(\alpha)\}$. b is the realized allocation if (τ, α) are such that $\tau < \Psi(\alpha)$. Finally, allocation c arises for (τ, α) such that $\Psi(\alpha) \leq \tau < \Theta(\alpha)$, a possibility which can arise under either outcome (ii) or (iii). Under outcome (ii), for every $\alpha \in (\hat{\alpha}, 1)$ there is a range of $\tau > 1$ for which allocation c results. Under outcome (iii), there is a range of $\tau > 1$ leading to allocation c for every $\alpha \in [0, 1)$. Upon inspection of Figure 1, it is straightforward to infer how the resulting allocation would change in each case as (τ, α) is altered.

5.3 Choice of Prizes by Event Organizer

At this point it is worthwhile to briefly provide an example illustrating that an event organizer may wish to choose prize levels resulting in this counter-intuitive allocation of fields. Specifically, consider a single ("monopolist") organizer of the two events, with a desire to realize a separating allocation while minimizing the total prizes paid to entrants across the two events.[12] That is, the promoter wants to minimize $(1+\alpha)(p_1+p_2)$ by choosing $\alpha \in [0, 1)$ and $p_1 \geq p_2 \geq 0$, while realizing either allocation b or allocation c.

The event organizer will only have to offer positive prizes if the entrants might choose to enter neither event. Generalizing the model to allow for such a choice, suppose that an agent i requires an expected payoff of at least $r_i \geq 0$ in order to enter an event. That is, the agent will prefer to not enter any event if each available option leads to an expected payoff below some reservation wage, r_i. With such participation constraints in place: allocation b will result when $\Pi_L^b > \Pi_L^a$, $\Pi_L^b \geq r_L$, and $\Pi_H^b \geq r_H$.[13] Likewise, in the presence of such participation constraints: allocation c will arise when $\Pi_L^a \geq \Pi_L^b$, $\Pi_L^a \geq r_L$, $\Pi_H^c > \Pi_H^a$, and $\Pi_H^c \geq r_H$.[14]

Consider $z = 1$, $\omega = 1$, and $\rho = \lambda = \frac{11}{20} = .55$. In this case, $\eta(\alpha) < 0$

[12] The promoter will place the same value on allocations b and c (the two separating allocations) if the value of a field depends only on the innate ability of the entrants, and not on the level of prizes or level of effort. Additionally, a desire to realize a separating allocation can be justified by making appropriate assumptions on the value of non-separating allocations.

[13] Allocation b also requires $\Pi_H^b \geq \Pi_H^c$, which is always satisfied as a result of Event 1 offering larger prizes.

[14] Allocation c also requires $\Pi_L^c \geq r_L$. However, recalling that in general $\Pi_L^c > \Pi_L^a$, this condition will clearly hold when $\Pi_L^a \geq r_L$.

for all $\alpha \in [0,1)$ (that is, outcome (iii) identified in subsection 5.2 occurs). Thus, for every $\alpha \in [0,1)$, there is a range of $\tau > 1$ for which allocation c will result. Further suppose $r_H = 1$ and $r_L = \frac{19}{20}$. A detailed discussion of the choice of α, p_1, and p_2 in this environment is presented in Appendix B.

As argued in Appendix B, the least costly way for this single event promoter to realize allocation b is to choose $p_1 = p_2 = 1$ and $\alpha = \frac{259}{279} \approx .9283$. This choice results in total payments of $(1+\alpha)(p_1+p_2) = \frac{1{,}076}{279} \approx 3.8566$. In contrast, the least costly way to realize allocation c is by setting $p_1 = \frac{380}{\sqrt{90{,}440}} \approx 1.2636$, $p_2 = 1$, and $\alpha = \frac{\sqrt{90{,}440}-81}{319} \approx .6888$. Such a choice leads to total payments of $(1+\alpha)(p_1+p_2) = \left(1+\frac{\sqrt{90{,}440}-81}{319}\right)\left(1+\frac{380}{\sqrt{90{,}440}}\right) \approx 3.8228$.

Thus, a monopolist event promoter wishing to realize a separating allocation while minimizing the total payments across the two events may need to choose prizes so that allocation c results. As such, recognizing that H may choose to enter Event 2 (and subsequently understanding why this choice is made and how the entry decision of the agents will vary for different prize structures) is important not only to the tournament entrants, but to the promoter as well.

6 Adding Structure to the Probabilities

Thus far no restrictions have been placed on the outcome probabilities (ω, λ, and ρ) other than the "intuitive restrictions" implied by the loose notions of "H being better than L" and "L being better than N." It would be worthwhile to determine if the counterintuitive possibility discussed in the previous sections can arise if additional structure is placed on these probabilities.

Consider a tournament in which two agents, A and B, compete by choosing effort levels e_A and e_B. Suppose the performance of A is determined by the realization of a random variable X_A and the performance of B is determined by the realization of a random variable X_B. A outperforms B when the realized values of X_A and X_B (x_A and x_B respectively) are such that $x_A > x_B$ (and similarly, B outperforms A when $x_B > x_A$). For any arbitrary level of effort $\tilde{e} \geq 0$, suppose that when $e_A = e_B = \tilde{e}$: X_A is drawn from a cumulative distribution function $F_A(x)$ and X_B is drawn from a cumulative distribution function $F_B(x)$, with neither distribution function depending upon the value of $\tilde{e}$.

In the current context we have that: when H competes in a tournament by choosing the same level of effort as his rival, his performance is determined by X_H which is drawn from $F_H(x)$; when L competes in a tournament by choosing the same level of effort as his rival, his performance is determined by X_L which is drawn from $F_L(x)$; and when N competes in a tournament by choosing the same level of effort as his rival, his performance is determined by X_N which is drawn from $F_N(x)$. Assume $F_H(0) = F_L(0) = F_N(0) = 0$ and $F_H(1) = F_L(1) = F_N(1) = 1$.[15]

With performance determined in this manner, the notion of "H being of higher ability than L, H being of higher ability than N, and L being of higher ability than N" can be strengthened. A natural notion would be that X_H, X_L, and X_N can be stochastically ordered.

6.1 First Order Stochastic Dominance is Not Sufficient

Working in this direction, first order stochastic dominance is not sufficient to rule out the possibility of allocation c arising. Consider

$$F_H(x) = \begin{cases} 0, & 0 \leq x < .5 \\ 51x - 25.5, & .5 \leq x < .51 \\ x, & .51 \leq x \leq 1 \end{cases},$$

$$F_L(x) = \begin{cases} x, & 0 \leq x \leq 1 \end{cases},$$

and

$$F_N(x) = \begin{cases} x, & 0 \leq x < .49 \\ 51x - 24.5, & .49 \leq x < .5 \\ 1, & .5 \leq x \leq 1 \end{cases}.$$

[15] Appropriate assumptions should be made so that "ties" occur with probability zero.

For the corresponding random variables, $X_H \succeq_{FSD} X_L \succeq_{FSD} X_N$. Further, $\omega = 1$ and $\lambda = \rho = \frac{251}{400} = .6275$.[16] If $z = 1$, $\Psi(.5) = \frac{223{,}001}{182{,}201} \approx 1.2239$ and $\Theta(.5) = \frac{320{,}000}{223{,}001} \approx 1.4350$, which imply that for $\alpha = .5$ along with any

$$1.2239 \approx \frac{223,001}{182,201} < \tau < \frac{320,000}{223,001} \approx 1.4350,$$

the resulting allocation of fields is allocation c.[17]

From here a natural question is whether the possibility of allocation c resulting can be ruled out when performance is determined by the realization of random variables that can be ordered by some stronger stochastic ordering, such as the "monotone hazard-rate condition" (written $X \succeq_{MHRC} Y$) or the "monotone likelihood-ratio condition" (written $X \succeq_{MLRC} Y$).[18] In the preceding example X_H, X_L, and X_N are not ordered by either the "monotone hazard-rate condition" or the "monotone likelihood-ratio condition."

6.2 Performance Drawn from Power Functions

Suppose $F_H(x) = x^H$, $F_L(x) = x^L$, and $F_N(x) = x^N$, with $0 < N < L < H < \infty$. For such random variables $X_H \succeq_{MLRC} X_L \succeq_{MLRC} X_N$, implying $X_H \succeq_{MHRC} X_L \succeq_{MHRC} X_N$ and $X_H \succeq_{FSD} X_L \succeq_{FSD} X_N$.

In general, if each distribution function is continuous and differentiable on the interval $[0, 1]$, then:

$$\omega = \int_0^1 \int_0^v f_N(y)dy f_H(v)dv,$$

$$\lambda = \int_0^1 \int_0^v f_L(y)dy f_H(v)dv,$$

[16]Recall that these are the values which were given in subsection 5.2 to illustrate that outcome (ii) could occur.

[17]For these values of ω, λ, and ρ, a similar outcome could be shown to arise even if $z = 0$ (and therefore, may arise for any $z \in [0, 1]$).

[18]X is stochastically larger than Y: in the sense of MHRC if $\frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)}$ for all x; and in the sense of MLRC if $\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}$ for all $x \leq y$ (where $F(\cdot)$ is the CDF and $f(\cdot)$ is the PDF of X, and $G(\cdot)$ is the CDF and $g(\cdot)$ is the PDF of Y).

and

$$\rho = \int_0^1 \int_0^v f_N(y)dyf_L(v)dv.$$

In this example: $\lambda = \frac{H}{L+H}$, $\rho = \frac{L}{N+L}$, and $\omega = \frac{H}{N+H}$. Thus, Condition 2 is:

$$\frac{H}{L+H} \geq \frac{\left(\frac{H}{N+H}\right)^2 - \left(\frac{L}{N+L}\right)^2 + 1}{2}.$$

Upon simplification, this condition can be expressed as

$$N^4(H-L) + 2N^2HL(H-L) + H^2L^2(H-L) + 2N^3\left[H^2 - L^2\right] \geq 0,$$

which clearly holds since $H > L$. As a result, when performance is drawn from power functions, allocation c can never arise.[19]

An example of random variables which are ordered by the "monotone likelihood-ratio condition" has been considered, for which the counterintuitive allocation of fields cannot arise. This framework should be examined more generally, in order to determine restrictions on $F_H(x)$, $F_L(x)$, and $F_N(x)$ which ensure that allocation c cannot occur. This can be done by determining precise conditions for which $\lambda \geq \frac{\omega^2-\rho^2+1}{2}$.

7 Conclusion and Future Research

Rank order tournaments, in which the payment made to an agent is based upon relative observed performance, are a common way to structure employee compensation. Such schemes often induce agents to exert effort when the exact level of effort is not easily observable. In many situations, agents will potentially compete in a series of rank order tournaments and must decide which events to enter. The decision of which tournaments to enter should depend upon the compensation packages offered by the different events. That is, "prizes" can be used not only to induce agents to exert effort in a particular tournament, but also to induce agents to enter a particular tournament.

[19]Observe that $N = .5$, $L = 1$, and $H = 2$ lead to $\omega = .8$ and $\lambda = \rho = \frac{2}{3}$, which were the probabilities given in subsection 5.2 to illustrate that outcome (i) could occur.

The existing theoretical literature mainly focuses on incentives for agents to exert effort in a single tournament, while neglecting such an entry decision across tournaments. The primary focus of the present study is the entry decision of heterogeneous agents in a series of rank order tournaments. Intuitively, one would expect that "larger prizes" (for a particular tournament) would attract entrants of "higher quality" (to that particular event). However, it is shown that a field of higher quality may be attracted to an event offering smaller prizes. Conditions are determined under which this counterintuitive outcome can occur.

Ultimately, an empirical analysis should be conducted as well. It would be of interest to determine whether or not "larger prizes" attract participants of "higher quality." Both Ehrenberg and Bognanno (1990) and Bronars and Oettinger (2001) provide some preliminary insights into the entry decisions of professional golfers. Bronars and Oettinger observe that an increase in the total prizes at only tournament k will lead more "unconstrained" golfers to enter tournament k, where "unconstrained" golfers are those that can enter any event during the current season. Ehrenberg and Bognanno find evidence to suggest that "exempt" golfers are more likely than "non-exempt" golfers to enter tournaments with larger prizes, where "exempt" golfers are those which are assured entrance into all tour events during the following year, regardless of performance during the current year.[20] In the end, both of these studies suggest that in some sense "higher prizes" do attract participants of "higher quality." However, neither study makes a fine enough distinction among the types of participants to address the questions raised in the present analysis. Specifically, neither makes both a clear distinction between constrained and unconstrained participants, along with a further distinction regarding the quality of the unconstrained participants.

[20]It should be noted that the "non-exempt" players considered by Ehrenberg and Bognanno may be either "constrained" or "unconstrained" as defined by Bronars and Oettinger.

Appendix A

The competition between two agents, A and B, in a particular tournament (as described in Section 3) is analyzed in this appendix. Recall that the structure of prizes is such that the "winner" of the tournament receives a prize of p, while the "runner-up" receives a prize of αp. It follows that the payoff of agent A is $\Pi_A = \alpha p + (1-\alpha)p\delta\left(e_A, e_B\right) - c(e_A)$, while the payoff of agent B is $\Pi_B = p - (1-\alpha)p\delta\left(e_A, e_B\right) - c(e_B)$.

From here:

$$\frac{\partial \Pi_A}{\partial e_A} = (1-\alpha)p\frac{\partial \delta\left(e_A, e_B\right)}{\partial e_A} - c'(e_A)$$

and

$$\frac{\partial \Pi_B}{\partial e_B} = -(1-\alpha)p\frac{\partial \delta\left(e_A, e_B\right)}{\partial e_B} - c'(e_B),$$

which lead to

$$\frac{\partial^2 \Pi_A}{\partial e_A^2} = (1-\alpha)p\frac{\partial^2 \delta\left(e_A, e_B\right)}{\partial e_A^2} - c''(e_A) \leq 0$$

and

$$\frac{\partial^2 \Pi_B}{\partial e_B^2} = -(1-\alpha)p\frac{\partial^2 \delta\left(e_A, e_B\right)}{\partial e_B^2} - c''(e_B) \leq 0.$$

If performance does not depend upon effort (which is the case when $\delta\left(e_A, e_B\right) = \delta$ for all $(e_A, e_B) \in [0,\infty) \times [0,\infty)$), then $\frac{\partial \delta(e_A, e_B)}{\partial e_A} = \frac{\partial \delta(e_A, e_B)}{\partial e_B} = 0$, implying that $\frac{\partial \Pi_A}{\partial e_A} < 0$ for all $0 \leq e_A < \infty$ and $\frac{\partial \Pi_B}{\partial e_B} < 0$ for all $0 \leq e_B < \infty$. As a result, we arrive at the intuitive "corner solution" in which each agent optimally chooses to exert zero effort.

If instead $\delta(e_A, 0) = 1$ for all $e_A > 0$ and $\delta(0, e_B) = 0$ for all $e_B > 0$, then there cannot be a pure strategy equilibrium in which either agent exerts zero effort. To see this, consider the choice of effort by agent i if his rival (agent j) chooses $e_j = 0$. Suppose i chooses $e_i = \epsilon > 0$. Since any $e_i > 0$ results in agent i outperforming agent j with probability one, it follows that agent i would be better off choosing any $e_i \in (0, \epsilon)$. Since this is true for any arbitrary $\epsilon > 0$, it follows that there is no pure strategy equilibrium characterized by $e_j = 0$ and $e_i > 0$. Further, a choice of zero effort by both agents cannot be an equilibrium. To see this, consider the choice of effort by the low ability agent (agent B) when the high ability agent (agent A) exerts zero effort. Since $\delta(0,0) = \delta \in \left[\frac{1}{2}, 1\right]$, while $\delta(0, e_B) = 0$ for all $e_B > 0$, agent B realizes a greater expected payoff from $e_B > 0$ so long as $c(e_B) < (1-\alpha)p\delta$. This condition clearly holds for e_B sufficiently close to zero. Thus, in response to a choice of $e_A = 0$ by A, agent B would optimally choose to exert a positive level of effort ($e_B > 0$). Thus, there are no pure strategy equilibria characterized by a choice of zero effort by either agent. From here, we should attempt to identify an "interior solution," in which each agent exerts a positive amount of effort.

At an interior solution, e_A and e_B should be such that $\frac{\partial \Pi_A}{\partial e_A} = 0$ and $\frac{\partial \Pi_B}{\partial e_B} = 0$ simultaneously. For this to be the case, it must be that

$$-\frac{\left(\frac{\partial \delta(e_A, e_B)}{\partial e_A}\right)}{\left(\frac{\partial \delta(e_A, e_B)}{\partial e_B}\right)} = \frac{c'(e_A)}{c'(e_B)}.$$

Since $\delta\left(e_A, e_B\right)$ is homogeneous of degree zero, it follows that

$$e_A \frac{\partial \delta\left(e_A, e_B\right)}{\partial e_A} + e_B \frac{\partial \delta\left(e_A, e_B\right)}{\partial e_B} = 0 \Rightarrow -\frac{\left(\frac{\partial \delta(e_A, e_B)}{\partial e_A}\right)}{\left(\frac{\partial \delta(e_A, e_B)}{\partial e_B}\right)} = \frac{e_B}{e_A}.$$

As a result, in order for $\frac{\partial \Pi_A}{\partial e_A} = 0$ and $\frac{\partial \Pi_B}{\partial e_B} = 0$ simultaneously, it must be that $\frac{e_B}{e_A} = \frac{c'(e_A)}{c'(e_B)}$ or equivalently $e_A c'(e_A) = e_B c'(e_B)$. Since $ec'(e)$ is increasing in e, this requires $e_A = e_B$. That is, any interior solution must be characterized by agents A and B choosing equal effort levels.

$\frac{\partial \Pi_A}{\partial e_A} = (1-\alpha)p\frac{\partial \delta(e_A, e_B)}{\partial e_A} - c'(e_A)$ can now be examined, restricting attention to $e_A = e_B = e$. Since $\delta\left(e_A, e_B\right)$ is homogenous of degree zero, $\frac{\partial \delta(e_A, e_B)}{\partial e_A}$ is homogeneous of degree -1, implying that $\left.\frac{\partial \delta(e_A, e_B)}{\partial e_A}\right|_{(e_A, e_B)=(e,e)}$ will: tend to ∞ as $e \to 0$, tend to zero as $e \to \infty$, and strictly decrease as e is increased. Recall that $c'(e)$ is positive, finite, and non-decreasing. Together, these insights imply that $\left.\frac{\partial \Pi_A}{\partial e_A}\right|_{(e_A, e_B)=(e,e)}$ is: decreasing in e, positive for e sufficiently small, and negative for e sufficiently large. Thus, there exists a unique $e^* \in (0, \infty)$ for which $\left.\frac{\partial \Pi_A}{\partial e_A}\right|_{(e_A, e_B)=(e^*, e^*)} = 0$ and $\left.\frac{\partial \Pi_B}{\partial e_B}\right|_{(e_A, e_B)=(e^*, e^*)} = 0$ simultaneously. In this case, there exists a unique equilibrium, characterized by $e_A^* = e_B^* = e^* > 0$. This unique equilibrium level of effort, must satisfy

$$\frac{1}{c'(e^*)}\left\{\left.\frac{\partial\delta\left(e_A, e_B\right)}{\partial e_A}\right|_{(e_A,e_B)=(e^*,e^*)}\right\} = \frac{1}{(1-\alpha)p}.$$

With $e_A^* = e_B^* = e^*$, in equilibrium $\delta(e_A^*, e_B^*) = \delta$.

Suppose $\delta\left(e_A, e_B\right) = \frac{\delta e_A^z}{(1-\delta)e_B^z+\delta e_A^z}$, with $z \in (0, 1]$. For this functional form of $\delta\left(e_A, e_B\right)$, $\frac{\partial\delta(e_A,e_B)}{\partial e_A} = \frac{z\delta(1-\delta)e_B^z}{e_A^{1-z}\left[(1-\delta)e_B^z+\delta e_A^z\right]^2}$. Under this assumption, e^* must satisfy $e^*c'(e^*) = z\delta(1-\delta)(1-\alpha)p$. Recalling that $ec'(e)$ is increasing in e, it follows that the optimal level of effort is an increasing function of $z\delta(1-\delta)(1-\alpha)p$. From here it is straightforward to determine how e^* will change as z, p, α, or δ are individually varied. Specifically, we arrive at the standard conclusions that the optimal level of effort will be higher if: the outcome is more dependent upon the chosen levels of effort (which is the case for larger values of z); the agents are of relatively similar ability (which, since $\delta \in \left[\frac{1}{2}, 1\right]$, is the case for smaller values of δ); or the difference between the first place prize and second place prize is greater (which is the case for larger values of $(1-\alpha)p$).

Further, if the cost of exerting effort is given by $c(e) = e$, we have

$$e^* = zp(1-\alpha)\delta(1-\delta).$$

Under this further assumption, it follows that the payoffs of the agents are

$$\Pi_{A,\{A,B\}} = p\left\{\alpha + (1-\alpha)\delta - z(1-\alpha)\delta(1-\delta)\right\} \geq 0$$

and

$$\Pi_{B,\{A,B\}} = p\left\{1 - (1-\alpha)\delta - z(1-\alpha)\delta(1-\delta)\right\} \geq 0.$$

Appendix B

Consider a single ("monopolist") tournament organizer, wishing to realize a separating allocation while minimizing the total prizes paid across the two events. As noted, if agent i will choose to participate in a tournament only if his expected payoff from doing so is at least as large as r_i, this imposes additional restrictions on the prizes which will achieve either allocation b or allocation c. In the presence of such participation constraints: allocation b (in which H enters Event 1 and L enters Event 2) will result when $\Pi_L^b > \Pi_L^a$, $\Pi_L^b \geq r_L$, and $\Pi_H^b \geq r_H$; while allocation c (in which L enters Event 1 and H enters Event 2) will arise when $\Pi_L^a \geq \Pi_L^b$, $\Pi_L^a \geq r_L$, $\Pi_H^c > \Pi_H^a$, and $\Pi_H^c \geq r_H$.

As an example, suppose $z = 1$, $\omega = 1$, $\rho = \lambda = \frac{11}{20} = .55$, $r_H = 1$, and $r_L = \frac{19}{20}$. For these values the minimum costs of realizing allocation c are less than the minimum costs of realizing allocation b .

Minimum Costs of Realizing Allocation b: Under the assumptions thus far, the conditions which must hold for allocation b to result are:

$$\Pi_L^b > \Pi_L^a \Leftrightarrow p_2 > \frac{1-(1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)}{\alpha+(1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)} p_1 \Leftrightarrow p_2 > \frac{319\alpha+81}{279\alpha+121} p_1,$$

$$\Pi_L^b \geq r_L \Leftrightarrow p_2 \geq \frac{r_L}{\alpha+(1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)} \Leftrightarrow p_2 \geq \frac{380}{279\alpha+121},$$

and

$$\Pi_H^b \geq r_H \Leftrightarrow p_1 \geq \frac{r_H}{\alpha+(1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)} \Leftrightarrow p_1 \geq 1.$$

Additionally, it must be that $p_2 \leq p_1$.

Considering an arbitrary value of $\alpha \in [0,1)$, the values of p_1 and p_2 which minimize $(1+\alpha)(p_1+p_2)$ can be determined as the solution to a linear programming problem. From here, the minimum amount of prizes paid across the two events can be treated as a function of α, denoted as $P_b(\alpha) = (1+\alpha)(p_1+p_2)$. Doing so,

- for $\alpha \in \left[0, \frac{259}{279}\right)$: $p_1 = p_2 = \frac{380}{279\alpha+121}$, so that $P_b(\alpha) = \frac{760(1+\alpha)}{279\alpha+121}$;
- for $\alpha \in \left[\frac{259}{279}, \frac{299}{319}\right)$: $p_1 = 1$ and $p_2 = \frac{380}{279\alpha+121}$, so that $P_b(\alpha) = \frac{(1+\alpha)(279\alpha+501)}{279\alpha+121}$;
- for $\alpha \in \left[\frac{299}{319}, 1\right)$: $p_1 = 1$ and $p_2 = \frac{319\alpha+81}{279\alpha+121} + \frac{\epsilon}{1+\alpha}$, so that $P_b(\alpha) = \frac{(1+\alpha)(598\alpha+202)}{279\alpha+121} + \epsilon$, with $\epsilon > 0$.

From here, $P_b'(\alpha) = \frac{-120{,}080}{(279\alpha+121)^2} < 0$ for $\alpha \in \left[0, \frac{259}{279}\right)$. Further, $P_b'(\alpha) = 1 - \frac{60{,}040}{(279\alpha+121)^2} > 0$ for $\alpha \in \left[\frac{259}{279}, \frac{299}{319}\right)$. Finally, for $\alpha \in \left[\frac{299}{319}, 1\right)$ we have $P_b'(\alpha) =$

$\frac{598\alpha+202}{279\alpha+121} + \frac{16{,}000(1+\alpha)}{(279\alpha+121)^2} > 0$. It follows that in order to realize allocation b at the lowest possible cost, the promoter should choose $\alpha = \frac{259}{279} \approx .9283$, along with $p_1 = p_2 = 1$. The resulting costs are $P_b^* = P_b\left(\frac{259}{279}\right) = \frac{1{,}076}{279} \approx 3.8566$.

Minimum Costs of Realizing Allocation c**:** Similarly, the conditions required to realize allocation c are:

$$\Pi_L^a \geq \Pi_L^b \Leftrightarrow p_2 \leq \frac{1-(1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)}{\alpha+(1-\alpha)\rho - z(1-\alpha)\rho(1-\rho)}p_1 \Leftrightarrow p_2 \leq \frac{319\alpha+81}{279\alpha+121}p_1,$$

$$\Pi_L^a \geq r_L \Leftrightarrow p_1 \geq \frac{r_L}{1-(1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)} \Leftrightarrow p_1 \geq \frac{380}{319\alpha+81},$$

$$\Pi_H^c > \Pi_H^a \Leftrightarrow p_2 > \frac{\alpha+(1-\alpha)\lambda - z(1-\alpha)\lambda(1-\lambda)}{\alpha+(1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)}p_1 \Leftrightarrow p_2 > \frac{279\alpha+121}{400}p_1,$$

and

$$\Pi_H^c \geq r_H \Leftrightarrow p_2 \geq \frac{r_H}{\alpha+(1-\alpha)\omega - z(1-\alpha)\omega(1-\omega)} \Leftrightarrow p_2 \geq 1.$$

Again, for an arbitrary value of $\alpha \in [0,1)$, it is straightforward to determine the optimal values of p_1 and p_2 by solving a linear programming problem. Once these values are determined, the minimum prizes paid across the two tournaments can be expressed as a function of α as $P_c(\alpha) = (1+\alpha)(p_1+p_2)$. Analyzing this problem, we have,

- for $\alpha \in \left[0, \frac{679}{1{,}079}\right]$: $p_1 = \frac{7{,}600}{6{,}380\alpha+1{,}620}$ and $p_2 = \frac{5{,}301\alpha+2{,}299}{6{,}380\alpha+1{,}620} + \frac{\epsilon}{1+\alpha}$, so that $P_c(\alpha) = \frac{(1+\alpha)(5{,}301\alpha+9{,}899)}{6{,}380\alpha+1{,}620} + \epsilon$, with $\epsilon > 0$;
- for $\alpha \in \left(\frac{679}{1{,}079}, \frac{259}{279}\right]$: $p_1 = \frac{380}{319\alpha+81}$ and $p_2 = 1$, so that $P_c(\alpha) = \frac{(1+\alpha)(319\alpha+461)}{319\alpha+81}$;
- for $\alpha \in \left(\frac{259}{279}, 1\right)$: $p_1 = \frac{279\alpha+121}{319\alpha+81}$ and $p_2 = 1$, so that $P_c(\alpha) = \frac{(1+\alpha)(598\alpha+202)}{319\alpha+81}$.

For $\alpha \in \left[0, \frac{679}{1{,}079}\right]$ we have $P_c'(\alpha) = \frac{33{,}820{,}380\alpha^2+17{,}175{,}240\alpha-38{,}531{,}620}{(6{,}380\alpha+1{,}620)^2} < 0$. Additionally, examining the behavior of $P_c(\alpha)$ for $\alpha \in \left(\frac{679}{1{,}079}, \frac{259}{279}\right]$, we have $P_c'(\alpha) = 1 - \frac{90{,}440}{(319\alpha+81)^2}$ which is negative for $\alpha = \frac{679}{1{,}079}$ and positive for $\alpha = \frac{259}{279}$. Since $P_c''(\alpha) = \frac{57{,}700{,}720}{(319\alpha+81)^3} > 0$ for $\alpha \in \left(\frac{679}{1{,}079}, \frac{259}{279}\right]$, it follows that over this range $P_c(\alpha)$ is minimized by $\alpha = \frac{\sqrt{90{,}440}-81}{319}$, the unique value for which $P_c'(\alpha) = 0$. Finally, $P_c'(\alpha) = \frac{190{,}762\alpha^2+96{,}876\alpha+362}{(319\alpha+81)^2} > 0$ for $\alpha \in \left(\frac{259}{279}, 1\right)$. From these insights, the least costly way to realize allocation c is by setting $\alpha = \frac{\sqrt{90{,}440}-81}{319} \approx .6888$, along with $p_1 = \frac{380}{\sqrt{90{,}440}} \approx 1.2636$ and $p_2 = 1$. This choice results in payments of $P_c^* = P_c\left(\frac{\sqrt{90{,}440}-81}{319}\right) = \left(1+\frac{\sqrt{90{,}440}-81}{319}\right)\left(1+\frac{380}{\sqrt{90{,}440}}\right) \approx 3.8228$.

Since $3.8228 \approx P_c^* < P_b^* \approx 3.8566$, for this example the least costly way to realize a separating allocation is to offer prizes so that allocation c results.

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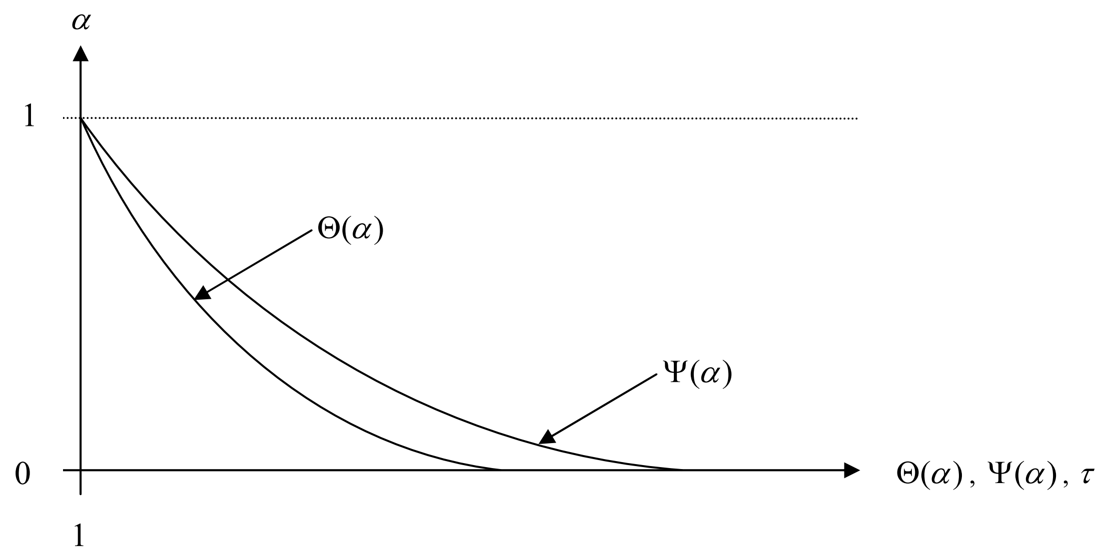
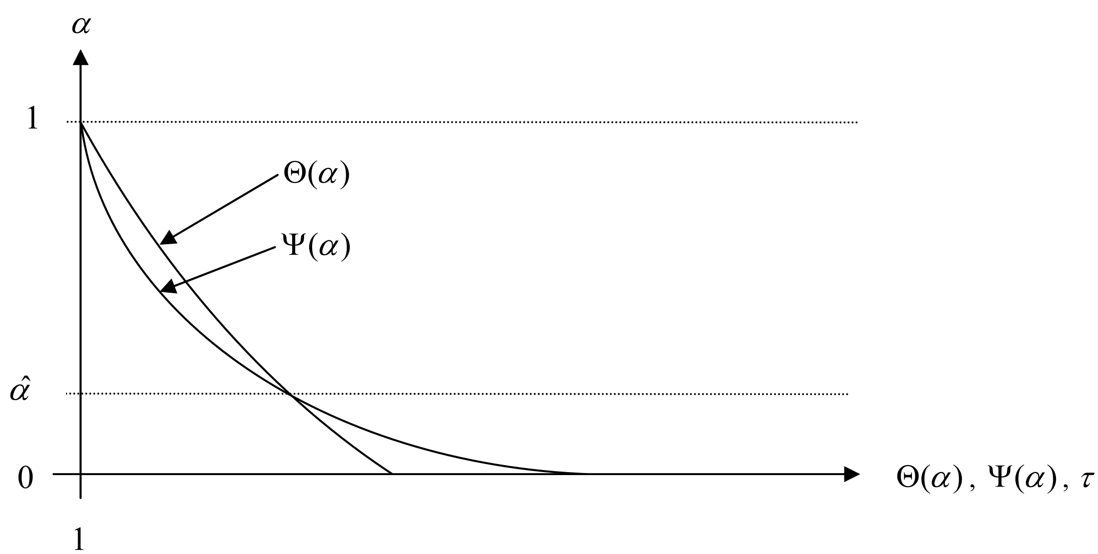
FIGURE 1

Outcome (i). $\eta(\alpha) > 0$ for all $\alpha \in [0,1)$.



Outcome (ii). $\eta(\alpha) > 0$ for $\alpha \in [0,\hat{\alpha})$.



(FIGURE 1 Continued)

Outcome (iii). $\eta(\alpha) < 0$ for all $\alpha \in [0,1)$.