# Strategic approval voting in a large electorate

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**Abstract**

The paper considers approval voting for a large population of voters. It is proven that, based on statistical information about candidate scores, rational voters vote sincerly. It is also proven that if a Condorcet-winner exists, this candidate is elected.

## 1 Introduction

Approval Voting (AV) is the method of election according to which a voter can vote for as many candidates as she wishes, the elected candidate beeing the one who receives the most votes. In this paper two results are established about AV in the case of a large electorate when voters behave strategically: the sincerity of individual behavior (rational voters choose sincere ballots) and the Condorcet-consistency of the choice function defined by Approval Voting (whenever a Condorcet winner exists, it is the outcome of the vote).

Under AV, a ballot is a subset of the set candidates. A ballot is said to be sincere, for a voter, if it shows no "hole" with respect to the voter's preference ranking: if the voter sincerely approves of a candidate $x$ she also approves of any candidate she prefers to $x$. Therefore, under AV, a voter has several sincere ballots at her disposal: she can vote for her most preferred candidate, or for her two, or three, or more most preferred candidates.[1]

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[1] While some scholars see this feature as a drawback (Saari and Van Newenhizen, 1988), observation shows that people appreciate to have this degree of freedom: see Balinski et al. (2003) and Laslier and Van der Straeten (2004) for an experiment, and the survey Brams and Fishburn (2003) on the practice of AV.

It has been found by Brams and Fishburn (1983) that a voter should always vote for her most-preferred candidate and never vote for her least-preferred one. Notice that this observation implies that strategic voting is sincere in the case of three candidates. The debate about strategic voting under AV was made vivid by a paper by Niemi (1984). Niemi argued that, because there is more than one sincere approval ballot, the rule "almost begs the voter to think and behave strategically, driving the voter away from *honest* behavior" (Niemi's emphasis, p. 953). Niemi then gave some examples showing that an approval game cannot be solved in dominant strategies. Brams and Fishburn (1985), responded to this view, but the debate was limited by the very few results available about equilibria of voting games in general and strategic approval voting in particular.

For instance in one chapter of their book, Brams and Fishburn discuss the importance of pre-election polls. They give an example to prove that, under AV, adjustment caused by continual polling can have various effect and lead to cycling even when a Condorcet winner exists (example 7, p.120). But with no defined notion of rational behavior they have to postulate specific (and changing) adjustment behavior from the voters.

With the postulate that voters use sincere and undominated ("admissible") strategies but can use any of these, Brams and Sanver (2003) describe the set of possible winners of an AV election. They conclude that a plethora of candidates pass this test.

None of the above approaches uses an equilibrium theory of approval voting. The rationality hypothesis in a voting situation may have odd implications. For instance it implies that a voter is indifferent between all her strategies as soon as her votes cannot change the result of the election. As a consequence, in an election held under plurality or AV rule, any situation in which one candidate is slightlly (three votes) ahead of the others is trivially a Nash equilibrium.

A breakthrough in the rational theory of voting occurred when it was realized that considering large numbers of voters was technically possible and offered a more realistic account of political elections. This approach was pioneered by Myerson and Weber (1993). In the same paper, AV and other rules are studied on an example with three types of voters and three candidates (a Condorcet cycle). Subsequent papers by Myerson improved the techniques and tackled several problems in the Theory of Voting (Myerson 1998, 2000, 2002). The present paper applies similar techniques to approval voting with no restriction on the number of candidates or voter types.

Rationality implies that a voter can decides of her vote by limiting her conjecture on those events in which her vote is pivotal. In a large electorate, this is a very rare event, and it may seem unrealistic that actual voters deduce their choices from implausible premisses. This might be a wise remark in general, but in the case of approval voting, the rational behavior turns out to be very simple. It can be described as follows. For a rational voter, let $x_1$ be the candidate who she thinks is the most likely to win. This voter will approve of any candidate she prefers to $x_1$. To decide wether she will approve of $x_1$ or not, she compares $x_1$ to the second most likely winner (the "most serious contender"). She will never votes for a candidate she prefers $x_1$ to. This behavior recommends a sincere ballot and its implementation does not require sophisticated computations.

The paper is organized as follows. After this introduction, Section 2 described all the essential features of the model. Section 3 contains the results: the description of rational voting (theorem 1) from which we deduce sincerity, and the description of equilibrium approval scores (theorem 2) from which we deduce Condorcet consistency. Section 4 is a short conclusion. Some computations are provided in an Appendix.

## 2 The model

*Candidates and voters*

Let $X$ denote the finite set of candidates. The number of voters is denoted by $n$ and is supposed to be large. We follow Myerson and Weber (1993) in considering that there exists a finite number of different voter types $\tau \in T$. A voter of type $\tau$ evaluates the utility of the election of candidate $x \in X$ according to a von Neuman and Morgenstern utility index $u_\tau(x)$. A preference on $X$ is a transitive and complete binary relation. In this paper all preferences are supposed to be strict: no voter is indifferent between two candidates (this is essentially for the purpose of simplicity.) Preferences are denoted in the usual way: $x\ P_\tau\ y$ means that $\tau$-voters prefer candidate $x$ to candidate $y$, thus:

$$x\ P_\tau\ y \iff u_\tau(x) > u_\tau(y).$$

Notice that one has to assume utility functions besides preference relations, because voters take decisions under uncertainty. But it turns out that the obtained results can all be phrased in terms of preferences. That is: It will be proven that rational behavior in the considered situation only depends on preferences.

The preference profile is described by a probability distribution over types; let $p_\mathcal{T}$ denotes the fraction of type-$\tau$ voters, with:

$$\sum_{\tau \in T} p_\mathcal{T} = 1.$$

The piece of notation $p[x, y]$ will be used to denote the fraction of voters who prefer $x$ to $y$:

$$p[x, y] = \sum_{\tau \colon x \ P_\tau \ y} p_\mathcal{T}$$

so that for all $x \neq y$, $p[x, y] + p[y, x] = 1$. The number of voters who prefer $x$ to $y$ is $np[x, y]$.

*Voting*

An approval voting ballot is just a subset of the set of candidates. If type-$\tau$ voters choose ballot $B_\tau \subseteq X$, the (relative) **approval score** $s(x)$ of candidate $x$ is the fraction of the electorate that approves of $x$, that is the sum of $p_\mathcal{T}$ over types $\tau$ such that $x \in B_\tau$:

$$s(x) = \sum_{\tau \colon x \in B_\tau} p_\mathcal{T}.$$

Here $0 \leq s(x) \leq 1$, and the number of votes in favor of $x$ is $ns(x)$.

*Errors*

We consider the following perturbations. For each voter and each candidate there is a (small) probability $\mu$ that this vote of the voter with respect to this candidate is not recorded properly. We suppose that these mistake occurs independently of the voter, of the candidate, and of the voter approving or not the candidate[2]. Thus, given a strategy profile, the (intended) number of votes for candidate $x$ is $ns(x)$, and the realized number of votes is a random variable $NV(x)$. Denote by $i = 1, ..., ns(x)$ the voters who approve of $x$ and by $i = ns(x) + 1, ..., n$ the voters who disaprove of $x$. Let $\mu_i$ be equal to 1 with probability $\mu$ and to 0 with the complement probability, then:

$$NV(x) = \sum_{i=1}^{ns(x)} (1 - \mu_i) + \sum_{i=ns(x)+1}^{n} \mu_i.$$

[2]The model here differs from Myerson's Poisson models, in which the population of voters is uncertain. If uncertainty about candidate scores were to arise from uncertainty about the population of voters, then errors would be correlated.

The expected value and the variance of $NV(x)$ are:

$$\begin{aligned} E[NV(x)] &= (1-2\mu)ns(x)+n\mu \\ V[NV(x)] &= n\mu(1-\mu). \end{aligned}$$

Denote by $b(x)$ the realized score:

$$b(x)=\frac{NV(x)}{n}.$$

For $n$ large, the central limit theorem implies that the score $b(x)$ is approximately normal. One can write:

$$\begin{aligned} b(x) &\rightsquigarrow \mathcal{N}\left(a(x),\ \frac{v}{n}\right), \\ a(x) &= (1-2\mu)s(x)+\mu, \\ v &= \mu(1-\mu). \end{aligned}$$

Up to an affine transformation, the realized score is equal to the intended score $s(x)$ plus a normal noise of variance $\frac{v}{n}$.

*More or less serious races*

The elected candidate is the one with highest score. Ties are resolved by a fair lottery. Given a strategy profile and its score vector $s$, the most probable event is that the candidate with highest score wins, but it may be the case that mistakes are such that another candidate does, and it may be the case that two (or more) candidates are so close that one vote can be decisive. In what follows, it will be needed to evaluate the probabilities of some of these events when the number of voters is large.

With $n$ voters, one ballot may have consequences on the result of the election only if the two (or more) first ranked candidates have scores that are within $1/n$ of each other. The probability of such a pivotal event is small if $n$ is large, but some of these events are even much less probable than others. It will be proved that three (or more)-way ties are negligible in front of two-way ties, and that different two-way ties are negligible one in front of the other in a simple manner: The most probable one is a tie between candidates $x_1$ and $x_2$: that is the most "serious race", the second most serious race is $x_1$ against $x_3$, which is less serious than $x_1$ against $x_4$, etc. (The lemma 1 states this point precisely.)

This observation turns out to be sufficient to infer rational behavior. A rational voter will obey a simple heuristic and consider in a sequential way the different occurrences of her being pivotal, according to the magnitude of these events.

*Rational behavior and the Law of Lexicographic Maximization:*

The heuristic that allows voters to compute their optimal behavior can be described in very general terms. Let $D$ be a finite set of possible decisions and $\Omega = \{\omega_1, ..., \omega_N\}$ a finite partition of events, with $\pi$ the probability measure on $\Omega$ and $u$ a von Neuman-Morgenstern utility. The maximization problem can be written

$$\max_{d \in D} \sum_{k=1}^{N} \pi(\omega_k) \; \mathrm{E}u(d, \omega_k),$$

where $\mathrm{E}u(d, \omega_k)$ denotes the expected utility of decision $d$ conditional on event $\omega_k$. Suppose that $\pi$ is such that, for $1 \leq k < k' \leq N$, the probability $\pi(\omega_k)$ is large compared to the probability $\pi(\omega_{k'})$:

$$\pi(\omega_1) \gg \pi(\omega_2) \gg ... \gg \pi(\omega_N).$$

Then the above maximization problem is solved recursively by the following algorithm:

- $D_0 = D$.
- For $k = 1$ to $N$, $D_k = \arg\max_{d \in D_{k-1}} \mathrm{E}u(d, \omega_k)$.

For instance, if the utility $\mathrm{E}u(d, \omega_1)$ in the most probable event $\omega_1$ is maximized at a unique decision, that is to say if $D_1$ is a singleton, then this decision is the right one. And if $D_1$ contains several elements, then searching for the best decision can proceed by going to the next most probable event and neglecting all decisions which are not in $D_1$. The algorithm proceeds until a single decision is reached, or until all events have been considered and thus the remaining decisions give the same utility in all events.

With the above noisy announcement model and a large enough number of voters, it will be proved that the law of lexicographic maximization applies to any voter, so that the voter's rational behavior can be described as follows. Let $x_1$ be the candidate with highest announced score. A rational voter votes for all the candidates that she prefers to $x_1$, she votes against all the candidates she prefers $x_1$ to, and she votes for $x_1$ if and only if she prefers $x_1$ to the candidate with the second highest announced score. In other words: all candidates are judged in comparison with the announced winner, and the announced winner himself is judged in comparison with the announced second-place winner. (This is the content of theorem 1.)

## 3 Results

We start by the lemma which describes how small the probability of a pivotal event is. To do so, some notation is helpful.

**Definition 1** *For each non-empty subset $Y$ of candidates, denote by* $\mathsf{pivot}(n, Y)$ *the event:*

$$\begin{aligned}\forall\, y &\in Y,\ b(y) \geq \max_{x\in X} b(x) - \frac{1}{n}\\ \forall\, y &\notin Y,\ b(y) < \max_{x\in X} b(x).\end{aligned}$$

*Given a score vector $s$, denote by $x_i$, for $i = 1, ..., K$ the candidates ordered so that $s(x_1) \geq s(x_2) \geq ... \geq s(x_K)$; for $i \neq j$, the **magnitude** of the race $\{x_i, x_j\}$ is:*

$$\beta_{i,j} = \lim_{n\to+\infty} \frac{1}{n} \log \Pr[\mathsf{pivot}(n, \{x_i, x_j\})].$$

**Lemma 1** *Suppose that there are no ties in the score vector $s$, then the magnitudes of the two-candidate races involving a given candidate $x_i$ are ordered:*

$$\beta_{i,K} < \beta_{i,K-1} < ... < \beta_{i,i+1} < \beta_{i,i-1} < ... < \beta_{i,1}.$$

*Moreover, if $Y$ contains three or more candidates then*

$$\lim_{n\to+\infty} \frac{1}{n} \log \Pr[\mathsf{pivot}(n, Y)] < \beta_{i,j}$$

*for $x_i, x_j$ two of them.*

(The lemma is proved in the appendix.) Remark that the lemma does not say, for instance, which of the two races $\{1, 4\}$ and $\{2, 3\}$ has the largest magnitude, but this point will not be needed in the sequel. The next lemma describes a voter's best responses without uncertainty.

**Lemma 2** *Suppose a given voter knows how the $(n - 1)$ other voters votes. Let $s^*$ be the highest score, let $Y_1$ be the set of candidates with score $s^*$, and let $Y_2$ be the set of candidates with score $s^* - 1$ ($Y_2$ can be empty). The best responses for this voter only depend on $Y_1$ and $Y_2$. Denoting by $\mathcal{B} = 2^X$ the set of ballots $B \subseteq X$, let $\phi(Y_1, Y_2) \subseteq \mathcal{B}$ be set of best responses.*

- *If $Y_1 = \{x_i\}$ and $Y_2 = \emptyset$: $\phi(Y_1, Y_2) = 2^{\mathcal{B}}$.*

- *If* $Y_1 = \{x_i, x_j\}$, $Y_2 = \emptyset$ *and the voters prefers* $x_j$ *to* $x_i$*:* $\phi(Y_1, Y_2) = \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$.

- *If* $Y_1 = \{x_i\}$, $Y_2 = \{x_j\}$ *and the voters prefers* $x_j$ *to* $x_i$*:* $\phi(Y_1, Y_2) = \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$.

- *If* $Y_1 = \{x_i\}$, $Y_2 = \{x_j\}$ *and the voters prefers* $x_i$ *to* $x_j$*:* $\phi(Y_1, Y_2) = B \setminus \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$.

**Proof.** Clearly the voter's behavior only depends on the scores of the candidates that can get elected, thus the voter can condition her decision on the different possibilities for $Y_1$ and $Y_2$.

If $Y_1$ contains a single candidate and $Y_2$ is empty, then the voter can have no effect on who is elected so that all ballots are identical for him. One can write $\phi(Y_1, Y_2) = 2^{\mathcal{B}}$. Consider now the cases with two candidates.

If $Y_1 = \{x_i, x_j\}$ and $Y_2 = \emptyset$, then the voter should vote for the candidate she prefers among $x_i$ and $x_j$, say $x_j$, and not for the other one. It does not matter for her wether she votes or not for any other candidate since they will not be elected: $\phi(Y_1, Y_2) = \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$.

If $Y_1 = \{x_i\}$ and $Y_2 = \{x_j\}$, then the voter can have either $x_i$ elected or produce a tie between $x_i$ and $x_j$. (1) If she prefers $x_j$ to $x_i$, she also prefers a tie between them to $x_i$ winning, so that a best response for her is any ballot that contains $x_j$ and not $x_i$: $\phi(Y_1, Y_2) = \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$. (2) If she prefers $x_i$ to $x_j$, she should either vote for both $x_i$ and $x_j$, or not vote for $x_j$ and it does not matter wether or not she votes for any other candidate. This means that she must avoid ballots that contain $x_j$ and not $x_i$. Her set of best responses is the complement of the previous set: $\phi(Y_1, Y_2) = B \setminus \{B \in \mathcal{B} : x_i \notin B,\ x_j \in B\}$.

If $Y_1 \cup Y_2$ contains three or more candidates then $\phi(Y_1, Y_2)$ is some subset of $\mathcal{B}$ that we do not need to specify. Notice that, in all cases, the set of best responses $\phi(Y_1, Y_2)$ does not depend on $n$. ∎

We can now state and prove the key result in this paper.

**Theorem 1** *Let* $s$ *be a score vector with two candidates at the two first places and no tie:* $x_1$ *and* $x_2$ *such that* $s(x_1) > s(x_2) > s(y)$ *for* $y \in X$, $y \neq x_1, x_2$*. There exists* $n_0$ *such that, for all* $n > n_0$*, type-*$\tau$ *voter has a unique best-response* $B^*_\tau$*:*

- *for* $\tau$ *such that* $u_\tau(x_1) > u_\tau(x_2)$, $B^*_\tau = \{x \in X : u_\tau(x) \geq u_\tau(x_1)\}$,

- *for* $\tau$ *such that* $u_\tau(x_1) < u_\tau(x_2)$, $B^*_\tau = \{x \in X : u_\tau(x) > u_\tau(x_1)\}$.

**Proof.** Suppose first that $u_\tau(x_1) > u_\tau(x_2)$. To prove that $B^*_\tau$ is the unique best response, we prove that any other ballot $B$ is not.

(i) Suppose that there exits $x \in X$ such that $u_\tau(x) > u_\tau(x_1)$ and $x \notin B$. Let then $B' = B \cup \{x\}$.

To compare ballots $B$ and $B'$, the voter computes the difference

$$\Delta = \sum_Y \Pr[\mathsf{pivot}(n, Y)] \, \mathrm{E}\left(u_\tau(B', Y) - u_\tau(B, Y)\right),$$

where $\mathrm{E}u_\tau(B, Y)$ denotes the expected utility for her of choosing ballot $B$ knowing the event $\mathsf{pivot}(n, Y)$. The events $\mathsf{pivot}(n, Y)$ with $x \notin Y$ have no consequences therefore the sum in $\Delta$ can run on the subsets $Y$ such that $x \in Y$. Among these is $\{x, x_1\}$. From the lemma 2, it follows that in all three case ($Y_1 = \{x, x_1\}$ and $Y_2 = \emptyset$, $Y_1 = \{x\}$ and $Y_2 = \{x_1\}$, $Y_1 = \{x_1\}$ and $Y_2 = \{x\}$), the voter's utility is strictly larger voting for $x$ than not. Thus $\mathrm{E}\left(u_\tau(B', \{x, x_1\}) - u_\tau(B, \{x, x_1\})\right) > 0$.

As proved in the lemma 1, the probability of $\mathsf{pivot}(n, Y)$ is exponentially decreasing in $n$ if $\#Y \geq 2$, and comparison of magnitudes shows that for all $Y \subseteq X$ with $x \in Y$, $Y \neq \{x, x_1\}$ and $\#Y \geq 2$,

$$\lim_{n \to \infty} \frac{\Pr[\mathsf{pivot}(n, Y)]}{\Pr[\mathsf{pivot}(n, \{x, x_1\})]} = 0.$$

One can thus factor out $\Pr[\mathsf{pivot}(n, \{x, x_1\})]$ in $\Delta$ and it follows that, for $n$ large enough, $\Delta > 0$. This establish that $B$ is not a best response.

(ii) Suppose that there exists $x \in X$ such that $u_\tau(x) < u_\tau(x_1)$ and $x \in B$. Let then $B' = B \setminus \{x\}$. The reasoning is the same as in the previous case: The relevant events are again $\mathsf{pivot}(n, Y)$ for $x \in Y$, the voter's utility is strictly larger not voting for $x$, and the relevant race is again $\{x, x_1\}$.

(iii) Suppose that $x_1 \notin B$. The conclusion follows considering $B' = B \cup \{x_1\}$, the race $\{x_1, x_2\}$ is here relevant and has the highest magnitude.

From items (i), (ii) and (iii) it follows that the voter's best response must satisfy (i) $u_\tau(x) \geq u_\tau(x_1)$ if $x \in B$, (ii) $u_\tau(x) \leq u_\tau(x_1)$ if $x \notin B$, and (iii) $x_1 \in B$. Therefore $B^*_\tau = \{x \in X : u_\tau(x) \geq u_\tau(x_1)\}$ as stated. The argument is identical in the case $u_\tau(x_1) < u_\tau(x_2)$. ■

Notice that the previous result implies that, for $n$ large enough, all voters of a given type use the same strategy when responding to a score vector that satisfy the mentionned properties. The next definition is standard in the study of approval voting.

**Definition 2** *A ballot $B$ is* ***sincere*** *for a type-$\tau$ voter if $u_\tau(x) > u_\tau(y)$ for all $x \in B$ and $y \notin B$.*

As a direct consequence of the previous theorem, one gets the following corollary.

**Corollary 1** *In the absence of tie, rational behavior is sincere.*

It should be emphasized that the previous theorem and corollary are true even if the announcement is not an equilibrium, they actually describe the voter's response to a conjecture she holds about the candidate scores.

**Theorem 2** *Let $s$ be a score vector with two candidates at the two first places and no tie: $x_1$ and $x_2$ such that $s(x_1) > s(x_2) > s(y)$ for $y \in X$, $y \neq x_1, x_2$. There exists $n_0$ such that, for all $n > n_0$, if $s$ is the score vector of an equilibrium of the game with $n$ voters, then*

- *the score of the first-ranked candidate is his majoritarian score against the second-ranked candidate:*

$$s(x_1) = p\left[x_1, x_2\right],$$

- *the score of any other candidate is his majoritarian score against the first-ranked candidate:*

$$x \neq x_1 \Rightarrow s(x) = p\left[x, x_1\right].$$

**Proof.** This is a direct consequence of theorem 1. Each voter approves of $x_1$ if and only of she prefers $x_1$ to $x_2$. For $x \neq x_1$, she approves of $x$ if and only if she prefers $x$ to $x_1$. ■

**Definition 3** *A **Condorcet winner** at profile $p$ is a candidate $x_*$ such that $p\left[x_*, x\right] > 1/2$ for all $x \neq x_*$.*

**Corollary 2** *If there is an equilibrium with no tie, the winner of the election is a Condorcet winner.*

**Proof.** Notice that, as a consequence of theorem 1, no voter votes simultaneously for $x_1$ and $x_2$, and all of them vote for either $x_1$ or $x_2$. Here,

$$s(x_2) = 1 - s(x_1) < s(x_1)$$

implies $s(x_2) < 1/2$, thus, for $x \neq x_1$

$$p\left[x, x_1\right] = s(x) \leq s(x_2) < \frac{1}{2}.$$

■

## 4 Concluding remarks

Notice that the corollary 2 implies that for a preference profile with no Condorcet winner, there can be no equilibrium no-tie announcement. Approval voting is by no mean a solution to the so-called Condorcet paradox. Indeed, approval voting retains the basic disequilibrium property of majority rule: if there is no Condorcet winner then any announced winner will be defeated, according to approval voting, by another candidate, preferred to the former by more than half of the population.

The key to this result is that, with a large population of voters, voters strategic thinking has to put special emphasis on pairwise comparisons of candidates, even if the voting rule itself is not defined in terms of pairwise comparisons. Therefore a natural avenue of research is to check wether the arguments here provided in the case of approval voting can be extended to other voting rules.

## A Appendix: magnitude of pivotal events

Let $b_k$, for $k = 1, ..., K$, independent normal random variables. The mean value of $b_k$ is denoted by $a_k$ and one supposes $a_1 \geq a_2 \geq ... \geq a_K$. The variance of $b_k$ is $v/n$, where $v$ is a fixed parameter and $n$ is a (large) number. All variables $b_k$ have the same variance. For convenience we denote

$$a_{ij} = \frac{a_i + a_j}{2}.$$

The computations are explained in this appendix under the assumption that for no $i \neq j$ there exists $k$ such that $a_k = a_{ij}$. It is not more difficult to arrive at the conclusion through the same type of computations in the case where for some $i \neq j$ there exists $k$ such that $a_k = a_{ij}$.

Consider the event $\mathsf{pivot}(n, \{i, j\})$ of a race between two candidates:

$$b_j \in [b_i - 1/n,\ b_i + 1/n[\ ,$$
$$\forall k \neq i, j\ ,\ \ b_k + 1/n < b_i, b_j$$

We will prove that

$$\lim_{n \to +\infty} \frac{1}{n} \log \Pr [\mathsf{pivot}(n, \{i, j\})] = -\sum_{k=1}^{k_{ij}} \frac{1}{2v} (a_k - a_{ij})^2 - \frac{(a_i - a_j)^2}{4v}, \quad (1)$$

where $k_{ij}$ is the last integer $k$ such that $a_k > a_{ij}$.

**Tie between the two first candidates**

To start by the simplest case, consider the event $\mathsf{pivot}(n, \{x_1, x_2\})$: It is the disjoint union of the two events

$$\forall k = 3, ..., K\ ,\ \ b_k < b_1 - 1/n < b_2 < b_1$$

and

$$\forall k = 3, ..., K\ ,\ \ b_k < b_2 - 1/n < b_1 < b_2$$

The probability of the former writes :

$$\int_{b_1=-\infty}^{+\infty} f(b_1; a_1, \frac{v}{n}) \int_{b_2=b_1-1/n}^{b_1} f(b_2; a_2, \frac{v}{n}) \prod_{k=3}^{K} F(b_1 - \frac{1}{n}; a_k, \frac{v}{n})\ db_2\ db_1,$$

where $f$ and $F$ denote the normal density and cumulative functions

$$f(t; \mu, v) = \frac{1}{\sqrt{2v\pi}} \exp -\frac{1}{2v}(t - \mu)^2,$$
$$F(t; \mu, v) = \int_{u=-\infty}^{t} f(u; \mu, v)\ du.$$

Because the integral

$$\int_{b_2=b_1-1/n}^{b_1+1/n} f(b_2; a_2, v/n)\; db_2$$

on $b_2$ is close to $\frac{1}{n} f(b_1; a_2, v/n)$, the probability of the former event is close to $A_{12}/2$, with:

$$A_{12} = \frac{2}{n} \int_{b_1=-\infty}^{+\infty} \prod_{k=3}^{K} F(b_1 - 1/n; a_k, v/n)\; f(b_1; a_1, v/n)\; f(b_1; a_2, v/n)\; db_1.$$

The same approximation is valid for the complementary event, so that the probability of the race $\{i, j\}$ is approximately

$$\Pr\left[\mathsf{pivot}(n, \{i, j\})\right] \simeq A_{12}$$

The product of two normal densities can be written

$$\begin{aligned} & f(b_1; a_2, v/n)\; f(b_1; a_1, v/n) \\ = \; & \frac{n}{2v\pi} \exp -\frac{n}{2v} \left[(b_1 - a_1)^2 + (b_1 - a_2)^2\right] \\ = \; & \frac{n}{2v\pi} \exp -\frac{n}{v} \left[(b_1 - a_{12})^2 + \frac{(a_1 - a_2)^2}{4}\right] \\ = \; & \frac{1}{2}\sqrt{\frac{n}{v\pi}} \left(\exp -\frac{n\,(a_1 - a_2)^2}{4v}\right) f(b_1; a_{12}, \frac{v}{2n}) \end{aligned}$$

so that one gets:

$$A_{12} = \alpha_{12} \int_{b_1=-\infty}^{+\infty} \prod_{k=3}^{K} F(b_1 - 1/n; a_k, v/n)\; f(b_1; a_{12}, \frac{v}{2n})\; db_1,$$

with

$$\alpha_{12} = \frac{1}{\sqrt{nv\pi}} \exp -\frac{n\,(a_1 - a_2)^2}{4v}.$$

For $n$ large, $F(b_1 - 1/n; a_k, v/n)$ tends to 1 if $b_1 > a_k$ and to 0 if $b_1 < a_k$, and the density $f(b_1; a_{12}, \frac{v}{2n})\; db_1$ tends to a Dirac mass at point $b_1 = a_{12}$. Here $a_k < a_{12}$, so that the integral in $A_{12}$ tends to 1 and one finally gets:

$$\log A_{12} \simeq -\, n\, \frac{(a_1 - a_2)^2}{4v}.$$

**General case**

More generally, consider $i$ and $j$ such that $1 < i < j$. The probability of the event $\mathsf{pivot}(n, \{i, j\})$ is approximately

$$A_{ij} = \alpha_{ij} \int_{b_i=-\infty}^{+\infty} \prod_{k \neq i,j} F(b_i - 1/n; a_k, v/n)\ f(b_i; a_{ij}, \frac{v}{2n})\ db_i,$$

with

$$\alpha_{ij} = \frac{1}{\sqrt{nv\pi}} \exp -\frac{n\,(a_i - a_j)^2}{4v}.$$

One still has that the density $f(b_i; a_{ij}, \frac{v}{2n})\ db_i$ tends to a Dirac mass at point $b_i = a_{ij}$, but now $F(a_{ij}; a_k, v/n)$ tends to 1 only for those $k$ such that $a_k < a_{ij}$. Denote them by $k = k_{ij} + 1, k_{ij} + 2, ..., K$. For $k = 1, ..., k_{ij}$, one uses the standard approximation of the tail of the normal distribution. Recall that, for $t >> 1$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} e^{-\frac{1}{2}u^2}\ du &\simeq \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} e^{\frac{1}{2}t^2 - tu}\ du \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_t^{+\infty} e^{-tu}\ du \\ &= \frac{1}{t\sqrt{2\pi}} e^{-\frac{1}{2}t^2}. \end{aligned}$$

One so gets that for $k = 1, ..., k_{ij}$,

$$F(a_{ij}; a_k, v/n) = \frac{1}{(a_k - a_{ij})} \sqrt{\frac{v}{2n\pi}} \exp -\frac{n}{2v}\,(a_k - a_{ij})^2\,.$$

Then

$$\log A_{ij} \simeq -\sum_{k=1}^{k_{ij}} \frac{n}{2v}\,(a_k - a_{ij})^2 - \frac{n\,(a_i - a_j)^2}{4v}$$

and the expression (1) follows. The first part of lemma 1 is easily deduced from these formulae. The second part of the lemma (about three way ties or more) is easily obtained by the same kind of arguments.

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