Long Persuasion in Sender-Receiver Games*

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Abstract

This paper characterizes the set of all Nash equilibrium payoffs achievable with unmediated communication in "sender-receiver" games (i.e., games with an informed expert and an uninformed decisionmaker) in which the expert's information is certifiable.

KEYWORDS: Cheap talk, communication, disclosure of certifiable information, jointly controlled lotteries, long conversation, persuasion, verifiable types.

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1 Introduction

As is now well known in the literature on cheap talk games (i.e., games with costless, non-binding, and unmediated communication), repeated communication generally allows to reach outcomes that cannot be implemented with unilateral or single-period communication, even if only one player is privately informed (see Aumann and Hart, 2003, Forges, 1984, 1990a, Krishna and Morgan, 2004, Krishna, 2004, and Simon, 2002). In this paper we study this feature in "sender-receiver" communication games with partially verifiable types, i.e., games in which the informed player (the expert, or "sender") has the ability to voluntarily certify partial or full information to the uninformed decisionmaker (the "receiver"). We characterize the set of all Nash equilibrium payoffs achievable with unmediated communication, by allowing players to talk and negotiate for many periods. At each stage of this communication phase, the sender can certify part of his information.

This possibility of certifying information, in addition to make cheap talk claims, is justified in many concrete interactive decision situations. For example, players may present physical proofs such as documents, observable characteristics of a product, endowments or costs. Alternatively, in economic or legal interactions there may be labels, penalties for perjury, false advertising and warranty violations, or accounting principles that allow agents to submit substantive evidence of their information. Interesting phenomena similar to those obtained in the cheap talk case arise in games with strategic information certification. We show that several bilateral communication stages and delayed information certification allow to convey substantive information and lead to equilibrium outcomes that are not achievable when only one signalling stage is permitted. A leading example is analyzed in Section 2.

Our study is closely related to Aumann and Hart (2003) who characterized Nash equilibria of long cheap talk games, i.e., the subset of communication equilibria (Forges, 1986a, 1990b; Myerson, 1982, 1986) that use only plain conversation. (A communication equilibrium is a Nash equilibrium of an extension of the game allowing the players to communicate for several periods, with the help of a mediator, before they make their decisions.) Here, we characterize the analog of that subset for certification equilibria (Forges and Koessler, 2005). (A certification equilibrium is defined as a communication equilibrium, except that each player can also transmit reports from a type-dependent set, i.e., can send certified information into the communication system.)

Our general model, presented in Section 3, is a one-side incomplete information game with an expert (the informed player) and a decision maker (the uninformed player). The expert has a finite set of types, or private signals, with a common prior probability distribution. The payoff of each player depends both on the expert's type and on the decision maker's action. The decision maker chooses his action without observing the expert's type. However, before the action phase, but after the expert learns his type, players are able to directly communicate with each other. Communication is assumed strategic, non-binding (no commitment and no contract are allowed), costless, payoff-irrelevant, and unmediated (decentralized). In addition, players are not able to observe private payoff-irrelevant signals ("private sunspots") and there is no extraneous noise in communication, which thus takes place "face-to-face". However, mixed (randomized) strategies are allowed in both the communication and action phases.

Contrary to usual cheap talk games (Crawford and Sobel, 1982; Ben-Porath, 2003; Gerardi, 2004; Krishna and Morgan, 2004), the set of messages available to the expert is type-dependent, reflecting his ability to certify his information. We will assume that the expert has always the opportunity to remain silent, i.e., to send a meaningless message to the decision maker. Furthermore, to guarantee that our geometric characterization be sufficient for an equilibrium, we will require that players have access to a rich language. More precisely, we make the following assumption: for any set of types containing his real type, the expert has a sufficiently large set of messages allowing him to certify that is real type belongs to that set.

In the associated one-shot communication game the expert learns his type and sends a message to the decision maker, who then chooses an action. Such games are sometimes called persuasion or disclosure games (see, e.g., Milgrom, 1981; Milgrom and Roberts, 1986; Seidmann and Winter, 1997). To the best of our knowledge, this literature has always focused on one-shot information revelation with very specific assumptions on players' preferences, like single-peakedness, strict concavity and monotonicity. Our first result (Theorem 1) is a full characterization of Nash equilibrium payoffs of one-shot communication games with certifiable information. Roughly, equilibrium payoff vectors are obtained by convexifying the graph of an extended set of equilibrium payoffs of the basic game without communication (the silent game), by keeping the payoff of the informed player constant and individually rational. Several geometric illustrations involving full, partial and/or no information revelation are provided.

In a multistage communication game, the talking phase has an arbitrary large number of periods. In each communication period *both* players simultaneously send a message that depends on the history of play up to that period. The informed player's message may also depend on his private information. As in Hart (1985) and Aumann and Hart (2003), our equilibrium characterization makes use of the mathematical concepts of diconvexification and dimartingale. In Theorem 2 we show that the set of equilibrium payoffs of any multistage communication game can be characterized in terms of starting points of di-martingales converging to the graph of an extended set of equilibrium payoffs of the silent game, and staying in an adapted set of individually rational payoffs during the whole process. This theorem is illustrated with our leading example. An equivalent representation in terms of diconvexification is provided.

The paper is organized as follows. In the next section we present our leading example. Section 3 describes the model. Section 4 formulates the geometric characterizations of the equilibrium payoffs and then explains and illustrates them through examples. Formal proofs for Theorem 1 (signalling) and Theorem 2 (persuasion with a deadline) are provided in Sections 5 and 6, respectively. We conclude and discuss extensions of the model in Section 7. The Appendix contains several additional examples.

2 An Example

In this section we introduce and motivate our study of multistage and bilateral communication with certifiable information through an example. This example motivates our study concerning two aspects. First, the example illustrates how by certifying their information players can reach equilibrium outcomes that cannot be achieved by any communication system with non-certifiable information (even with a mediator). Second, the example shows that delayed information revelation and multiple rounds of communication and compromises may be required to achieve some equilibrium payoffs, even if only one player has substantive information.

Consider two players, player 1 (the expert) and player 2 (the decisionmaker), who are playing a strategic form game which depends on the true state of Nature, k_1 or k_2, with probability 1/2 each (see Figure 1). Player 1 knows the true state of Nature but player 2 does not know the actual game being played. Player 2 must choose action j_1, j_2, j_3, j_4 or j_5, and player 1 has no choice. The expected payoff of player 2 depending on his action and his belief $p \in [0, 1]$ about the state of Nature k_1 is represented by Figure 2 on the next page (the thick lines denote his best-reply payoff).

	j_1	j_2	j_3	j_4	j_5
k_1	$5, 0$	$3, 4$	$0, 7$	$4, 9$	$2, 10$
k_2	$1, 10$	$3, 9$	$0, 7$	$5, 4$	$6, 0$

Figure 1: Introductory Example.

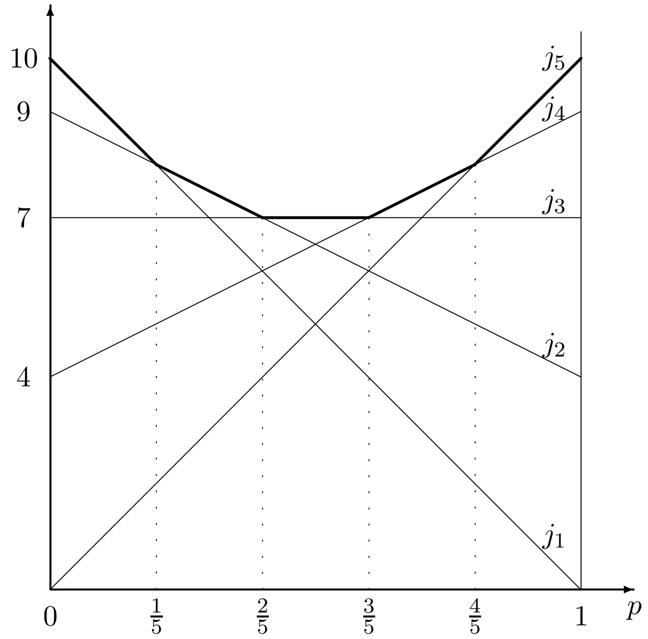

Figure 2: Player 2's Expected Payoffs (thin lines) and Best-Reply Expected Payoffs (thick lines) in the Introductory Example.

Without communication possibilities (in the "silent game"), the only equilibrium payoff is $(0, 7)$ since action j_3 yields the best expected payoff for player 2 given his prior belief $p = 1/2$. If, before choosing his action, player 2 is able to talk with player 1, and no information can be certified (or verified) concerning the true state of Nature then, whatever the communication possibilities (even repeated), the unique equilibrium payoff remains $(0, 7)$. Information transmission is not possible here because if player 2 chooses his action conditionally on the messages sent by player 1 then, whatever the true state of Nature, player 1 has always an incentive to use the messages he should have sent at the other state (since information is not certifiable, the set of available messages of player 1 is the same whatever the true state of Nature). In other words, information which is transmitted to player 2 is never credible, even if in every state it is to the advantage of both players that player 1 tells player 2 the truth, and that this latter believes him. Notice that allowing unboundedly long communication, or even adding a mediator, cannot help here: one can verify that the unique communication equilibrium[1] is the singleton equilibrium $\{j_3\}$ of the silent game.

Assume now that player 1 can voluntarily certify his information concerning the real state of Nature. That is, his informational reports are assumed truthful (the making of

[1] See, e.g., Myerson (1994).

false statements is prohibited), but he may withhold his information since it is not required that he makes positive disclosures. To begin with, assume that player 1 can only send a single message and that player 2 cannot send any message. More precisely, assume that player 1 can choose between two types of reports: either he certifies his information (he sends the message $\mathsf{m} = \mathsf{c}^1$ if the real state is k_1 and the message $\mathsf{m} = \mathsf{c}^2$ if the real state is k_2), or he certifies no information[2] (he sends the message $\mathsf{m} = \overline{\mathsf{m}}$ which is available whatever the true state). It is easy to see that full revelation of information is now possible: player 2 chooses action j_5 if player 1 reveals that the true state is k_1, he chooses j_1 if player 1 reveals that the true state is k_2, and chooses j_3 if player 1 reveals nothing. In such a situation, player 1 has no incentive not to reveal his information because his payoff would be zero instead of 2 in state k_1 and 1 in state k_2. Obviously, player 2 also behaves rationally because he chooses the best action for him in each state of Nature.

As in usual cheap talk games, the non-revealing outcome is also an equilibrium since player 2 can always ignore what player 1 says and choose action j_3.[3]

The two equilibrium outcomes described above (the perfectly revealing one and the non-revealing one) are *not* the only Nash equilibria of the one-shot communication game with certifiable information. Indeed, if we allow player 1 to randomize, then there are two other *partially revealing* equilibria. One of them is better for player 1 than any of the previous pure strategy equilibria since it gives him a payoff of 2 whatever his type. In this equilibrium, denoted by PRE1, player 1 certifies his type (sends the message c^1) with probability 1/3 and remains silent (sends the message $\overline{\mathsf{m}}$) with probability 2/3 in k_1, and he always remain silent (sends the message $\overline{\mathsf{m}}$) in state k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{\mathsf{m}}) = \frac{\Pr(\overline{\mathsf{m}}|k_1)\Pr(k_1)}{\Pr(\overline{\mathsf{m}})} = \frac{2/6}{2/6+1/2} = 2/5$ and $\Pr(k_1 \mid \mathsf{c}^1) = 1$, so he plays action j_5 when he receives the message c^1 and is indifferent between j_2 and j_3 when he receives the message $\overline{\mathsf{m}}$. If he plays j_2 with probability 2/3 and j_3 with probability 1/3 after $\overline{\mathsf{m}}$, and if he plays j_1 after the off-equilibrium message c^2 then player 1 has no incentive to deviate: in k_1 he gets a payoff of 2 if he sends the message c^1 and also $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends the message $\overline{\mathsf{m}}$, so he is indifferent between the two messages; in k_2 he gets a payoff of 1 if he sends the message c^2 and $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends the message $\overline{\mathsf{m}}$, so he strictly prefers to send message $\overline{\mathsf{m}}$.

In the second partially revealing equilibrium with randomized certification, denoted by PRE2, player 1 always remains silent (sends the message $\overline{\mathsf{m}}$) in state k_1 and he certifies his type (sends the message c^2) with probability 1/3 and remains silent (sends the message

[2] Or, equivalently, he remains silent.

[3] However, notice that contrary to the fully revealing equilibrium, the non-revealing equilibrium is based on irrational choices outside the equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature.

$\overline{\mathsf{m}}$) with probability 2/3 in k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{\mathsf{m}}) = 3/5$ and $\Pr(k_1 \mid \mathsf{c}^2) = 0$, so he plays action j_1 when he receives the message c^2 and is indifferent between j_3 and j_4 when he receives the message $\overline{\mathsf{m}}$. If he plays j_3 with probability 4/5 and j_4 with probability 1/5 after message $\overline{\mathsf{m}}$, and if he plays j_3 after the off-equilibrium message c^1 then it can be checked as before that player 1 has no incentive to deviate: in k_1 he gets a payoff of 0 if he sends the message c^1 and $(4/5) \times 0 + (1/5) \times 4 = 4/5$ if he sends the message $\overline{\mathsf{m}}$, so he strictly prefers to send message $\overline{\mathsf{m}}$; in k_2 he gets a payoff of 1 if he sends the message c^2 and $(4/5) \times 0 + (1/5) \times 5 = 1$ if he sends the message $\overline{\mathsf{m}}$, so he is indifferent between the two messages.[4]

Now, we show that if player 1 is able to talk with player 2 during several bilateral communication rounds, then he is able to reach even a higher equilibrium payoff of 3 whatever his type. This equilibrium can be achieved in three communication stages. In the first two communication stages there is no information certification, and in the last communication stage player 1 will reveal (and certify) his information to player 2 conditionally on what both players said in the previous communication stages. The whole communication phase can work as follows.

Equilibrium (i).

In the first communication stage player 1 partially reveals (without certifying) his information by using a random communication strategy which transmits the correct information with probability 3/4 so as to leave some doubt in player 2's mind. That is, he sends a message $m = \mathsf{a}$ with probability 3/4 if the real state is k_1 and with probability 1/4 if the real state if k_2. Symmetrically, he sends a message $m = \mathsf{b}$ with probability 3/4 if the real state is k_2 and with probability 1/4 if the real state if k_1 (the labeling of these two messages is irrelevant but both messages a and b are cheap talk messages: they must be available to player 1 whatever his type). From Bayes' rule, player 2 will believe state k_1 with probability 3/4 if he receives the message a and with probability 1/4 if he receives the message b. Hence, substantive but only partial information is conveyed, without any information certification. Communication cannot stop now since, as seen before, player 1 would have an incentive to deviate by *always* sending the message a at k_1 and the message b at k_2. Assume that player 2 chooses action j_2 whenever he receives the message b. This choice is rational given his beliefs. Otherwise, when the message a is sent, they agree on a jointly controlled $\frac{1}{2} - \frac{1}{2}$ lottery to reach the following compromise (this second

[4]Notice that contrary to the previous partially revealing equilibrium, this equilibrium is based on irrational choices outside the equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature (the equilibrium is not subgame perfect).

communication stage conveys no substantive information, i.e., no information about the fundamentals of the game).[5] If head occurs, then communication stops and thus player 1 chooses action j_4. On the contrary, if tail occurs, then player 1 certifies his information in the last communication stage (he sends the message c^k if the real state is k). Then, player 2 chooses action j_5 if c^1 is sent and action j_1 if c^2 is sent. Player 1 has no incentive to deviate if, for example, player 2 chooses action j_3 when player 1 deviates in the last communication stage by remaining silent. The whole communication and decision process in this equilibrium is summarized by Figure 3 (where "JCL" stands for "jointly controlled lottery"). Player 2's expected payoff is

$$\frac{1}{2}\left[\frac{3}{4}\left(\frac{1}{2}\,9+\frac{1}{2}\,10\right)+\frac{1}{4}\,4\right]+\frac{1}{2}\left[\frac{1}{4}\left(\frac{1}{2}\,4+\frac{1}{2}\,10\right)+\frac{3}{4}\,9\right]=\frac{133}{16}=8.3125,$$

and player 1's expected payoff is 3 whatever his type.

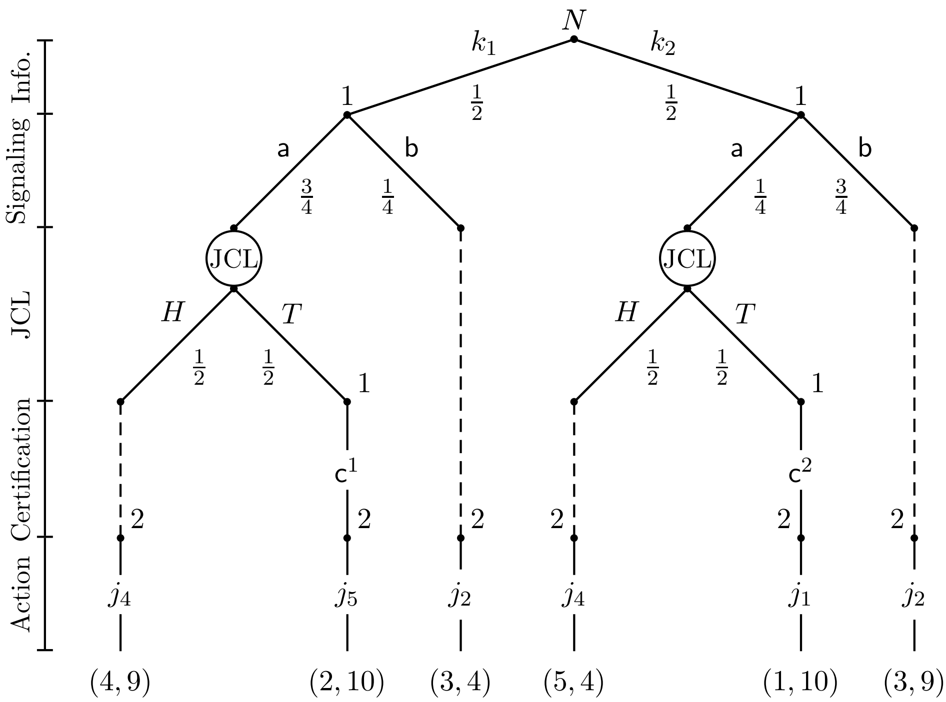

Figure 3: An Equilibrium Communication and Decision Tree for the Introductory Example (Equilibrium (i)).

[5] A jointly controlled lottery is a mechanism that generates a uniform probability distribution on any finite set from private random communication strategies so that a unilateral deviation does not change the probability distribution. For example, a $\frac{1}{2}-\frac{1}{2}$ lottery can be generated as follows: each player chooses a message in $\{\mathsf{a},\mathsf{b}\}$ at random, both players announce their choices simultaneously and the outcome is head (H) if the messages coincide and tail (T) otherwise.

Equilibrium (ii).

The communication and decision process is similar to the previous one except that in the first signaling stage the messages are sent with probability $(3/5, 2/5)$ instead of $(3/4, 1/4)$.

In Section 4 we will provide geometric characterizations of all possible equilibria of unilateral or bilateral communication games with certifiable information. For example, the previous fully revealing equilibrium (FRE) and the two partially revealing equilibria (PRE1 and PRE2) of the unilateral communication game are simply characterized by the intersection points FRE, PRE1 and PRE2 in Figure 7 on page 19. The non-existence of informative equilibrium in direct communication games (of bounded length) in which information is not certifiable is simply characterized by the fact that the gray solid lines in Figure 7 never intercept. The geometric characterization of the equilibria described above requiring information certification as well as multiple and bilateral communication stages is slightly more complex, and will be illustrated in subsection 4.3.

3 Model

We consider two players: player 1 (the "sender", informed player, interested party or expert) and player 2 (the "receiver" or uninformed decisionmaker (DM)). J ($|J| \geq 2$) is the finite action set of player 2 (player 1 has no action). K ($|K| \geq 2$) is the set of possible states (or types of player 1), with a common prior probability distribution $p = (p^1, \ldots, p^k, \ldots, p^K) \in \Delta(K)$. The normal form "game" Γ^k in state $k \in K$ is given by two payoff vectors $A^k \equiv [A^k(j)]_{j \in J}$ and $B^k \equiv [B^k(j)]_{j \in J}$. That is, $A^k(j)$ and $B^k(j)$ are the payoffs to player 1 and player 2, respectively, when player 2 chooses action $j \in J$ and the state is $k \in K$.

3.1 Silent Game: Non-Revealing Equilibria

The *silent game*, denoted by $\Gamma(p)$, consists of two phases. In the *information phase* a state $k \in K$ is picked at random according to the probability distribution p. Player 1 is perfectly informed about the true state k, while player 2 is not. In the *action phase*, player 2 chooses an action $j \in J$ and players are paid off in accordance with the normal form game $\Gamma^k = (A^k, B^k)$. That is, player 1 and player 2 receive payoffs $A^k(j)$ and $B^k(j)$, respectively, where k is the true state.

A mixed strategy of player 2 in the silent game $\Gamma(p)$ is simply a mixed action $y \in \Delta(J)$. We extend payoff functions linearly to mixed actions: $A^k(y) = A^k \cdot y = \sum_{j \in J} y(j)\, A^k(j)$ and $B^k(y) = B^k \cdot y = \sum_{j \in J} y(j)\, B^k(j)$.

The set of (Bayesian) Nash equilibria of the silent game $\Gamma(p)$ is simply the set of optimal mixed actions for player 2 in the silent game $\Gamma(p)$:

$$Y(p) \equiv \arg\max_{y \in \Delta(J)} \underbrace{\sum_{k \in K} p^k \, B^k(y)}_{p\,B(y)} = \left\{ y \in \Delta(J) : \sum_{k \in K} p^k \, B^k(y) \geq \sum_{k \in K} p^k \, B^k(j), \ \forall \, j \in J \right\}.$$

It is also called the set of *non-revealing equilibrium outcomes* at p.

Remark 1 A pure action is always sufficient to maximize the decisionmaker's payoff. So, for all $j,\ j' \in \operatorname{supp}[Y(p)]$ and $y \in \Delta(J)$ we have $p\,B(j) = p\,B(j') \geq p\,B(y)$. However, mixed actions might be used by player 2 for two reasons: (i) on the equilibrium path to make player 1 indifferent between several messages, and (ii) off the equilibrium path to punish (minmax) player 1.

The resulting equilibrium payoffs are the $(K+1)$-dimensional vectors (a, β), where $a = (a^1, \ldots, a^K) \in \mathbb{R}^K$, $a^k = A^k(y)$ is the payoff of player 1 of type k and the scalar $\beta = p\,B(y) \in \mathbb{R}$ is player 2's expected payoff (expectation over k). Let $\mathcal{E}(p) \subseteq \mathbb{R}^K \times \mathbb{R}$ be the set of equilibrium payoffs of $\Gamma(p)$, also called the set of *non-revealing equilibrium payoffs* at p. That is,

$$\mathcal{E}(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : \exists \, y \in Y(p), \ a = A(y), \ \beta = p\,B(y)\}.$$

3.2 Unilateral Communication Game: Signalling

Here, we consider only direct (unmediated and noiseless) and unilateral communication, from player 1 to player 2, as in standard sender-receiver/persuasion/signalling games. The set of available messages ("keystrokes") of player 1 is state-dependent and is denoted by $\mathsf{M}(k)$ when his type is k (so, the "keyboard" of player 1 depends on his type). Let $\mathsf{M}^1 = \bigcup_{k \in K} M(k)$ be the set of all messages that player 1 could send. The set $\bigcap_{k \in K} M(k)$ is the set of all cheap talk messages available to player 1, i.e., the set of all messages that player 1 can send whatever his type.

We assume that the set of cheap talk messages available to player 1 is nonempty. That is, there exists $\overline{m} \in M^1$ such that $M^{-1}(\overline{m}) = K$. This "right to remain silent" assumption will be needed for the "only if" part (from equilibrium to dimartingales) of the Theorems (see Theorems 1 and 2). For the "if" part (from dimartingales to equilibrium) we will further assume that the message space and certifiability possibilities of the sender are sufficiently rich. That is, whatever his type k, and for each event $L \subseteq K$ containing k, player 1 can choose among a sufficiently large set of messages certifying that his real type is in L. Formally, we assume that

$$|\{m \in M^1 : M^{-1}(m) = L\}| \geq L + 1, \quad \text{for all } L \subseteq K.$$

Notice that this "rich langage/certifiability" assumption implies the previous assumption that the set $\bigcap_{k \in K} M(k)$ is nonempty (simply take $L = K$). Note also that assuming full certifiability only for singleton events $L = \{k\}$ will *not* be sufficient for the "if" part of the theorems.

The signalling game determined by Γ and p, denoted by $\Gamma_S(p)$, is obtained by adding a unilateral talking phase to the silent game $\Gamma(p)$ before the action phase but after the information phase. Therefore, the signalling game corresponds to a standard "persuasion game" (Milgrom, 1981; Shin, 1994; Seidmann and Winter, 1997) and has three phases (see Figure 4).

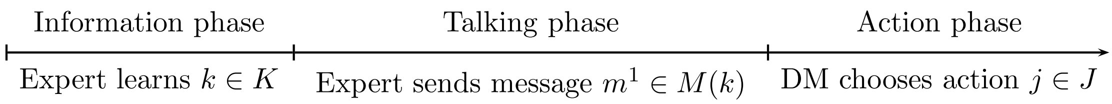

Figure 4: Signalling Game $\Gamma_S(p)$.

In the *information phase* a normal form "game" Γ^k is picked at random from $\{\Gamma^1, \ldots, \Gamma^K\}$ according to the probability distribution p. Player 1 is informed of k and player 2 is not.

The *talking phase* has only one time period in which type $k \in K$ of player 1 (publicly) sends a message $\mathsf{m}^1 \in \mathsf{M}(k)$. Finally, in the *action phase*, player 2 chooses an action and they are paid off in accordance with the normal form game Γ^k. The extensive form representation of the signalling game with only two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$) is given in Figure 5.

A (mixed) strategy for player 1 in the signalling game is a profile $\sigma = (\sigma^k)_{k \in K}$, with $\sigma^k \in \Delta(\mathsf{M}(k))$ for all k. A (mixed) strategy for player 2 is a function $\tau : \mathsf{M}^1 \to \Delta(J)$. A pair of strategies (σ, τ) of the signalling game generates expected payoffs $(a^1_{\sigma,\tau}, \ldots, a^K_{\sigma,\tau})$ and $\beta_{\sigma,\tau}$ for player 1 and player 2, respectively. As usual, a (Bayesian) Nash equilibrium of the signalling game is a pair of mixed strategies (σ, τ) satisfying

$$a^k_{\sigma,\tau} = \max_{\widetilde{\sigma}} a^k_{\widetilde{\sigma},\tau} \text{ for all } k \in K; \text{ and}$$

$$\beta_{\sigma,\tau} = \max_{\widetilde{\tau}} \beta_{\sigma,\widetilde{\tau}}.$$

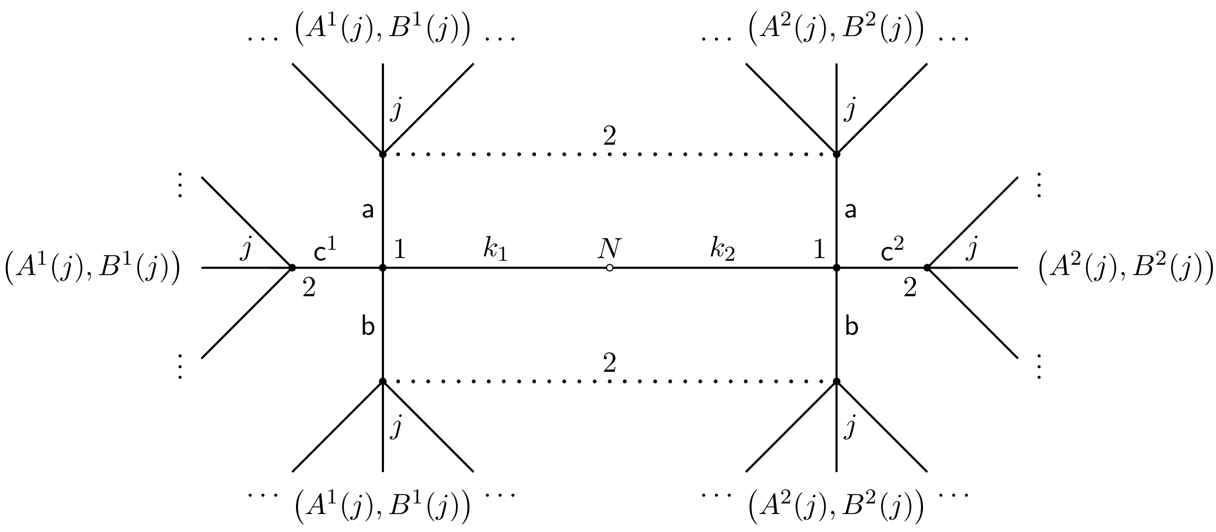

Figure 5: Extensive form of the signalling game $\Gamma_S(p)$ with two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$).

Let $\mathcal{E}_S(p)$ be the set of Nash equilibrium payoffs of $\Gamma_S(p)$.

3.3 Bilateral and Bounded Communication Game: Persuasion with a Deadline

We consider an arbitrary large but finite number $n \geq 1$ of communication rounds. In each communication round $t = 1, \dots, n$ each player can directly send a message to the other. As in the signalling game, the set of available messages of player 1 is denoted by $\mathsf{M}(k)$ when his type is k, $\mathsf{M}^1 = \bigcup_{k \in K} M(k)$ is the set of all messages that player 1 could send, and $\bigcap_{k \in K} M(k) \neq \emptyset$ is the set of all cheap talk messages available to player 1. The set of available messages of player 2 is denoted by M^2, with $|M^2| \geq 2$.

As in the signalling game we assume that $|\{m \in M^1 : M^{-1}(m) = L\}| \geq L + 1$ for all $L \subseteq K$. However, notice that in the multistage communication game it would be sufficient to have two cheap talk messages and that a combination of several certificates allows to certify any event $L \subseteq K$.[6] The above specific assumption on the richness of the message space is only for convenience.

The direct (unmediated and noiseless) communication game with n communication stages, determined by Γ and p, is denoted by $\Gamma_n(p)$. It is obtained by adding a talking phase with n bilateral communication rounds to the silent game $\Gamma(p)$ before the action

[6] That is, it would be sufficient to assume that $|\bigcap_{k \in K} M(k)| \geq 2$, and $\forall\ k$, $\forall\ k \neq k'$, $\exists\ m \in M(k)$, $M^{-1}(m) = K \backslash \{k'\}$.

phase but after the information phase(see Figure 6).

Information phase	Talking phase ($n \geq 1$ rounds)	Action phase
Expert learns $k \in K$	Expert and DM send $(m_t^1, m_t^2) \in M(k) \times M^2$ $(t = 1, \ldots n)$	DM chooses $j \in J$

Figure 6: n-Stage Communication Game $\Gamma_n(p)$.

In the *information phase* a normal form "game" Γ^k is picked at random from $\{\Gamma^1, \ldots, \Gamma^K\}$ according to the probability distribution p. Player 1 is informed of k and player 2 is not. At each period $t = 1, \ldots, n$ of the *talking phase*, type $k \in K$ of player 1 (publicly) sends a message $\mathsf{m}_t^1 \in \mathsf{M}(k)$ and player 2 (publicly) sends a message $\mathsf{m}_t^2 \in \mathsf{M}^2$. Messages are sent simultaneously. Finally, in the *action phase* (in period $n+1$), player 2 chooses an action and they are paid off in accordance with the normal form game Γ^k.

A t-period *history*, $t = 0, 1, \ldots, n$, is a sequence consisting of t pairs of messages,

$$h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2) \in (\mathsf{M}^1 \times \mathsf{M}^2)^t.$$

The set of all t-period histories is denoted by $\mathsf{M}_t = (\mathsf{M}^1 \times \mathsf{M}^2)^t$.

A (behavioral[7]) *strategy* σ of player 1 in the direct n-period communication game $\Gamma_n(p)$ consists of a sequence of functions $\sigma_1, \ldots, \sigma_n$, where $\sigma_t = (\sigma_t^1, \ldots, \sigma_t^K)$ and $\sigma_t^k : \mathsf{M}_{t-1} \to \Delta(\mathsf{M}(k))$ for $k \in K$ and $t = 1, \ldots, n$.

A (behavioral) strategy τ of player 2 consists of a sequence of functions $\tau_1, \ldots, \tau_n$, and a function τ_{n+1}, where $\tau_t : \mathsf{M}_{t-1} \to \Delta(\mathsf{M}^2)$ for $t = 1, \ldots, n$, and $\tau_{n+1} : \mathsf{M}_n \to \Delta(J)$.

A pair of strategies (σ, τ) of the communication game generates expected payoffs $a_{\sigma,\tau} = (a_{\sigma,\tau}^1, \ldots, a_{\sigma,\tau}^K)$ and $\beta_{\sigma,\tau}$ for player 1 and player 2, respectively. A (Bayesian) Nash equilibrium of the direct n-period communication game $\Gamma_n(p)$ is a pair of behavioral strategies (σ, τ) satisfying

$$\begin{aligned} a_{\sigma,\tau}^k &= \max_{\widetilde{\sigma}} a_{\widetilde{\sigma},\tau}^k \text{ for all } k \in K; \text{ and} \\ \beta_{\sigma,\tau} &= \max_{\widetilde{\tau}} \beta_{\sigma,\widetilde{\tau}}. \end{aligned}$$

Let $\mathcal{E}_n(p)$ be the set of Nash equilibrium payoffs of $\Gamma_n(p)$. Notice that $\mathcal{E}_S(p) \subseteq \mathcal{E}_n(p) \subseteq \mathcal{E}_{n+1}(p)$ for all $n \geq 1$. Let $\mathcal{E}_B(p) = \bigcup_n \mathcal{E}_n(p)$ be the set of Nash equilibrium payoffs of all bilateral/bounded communication games determined by Γ and p.

The next section is aimed at characterizing the sets $\mathcal{E}_S(p)$ and $\mathcal{E}_B(p)$.

[7] We focus on finite games with perfect recall. Hence, by Kuhn's (1953) theorem behavioral strategies are without loss of generality (see also Subsection 7.2).

4 Characterization of Equilibrium Payoffs $\mathcal{E}_S(p)$ and $\mathcal{E}_B(p)$

4.1 Statement of the Results

When some coordinates of p vanish, Aumann and Hart (2003) consider the modified equilibrium payoffs $\mathcal{E}^+(p)$ of the silent game $\Gamma(p)$, which is the same as $\mathcal{E}(p)$ except that when the probability of one of player 1's type vanishes, then the corresponding type of player 1 can get *more* than his equilibrium payoff. That is, the set of *modified equilibrium payoffs* of the silent game $\Gamma(p)$ is the set of all payoffs $(a, \beta) \in \mathbb{R}^K \times \mathbb{R}$ such that there exits an equilibrium $y \in Y(p)$ of the silent game $\Gamma(p)$ satisfying

(i) $a^k \geq A^k(y)$, for all $k \in K$;

(ii) $a^k = A^k(y)$ if $p^k \neq 0$;

(iii) $\beta = \sum_{k \in K} p^k B^k(y)$.

So, if p has full support we have $\mathcal{E}^+(p) = \mathcal{E}(p)$. The *graph* of the modified equilibrium payoff correspondence is

$$G \equiv \operatorname{gr} \mathcal{E}^+ \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a, \beta) \in \mathcal{E}^+(p)\}.$$

Here, we consider a larger set of modified equilibrium payoffs, denoted by $\mathcal{E}^{++}(p)$, and called the set of *extended equilibrium payoffs* of the silent game $\Gamma(p)$. It is also obtained from equilibrium payoffs $\mathcal{E}(p)$ of the silent game $\Gamma(p)$, but when the probability of one of player 1's type vanishes we allow the corresponding type of player 1 to get *any* payoff, which of course may be less than his equilibrium payoff. That is, the set of extended equilibrium payoffs $\mathcal{E}^{++}(p)$ is the set of all payoffs $(a, \beta) \in \mathbb{R}^K \times \mathbb{R}$ such that there exits an equilibrium $y \in Y(p)$ of the silent game $\Gamma(p)$ satisfying only (ii) and (iii). Clearly, $\mathcal{E}(p) \subseteq \mathcal{E}^+(p) \subseteq \mathcal{E}^{++}(p)$ and if p has full support then all these sets coincide.

Remark 1 Notice that the sets $\mathcal{E}(p)$, $\mathcal{E}^+(p)$ and $\mathcal{E}^{++}(p)$ are convex for all p, and if (a, β) and (a', β') belong to one of these sets then $\beta = \beta'$. This is because we consider only one decisionmaker (in general, if both players are decisionmakers as in Aumann and Hart, 2003, then the sets are not convex and player 2's equilibrium payoff is not unique).

The *graph* of this extended equilibrium payoff correspondence is denoted by

$$H = \operatorname{gr} \mathcal{E}^{++} \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a, \beta) \in \mathcal{E}^{++}(p)\}.$$

For any (nonempty) set of types $L \subseteq K$, let

$$\text{INTIR}_L \equiv \{a \in \mathbb{R}^K : \exists\, \overline{y} \in \Delta(J),\; a^k \geq A^k(\overline{y})\; \forall\, k \in L\},$$

be the set of payoffs that are interim individually rational for player 1 when we restrict the individual rationality constraint to a subset L of player 1's set of types. Remark that $\text{INTIR}_L \subseteq \text{INTIR}_{L'}$ whenever $L' \subseteq L$.

Let $H_1 \equiv \text{conv}_a(H) \cap \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_K\}$ be the set of expected payoffs obtained from H by convexifying in (β, p) when the payoff of player 1, a, is kept constant and is interim individually rational (on the whole set of types K) for player 1. We show that H_1 fully characterizes the set of equilibrium payoffs of $\Gamma_S(p)$, the talking game determined by Γ and $p \in \Delta(K)$ with only one step of signalling. Let $H_1(p) \equiv \text{Proj}_p(H_1) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in H_1\}$ be the p-section of H_1.

Theorem 1 (Signalling) *Let $p^k > 0$ for all $k \in K$. We have,*

$$\mathcal{E}_S(p) = H_1(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in H_1\}.$$

In addition, any Nash equilibrium payoff of $\Gamma_S(p)$ can be obtained with at most $K + 1$ messages.

That is, $(a, \beta) \in \mathbb{R}^{K+1}$ is the payoff to a Nash equilibrium in a signalling game $\Gamma_S(p)$ if and only if (a, β, p) is in $H_1 \equiv \text{conv}_a(H) \cap \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_K\}$, the set of payoffs obtained from $H \equiv \text{gr}\, \mathcal{E}^{++}$ by convexifying in (β, p) when a is kept constant and is interim individually rational for player 1.

From the proof of this "if" part of the Theorem (the construction of the sender's strategy; see Section 5.2), the following proposition is immediate:

Proposition 1 *Every equilibrium of the signalling game $\Gamma_S(p)$ is outcome equivalent (i.e., it induces the same probability distribution over player 2's decision conditional on k) to a "canonical" equilibrium (σ, τ) with the following property:*

For all $m \in M^1$, if $\sigma^k(m) > 0$ for some $k \in K$, then $\sigma^{k'}(m) > 0$ for all $k' \in M^{-1}(m)$.

In particular, if a cheap talk message $\overline{m} \in \bigcap_{k \in K} M(k)$ is sent with strictly positive probability by player 1, then all types of player 1 send this message with strictly positive probability. More generally, the proposition says that in equilibrium we can assume without loss of generality that if player 2's posterior about a certain type k of player 1 is null after some message m sent with strictly positive probability ($p^k_m = 0$ with $P(m) > 0$), then $k \notin M^{-1}(m)$, i.e., the message m certifies that k is not realized. In particular, all types have strictly positive posterior probability after a cheap talk message (sent with strictly positive probability in equilibrium). Without using the geometric characterization of Theorem 1, the intuition of the proposition is as follows. Assume that type k' does not send a message m but could have sent it (i.e., $m \in M(k')$). Then, the types who send message m could have sent another message instead of m that certifies that k' is not realized, without changing player 2's posteriors and so without changing the equilibrium outcome.

To get the equilibrium payoffs for talking games with several bilateral communication rounds, let H_2 be the set of payoffs obtained from H_1 by first convexifying in (a, β) when player 2's belief p is fixed, then in (β, p) when player 1's payoff a is fixed, and by convexifying again in (a, β) when player 2's belief p is fixed, with the restriction that in each step of the process of diconvexification the payoff of player 1 is interim individually rational for the types with a strictly positive posterior in that step. The p-section of the set H_2 will correspond to the set of equilibrium payoffs of talking games with four communication rounds: a jointly controlled lottery, a step of signalling, a second jointly controlled lottery, and a second step of signalling. Next, let H_3 be the set of payoffs obtained from H_2 by convexifying in (β, p) when player 1's payoff a is fixed, and then by convexifying in (a, β) when player 2's belief p is fixed, with again the restriction that in each step of the process of diconvexification the payoff of player 1 is interim individually rational for the types with a strictly positive posterior in that step. The p-section of the set H_3 will correspond to the set of equilibrium payoffs of talking games with six communication rounds. The limit of the increasing sequence $H_1,\ H_2,\ \ldots H_l$ constructed in this way is denoted by $\text{di-co}^{\text{IR}}(H) \equiv \bigcup_{l=1}^{\infty} H_l$. Points in $\text{di-co}^{\text{IR}}(H)$ will correspond to all equilibrium payoffs of talking games of bounded length. Let $H_B(p) \equiv \text{Proj}_p(\text{di-co}^{\text{IR}}(H)) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in \text{di-co}^{\text{IR}}(H)\}$ be the p-section of $\text{di-co}^{\text{IR}}(H)$.

The set $\text{di-co}^{\text{IR}}(H)$ can also be expressed as the set of starting points of particular martingales that converge to H, as in the next Theorem.

Theorem 2 (Bounded Persuasion) *Let $p^k > 0$ for all $k \in K$. We have,*

$$\mathcal{E}_B(p) = H_B(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in \text{di-co}^{\text{IR}}(H)\}.$$

Equivalently, $(a, \beta) \in \mathbb{R}^{K+1}$ is a Nash equilibrium payoff of the bounded persuasion game

$\Gamma_n(p)$ *for some* $n \geq 1$*, i.e.,*

$$(a, \beta) \in \mathcal{E}_B(p),$$

if and only if there exists a martingale $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$*, with* $z_s = (a_s, \beta_s, p_s) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K)$ *satisfying the following properties:*

(1) $\boldsymbol{z}_0 = (a, \beta, p)$*. That is, the starting point (and expectation) of the martingale is the Nash equilibrium payoff under consideration.*

(2) $\boldsymbol{z}_N \in \operatorname{gr} \mathcal{E}^{++} \equiv H$*. That is, the martingale converges to the extended equilibrium payoffs of the silent game:* $(\boldsymbol{a}_N, \boldsymbol{\beta}_N) \in \mathcal{E}^{++}(\boldsymbol{p}_N)$*.*

(3) $\boldsymbol{a}_{s+1} = \boldsymbol{a}_s$ *for all even* s *and* $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ *for all odd* s*. That is, the martingale is a dimartingale.*

(4) For all $s = 0, 1, \ldots, N$*,* $\boldsymbol{a}_s \in \mathit{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]}$*, where*

$$\mathit{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]} \equiv \{a \in \mathbb{R}^K : \exists\, \overline{y} \in \Delta(J),\ \boldsymbol{a}_s^k \geq A^k(\overline{y})\ \forall\, k \in \mathrm{supp}[\boldsymbol{p}_s]\},$$

and $\mathrm{supp}[\boldsymbol{p}_s] \equiv \{k \in K : \boldsymbol{p}_s^k > 0\}$*. That is, the vector payoff of player 1's types with strictly positive probability is individually rational along the communication process.*

Remark 2 Notice that if $\boldsymbol{z}_N \in H$ and $\boldsymbol{a}_N \in \mathrm{INTIR}_K$ then (2) and (4) are satisfied. But the converse is not true: it is easy to construct an example with an equilibrium payoff $(a, \beta) \in \mathcal{E}_B(p)$ but $\boldsymbol{a}_N \notin \mathrm{INTIR}_K$, $K \neq \mathrm{supp}[\boldsymbol{p}_N]$. Notice also that condition (4) only for $s = 0$ is not sufficient. Indeed, one can easily construct a dimartingale with $a_0 \in \mathrm{INTIR}_K$, $(\boldsymbol{a}_N, \boldsymbol{\beta}_N, \boldsymbol{p}_N) \in H$, but $(a, \beta) \notin \mathcal{E}_B(p)$ ($\boldsymbol{a}_s \notin \mathrm{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]}$ for some history at s).[8]

[8]Examples are available from the authors upon request.

4.2 Illustration of Theorem 1 (Signalling)

For the introductory example, the graph of the modified equilibrium payoff correspondence, $G = \mathrm{gr}\,\mathcal{E}^{+}$, is represented on the (a^1, a^2) coordinates by grey solid lines in Figure 7. The graph of the extended equilibrium payoff correspondence, $H = \mathrm{gr}\,\mathcal{E}^{++}$, is represented in the same figure by all the grey lines, including the dashed ones. The sets G and H are also described in Table 1. Since all points at the north-east of $(0, 0)$ are interim individually rational for player 1, convexifying the set H by keeping a constant and interim individually rational yields three new points at $p = 1/2$: FRE, PRE1 and PRE2, which are exactly the three Nash equilibrium payoffs found in Section 2, in addition to the non-revealing equilibrium (NRE). Indeed, each of these points corresponds to two extended non-revealing equilibria, at two different p's forming an interval that includes $p = 1/2$, giving the same payoff to player 1. Notice that, for example, the intersection point PRE3 is not an equilibrium payoff for $p = 1/2$ because 1/2 lies outside the interval $[3/5, 1]$.

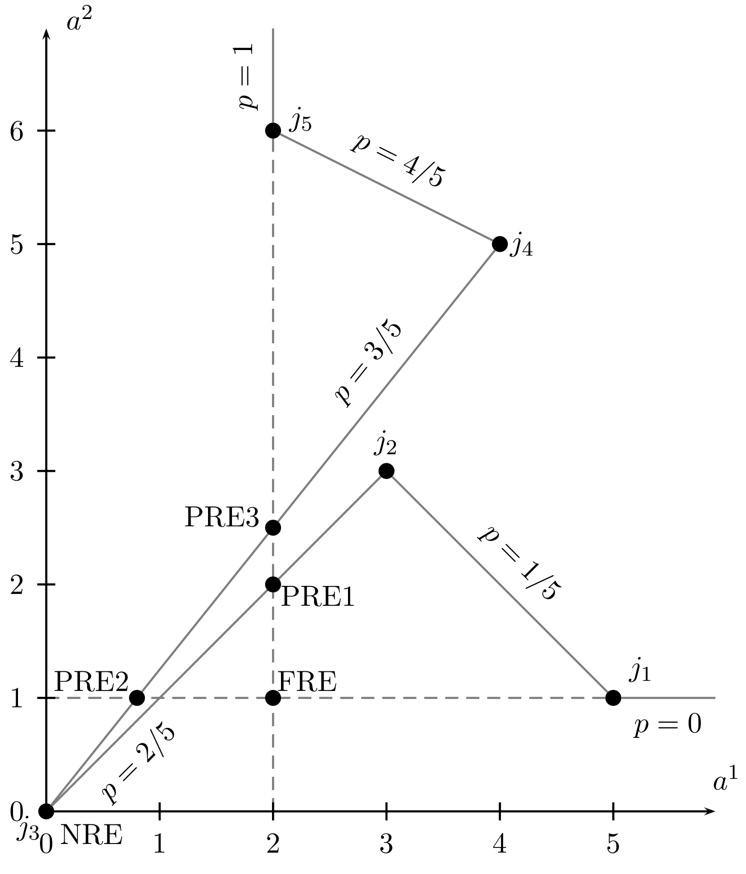

Figure 7: Extended equilibrium payoffs of the expert in the introductory example.

4.3 Illustration of Theorem 2 (Persuasion with a Deadline)

The dimartingale corresponding to Equilibrium (i) of the introductory example (see Figure 3) is represented by Figure 8. It yields to the point j_2 at $p = 1/2$ in Figure 7, which is not achievable at $p = 1/2$ with only one step of signalling/diconvexification. The dimartingale corresponding to Equilibrium (ii) is similar.

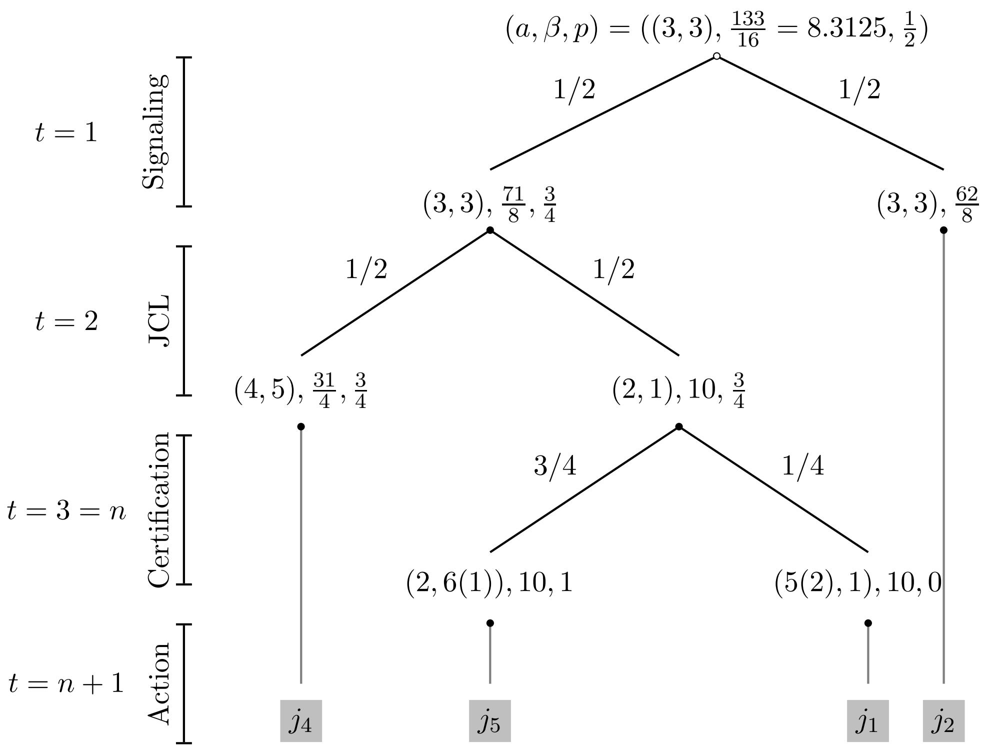

Figure 8: Dimartingale/diconvexification Corresponding to Equilibrium (i) of the Introductory Example.

Adding a jointly controlled lottery before a signalling stage allows a convexification by keeping p fixed. This yields to the graph $H_1^* = \text{conv}_p(H_1)$ described on the a-coordinates in the fourth column of Table 1. For example, adding a jointly controlled lottery before a signalling stage at $p = 1/2$ yields to all convex combinations of equilibrium payoffs of the signalling game, $[j_3, \text{FRE}, \text{PRE1}, \text{PRE2}]$. Adding a second signalling stage allows a second convexification by keeping a fixed. One can check that this does not yield new equilibrium payoffs, except for $p \in (2/5, 3/5)$. Indeed, for $p \in (2/5, 3/5)$ one can combine the sets $H_1^*(p') = [j_2, \text{PRE2}, \text{FRE}]$, $p' \in (1/5, 2/5)$, and $H_1^*(p'') = [j_4, \text{PRE3}, \text{FRE}]$, $p'' \in (3/5, 4/5)$, which yields to the payoffs in the triangle $[j_2, \text{PRE1}, \text{FRE}]$, which were not achievable at $p \in (2/5, 3/5)$ with only 2 communication stages. Hence, for $p \in (2/5, 3/5)$, $H_2(p) =$

$H_1^*(p) \cup [j_2, \text{PRE1,FRE}] = [j_3, \text{PRE2}, j_2, \text{FRE}]$. It is easy to verify that one cannot get new points after two steps of diconvexification on both directions, so $H_2 = H_3 = \cdots = H_\infty$.

p	G	H	$H_1^* = \text{conv}_p(H_1)$	H_2
0	$(a^1 \geq 5, 1)$	$(a^1 \geq 0, 1)$	$\cdots$	$\cdots$
$(0, \frac{1}{5})$	j_1	j_1	$[j_1, \text{PRE2}]$	$\cdots$
$\frac{1}{5}$	$[j_1, j_2]$	$[j_1, j_2]$	$[j_1, j_2, \text{PRE2}]$	$\cdots$
$(\frac{1}{5}, \frac{2}{5})$	j_2	j_2	$[j_2, \text{PRE2}, \text{FRE}]$	$\cdots$
$\frac{2}{5}$	$[j_2, j_3]$	$[j_2, j_3]$	$[j_2, \text{PRE2}, j_3, \text{FRE}]$	$\cdots$
$(\frac{2}{5}, \frac{3}{5})$	j_3	j_3	$[j_3, \text{FRE,PRE1,PRE2}]$	$[j_3, \text{PRE2}, j_2, \text{FRE}]$
$\frac{3}{5}$	$[j_3, j_4]$	$[j_3, j_4]$	$[j_3, j_4, \text{FRE}]$	$\cdots$
$(\frac{3}{5}, \frac{4}{5})$	j_4	j_4	$[j_4, \text{PRE3,FRE}]$	$\cdots$
$\frac{4}{5}$	$[j_4, j_5]$	$[j_4, j_5]$	$[j_4, j_5, \text{FRE}]$	$\cdots$
$(\frac{4}{5}, 1)$	j_5	j_5	$[j_5, \text{FRE}]$	$\cdots$
1	$(2, a^2 \geq 6)$	$(2, a^2 \geq 0)$	$\cdots$	$\cdots$

Table 1: "$\cdots$" means "as in the previous column".

5 Proof of Theorem 1 (Signalling)

5.1 From equilibrium to constrained convexification (only if): $\mathcal{E}_S(p) \subseteq H_1(p)$

Let (σ, τ) be any Nash equilibrium of the signalling game $\Gamma_S(p)$, where $p^k > 0$ for all $k \in K$, and let $(a, \beta) \in \mathcal{E}_S(p)$ be the associated equilibrium payoffs. We must show that (a, β, p) is in H_1, i.e., (a, β, p) can be obtained as a convex combination of points in $H = \operatorname{gr} \mathcal{E}^{++}$ by keeping a constant and interim individually rational ($a \in \mathrm{INTIR}_K$).

Let $P = P_{\sigma,\tau,p}$ be the probability distribution on $\Omega = K \times M^1 \times J$ generated by players' strategies and the priors. So,

$$P(m) = \sum_{k \in K} p^k \, \sigma^k(m),$$

is the (ex ante) probability that player 1 sends the message $m \in M^1$. Let $M^* = \{m \in M^1 : P(m) > 0\}$. For all $m \in M^*$, let

$$p_m^k = P(k \mid m) = \frac{p^k \, \sigma^k(m)}{P(m)},$$

be player 2's posterior about player 1's type after receiving the message m, let $p_m = (p_m^k)_{k \in K}$, and let

$$\beta_m = \sum_{k \in K} p_m^k \, B^k(\tau(m)),$$

be the resulting expected payoff for player 2 when m is reached.

Since $p^k = \sum_{m \in M^*} P(m) \, p_m^k$ for all $k \in K$ and $\beta = \sum_{m \in M^*} P(m) \, \beta_m$, we have

$$(a, \beta, p) = \sum_{m \in M^*} P(m) \, (a, \beta_m, p_m).$$

So, to show that (a, β, p) is a convex combination of points in H be keeping a constant it suffices to show that $(a, \beta_m, p_m) \in H$ for all $m \in M^*$, i.e., $(a, \beta_m) \in \mathcal{E}^{++}(p_m)$ for all $m \in M^*$. Player 2's equilibrium condition implies that $\tau(m) \in Y(p_m)$ for all $m \in M^*$, so condition (iii) in the definition of $\mathcal{E}^{++}(p_m)$ (see page 15) is satisfied for all $m \in M^*$. Player 1's equilibrium condition implies that $A^k(\tau(m)) = A^k(\tau(m'))$ whenever $\sigma^k(m) > 0$ and $\sigma^k(m') > 0$ (player 1 of type k should be indifferent between all messages that he sends with strictly positive probability), so

$$a^k = \sum_{m \in M^*} \sigma^k(m) \, A^k(\tau(m)) = A^k(\tau(m)),$$

for all m such that $\sigma^k(m) > 0$ (which is equivalent to $p_m^k > 0$ because $p^k > 0$), so condition (ii) in the definition of $\mathcal{E}^{++}(p_m)$ is also satisfied for all $m \in M^*$.

Remark 3 Notice that when $p_m^k = 0$ we may have $a^k < A^k(\tau(m))$ (because type k *cannot* send the message m when $m \notin M(k)$), so when some coordinates of p_m vanish it is possible that $(a, \beta_m, p_m) \notin G \equiv \mathrm{gr}\, \mathcal{E}^+$, contrary to the case of cheap talk with unverifiable information (Aumann and Hart, 2003).

It remains to show that $a \in \mathrm{INTIR}_K$. Consider a message $\overline{m} \in \bigcap_{k \in K} M(k)$ (which exists by assumption), and let $\overline{y} = \tau(\overline{m})$ ($\overline{m}$ may or may not be a message sent by player 1 with positive probability, so there may be no rationality condition on $\overline{y}$ for player 2 as long as no equilibrium refinement is introduced). By player 1's equilibrium condition, for all $k \in K$ and m such that $\sigma^k(m) > 0$ we have $a^k = A^k(\tau(m)) \geq A^k(\overline{y})$, which proves that $a \in \mathrm{INTIR}_K$. ∎

5.2 From constrained convexification to equilibrium (if): $H_1(p) \subseteq \mathcal{E}_S(p)$

We start from (a, β, p), a convex combination of points in H by keeping a constant, with $a \in \mathrm{INTIR}_K$ and $p^k > 0$ for all $k \in K$, and we construct an equilibrium (σ, τ) of the signalling game $\Gamma_S(p)$ with expected payoffs (a, β).

Since $(a, \beta, p) \in \mathrm{conv}_a(H)$, we can write

$$(a, \beta, p) = \sum_{w \in W} \pi(w)\,(a, \beta_w, p_w),$$

with $\pi \in \Delta(W)$ and $(a, \beta_w, p_w) \in H$ for all $w \in W$. Without loss of generality we assume that π has full support. In addition, from Carathéodory's theorem we can let $|W| \leq K + 1$ since the dimension of $(\beta, p) \in \mathbb{R} \times \Delta(K)$ is equal to K.

For all $w \in W$, we associate a set of types $\mathrm{supp}[p_w] \equiv \{k \in K : p_w^k > 0\}$ and a message $m_w \in M^1$ with $m_w \neq m_{w'}$ for $w \neq w'$, and $M^{-1}(m_w) = \mathrm{supp}[p_w]$. This is possible given our assumption on the richness of the message space.

Player 1's strategy σ. For all $k \in K$ and $w \in W$ define

$$\sigma^k(m_w) = \frac{\pi(w)\,p_w^k}{p^k} \quad \text{(and } \sigma^k(m) = 0 \text{ if } m \neq m_w \text{ for all } w \in W\text{)}.$$

Player 2's strategy τ. Since by assumption $(a, \beta_w) \in \mathcal{E}^{++}(p_w)$, for all $w \in W$ we can define (see condition (ii) and (iii) of $\mathcal{E}^{++}(p_w)$),

$$y_w = \tau(m_w) \in Y(p_w) \quad \text{such that} \quad \begin{cases} a^k = A^k(\tau(m_w)) \text{ if } p_w^k > 0 \\ \beta_w = \sum_{k \in K} p_w^k\, B^k(\tau(m_w)). \end{cases}$$

For the other messages $m \neq m_w$, $w \in W$, since by definition $a \in \mathrm{INTIR}_K$, we can define

$$\tau(m) = \overline{y} \quad \text{such that} \quad a^k \geq A^k(\overline{y}) \text{ for all } k \in K.$$

Payoffs. We first verify that (a, β) is the payoff generated by the strategy profile (σ, τ) defined just before. Let $P = P_{\sigma,\tau,p}$ be the probability distribution on $\Omega = K \times M^1 \times J$ generated by those strategies and the prior, and let $E = E_{\sigma,\tau,p}$ be the associated expectation operator.

First, we check that $P(m_w) = \pi(w)$ for all $w \in W$:

$$P(m_w) = \sum_{k \in K} p^k\,\sigma^k(m_w) = \sum_{k \in K} p^k \frac{\pi(w)\,p_w^k}{p^k} = \sum_{k \in K} \pi(w)\,p_w^k = \pi(w) \sum_{k \in K} p_w^k = \pi(w) \times 1.$$

By construction, player 1's expected payoff when his type is k is given by

$$\begin{aligned} E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k] &= \sum_{w\in W} P[\boldsymbol{m}=m_w \mid \boldsymbol{k}=k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k, \boldsymbol{m}=m_w] \\ &= \sum_{w\in W} \sigma^k(m_w) \sum_{j\in J} \tau(m_w)(j) A^k(j) = \sum_{w\in W} \sigma^k(m_w)\, A^k(\tau(m_w)) = a^k, \end{aligned}$$

the last equality following from the construction of player 2's strategy: $A^k(\tau(m_w)) = a^k$ whenever $\sigma^k(m_w) > 0$ ($\Leftrightarrow p^k_w > 0$ because $p^k > 0$).

Finally, player 2's expected payoff is

$$\begin{aligned} E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k\in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k] \\ &= \sum_{k\in K} p^k \sum_{w\in W} P[\boldsymbol{m}=m_w \mid \boldsymbol{k}=k] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}=k, \boldsymbol{m}=m_w] \\ &= \sum_{k\in K} p^k \sum_{w\in W} \sigma^k(m_w) \sum_{j\in J} \tau(m_w)(j) B^k(j) = \sum_{k\in K} p^k \sum_{w\in W} \frac{\pi(w) p^k_w}{p^k} B^k(\tau(m_w)) \\ &= \sum_{w\in W} \pi(w) \sum_{k\in K} p^k_w\, B^k(\tau(m_w)) = \sum_{w\in W} \pi(w)\, \beta_w = \beta. \end{aligned}$$

Equilibrium condition for player 2. Next, we verify that τ is a best reply for player 2 to player 1's strategy σ. Since we have defined $\tau(m_w) \in Y(p_w)$ for all $w \in W$, and since the messages $(m_w)_{w\in W}$ are the only messages sent with strictly positive probability by player 1, it suffices to verify that p_w is the correct posterior belief of player 2 when he receives the message m_w (again, remember that we use no equilibrium refinement). This is immediately obtained by Bayes's rule given the definition of the strategy σ of player 1:

$$P[\boldsymbol{k}=k \mid \boldsymbol{m}=m_w] = \frac{P[\boldsymbol{m}=m_w \mid \boldsymbol{k}=k] P[\boldsymbol{k}=k]}{P[\boldsymbol{m}=m_w]} = \frac{\sigma^k(m_w) p^k}{\pi(w)} = p^k_w.$$

Equilibrium condition for player 1. Finally, we verify that σ^k is a best reply for player 1 of type k to player 2's strategy τ. Player 1 of type k sends each message m_w, $w \in W$, satisfying $p^k_w > 0$ ($\Leftrightarrow \sigma^k(m_w) > 0$ because $p^k > 0$) with strictly positive probability. By construction of player 2's strategy we have $A^k(\tau(m_w)) = a^k$ (see the previous paragraph "payoffs") for all such messages, so type k is indeed indifferent between all these messages. Next, remark that type k *cannot* send the other messages m_w satisfying $p^k_w = 0$ because such messages are such that $M^{-1}(m_w) = \mathrm{supp}[p_w]$, with $k \notin \mathrm{supp}[p_w]$ (by the definition of $\mathrm{supp}[p_w]$ since $p^k_w = 0$), so $m_w \notin M(k)$. Finally, if player 1 sends a message off the equilibrium path, $\overline{m} \neq m_w$ for all $w \in W$ (so $P(\overline{m}) = 0$), then he gets $A^k(\tau(\overline{m})) =$

$A^k(\overline{y}) \leq a^k = A^k(\tau(m_w))$ for $\sigma^k(m_w) > 0$, so he does not deviate. This completes the proof of Theorem 1. $\blacksquare$

6 Proof of Theorem 2 (Persuasion with a Deadline)

6.1 From equilibrium to constrained dimartingales (only if): $\mathcal{E}_B(p) \subseteq H_B(p)$

Except for the construction of player 1's sequence of virtual payoffs and the fact that we consider martingales that are bounded in length, this part of the proof is similar to the proof of Hart (1985) and Aumann and Hart (2003).

Let (σ, τ) be any Nash equilibrium of the communication game $\Gamma_n(p)$ for some finite $n \geq 1$, where $p^k > 0$ for all $k \in K$, with payoffs $a = (a^1, \ldots, a^K) \in \mathbb{R}^K$ for player 1 and $\beta \in \mathbb{R}$ for player 2. We construct a sequence of random variables $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $N = 2n$, satisfying properties (1) to (4) of Theorem 2 and the martingale property:

$$E[\boldsymbol{z}_{s+1} \mid \boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_s] = \boldsymbol{z}_s, \quad s = 0, 1, \ldots, N.$$

We work on the probability space

$$\Omega = K \times \underbrace{(M^1 \times M^2)^n}_{M_n} \times J.$$

A realization $\omega = (k, m_1^1, m_1^2, \ldots, m_t^1, m_t^2, \ldots, m_n^1, m_n^2, j) \in \Omega$ consists in a type for player 1, a final communication history, and an action for player 2. All random variables (denoted in bold letters when there may be a risk of confusion) are defined on Ω. Let $P = P_{\sigma,\tau,p}$ be the probability distribution on Ω generated by players' strategies and the prior probability distribution on player 1's set of types, and let $E = E_{\sigma,\tau,p}$ be the corresponding expectation operator. For example, $P[\boldsymbol{k} = k] = p^k$ and

$$P[\boldsymbol{m}_t^1 = m \mid \boldsymbol{h}_{t-1} = h_{t-1}, \boldsymbol{k} = k] = \sigma_t^k(h_{t-1}).$$

As in Aumann and Hart (2003), for $s = 0, \ldots, N$ we construct a new random variable on Ω, $\boldsymbol{g}_s$, that corresponds to every history of talk, plus every history of talk followed by player 1's message in the next period. Formally,

$$g_s \equiv \begin{cases} h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2), & \text{if } s = 2t \text{ is even, } t = 0, \ldots, n \\ (h_t, m_{t+1}^1), & \text{if } s = 2t+1 \text{ is odd, } t = 0, \ldots, n-1. \end{cases}$$

So, $g_0 = h_0 = \emptyset$, $g_N = g_{2n} = h_n$, when s is even the last message in g_s is from player 2, and when s is odd the last message in g_s is from player 1. We consider this new random variable in order to have the dimartingale property (property (3) of the Theorem).

Sequence of posteriors $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$**.** For each $k \in K$ and $s = 0, \ldots, N$, define

$$\boldsymbol{p}_s^k \equiv P[\boldsymbol{k} = k \mid \boldsymbol{g}_s],$$

and $\boldsymbol{p}_s = (\boldsymbol{p}_s^k)_{k \in K} \in \Delta(K)$.

Lemma 1 The sequence $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying

(i) $\boldsymbol{p}_0 = p$;

(ii) $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ for all odd s.

Proof. The martingale property is simply due to the fact that $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ is a sequence of posteriors by conditioning on more and more information (it is adapted to the sequence of fields $(\mathcal{G}_s)_{s=0,\ldots,N}$ generated by $(\boldsymbol{g}_s)_{s=0,\ldots,N}$). (i) is immediate: $\boldsymbol{p}_0^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_0] = P[\boldsymbol{k} = k] = p^k$. To prove (ii), let $s = 2t + 1$ be an odd number. For each $k \in K$ we have

$$\boldsymbol{p}_{s+1}^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_{s+1}] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1, \boldsymbol{m}_{t+1}^2] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1] = \boldsymbol{p}_s^k$$

the last but one equality following from the fact that, conditional on $(\boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1)$, $\boldsymbol{m}_{t+1}^2$ and $\boldsymbol{k}$ are independent. ■

Sequence of player 2's payoff $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$**.** For each $s = 0, \ldots, N$, define

$$\boldsymbol{\beta}_s \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

and let $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N)$.

Lemma 2 The sequence $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying

(i) $\boldsymbol{\beta}_0 = \beta$;

(ii) $\boldsymbol{\beta}_N = \sum_{k \in K} \boldsymbol{p}_N^k \, B^k(\boldsymbol{y})$, with $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. The martingale property is due to the fact that $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ is a sequence of conditional expectations of a fixed random variable by conditioning on more and more information. (i) is immediate by the definition of β: $\boldsymbol{\beta}_0 = E[B^{\boldsymbol{k}}(\boldsymbol{j})] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}]\right] =$

$\sum_{k\in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = \beta$. Next, we have

$$\boldsymbol{\beta}_N \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k}]\right] = \sum_{k\in K} P[\boldsymbol{k} = k \mid \boldsymbol{g}_N] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k]$$
$$= \sum_{k\in K} \boldsymbol{p}_N^k E[B^k(\boldsymbol{j}) \mid \boldsymbol{g}_N] = \sum_{k\in K} \boldsymbol{p}_N^k B^k(\tau_{n+1}(\boldsymbol{g}_N)),$$

the last but one equality following from the fact that, conditional on $\boldsymbol{g}_N$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent.[9] The equilibrium condition of player 2 implies that $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N)$. This completes the proof of the Lemma. ■

At this stage, we have constructed $(\boldsymbol{p}_s)_{s=0,1,\dots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\dots,N}$ that have all the properties required by the Theorem. It remains to construct an appropriate sequence of player 1's payoffs, which is more delicate.

Sequence of player 1's vector payoff $(\boldsymbol{a}_s^k)_{s=0,1,\dots,N}$, $k \in K$. A first definition that could come to mind for the characterization of the sequence of player 1's payoffs is to simply take

$$E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

which is always well defined. However, it is not relevant, in general, for type k (except when $s = N$). To see this, consider a very simple example with one unilateral communication period ($N = 1$), two types of equal probability ($K = \{k_1, k_2\}$, $p^1 = p^2 = 1/2$), and assume that in the first talking period type k_1 sends message m with probability one and type k_2 sends message m' with probability one. After message m, player 2 chooses action j_1, and after message m' he chooses action j_2. Then, we would have $E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_0] = (1/2)A^k(j_1) + (1/2)A^k(j_2)$, which is not meaningful for any type k.

A more meaningful definition of k's expected payoff is

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k].$$

Unfortunately, it is not well defined when $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] = 0$, and this can happen even when $P[\boldsymbol{g}_s = g_s] > 0$. This can be seen easily in the previous example, where $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_1 = m', \boldsymbol{k} = k_1]$ is not well defined albeit $P[\boldsymbol{g}_1 = m'] = 1/2 > 0$.

Finally, it is worth noticing that the definition used by Aumann and Hart (2003) does not work in our setup. Indeed, they define the (highest) payoff that player 1 of type k can

[9] For the last equality, remember that we have extended B^k linearly to mixed actions.

achieve against player 2's strategy τ after the history $\boldsymbol{g}_s$ as

$$\sup_{\tilde{\sigma}} E_{\tilde{\sigma},\tau,p}[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

where the supremum is over all strategies $\tilde{\sigma}$ of player 1 such that $P_{\tilde{\sigma},\tau,p}[\boldsymbol{g}_s \mid \boldsymbol{k} = k] > 0$. But this is not necessarily well defined in our setup even when $P[\boldsymbol{g}_s = g_s] > 0$ because a history g_s may contain a message (certificate) that *cannot* be sent by type k (for example, $g_1 = m \notin M(k)$).

Hence, we follow a different, and somehow simpler, approach. For each $k \in K$, we construct the sequence of type k's (virtual) payoff $(\boldsymbol{a}_s^k)_{s=0,1,\ldots,N}$ as follows. Let $\boldsymbol{a}_s^k = a_s^k(\boldsymbol{g}_s)$. When

$$P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0,$$

we define

$$a_s^k(g_s) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k],$$

which is unambiguously k's expected payoff given the history g_s (and k). Clearly, for $s = 0$, $a_s^k(g_s)$ is always well defined: $a_0^k(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$. More generally, assume inductively that $a_s^k(g_s)$ is well defined, i.e., assume that $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0$. If $s = 2t - 1$ is odd, then $g_{s+1} = (g_s, m_t^2)$, so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] > 0$ when $P[\boldsymbol{m}_t^2 = m_t^2 \mid \boldsymbol{g}_s = g_s] > 0$, which implies that $a_{s+1}^k(g_{s+1})$ remains well defined. If $s = 2t$ is even, then we may have a problem to define $a_{s+1}^k(g_{s+1})$ because now it is player 1's message that is added to the history: $g_{s+1} = (g_s, m_{t+1}^1)$. Indeed, we may have $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] = \sigma_{t+1}^k(m_{t+1}^1 \mid h_t) = 0$ (even when $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s] > 0$), so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] = 0$. It that situation, we let

$$a_{s+1}^k(g_s, m_{t+1}^1) = a_s^k(g_s).$$

First, notice that the equilibrium condition of player 1 implies $a_s^k(g_s) = a_{s+1}^k(g_s, m)$ for all m such that $\sigma_{t+1}^k(m \mid g_s) > 0$. Second notice that we will have the same problem in all histories following (g_s, m_{t+1}^1) (they have probability 0 conditional on k), so we fix more generally k's payoff for all these histories:

$$a_{s+l}^k(g_s, m_{t+1}^1, \ldots) = a_s^k(g_s), \quad l = 1, 2 \ldots.$$

All this construction can be summarized formally as follows. For each $s = 0, \ldots, N$ and $k \in K$ define the random variable

$$\boldsymbol{f}_s^k$$

as the longest subhistory of $\boldsymbol{g}_s$ satisfying $P[\boldsymbol{f}_s^k \mid \boldsymbol{k} = k] > 0$ (notice that this history necessarily ends with player 2's message, or is equal to $\boldsymbol{g}_s$), and let

$$\boldsymbol{a}_s^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{f}_s^k, \boldsymbol{k} = k].$$

This definition is equivalent to,

$$\boldsymbol{a}_s^k = \begin{cases} E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k], & \text{if } \boldsymbol{p}_s^k > 0 \\ \boldsymbol{a}_{\boldsymbol{r}}^k, & \text{if } \boldsymbol{p}_s^k = 0, \end{cases}$$

where $\boldsymbol{r}$ is a random variable (stopping time) which is equal to the largest r such that $\boldsymbol{p}_r^k > 0$.

Lemma 3 For every $k \in K$, the sequence $(\boldsymbol{a}_s^k)_{s=0,\dots,N}$ is a (bounded) martingale satisfying

(i) $\boldsymbol{a}_0^k = a^k$;

(ii) $\boldsymbol{a}_{s+1}^k = \boldsymbol{a}_s^k$ for all even s;

(iii) If $\boldsymbol{p}_N^k > 0$, then $\boldsymbol{a}_N^k = A^k(\boldsymbol{y})$, with $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. To prove the martingale property we must show that

$$E[\boldsymbol{a}_{s+1}^k \mid \boldsymbol{g}_s] = \boldsymbol{a}_s^k, \qquad \text{for all } s = 0, 1, \dots, N.$$

If $\boldsymbol{p}_{s+1}^k = 0$, then this property is immediate because by construction we have $\boldsymbol{a}_{s+1}^k = \boldsymbol{a}_s^k = \boldsymbol{a}_r^k$, where $r \leq s$ is the largest number such that $\boldsymbol{p}_r^k > 0$. Now, consider the case $\boldsymbol{p}_{s+1}^k > 0$, and let $s = 2t - 1$ be odd (when s is even, the martingale property will follow from (ii)). Thus, $\boldsymbol{p}_s^k > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}_t^2)$, which implies

$$\begin{cases} \boldsymbol{a}_{s+1}^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_t^2, \boldsymbol{k} = k] \\ \boldsymbol{a}_s^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

So,

$$\begin{aligned} E[\boldsymbol{a}_{s+1}^k \mid \boldsymbol{g}_s] &= \sum_{m \in \text{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}_t^2 = m \mid \boldsymbol{g}_s] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_t^2 = m, \boldsymbol{k} = k] \\ &= \sum_{m \in \text{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}_t^2 = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_t^2 = m, \boldsymbol{k} = k] \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k] = \boldsymbol{a}_s^k, \end{aligned}$$

the second equality following from the fact that $\boldsymbol{m}_t^2$ and $\boldsymbol{k}$ are independent conditional on $\boldsymbol{g}_s$. This proves the martingale property for all odd s. Property (i) is immediate: $\boldsymbol{a}_0^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$ by the definition of a^k. To prove (ii) let $s = 2t$ be even, so $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}_{t+1}^1)$. As before, when $\boldsymbol{p}_{s+1}^k = 0$ the property is immediate because $\boldsymbol{a}_{s+1}^k = \boldsymbol{a}_s^k = \boldsymbol{a}_r^k$, with $r \leq s$. When $\boldsymbol{p}_{s+1}^k > 0$, then $\boldsymbol{p}_s^k > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}_{t+1}^1)$, so

$$\begin{cases} \boldsymbol{a}_{s+1}^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_{t+1}^1, \boldsymbol{k} = k] \\ \boldsymbol{a}_s^k = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

In such a situation these two terms are equal by the equilibrium condition of player 1 since every message m_{t+1}^1 player 1 of type k sends with strictly positive probability given $\boldsymbol{g}_s$ (and $\boldsymbol{k} = k$) should yield the same expected payoff to player 1 of type k:

$$\begin{aligned} \boldsymbol{a}_s^k &= \sum_{m \in \operatorname{supp}[\sigma_{t+1}^k(\boldsymbol{g}_s)]} P[\boldsymbol{m}_{t+1}^1 = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_{t+1}^1 = m, \boldsymbol{k} = k] \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}_{t+1}^1 = m, \boldsymbol{k} = k], \quad \text{for all } m \in \operatorname{supp}[\sigma_{t+1}^k(\boldsymbol{g}_s)] \\ &= \boldsymbol{a}_{s+1}^k. \end{aligned}$$

Finally, to prove (iii), assume that $\boldsymbol{p}_N^k > 0$, so

$$\begin{aligned} \boldsymbol{a}_N^k &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k] = E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_N] \\ &= A^k(\tau_{n+1}(\boldsymbol{g}_N)) = A^k(\boldsymbol{y}), \text{ with } \boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N), \end{aligned}$$

the second equality following from the fact that $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent conditional on $\boldsymbol{g}_N$, and the last from the equilibrium condition of player 2. ■

Lemma 4 For every $s = 0, 1, \ldots, N$ we have,

$$\boldsymbol{a}_s \in \mathrm{INTIR}_{\operatorname{supp}[\boldsymbol{p}_s]}.$$

Proof. Let us fix a history g_s such that $P[\boldsymbol{g}_s = g_s] > 0$ and let $\operatorname{supp}[p_s] \subseteq K$, $\operatorname{supp}[p_s] \neq \emptyset$, be the set of types with a strictly positive posterior probability: $p_s^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_s = g_s] > 0$ for all $k \in \operatorname{supp}[p_s]$. We must show that there exists $\overline{y} \in \Delta(J)$ such that

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq A^k(\overline{y}), \text{ for all } k \in \operatorname{supp}[p_s].$$

Player 1's equilibrium condition implies (in particular) that, whatever his type $k \in \operatorname{supp}[p_s]$, if he sends the same message $\overline{m} \in \bigcap_{k \in K} M(k)$ in all upcoming periods $t' \geq \overline{t}$ (where

$\bar{t} = (s+2)/2$ is s is even, and $\bar{t} = (s+3)/2$ is s is odd), then his expected payoff in the current period ($s/2$ if s is even, and $(s+1)/2$ if s is odd) is not increased, so

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}, \boldsymbol{k} = k], \text{ for all } k \in \operatorname{supp}[p_s].$$

Next, remark that, given $\boldsymbol{g}_s = g_s$ and $\boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}$, which specifies the sequence of all player 1's messages in the talking phase, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent. This implies

$$\begin{aligned} E\left[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}, \boldsymbol{k} = k\right] &= E\left[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}\right] \\ = A^k\left(E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}\right]\right). \end{aligned}$$

(Remember that we have extended linearly A^k to mixed actions.) Hence, by letting

$$\overline{y} = E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}^1_{t'} = \overline{m}\ \forall\, t' \geq \bar{t}\right],$$

which does not depend on k (conditional on $\boldsymbol{g}_s$), we have completed the proof of the Lemma. ■

As we have already mentioned, $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$ have all the properties required by Theorem 2 by Lemma 1 and Lemma 2. By Lemma 3 and Lemma 4, the sequence $(\boldsymbol{a}_s)_{s=0,1,\ldots,N}$ also satisfies all the properties of the Theorem. This completes the proof of the "only if" part of Theorem 2. ■

6.2 From constrained dimartingales to equilibrium (if): $H_B(p) \subseteq \mathcal{E}_B(p)$

Let $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$ be a martingale over some probability space $(F, \mathcal{F}, \pi)$ and (finite) sub σ-fields $(\mathcal{F}_t)_{t=1,\ldots,N}$, satisfying the four properties of Theorem 2, with $p^k > 0$ for all $k \in K$, and $N = n$.

We construct a Nash equilibrium (σ, τ) of the n-stage communication game $\Gamma_n(p)$ with expected payoffs (a, β).

First, for convenience we introduce a set W with $K+1$ elements ($|W| = K+1$), write F as W^N, and the atoms of $\mathcal{F}_t$ as elements g_t of W^t. We thus describe the martingale $\boldsymbol{z}$ as

$$\boldsymbol{z} = (z_t(\boldsymbol{g}_t))_{t=0,1,\ldots,n},$$

where for each $t = 0, 1, \ldots, n$, $\boldsymbol{g}_t \in W^t$, and

$$z_t(g_t) = (a_t(g_t), \beta_t(g_t), p_t(g_t)) = \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, z_{t+1}(g_t, w),$$

for all $g_t \in W^t$ satisfying $\pi(g_t) > 0$ (this is the martingale property). Notice that this implies

$$E[\boldsymbol{z}_t] = E[z_t(\boldsymbol{g}_t)] = \sum_{g_t \in W^t} \pi(g_t)\, z_t(g_t) = z_0, \quad t = 0, 1, \ldots, n.$$

The four properties of the martingale in Theorem 2 can be restated as follows:

(1) $z_0(g_0) = z_0 = (a, \beta, p)$.

(2) If $\pi(g_n) > 0$, then $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}^{++}(p_n(g_n))$.

(3) $a_{t+1}(g_{t+1}) = a_t(g_t)$ for all even t and $p_{t+1}(g_{t+1}) = p_t(g_t)$ for all odd t, if $\pi(g_t) > 0$ and $\pi(g_{t+1}) > 0$.

(4) For all $t = 0, 1, \ldots, n$, if $\pi(g_t) > 0$, then

$$a_t(g_t) \in \mathrm{INTIR}_{\mathrm{supp}[p_t(g_t)]} \equiv \{a \in \mathbb{R}^K : \exists\, \overline{y} \in \Delta(J),\ a_t^k(g_t) \geq A^k(\overline{y})\ \forall\, k \in \mathrm{supp}[p_t(g_t)]\},$$

where $\mathrm{supp}[p_t(g_t)] \equiv \{k \in K : p_t^k(g_t) > 0\}$. Notice that $\mathrm{supp}[p_t(\boldsymbol{g}_t)] = \mathrm{supp}[\boldsymbol{p}_t] = \{k \in K : \boldsymbol{p}_t^k > 0\}$, in accordance with the definitions used in Theorem 2.

In odd periods t, w_t is associated to a message $m_t^1 \in M^1$ of player 1 (player 2's message does not affect players's decisions), and in even periods t, w_t is directly associated to a jointly controlled lottery (possibly a series of jointly controlled lotteries), which is not explicitly formalized here.[10] Therefore, a history of messages h_n consists, with some abuse

[10]The technique is standard; see, e.g., Aumann and Maschler (1995) and Aumann and Hart (2003). Note that irrational probabilities might lead to infinitely many jointly controlled lotteries (see Subsection 7.2). For simplicity, the reader may simply consider w_t as a signal publicly observed in even periods.

of notation, in a message $m_t^1 \in M^1$ of player 1 in each odd period t, and in a realization $w_t \in W$ of one or several jointly controlled lotteries in each even period t.

Accordingly, in the remaining of the proof we only construct explicitly player 1's strategy σ_{t+1}^k, $k \in K$, when t is even, and player 2's strategy in the action phase, τ_{n+1}.

The set of histories of the talking phase up to period t is

$$M_t = \begin{cases} (M^1 \times W)^{t/2} & \text{if } t \text{ is even,} \\ (M^1 \times W)^{(t-1)/2} \times W & \text{if } t \text{ is odd.} \end{cases}$$

To each sequence $g_t = (w_1, \ldots, w_t) \in W^t$ we associate a history $\phi_t(g_t) \in M_t$, with $\phi_t(g_t) \neq \phi_t(g_t')$ whenever $g_t \neq g_t'$, as follows:

$$\begin{aligned} \phi_t(g_t) &= \phi_t(w_1, w_2, w_3, w_4 \ldots, w_t) \\ &= (m_1(w_1), w_2, m_3(g_3), w_4, \ldots), \end{aligned}$$

where $g_r = (w_1, \ldots, w_r)$, $r < t$, is a subsequence of g_t, and for all odd t, $m_t(g_t) \in M^1$, $m_t(g_{t-1}, w_t) \neq m_t(g_{t-1}, w_t')$ whenever $w_t \neq w_t'$, and

$$M^{-1}(m_t(g_t)) = \text{supp}[p_t(g_t)].$$

This is possible given our assumption on the richness of the message space.

Player 1's strategy σ. For each even period $t = 0, 2, 4, \ldots$, each sequence $g_t \in W^t$ with strictly positive probability and each type $k \in \text{supp}[p_t(g_t)]$ we construct player 1's local strategy $\sigma_{t+1}^k(\phi_t(g_t))$ (player 1's strategy is irrelevant off the equilibrium path).

For each $w \in \text{supp}[\pi(\cdot \mid g_t)]$, define

$$\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t)) = \frac{\pi(w \mid g_t)\, p_{t+1}^k(g_t, w)}{p_t^k(g_t)},$$

and $\sigma_{t+1}^k(m \mid \phi_t(g_t)) = 0$ if $m \neq m_{t+1}(g_t, w)$ for all $w \in W$.

Player 2's strategy τ. We construct the local strategy $\tau_{n+1}(h_n)$ of player 2 for each final history of talk $h_n \in M_n$, with *and* without strictly positive probability (players' strategies in the talking phase are irrelevant off the equilibrium path, but player 2's strategy in the action phase is very important even after 0-probability histories).

If $\pi(g_n) > 0$ for $g_n \in W^n$, then by the second property of the martingale assumed in the Theorem, $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}^{++}(p_n(g_n))$, so we can define,

$$y(g_n) = \tau_{n+1}(\phi_n(g_n)) \in Y(p_n(g_n)) \text{ such that } \begin{cases} a_n^k(g_n) = A^k(y(g_n)) \text{ if } p_n^k(g_n) > 0 \\ \beta_n(g_n) = \sum_{k \in K} p_n^k(g_n)\, B^k(y(g_n)). \end{cases}$$

If $\pi(g_n) = 0$ for $g_n \in W^n$, then consider the shortest subsequence $g_t = (w_1, w_2, \ldots, w_t)$ of $g_n = (w_1, w_2, \ldots, w_n)$ (note: t may be 0) such that $\pi(g_t) > 0$ and define

$$\tau_{n+1}(\phi_n(g_n)) = \overline{y} \text{ such that } a_t^k(g_t) \geq A^k(\overline{y}) \text{ for all } k \in \operatorname{supp}[p_t(g_t)].$$

This is possible by the forth property of the martingale.

The strategy profile (σ, τ) of the communication game $\Gamma_n(p)$ is now completely defined (except, as explained above, for the JCL). We next check that it generates the appropriate expected payoffs and that it constitutes a Nash equilibrium of $\Gamma_n(p)$.

Let $P = P_{\sigma,\tau,p}$ be the probability distribution on

$$\Omega = K \times M_n \times J,$$

induced by (σ, τ) and p, and let $E = E_{\sigma,\tau,p}$ be the corresponding expectation operator. Note: Since JCL are not formalized, P and E also depend on π for the realizations $w_t \in W$ of JCL (public signals) in even periods.

The next lemma will be useful in several steps of the remaining of the proof.

Lemma 5 For all $t = 0, 1, \ldots, n$ and $g_t \in W^t$ we have:

(i) $P[\boldsymbol{h}_t = \phi_t(g_t)] = \pi(g_t)$;

(ii) $P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] = p_t^k(g_t)$ for all $k \in K$, $\pi(g_t) > 0$.

Proof. By induction on t. For $t = 0$ property (ii) is immediate: $P[\boldsymbol{k} = k] = p^k = p_0^k(g_0)$. For $t = 1$:

(i) We have:

$$\begin{aligned} P[\boldsymbol{h}_1 = \phi_1(g_1)] &= \sum_{k \in K} p^k P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] \\ &= \sum_{k \in K} p^k \sigma_1^k(\phi_1(g_1)) = \sum_{k \in K} p^k \sigma_1^k(m_1(g_1)) \\ &= \sum_{k \in K} p^k \frac{\pi(g_1)\, p_1^k(g_1)}{p^k} = \pi(g_1) \sum_{k \in K} p_1^k(g_1) = \pi(g_1). \end{aligned}$$

(ii) We have:

$$
\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_1 = \phi_1(g_1)] &= \frac{P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_1 = \phi_1(g_1)]} \\
&= \frac{\sigma_1^k(m_1(g_1))\, p^k}{P[\boldsymbol{h}_1 = \phi_1(g_1)]} = \frac{\sigma_1^k(m_1(g_1))\, p^k}{\pi(g_1)} \text{ by (i) just above} \\
&= \frac{\pi(g_1)\, p_1^k(g_1)}{p_0^k} \frac{p^k}{\pi(g_1)} = p_1^k(g_1).
\end{aligned}
$$

Now assume that properties (i) and (ii) are satisfied at t, and let us check them at $t+1$. We distinguish two cases: (a) t is odd, i.e., a JCL is added in $t+1$; (b) t is even, i.e., player 1's signal is added in $t+1$. Case (a) is simpler because we can exploit the fact that the JCL does not depend on k.

In the rest of the proof of the Lemma, let $g_{t+1} = (g_t, w_{t+1}) \in W^{t+1}$.

(a) (i) Since $t+1$ is even we have:

$$
\begin{aligned}
P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})] \\
&= P[\boldsymbol{h}_t = \phi_t(g_t)]\, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)] \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t), \quad \text{by property (i) at } t \\
&= \pi(g_t, w_{t+1}) = \pi(g_{t+1}).
\end{aligned}
$$

(a) (ii) Since $t+1$ is even we have:

$$
\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})] \\
&= P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] \text{ because } w_{t+1} \text{ and } k \text{ are independent} \\
&= p_t^k(g_t) \text{ by property (ii) at } t \\
&= p_{t+1}^k(g_{t+1}) \text{ by the third property of the martingale.}
\end{aligned}
$$

(b) (i) Since $t+1$ is odd we have:

$$\begin{aligned}
P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})] \\
&= P[\boldsymbol{h}_t = \phi_t(g_t)]\, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})) \mid \boldsymbol{h}_t = \phi_t(g_t)] \\
&= \pi(g_t)\, P[\boldsymbol{m}_{t+1} = m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)], \quad \text{by property (i) at } t \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t)\, \sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t) \frac{\pi(w_{t+1} \mid g_t)\, p_{t+1}^k(g_{t+1})}{p_t^k(g_t)} \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t) \sum_{k \in K} p_{t+1}^k(g_{t+1}) \\
&= \pi(g_t)\, \pi(w_{t+1} \mid g_t) = \pi(g_t, w_{t+1}) = \pi(g_{t+1}).
\end{aligned}$$

(b) (ii) Since $t+1$ is odd we have:

$$\begin{aligned}
&P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] = \frac{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1}) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k] P[\boldsymbol{h}_t = \phi_t(g_t) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{m}_{t+1} = m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k] P[\boldsymbol{h}_t = \phi_t(g_t) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{\pi(g_{t+1})} \\
&= \frac{\sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) P[\boldsymbol{h}_t = \phi_t(g_t)] P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)]}{\pi(g_{t+1})},
\end{aligned}$$

the last but one equality following from property (i) at $t+1$, which has been checked just before. By properties (i) and (ii) at t this yields:

$$\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= \frac{\sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) \pi(g_t) p_t^k(g_t)}{\pi(g_{t+1})} \\
&= \frac{\pi(w_{t+1} \mid g_t)\, p_{t+1}^k(g_{t+1})}{p_t^k(g_t)} \frac{p_t^k(g_t) \pi(g_t)}{\pi(g_{t+1})} = p_{t+1}^k(g_{t+1}).
\end{aligned}$$

This completes the proof of Lemma 5. ■

Lemma 6 We have:

(i) $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$ for all $k \in K$;

(ii) $E[B^{\boldsymbol{k}}(\boldsymbol{j})] = \beta$.

Proof. (i) We show by induction on t (starting from $t = n$) that, for $t = 0, 1, \ldots, n$,

$$a_t^k(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k], \quad \forall\, k \in \operatorname{supp}[p_t(g_t)]. \tag{1}$$

In particular, for $t = 0$, this will lead to what we are required to prove:

$$a^k = a_0^k(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_0 = \phi_0(g_0), \boldsymbol{k} = k] = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k].$$

Let $t = n$. If $k \in \operatorname{supp}[p_n(g_n)]$, then, by the construction of player 2's strategy,

$$\begin{aligned} a_n^k(g_n) &= A^k(\tau_{n+1}(\phi_n(g_n))) \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_n = \phi_n(g_n), \boldsymbol{k} = k], \end{aligned}$$

so property (1) is satisfied for $t = n$.

Now assume that the property is satisfied at $t + 1$ and let us check it at t. Let $k \in \operatorname{supp}[p_t(g_t)]$. By the martingale property, we have

$$a_t^k(g_t) = \sum_{w \in \operatorname{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, a_{t+1}^k(g_t, w).$$

We distinguish two cases: when t is odd and when t is even.

If t is odd. Then, $p_{t+1}(g_t, w) = p_t(g_t)$ for all $w \in \operatorname{supp}[\pi(\cdot \mid g_t)]$, which implies $\operatorname{supp}[p_{t+1}(g_t, w)] = \operatorname{supp}[p_t(g_t)]$, so $k \in \operatorname{supp}[p_{t+1}(g_t, w)]$ for all $w \in \operatorname{supp}[\pi(\cdot \mid g_t)]$. Therefore, by the induction hypothesis, for all $w \in \operatorname{supp}[\pi(\cdot \mid g_t)]$ we have

$$a_{t+1}^k(g_t, w) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

so

$$\begin{aligned} a_t^k(g_t) &= \sum_{w \in \operatorname{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\ &= \sum_{w \in \operatorname{supp}[\pi(\cdot|g_t)]} P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w) \mid \boldsymbol{h}_t = \phi_t(g_t)]\; E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\ &= \sum_{w \in \operatorname{supp}[\pi(\cdot|g_t)]} P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k]\; E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k] \\ &= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k]. \end{aligned}$$

If t is even. Then, $a_{t+1}^k(g_t, w) = a_t^k(g_t)$ for all $w \in \operatorname{supp}[\pi(\cdot \mid g_t)]$, which implies, by the induction hypothesis,

$$a_t^k(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

for all w such that $p_{t+1}^k(g_t, w) > 0$. Hence, $a_t^k(g_t)$ is also equal to any average of the previous value, so we get property (1) at t.

(ii) Player 2's expected payoff is

$$\begin{aligned}
E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k \in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] \\
&= \sum_{k \in K} p^k \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k} = k] \, E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{h}_n = h_n] \\
&= \sum_{k \in K} p^k \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k} = k] \sum_{j \in J} \tau_{n+1}(h_n)(j) B^k(j) \\
&= \sum_{k \in K} p^k \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k} = k] \, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n] \sum_{k \in K} P[\boldsymbol{k} = k \mid \boldsymbol{h}_n = h_n] \, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{g_n \in W^n} \pi(g_n) \sum_{k \in K} p_n^k(g_n) \, B^k(\tau_{n+1}(\phi_n(g_n))), \text{ by Lemma 5} \\
&= \sum_{g_n \in W^n} \pi(g_n) \beta_n(g_n), \text{ by the construction of player 2's strategy} \\
&= E[\boldsymbol{\beta}_n] = \beta_0 = \beta.
\end{aligned}$$

This completes the proof of Lemma 6. ■

Lemma 7 The strategy τ of player 2 is a best reply to the strategy σ of player 1 in the n-stage communication game $\Gamma_n(p)$.

Proof. Since $\tau_{n+1}(\phi_n(g_n)) \in Y(p_n(g_n))$ for $\pi(g_n) > 0$ it suffices to check that $p_n^k(g_n) = P[\boldsymbol{k} = k \mid \boldsymbol{h}_n = \phi_n(g_n)]$ for all $k \in K$. This as been proved in Lemma 5 (property (ii) with $t = n$). ■

Lemma 8 The strategy σ of player 1 is a best reply to the strategy τ of player 2 in the n-stage communication game $\Gamma_n(p)$.

Proof. (Sketch of the proof). There is no deviation to a message off the equilibrium path by the interim individually rational condition. There is no deviation by type k to an equilibrium message that is not sent by type k at equilibrium by the construction of player 1's strategy (if a message m is sent along the equilibrium path, but never sent by type k, then, by construction, message m cannot be sent by type k). Finally, the expected payoff of any type k is not modified if he changes the probabilities of the messages sent at equilibrium with strictly positive probability because type k is indifferent between all these messages. ■

By Lemmas 6, 7 and 8, we have constructed the appropriate strategy profile. This completes the proof of Theorem 2. ■

7 Discussion and Extensions

7.1 Mediated Communication

7.2 Persuasion without a Deadline

7.3 Partial Certifiability

7.4 Sequential Rationality

Appendix

A Simple Signalling Examples

Example 1 In the silent game of Figure 9 there is no incentive problem: the expert's preferences over the decisionmaker's beliefs are *positively correlated* with the truth. The optimal actions of the decisionmaker (the non-revealing equilibria) are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p > 3/4, \\ \{j_2\} & \text{if } p < 3/4, \\ \Delta(J) & \text{if } p = 3/4. \end{cases}$$

The corresponding interim individually rational extended equilibrium payoffs (a^1, a^2) of the expert are represented by Figure 10 in solid gray lines.

	j_1	j_2	
k_1	$1,1$	$0,0$	p
k_2	$0,0$	$3,3$	$(1-p)$

Figure 9: Silent Game of Example 1.

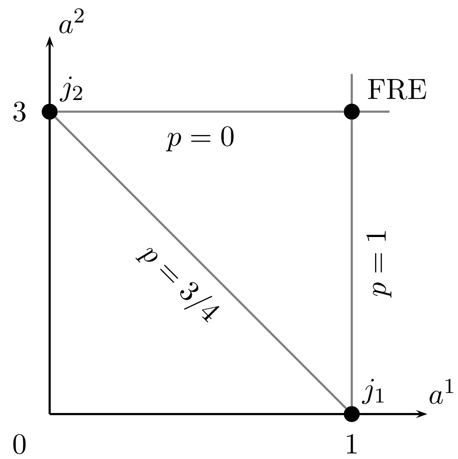

Figure 10: Extended equilibrium payoffs of the expert in Example 1.

Example 2 In the silent game of Figure 11 the expert's preferences over the decisionmaker's beliefs are *not correlated* with the truth since the expert's payoff $A^k(j)$ does not depend on k so he always want the decisionmaker to choose the same action whatever his type. The optimal actions of the decisionmaker (the non-revealing equilibria) are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p > 2/3, \\ \{j_2\} & \text{if } p < 2/3, \\ \Delta(J) & \text{if } p = 2/3. \end{cases}$$

The corresponding interim individually rational extended equilibrium payoffs of the expert are represented by Figure 12 in solid and dashed gray lines.

	j_1	j_2	
k_1	$3, 2$	$1, 0$	p
k_2	$3, 0$	$1, 4$	$(1-p)$

Figure 11: Silent Game of Example 2.

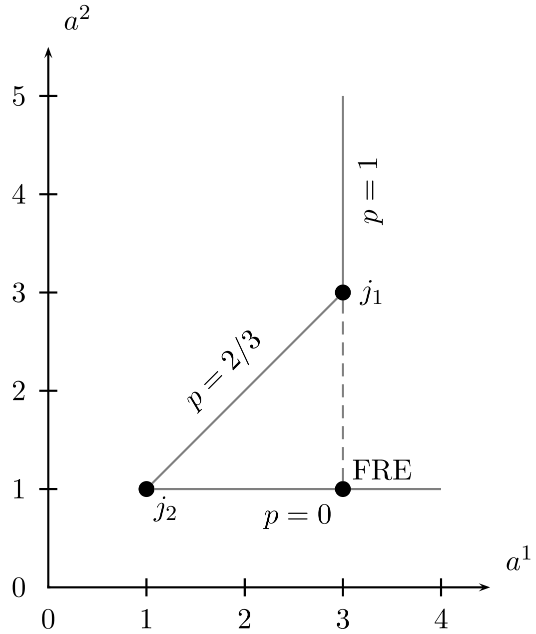

Figure 12: Extended equilibrium payoffs of the expert in Example 2.

Example 3 In the silent game of Figure 13 the correlation fails more dramatically than in Example 2: the expert's preferences over the decisionmaker's beliefs are *negatively correlated* with the truth. Cheap talk and information certification cannot matter here. The optimal actions of the decisionmaker are the same as in Example 2. The corresponding interim individually rational extended equilibrium payoffs of the expert are represented by Figure 14 in solid gray lines. The dotted lines do not belong to the set of interim individually rational payoffs, so the communication game does not admit a fully revealing equilibrium.

	j_1	j_2	
k_1	$3,2$	$4,0$	p
k_2	$3,0$	$1,4$	$(1-p)$

Figure 13: Silent Game of Example 3.

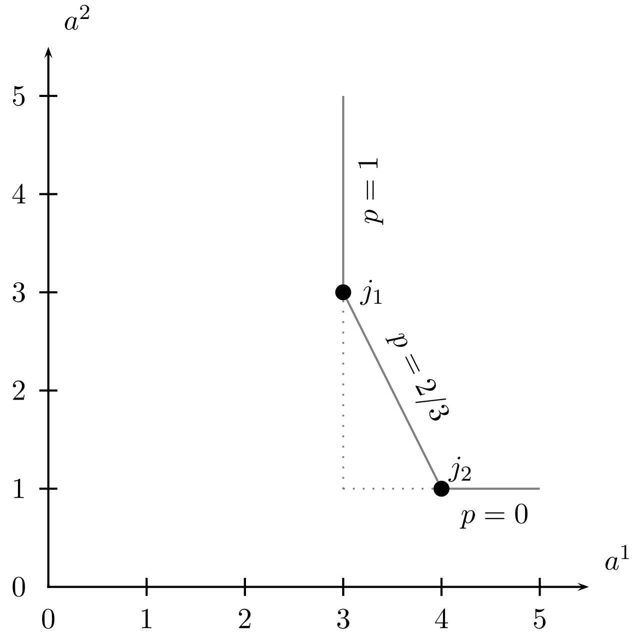

Figure 14: Extended equilibrium payoffs of the expert in Example 3.

Example 4 In the silent game of Figure 15, the optimal actions of the decisionmaker (the non-revealing equilibria) are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p < 3/10, \\ \Delta(\{j_1, j_2\}) & \text{if } p = 3/10, \\ \{j_2\} & \text{if } p \in (3/10, 7/10), \\ \Delta(\{j_2, j_3\}) & \text{if } p = 7/10, \\ \{j_3\} & \text{if } p \in (7/10, 4/5), \\ \Delta(\{j_3, j_4\}) & \text{if } p = 4/5, \\ \{j_4\} & \text{if } p > 4/5. \end{cases}$$

	j_1	j_2	j_3	j_4	
k_1	$4, 0$	$2, 7$	$5, 9$	$1, 10$	p
k_2	$1, 10$	$4, 7$	$4, 4$	$2, 0$	$1 - p$

Figure 15: Silent Game of Example 4.

The corresponding interim individually rational extended equilibrium payoffs (a^1, a^2) of the expert are represented by Figure 16 in solid gray lines. As in Example 3 this game does not admit a fully revealing equilibrium (the dotted lines are not interim individually rational), but it has a partially revealing equilibrium for $p \in (3/10, 4/5)$.

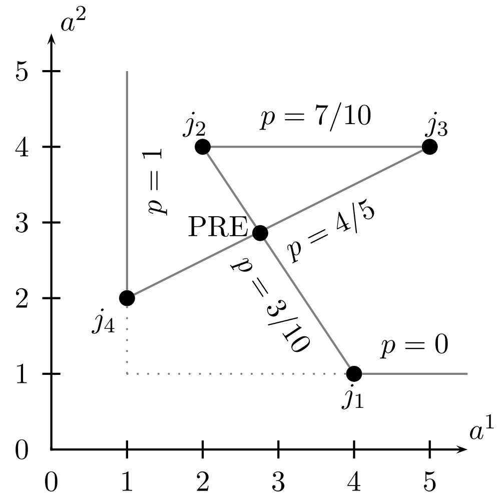

Figure 16: Extended equilibrium payoffs of the expert in Example 4.

Example 5 In the silent game of Figure 17,[11] the optimal actions of the decisionmaker (the non-revealing equilibria) are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p < 2/7, \\ \Delta(\{j_1, j_2\}) & \text{if } p = 2/7, \\ \{j_2\} & \text{if } p \in (2/7, 5/7), \\ \Delta(\{j_2, j_3\}) & \text{if } p = 5/7, \\ \{j_3\} & \text{if } p > 5/7. \end{cases}$$

	j_1	j_2	j_3	
k_1	$1, -2$	$3, 3$	$2, 5$	p
k_2	$2, 5$	$3, 3$	$1, -2$	$1 - p$

Figure 17: Silent Game of Example 5.

The corresponding interim individually rational extended equilibrium payoffs of the expert are represented by Figure 18 in solid gray lines.

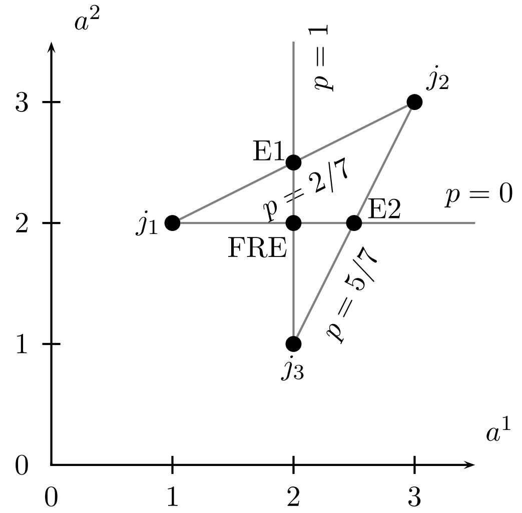

Figure 18: Extended equilibrium payoffs of the expert in Example 5.

[11] This game is taken from Farrell and Rabin (1996).

Example 6 (Forges, 1986b) In the silent game of Figure 19, the optimal actions of the decisionmaker are

$$Y(p) = \begin{cases} \{j_1\} & \text{if } p < 1/3, \\ \{j_2\} & \text{if } p \in (1/3, 1/2), \\ \{j_3\} & \text{if } p \in (1/2, 2/3), \\ \{j_4\} & \text{if } p > 2/3. \end{cases}$$

	j_1	j_2	j_3	j_4	j_5	
k_1	$3, -2$	$3, 0$	$0, 3$	$3, 4$	$1, 0$	p
k_2	$3, 4$	$0, 3$	$3, 0$	$3, -2$	$1, 0$	$1 - p$

Figure 19: Silent Game of Example 6.

The corresponding interim individually rational extended equilibrium payoffs of the expert are represented by Figure 20 in solid gray lines.

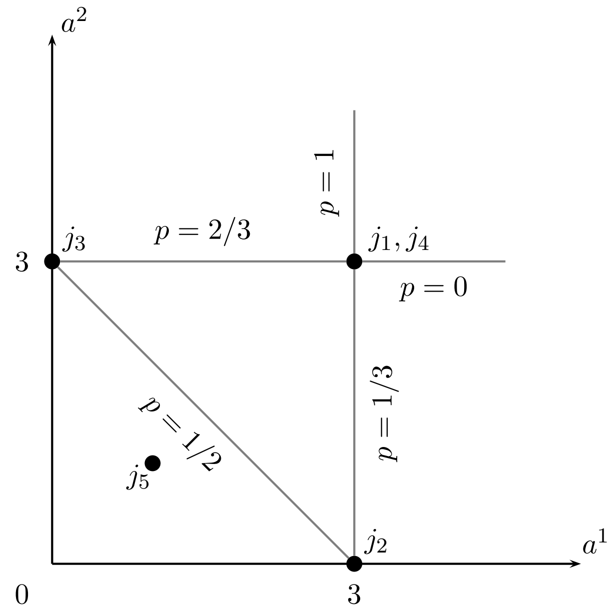

Figure 20: Extended equilibrium payoffs of the expert in Example 6.

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