Better Ways to Cut a Cake

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Abstract

Simple cake-cutting procedures used to divide a cake, which could be any heterogeneous good, are analyzed and compared. The well-known 2-person, 1-cut cake-cutting procedure, cut-and-choose, while envy-free and efficient, is not equitable, limiting the cutter to exactly 50% when the chooser, in general, can do better. A new *surplus procedure* (SP), which induces the players to be truthful in order to maximize their minimum allocations, leads to a more equitable division of the surplus—the part that remains after each person receives exactly 50%. However, SP is more information-demanding than cut-and-choose, requiring that the players report their value functions over the entire cake, not just indicate 50-50 points.

For 3 persons, there may be no envy-free division that is equitable. But there is a simple 3-person, 2-cut *squeezing procedure* that induces maximin players to make cuts that yield an envy-free division. By contrast, no 4-person, 3-cut envy-free procedure is known to exist. The applicability of the surplus and squeezing procedures to the fair division of a heterogeneous good, like land, are discussed.

Better Ways to Cut a Cake

1. Introduction

Economics, we are told, is the study of the allocation of scarce resources. If the resources constitute a heterogeneous, divisible good, parts of which are valued differently by different people, then the best way to divide it is of central importance.

Also important is the perceived fairness of a division. Divisions viewed as fair are more likely to be accepted as legitimate and, therefore, to be stable. But stability is also related to the difficulty of manipulating a procedure that produces a division.

In this paper, we analyze cake-cutting algorithms to divide a cake, which could be any heterogeneous good. These algorithms, which use a minimal number of cuts, not only establish the existence of fair divisions—defined by properties described below—but also provide step-by-step procedures for carrying them out. In addition, they give us insight into the difficulties underlying the simultaneous satisfaction of certain properties of fair division, including the incentive to be truthful about one's valuation of the cake.

We begin with the well-known 2-person cake-cutting procedure, "I cut, you choose," or cut-and-choose. It goes back at least to the Hebrew Bible (Brams and Taylor, 1999, p. 53) and satisfies two desirable properties:

1. *Envy-freeness*: Each person thinks that he or she receives at least a tied-for-largest piece and so does not envy the other person.
2. *Efficiency (Pareto-optimality)*: There is no other allocation that is better for one person and at least as good for the other person.

But cut-and-choose does not satisfy a third desirable property:

3. *Equitability*. Each person's subjective valuation of the piece that he or she receives is the same as the other person's subjective valuation.

The last property raises a question about the interpersonal comparison of utility, about which we will say more later.

We propose a new cake-cutting procedure that, while it does not satisfy equitability in an absolute sense, does approximate it in a relative sense: After ensuring that each person receives exactly 50%, it gives each person approximately the same proportion of the cake that remains, called the *surplus*. Thereby this procedure, which we call the *surplus procedure* (SP), gives each person more than 50% of the entire cake. By contrast, cut-and-choose limits the cutter to exactly 50% when he or she has no information about the other person's preferences.

As is usual in the cake-cutting literature, we postulate that the goal of each person is to maximize the value of the minimum-size piece (*maximin piece*) that he or she can guarantee, regardless of what the other person does. Thus, we assume that each person is *risk-averse*: He or she will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a maximin piece.

Remarkably, as we will show, maximin strategies under SP require that each person be truthful about his or her preferences for different portions of the cake. This is because the incentives to undervalue and to overvalue different portions conflict, creating a tension such that being truthful becomes the unique strategy that guarantees 50% shares plus a minimum percentage of the surplus.

In section 2, we describe cut-and-choose and illustrate it with an example. In section 3, we describe SP and show that it gives, initially, "proportional equitability" that is only approximate. But SP provides two persons with the incentive to negotiate a cut that makes proportional equitability exact, so each person in the end receives exactly the same proportion of the surplus.

If there are $n > 2$ persons, we show in section 4 that envy-freeness and equitability cannot always be achieved with $n - 1$ cuts (the minimal number). We describe a simple 3-person, 2-cut envy-free procedure, called the *squeezing procedure*. If $n > 3$, it is not known whether there exists an n-person, (n-1)-cut envy-free procedure.

In section 5, we discuss trade-offs in cake division. Whereas SP is more equitable than cut-and-choose, it is also more information-demanding. The squeezing procedure, on the other hand, does not require any prior revelation of information, but it does require the use of "moving knives" (Brams, Taylor, and Zwicker, 1995). We conclude by considering the applicability of these procedures to real-world problems of fair division.

2. Cut-and-Choose

Assume that two players, A and B, value a cake along a line that ranges from $x = 0$ to $x = 1$. More specifically, we postulate that the players have value functions, $v_A(x)$ and $v_B(x)$, where $v_A(x) \geq 0$ and $v_B(x) \geq 0$ for all x over [0, 1]. Analogous to probability density functions, or *pdfs*, we assume the total valuations of the players—the areas under $v_A(x)$ and $v_B(x)$—are 1. We also assume that only parallel, vertical cuts, perpendicular to the horizontal x-axis, are made, which we will illustrate later.

Under cut-and-choose, one player cuts the cake into two portions, and the other player chooses one. To illustrate, assume a cake is vanilla over [0, 1/2] and chocolate over (1/2, 1]. Suppose the cutter, player A, values the left half (vanilla) twice as much as the right half (chocolate). This implies that $v_A(x) = 4/3$ on [0, 1/2], and $v_A(x) = 2/3$ on (1/2, 1].

To guarantee envy-freeness when the players have no information or beliefs about each other's preferences, A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.[1] The two portions will be equal when A's valuation of the cake between 0 and x is equal to the sum of its valuations between x and 1/2 and between 1/2 and 1:

$$(4/3)(x - 0) = (4/3)(1/2 - x) + (2/3)(1 - 1/2),$$

which yields $x = 3/8$. In general, the only way that A, as the cutter, can ensure itself of getting half the cake is to give B the choice between two portions that A values at exactly 1/2 each.

To show that cut-and choose does not satisfy equitability, assume B values vanilla and chocolate equally. Thus, when A cuts the cake at $x = 3/8$, B will prefer the right portion, which it values at 5/8, and consequently will choose it. Leaving the left portion to A, B does better in its eyes (5/8) than A does in its eyes (1/2), rending cut-and-choose inequitable.

If the roles of A and B as cutter and chooser are reversed, the division remains inequitable. In this case, B will cut the cake at $x = 1/2$. A, by choosing the left half (all vanilla), will get 2/3 of its valuation, whereas B, getting the right half, will receive only 1/2 of its valuation. Because cut-and-choose selects the endpoints of the interval of envy-free cuts, any cut between 3/8 and 1/2 will be envy-free.

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[1] When players do have information or beliefs, a cutter may do better with a less conservative strategy (Brams and Taylor, 1996, 1999).

3. The Surplus Procedure (SP)

Here are the rules of SP, which we will refer to as *steps*:

1. Independently, A and B report their value functions, $f_A(x)$ and $f_B(x)$, over the cake, [0, 1], to a referee. These functions may be different from the players' true value functions, $v_A(x)$ and $v_B(x)$.

2. The referee determines the 50-50 points, a and b, of A and B—that is, the points on [0, 1] such that each player reports that half the cake, as it values it, lies to the left and half to the right (these points are analogous to the median points of pdfs).

3. If a and b coincide, the cake is cut at $a = b$. One player is randomly assigned the piece to the left of this cutpoint, the other player the piece to the right. The procedure ends.

4. Assume that a is to the left of b, as illustrated below:

0--------------------*a*----------------*b*---------------------1.

Then A receives the portion [0, a], and B the portion [b, 1], which each player values at 1/2 according to its reported value function.

5. Let c (for cutpoint) be the point in [a, b] at which the players receive the same *percentage* of the cake in this interval, as each values it:

0--------------------*a*----------*c*-------*b*---------------------1.

Determine the points, $a' \leq c$ and $b' \geq c$, that give A and B the maximum *common* percentage of the surplus, in deciles, that is possible when A gets [a, a'] and B gets [b', b].

0--------------------*a*----- *a*'---*c*--*b*'-----*b*---------------------1.

This decile percentage, *p*, will necessarily be less than or equal to the percentage that the players receive at *c*.

Remark 1. If A values the cake more near *a* and B values the cake more near *b*, *p* will be greater than 50% (e.g., 60%), whereas *p* will be less than 50% (e.g., 40%) if the opposite is the case.

6. If *p* occurs at *c*, *a*' = *c* = *b*'. Then the cake is cut at *c*, with A receiving the portion to the left and B the portion to the right.

7. Assume *p* does not occur at *c*. Then the player that values the subinterval [*a*', *b*'] more is awarded this portion of the cake. If this player is A, it receives *in toto* the portion [0, *b*'], and B receives the portion (*b*', 1]. If this player is B, it receives *in toto* the portion (*a*', 1], and A receives *in toto* the portion [0, *a*']. If the players equally value the subinterval [*a*', *b*'], then A gets the portion to the left of *c*, and B gets the portion to the right of *c*.

8. Before the referee informs the players of these assignments, they would be told the value of *p*. If *both* players agree to settle at *c*, the cake is cut at *c*, with A getting the portion [0, *c*] and B getting the portion (*c*, 1].

Remark 2. When the players are informed of the value of *p* (e.g., 50%), they can surmise that if they both agree, they both will both get a minimum of *p*.

To illustrate SP with the example in section 2, recall that the 50-50 points for A and B, respectively, are *a* = 3/8 and *b* = 1/2. To find the point *c* in [*a*, *b*] = [3/8, 1/2] at which A and B obtain the same percentage of the cake in this interval—as each player values it—note that A attaches value

$$V_A(x) = \int_a^b v_A(x)dx = \int_{3/8}^{1/2}(4/3)dx = 1/6 \tag{1}$$

to this interval, and B attaches value

$$V_B(x) = \int_a^b v_B(x)dx = \int_{3/8}^{1/2} dx = 1/8 \tag{2}$$

to it.

We solve for the point c at which the percentage of [3/8, 1/2] that A receives (to the left of c) is equal to the percentage that B receives (to the right):

$$\int_{3/8}^{c}[v_A(x)dx]/(1/6) = \int_{c}^{1/2}[v_B(x)dx]/(1/8),$$

which yields c = 7/16 = 0.4375, the midpoint of the interval, because both players have uniform value functions over [3/8, 1/2]. At this point, A receives a value of 1/12 from [3/8, 7/16], and B receives a value of 1/16 from (7/16, 1/2]. These values are exactly 1/2 the players' valuations of the [*a*, *b*] interval, [3/8, 1/2], as given by equations (1) and (2) above, so p = 0.50.

Because p has a decile value, the subinterval, [*a*', *b*'] = [*c*, *c*] has zero length and, therefore, is of no value to the players. Thus, SP ends in step 6, with A receiving a total value of 1/2 + 1/12 = 7/12 = .5833, and B receiving a total value of 1/2 + 1/16 = 9/16 = 0.5625.

Note that *both* players receive more than 50%, but not by the same amount, because they value [*a*, *b*] differently. Because A values this interval more than B does, it receives more, in absolute terms, since SP gives each player the same percentage of the interval, as each values it (50% in our example).[2] We will give another example later in this section in which the subinterval [a', b'] does not shrink to a point, because cutting at c does not happen to give a decile value to the players.

To show that maximin players will be truthful when they submit their value functions to a referee, we next show that A or B may do worse if they are not truthful in reporting the following:

1. *The locations of their 50-50 points, a and b.*

Assume B is truthful and A is not. If A misrepresents a and causes it to crisscross b, as illustrated by the location of α below,

0--------------------------a---b--α-------------------------1.

then A will obtain [α, 1] and, in addition, get some less-than-complete portion of (b, α). But this is less than 50% of the cake for A and, therefore, less than what A would obtain under SP if it was truthful.

2. *The locations of a' and b'.*

Again, assume B is truthful and A is not. Because A does not know the location of b, it does not know the location of a', much less b'. Without knowing the location of [a', b'], it cannot overvalue this subinterval with certainty in order to increase its chances of obtaining it under SP. If it overvalues, instead, the portion [a, a'], it may only succeed in moving a' leftward and do worse than if it were truthful, whether or not it obtains the subinterval [a', b']. The uncertainty about the location of [a', b'] robs A of the ability to report a value function $f_A(x) \neq v_A(x)$ that would assuredly give it a better outcome than truthfulness gives.

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[2] By comparison, an equitable cut at 3/7 $\approx$ 0.429 would give A a value of 4/7 $\approx$ 0.571 to the left and B a value of 0.571 to the right. This common value to the players is between what A and B receive under SP (0.5833 and 0.5625, respectively).

In summary, if A is not truthful in step 1 of SP, it may not obtain 50% of the cake or, if it does, it may obtain a smaller portion of [*a*, *b*] than it would obtain if it were truthful. To be sure, A may succeed in increasing the value of its portion over what it would obtain by being truthful. But there is no guarantee that this will occur and, indeed, the opposite outcome is possible, as we have shown. We conclude that truthfulness is the only strategy that ensures both players of at least 50%, and generally more, under SP.

Are there variants of SP that offer the same assurance? Consider the following:

1. Change step 7 of SP so that the player that more values the interval [*a*, *b*]—not just the subinterval [*a*', *b*']—receives it. This variant, which is effectively cut-and-choose (with the player that values [*a*, *b*] more the chooser), would limit one player to exactly 50%. Moreover, each player, knowing one bound of [*a*, *b*], would have an incentive to overvalue the cake near this bound to increase its chances of obtaining all of [*a, b*].
2. Change step 7 of SP so that the cake is divided at *c* without the assent of both players (step 8). This variant would give each player an incentive to undervalue, in proportional terms, the cake near its 50-50 point in order to push *c* toward the other player's 50-50 point (the players would still have an incentive to be truthful about their 50-50 points to ensure a minimum of 50%). By contrast, the reward of [*a*', *b*'] to the player that values it more counters this incentive to undervalue under SP, creating the tension needed to induce truthfulness. This tension persists even when the players reach agreement to cut at *c* in step 8, because each must assume that there may not be agreement, in which case the procedure would end at step 7.
3. Divide [*a*, *b*] not at *c* but at the point *e* (for equitable) where the players' valuations of their portions (A's from 0, B's from 1) are *exactly* the same, which would satisfy equitability rather than proportional equitability. This

variant would give each player an incentive to undervalue the cake, in absolute terms, near its 50-50 point in order to push *e* toward the other player's 50-50 point. Jones (2002) shows that the point *e* always exists.

For variant 3, we could define a subinterval around *e*—as we do around *c* for SP—and award it to the player that values it more. For example, if both players increase their valuations of the entire cake from 50% at *a* and *b* to 70% at *e*, the rule might be that they each would receive 3/4 of the additional 20% (i.e., 15% each). As for the subinterval that contains the remaining 5% for each, it would go to the player that values it more.

Like step 7 of SP, awarding the subinterval to the player that values it more would counter the incentive of the players to undervalue the cake near *a* and *b* in order to push *e* toward the 50-50 point of the other player. But it is possible that the player that values the subinterval more might value the entire interval, [*a*, *b*], less, so this variant could give more of [*a*, *b*], in absolute terms, to the player that values it less. By contrast, under SP this effect is mitigated, because each player gets approximately the same percentage of [*a*, *b*]—up to a decile—whether it wins or loses [*a*', *b*'], so the player that values [*a*, *b*] more is also likely to do better in absolute terms.

It is useful to illustrate SP with an example in which [*a*', *b*'] does not shrink to a point, as in our earlier example. Suppose A's valuation function is $v_A(x) = 2x$, and B's is $v_B(x) = 1$, on [0, 1]. Thus, A has a triangular distribution, with its value most concentrated at $x = 1$, and B has a uniform distribution. It is straightforward to show that $a = 1/\sqrt{2} \approx 0.707$, $b = 1/2 = 0.500$, and $c = \left(1+\sqrt{2}\right)\left(\sqrt{21-12\sqrt{2}}-1\right)/4 \approx 0.608$. At cutpoint *c*, A and B would each receive about $(.130)/(.280) \approx 52\%$ of $[a, b] \approx [.500, .707]$.

Under SP, however, $p = 50\%$, so B would receive exactly 50% of [*a*, *b*], as it values it. A would receive about 54% because of the higher value it places on $[a', b'] \approx [.604, .612]$, which it wins.

In sum, B would get [0, .604], giving it 60.4% of the entire cake, and A would get (.604, 1], giving it 63.5% of the entire cake. Thereby, SP slightly favors the player (A) that values [*a*, *b*]—as well as [*a*', *b*']—more. Nonetheless, both players, in receiving more than 60% each, are considerably above the 50% that cut-and-choose would give the cutter.

It is worth noting that if the cake were cut at the equitable point, *e*, both players would get 61.8% in our example, which is less for A (triangular distribution) and more for B (uniform distribution) than SP gives each. We think it fair that A benefits more under SP, because A values [*a*, *b*] as well as [*a*', *b*']—the portion of the surplus that it wins—more than B does. Still, both players do quite well under SP.

To return to our three properties, any 2-person, 1-cut procedure that gives at least 50% to each player, including cut-and-choose and SP, is envy-free and efficient. The main difference between these procedures is the closeness with which they approximate equitability, which involves an interpersonal comparison of utility between A and B: How does the subjective value that A attaches to its portion of the cake compare with the subjective value that B attaches to its portion?[3]

The equitability comparison, we believe, is perfectly legitimate in assessing how satisfied A and B are likely to be with their shares. Although analysts might differ over whether equitability or proportional equitability is the appropriate standard, our examples suggest that the difference in allocations that each standard yields will not usually be very great.

Proportional equitability awards [*a*', *b*'] to the player that values it more. An equitable allocation at *e* may reverse this. In our last example, *e* = 0.618, so B would receive [0, .618], which includes [*a*', *b*'] = [.604, .612]. Consequently, B would do better than under SP.

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[3] By comparison, A's possible envy of B does not depend on how B values its portion but, rather, on how A values B's portion. In this sense, envy-freeness does not involve an interpersonal comparison of utility.

Whether subintervals are defined around c, as SP does, or around e if one redefines SP in terms of equitability rather than proportional equitability, how big subintervals should be in comparison to the interval, $[a, b]$, is a question we leave open. For c, we suggested giving each player up to the largest decile possible; for e, it could be a fixed percentage of $[a, b]$, like 75%. In either case, it is the uncertainty about the location of the subinterval that induces the players not to undervalue the cake beyond their 50-50 points, because by doing so they risk losing the subinterval.

How might SP be extended to three or more players is unclear. In section 4, we will describe a procedure that induces maximin players to make envy-free cuts in 3-person cake division, but this division may not be equitable, even in theory.

4. Extensions to Three or More Players

We now show, via an example, that it is not always possible to divide a cake among three players into envy-free and equitable portions using two cuts. Because envy-freeness seems to us to be the more important property to satisfy if one has to make a choice between it and equitability, we henceforth focus on it.

Assume that A and B have (truthful) piecewise linear value functions that are symmetric and V-shaped,

$$v_A(x) = \begin{cases} -4x + 2 & \text{for } x \in [0, 1/2] \\ 4x - 2 & \text{for } x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} -2x + 3/2 & \text{for } x \in [0, 1/2] \\ 2x - 1/2 & \text{for } x \in (1/2, 1] \end{cases}.$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is "steeper" (higher maximum, lower minimum) than B's, as illustrated in Figure

1. In addition, suppose that a third player, C, has a uniform value function, $v_C(x) = 1$ for $x \in [0, 1]$.

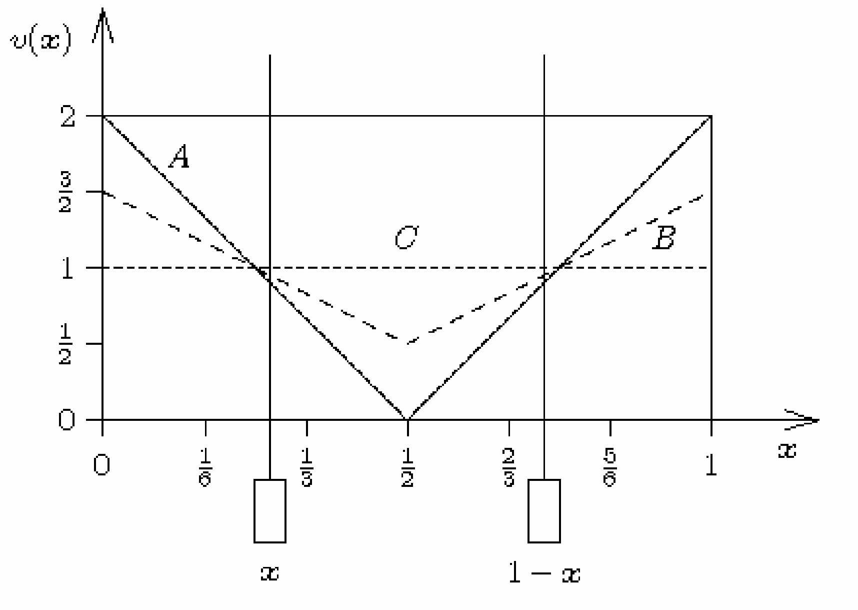

Figure 1. Impossibility of Envy-Free and Equitable Cuts for Three Players

In this situation, every envy-free allocation of the cake will be one in which A gets the portion to the left of x, B the portion to the right of $1 - x$ (A and B could be interchanged), and C the portion in the middle. If the horizontal lengths of A and B's portions are not the same (i.e., x), the player whose portion is shorter in length will envy the player whose portion is longer. But such an envy-free allocation will not be equitable, because A will receive a larger portion in its eyes than B receives in its eyes, violating equitability. Thus, an envy-free allocation cannot be equitable in this example, nor an equitable allocation envy-free, though both these allocations will be efficient with respect to parallel, vertical cuts.[4]

[4] This conflict also holds for proportional equitability. It is worth pointing out that an equitable allocation need not be efficient. Thus, if C were given an end piece and A or B the middle piece in the example, cutpoints could be found such all the players receive, in their own eyes, the same value. However, this value would be less than what another equitable allocation, in which C gets the middle piece and A and B the end pieces, yields. By contrast, an envy-free allocation that uses $n - 1$ parallel, vertical cuts is always efficient (Gale, 1993; Brams and Taylor, 1996, pp. 150-151).

Two envy-free procedures have been found for 3-person, 2-cut cake division. Whereas one of the envy-free procedures requires four simultaneously moving knives (Stromquist, 1980), the other requires only two simultaneously moving knives (Barbanel and Brams, 2004).

We describe the simpler procedure of Barbanel and Brams that involves "squeezing" pieces. It assumes that virtual cuts, or what Shishido and Zeng (1999) call "marks," can be made on the line segment defining the cake. These marks may subsequently be changed by another player before real cuts are made.

Players are given "instructions." If they are followed, an envy-free allocation results; if they are not, then a player may do worse, violating the maximin goal we assume of players.

Barbanel-Brams 3-person, 2-cut squeezing procedure. A referee slowly moves a knife from left to right across a cake. The players are instructed to call stop when the knife reaches the 1/3 point for each. Let the *first* player to call stop be player A. (If two players call stop at the same time, choose one randomly.)

Let A place a mark at the point where it calls stop (the right boundary of piece 1 in the diagram below), and a second mark to the right that bisects the remainder of the cake (the right boundary of piece 2 below). Thereby A indicates the two points that, for it, trisect the cake into pieces 1, 2, and 3:

```
        1               2               3
0---------------|---------------|---------------1.
                A               A
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Because neither player B nor player C called stop before A did, each of B and C thinks that piece 1 is at most 1/3. They are then asked whether they prefer piece 2 or piece 3. There are three cases to consider:

1. If B and C each prefers a different piece—one player prefers piece 2 and the other piece 3—we are done: A, B, and C can each be assigned a piece that they consider to be at least tied for largest.

2. Assume B and C both prefer piece 2. A referee places a knife at the right boundary of piece 2 and moves it to the left. Meanwhile, A places a knife at the left boundary of piece 2 and moves it to the right in such a way that the amounts of cake traversed on the left and right are equal for A. Thereby pieces 1 and 3 increase equally in A's eyes. At some point, piece 2 will be diminished sufficiently to piece 2'—in either B's or C's eyes—to tie with either piece 1' or piece 3', the enlarged 1 and 3 pieces. Assume B is the first, or the tied-for-first, player to call stop when this happens; then give C piece 2', which it still thinks is the largest or the tied-for-largest piece. Give B the piece it thinks ties for largest with piece 2' (say, piece 1'), and give A the remaining piece (piece 3'), which it thinks ties for largest with the other enlarged piece (piece 1'). Clearly, each player will think it gets at least a tied-for-largest piece.

3. Assume B and C both prefer piece 3. A referee places a knife at the right boundary of piece 2 and moves it to the right. Meanwhile, A places a knife at the left boundary of piece 2 and moves it to the right in such a way as to maintain the equality, in its view, of pieces 1 and 2 as they increase. At some point, piece 3 will be diminished sufficiently to piece 3'—in either B or C's eyes—to tie with either piece 1' or piece 2', the enlarged 1 and 2 pieces. Assume B is the first, or the tied-for-first, player to call stop

when this happens; then give C piece 3', which it still thinks is the largest or the tied-for-largest piece. Give B the piece it thinks ties for largest with piece 3' (say, piece 1'), and give A the remaining piece (piece 2'), which it thinks ties for largest with the other enlarged piece (1'). Clearly, each player will think it got at least a tied-for-largest piece.

Note that which player moves a knife or knives varies, depending on what stage is reached in the procedure. In the beginning, we assume a referee moves a single knife, and the first player to call stop (A) then trisects the cake. But at the next stage of the procedure, in cases (2) and (3), it is the referee and A that move two knives simultaneously, “squeezing” what players B and C consider to be the largest piece until it eventually ties, for one of them, with one of the two other pieces. While Barbanel and Brams show that squeezing can also be used to produce an “almost” envy-free 4-person, 3-cut division (at most one player is envious), absolute envy-freeness eludes them unless up to 5 cuts are allowed, which may require combining disconnected pieces.

Earlier, Brams, Taylor, and Zwicker (1997) gave a 4-person, envy-free procedure that requires up to 11 cuts; chore division for 4 players requires even more (16 cuts) (Peterson and Su, 2002). Because the Brams-Taylor-Zwicker 4-person procedure involves fewer cases than the Barbanel-Brams procedure, it is probably simpler, even though it requires more cuts (11 versus 5).

Beyond 4 players, no procedure is known that yields an envy-free division of a cake unless an unbounded number of cuts is allowed (Brams and Taylor, 1995, 1996; Robertson and Webb, 1998). While this number can be shown to be finite, it cannot be specified in advance—this will depend on the specific cake being divided. The complexity of what Brams and Taylor call the “trimming procedure” makes it of dubious practical value.

5. Conclusions

We have described a new 2-person, 1-cut cake-cutting procedure, called the surplus procedure (SP). Like cut-and-choose, it induces the players to be truthful when they have no information about each other's preferences. But unlike cut-and-choose, it produces a proportionally equitable division, which is approximate if the procedure ends in step 7, exact if it ends in step 3 or step 8.

SP is more information-demanding than cut-and-choose, requiring that the players submit to a referee their value functions over an entire cake, not just indicate a 50-50 point. Practically, players might sketch such functions, or choose from a variety of different-shaped functions, to indicate how they value a divisible good like land.

Thus, land bordering water might be more valuable to one person (A), whereas land bordering a forest might be more valuable to the other (B). Even if players know these basic preferences of each other, and hence that a will be closer to the water and b will be closer to the forest, SP creates sufficient uncertainty about a' and b' that it would be impossible for maximin players to exploit it without knowledge of the other player's value function.

For 3 persons, there may be no an envy-free division that is also equitable. But there is a simple 3-person squeezing procedure that induces maximin players to make cuts that yield an envy-free division with only 2 cuts. By contrast, no 4-person, 3-cut envy-free procedure is known to exist.

Most disputes over land or other divisible property boil down to two or three parties, so it is pleasing to have procedures that yield efficient and envy-free divisions, which are proportionally equitable (approximate or exact) in the case of two parties. If there are multiple divisible goods that must be divided, however, 2-person procedures like "adjusted winner" (Brams and Taylor, 1996, 1999) seem more applicable than cake-cutting procedures, though Jones (2002) shows that adjusted winner can be viewed as a cake-cutting procedure.

The fair division of indivisible goods poses significant new challenges that lead to certain paradoxes (Brams, Edelman, and Fishburn, 2001). But recently progress has been made in finding ways of dividing such goods (Brams and Fishburn, 2000; Edelman and Fishburn, 2001; Brams, Edelman, and Fishburn, 2003; Brams and King, 2004). Ideally, procedures that work for both divisible and indivisible goods will inspire new approaches to settling disputes at all levels, from interpersonal to international.

References

Barbanel, Julius B., and Steven J. Brams (2004). "Cake Division with Minimal Cuts: Envy-Free Procedures for 3 Person, 4 Persons, and Beyond." *Mathematical Social Sciences* 48, no. 4 (November): 251-269.

Brams, Steven J., Paul H. Edelman, and Peter C. Fishburn (2001). "Paradoxes of Fair Division." *Journal of Philosophy* 98, no. 6 (June): 300-314.

Brams, Steven J., Paul H. Edelman, and Peter C. Fishburn (2003). "Fair Division of Indivisible Goods." *Theory and Decision* 55, no. 2 (September): 147-180.

Brams, Steven J., and Peter C. Fishburn (2000). "Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-Freeness, Pareto-Optimality and Equity." *Social Choice and Welfare* 17, no. 2 (February): 247-267.

Brams, Steven J., and Daniel L. King (2004). "Efficient Fair Division: Help the Worst Off or Avoid Envy?" Preprint, Department of Politics, New York University.

Brams, Steven J., and Alan D. Taylor (1995). "An Envy-Free Cake Division Protocol." *American Mathematical Monthly* 102, no. 1 (January 1995): 9-18.

Brams, Steven J., and Alan D. Taylor (1996). *Fair Division: From Cake-Cutting to Dispute Resolution.* Cambridge, UK: Cambridge University Press.

Brams, Steven J., and Alan D. Taylor (1999). *The Win-Win Solution: Guaranteeing Fair Shares to Everybody*. New York: W.W. Norton.

Brams, Steven J., Alan D. Taylor, and William S. Zwicker (1995). "Old and New Moving-Knife Schemes." *Mathematical Intelligencer* 17, no. 4 (Fall): 30-35.

Brams, Steven J., Alan D. Taylor, and William S. Zwicker (1997). "A Moving-Knife Solution to the Four-Person Envy-Free Cake Division Problem." *Proceedings of the American Mathematical Society* 125, no. 2 (February): 547-554.

Gale, David (1993). "Mathematical Entertainments." *Mathematical Intelligencer* 15, no.

1 (Winter): 48-52.

Jones, Michael A. (2002). "Equitable, Envy-Free, and Efficient Cake Cutting for Two People and Its Application to Divisible Goods." *Mathematics Magazine* 75, no. 4 (October): 275-283.

Peterson, Elisha, and Francis Edward Su (2002). "Four-Person Envy-Free Chore Division." *Mathematics Magazine* 75, no. 2 (April): 117-122.

Robertson, Jack, and William Webb (1998). *Cake-Cutting Algorithms: Be Fair If You Can.* Natick, MA: A K Peters.

Shishido, Harunori, and Dao-Zhi Zeng (1999). "Mark-Choose-Cut Algorithms for Fair and Strongly Fair Division." *Group Decision and Negotiation* 8, no. 2 (March 1999): 125-137.

Stromquist, Walter (1980). "How to Cut a Cake Fairly." *American Mathematical Monthly* 87, no 8 (October): 640-644.