The Consensus Value for Games in Partition Function Form[1]

Yuan Ju[2]

CentER DP No. 2004-60

Revision: October 2004

[1]I thank Peter Borm, René van den Brink, Yuji Fujinaka, Yukihiko Funaki, Henk Norde and Dolf Talman for their valuable comments and helpful discussions.

[2]Address: Department of Economics, Keele University, Keele, Staffordshire, ST5 5BG, UK. Tel: +44-1782-583095, Fax: +44-1782-717577, E-mail: y.ju@econ.keele.ac.uk.

Abstract

This paper studies a generalization of the consensus value (cf. Ju, Borm and Ruys (2004)) to the class of partition function form games. The concepts and axioms, related to the consensus value, are extended. This value is characterized as the unique function that satisfies efficiency, complete symmetry, the quasi-null player property and additivity. By means of the transfer property, a second characterization is provided. Moreover, it is shown that this value satisfies the individual rationality under a certain condition, and well balances the tradeoff between coalition effects and externality effects. By modifying the stand-alone reduced game, a recursive formula for the value is established. A further generalization of the consensus value is discussed. Finally, two applications of the consensus value are given: one is for oligopoly games in partition function form and the other is about participation incentives in free-rider situations.

JEL classification codes: C71; C72; D43; D62.
Keywords: partition function form games; coalition structure; Shapley value; consensus value.

1 Introduction

The problem of sharing the joint gains of cooperation is well captured by cooperative game theory. The Shapley value (Shapley (1953)) has been proven to be the most studied and widely used single-valued solution concept for cooperative games with transferable utility (TU games) in characteristic function form as it satisfies some desirable properties. In some sense, the value captures the expected outcome of a game, and represents a distinct approach to the problems of complex strategic interactions that game theory seeks to illuminate (Roth (1988)).

However, considering an economy with externalities one cannot easily recommend a division of the joint profits in the same way as the final profits depend on the coalition structure which has been formed. This feature was first captured by Thrall and Lucas (1963) by the concept of *partition function form games*: A partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. The advantage of this model is that it takes both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperation outcomes into account and allows to go deeper into cooperation problems. Thus, it is closer to real life although more complex to analyse.

Values for such games can be found in Myerson (1977), Bolger (1989), Feldman (1994), Potter (2000), and Pham Do and Norde (2002). All of them are in some way extensions of the Shapley value (Shapley (1953)) for cooperative TU games in characteristic function form. Myerson (1977) introduced a value based on the extensions of the three axioms in the Shapley's original paper. Bolger's value assigns zero to dummies and assigns nonnegative values to players in monotone simple games. Potter (2000) added another axiom, coalitional symmetry, and reformed the regular definition of the dummy player such that the dummy player can get nonnegative worth. But note that a null player defined in this paper still gets zero worth by Potter's value. When $|N| = 3$, Potter's value coincides with the values introduced by Bolger and Feldman. But they are different when $|N| > 3$. The difference is due to the fact that Potter defined the worth of each embedded coalition as the average worth. Pham Do and Norde (2002) studied another extension of the Shapley value for the class of partition function form games, which is the average of a collection of marginal vectors. Fujinaka (2004) provided alternative characterizations for the Shapley value defined by Pham Do and Norde (2002) based on a marginality axiom and a monotonicity axiom. Moreover, he found an error in the proof of the axiomatization which is based on the axiom of additivity in Pham Do and Norde (2002) and amended it in his paper.

This paper takes a different perspective and aims to derive a solution concept which not only satisfies "reasonable" properties but also has a constructive sharing procedure. Following a simple and natural way of generalizing the standard solution for 2-person partition function form games into n-person cases, a new solution concept for partition function form games is obtained: *the consensus value*. It is, in fact, a natural extension of the consensus value for TU games in characteristic function form introduced in Ju, Borm and Ruys (2004). This value differs from all the previous values as it is characterized to be the unique function that satisfies efficiency, complete symmetry, additivity and the *quasi-null player property*. The first three requirements are relatively weak, especially the property of complete symmetry is a natural and obvious requirement. A quasi-null player is a player who has zero payoff in the complete breakdown situation (every player stands alone) and whose marginal contributions to all non-empty coalitions are also zero. Instead of the "regular" marginal contribution perspective requiring zero payoff to a quasi-null player (we may call it the *marginal* quasi-null player property) which is implicitly specified by the Shapley value in Pham Do and Norde (2002), this paper introduces the so-called quasi-null player property based on the positive or negative externalities that the quasi-null player might benefit or suffer from.

One may argue that the efficiency[1] postulate and the marginal quasi-null player property seem to be contradictory to each other when considering solution concepts for partition function form games because a quasi-null player, given the positive externalities she might enjoy, can hardly participate in coalitions where she contributes nothing and will get zero payoff. More generally, we have no reason to ignore the externality effect in partition function form games while the marginal quasi-null player property rules out the considerations on externalities and completely favors coalitions. That is, from the positive externality point of view, any quasi-null player could obtain nonnegative worth when standing alone, and analogously, she might get nonpositive worth in the presence of negative externality, which opens up the possibility to relax the marginal quasi-null player postulate. In this spirit, the quasi-null player property is introduced and discussed.

By defining *the expected stand-alone value*, we can determine, in some sense, the maximum and minimum that a quasi-null player might get in a game due to the positive and negative externalities[2], respectively. In order to balance the tradeoff between those two contrastive opinions, i.e. emphasizing coalitions or focusing on externalities, we make a fair compromise and take the average as the value for a quasi-null player, resulting in the

[1] A more general criticism on the efficiency postulate can be traced back to Luce and Raiffa (1957); and, more recently, is seen in Maskin (2003).

[2] More strictly, since the externalities from different coalitions imposed on a player could be both positive and negative in a game, the expected stand-alone value is just a value focusing on externalities, in contrast with the value derived from the contributions to coalitional values.

quasi-null player property. At the same time, introducing the quasi-null player property actually affects all other players in the same way such that any player's value is determined by her contributions to coalitions and the externalities imposed on her if stand-alone, which is further confirmed by the general formula of the consensus value: It is the average of the Shapley value introduced by Pham Do and Norde (2002) and the expected stand-alone value.

A novel feature of the consensus value for TU games in characteristic function form is its underlying sharing process. It is shown in this paper that such a process is particularly suitable for the settings of games in partition function form because, given a coalition structure, the standard solution for 2-person partition function form games can be well implemented. Here, given an ordering of players, we also take a bilateral perspective and consider to allocate the joint surplus of an existing coalition of players (i.e., the incumbents) and an entrant, which means that the unilateral viewpoint like the marginal contribution approach focusing on entrants is abandoned. By taking the incumbents as one party and the entrant as a second party, the standard solution for 2-person games can be applied all the way with consensus. That is, all the joint surpluses are always equally split between the corresponding two parties. Since no specific ordering is pre-determined, we average over all possible permutations. Hence, by this rule, not only the concern of all the possible orderings but also what happens in each ordering are mutually accepted: Consensus is obtained.

By means of the transfer property, a second characterization for the consensus value is provided. Based on a modification of the stand-alone reduced game introduced in Ju, Borm and Ruys (2004) and a related recursive formula, the consensus value for partition function form games is reformulated. Furthermore, by introducing a share parameter on the splitting of joint surpluses, a generalization of the consensus value is obtained. In particular, the Shapley value and the expected stand-alone value are the two polar cases of the generalized consensus value. Accordingly, characterizations for the expected stand-alone value are obtained. A special case of the partition function form games is that any player's stand-alone values are the same as that in the complete breakdown situation. Then, the consensus value is equivalent to that in TU games in characteristic function form, which equals to the average of the Shapley value and the equal surplus solution.

In addition to this section introducing the paper and reviewing the seminal works briefly, the remaining part proceeds as follows. In the next section, we briefly recall the basic features of partition function form games. In section 3, we address 2-person partition function form games and take the corresponding solutions as a standardization and define the consensus value for partition function form games. The consensus value is characterized in an axiomatic way in section 4. It is shown that the consensus value is the average of the Shapley value for partition function form games and the expected stand-alone value. Section 5 discusses a generalization of this solution concept. The final section shows the applications of the consensus value by providing two illustrative examples: one is about oligopoly games in partition function form and the other is about the participation incentives in free-rider situations.

2 Preliminaries

This section, based on Pham Do and Norde (2002), recalls some basic definitions and notations related to games in partition function form.

A *partition* κ of the player set N, a so-called *coalition structure*, is a set of mutually disjoint coalitions, $\kappa = \{S_1, ..., S_m\}$, so that their union is N. Let $\mathbb{P}(N)$ be the set of all partitions of N. For any coalition $S \subset N$, the set of all partitions of S is denoted by $\mathbb{P}(S)$. A typical element of $\mathbb{P}(S)$ is denoted by κ_S. Note that two partitions will be considered equal if they differ only by the insertion or deletion of $\emptyset$. That is, $\{\{1,2\},\{3\}\} = \{\{1,2\},\{3\},\emptyset\}$.

A pair (S, κ) consisting of a coalition S and a partition κ of N to which S belongs is called an *embedded coalition*, and is nontrivial if $S \neq \emptyset$. Let $\mathbb{E}(N)$ denote the set of embedded coalitions, i.e.

$$\mathbb{E}(N) = \left\{(S,\kappa) \in 2^N \times \mathbb{P}(N) | S \in \kappa\right\}.$$

Definition 2.1 *A mapping*

$$w : \mathbb{E}(N) \longrightarrow \mathbb{R}$$

that assigns a real value, $w(S,\kappa)$, to each embedded coalition (S,κ) is called a partition function. *By convention, $w(\emptyset,\kappa) = 0$ for all $\kappa \in \mathbb{P}(N)$. The ordered pair (N, w) is a* partition function form game. *The set of partition function form games with player set N is denoted by PG^N.*

The value $w(S,\kappa)$ represents the payoff of coalition S, given the coalition structure κ forms. For a given partition $\kappa = \{S_1, ..., S_m\}$ and a partition function w, let $\bar{w}(S_1, ..., S_m)$ denote the m-vector $(w(S_i,\kappa))_{i=1}^m$. It will be convenient to economize brackets and suppress the commas between elements of the same coalition. Thus, where no confusion can arise, we will write, for example, $w(\{i,j,k\},\{\{i,j,k\},\{l,h\}\})$ as $w(ijk,\{ijk,lh\})$, and $\bar{w}(\{i,j,k\},\{l,h\})$ as $\bar{w}(ijk,lh)$. For a partition $\kappa \in \mathbb{P}(N)$ and $i \in N$, we denote the coalition in κ to which player i belongs by $S(\kappa,i)$.

The typical partition which consists of *singleton coalitions* only, $\kappa = \{\{1\}, ..., \{n\}\}$, is denoted by $[N]$, whereas the partition, which consists of the grand coalition only is denoted by $\{N\}$. For any subset $S \subset N$, let $[S]$ denote the typical partition which consists of the singleton elements of S, i.e., $[S] = \{\{j\}|j \in S\}$

Definition 2.2 *A solution concept on PG^N is a function f, which associates with each game (N, w) in PG^N a vector $f(N, w)$ of individual payoffs in $\mathbb{R}^N$, i.e.,*

$$f(N, w) = (f_i(N, w))_{i \in N} \in \mathbb{R}^N.$$

Since the consensus value for partition function form games is related to the Shapley value defined by Pham Do and Norde (2002), it is necessary to recall that definition.

Let $\Pi(N)$ be the set of all bijections $\sigma : \{1, ..., |N|\} \longrightarrow N$. For a given $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$, we define the partition κ_k^σ associated with σ and k, by $\kappa_k^\sigma = \{S_k^\sigma\} \cup [N \backslash S_k^\sigma]$ where $S_k^\sigma := \{\sigma(1), ..., \sigma(k)\}$, and $\kappa_0^\sigma = [N]$. So, in κ_k^σ the coalition S_k^σ has already formed, whereas all other players still form singleton coalitions.

For a game $w \in PG^N$, define the marginal vector $m^\sigma(w)$ as the vector in $\mathbb{R}^N$ by

$$m^\sigma_{\sigma(k)}(w) = w(S_k^\sigma, \kappa_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma)$$

for all $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$.

Definition 2.3 *(Pham Do and Norde (2002)) The* Shapley value $\Phi(w)$ *of the partition function form game (N, w) is the average of the marginal vectors, i.e.*

$$\Phi(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(w).$$

3 The consensus value

One may notice that the Shapley value for partition function form games defined by Pham Do and Norde (2002) actually ignores all the new information provided in a partition function form game compared to a TU game in characteristic function form. For instance, this value is in dependent of $w(\{i\}, (\kappa_{N\backslash\{i\}}) \cup \{\{i\}\})$ for all $i \in N$ and for all $\kappa_{N\backslash\{i\}} \in$

$\mathbb{P}(N\backslash\{i\})$ such that $\kappa_{N\backslash\{i\}} \neq [N\backslash\{i\}]$. Put differently, partition functions play no role here and the Shapley value defined above does not take externality into account.

The idea to define the consensus value for TU games in characteristic function form (cf. Ju, Borm and Ruys (2004)) well fits the settings of partition function form games: It takes a bilateral perspective and allocates payoffs based on a fair compromise between coalition effect and externality effect. Hence, it reflects the role of coalition structures in determining players' final payoffs.

To illustrate the idea, we first consider an arbitrary 2-person partition function form game with player set $N = \{1, 2\}$ and partition function w determined by the values: $w(1, \{1, 2\})$, $w(2, \{1, 2\})$ and $w(12, \{12\})$. Note that, as mentioned in section 2, here we use shortcut notations, for example, $w(1, \{1, 2\})$ is for $w(\{1\}, \{\{1\}, \{2\}\})$, and $w(12, \{12\})$ is for $w(\{1, 2\}, \{\{1, 2\}\})$. A reasonable solution is that player 1 gets

$$w(1, \{1, 2\}) + \frac{w(12, \{12\}) - w(1, \{1, 2\}) - w(2, \{1, 2\})}{2}$$

and player 2 gets

$$w(2, \{1, 2\}) + \frac{w(12, \{12\}) - w(2, \{1, 2\}) - w(1, \{1, 2\})}{2}.$$

That is, the (net) surplus generated by the cooperation between player 1 and 2,

$$w(12, \{12\}) - w(2, \{1, 2\}) - w(1, \{1, 2\}),$$

is equally shared between the two players. This solution is called the *standard* solution for 2-person partition function form games.

Then, we consider a generalization of the standard solution for 2-person games into n-person cases. It follows the following line of reasoning.

Consider a 3-person game (N, w) with player set $N = \{1, 2, 3\}$. Suppose we have the ordering $(1, 2, 3)$: player 1 shows up first, then player 2, and finally player 3. When player 2 joins 1, we in fact have a 2-person situation where the surplus sharing problem is solved by the standard solution. Next, player 3 enters the scene, who would like to cooperate with player 1 and 2. Because coalition $\{12\}$ has already been formed before she joins, player 3 will actually cooperate with the existing coalition $\{12\}$ instead of simply cooperating with 1 and 2 individually. If $\{12\}$ agrees to cooperate with player 3 as well, the value of the grand coalition, $w(123, \{123\})$ will be generated. Now, the question is how to share it?

Again, following the standard solution to 2-person games, one can argue that both parties should get half of the joint surplus

$$w(123, \{123\}) - w(12, \{12, 3\}) - w(3, \{12, 3\})$$

in addition to their stand-alone payoffs. The reason is simple: coalition $\{12\}$ should be regarded as one player instead of two players because they have already formed a cooperating coalition. Internally, 1 and 2 will receive equal shares of the surplus because this part is obtained extra by the coalition $\{12\}$ cooperating with coalition $\{3\}$.

One can also tell the story in a reverse way, which yields the same outcome in terms of surplus sharing. Initially, three players cooperate with each other and $w(123, \{123\})$ is obtained. We now consider players leaving the existing coalitions one by one in the opposite order $(3, 2, 1)$. So player 3 leaves first. By the standard solution for 2-person games, player 3 should get half of the joint surplus plus her stand-alone payoff, i.e.

$$w(3, \{12, 3\}) + \frac{w(123, \{123\}) - w(3, \{12, 3\}) - w(12, \{12, 3\})}{2},$$

as 1 and 2 remain as one cooperating coalition $\{12\}$. Thus, the value left for coalition $\{12\}$, which we call the *standardized remainder* (the value left for the corresponding remaining coalition) for $\{12\}$, is

$$w(12, \{12, 3\}) + \frac{w(123, \{123\}) - w(12, \{12, 3\}) - w(3, \{12, 3\})}{2}.$$

In the same fashion, the standardized remainder for $\{1\}$ will be

$$w(1, \{1, 2, 3\}) + \frac{\frac{w(123,\{123\})+w(12,\{12,3\})-w(3,\{12,3\})}{2} - w(1, \{1, 2, 3\}) - w(2, \{1, 2, 3\}))}{2}.$$

Extending this argument to an n-person case, we then have a general method, which can be understood as a *standardized remainder rule* since we take the later entrant (or earlier leaver) and all her pre-entrants (or post-leavers) as two parties and apply the standard solution for 2-person games all the way. Furthermore, since no ordering is pre-determined for a partition function form game, we will average all possible orderings.

Formally, for a game in partition function form we shall define the standardized remainder as follows.

$$r(S_k^\sigma) = \begin{cases} w(N, \{N\}) & \text{if } k = |N| \\ w(S_k^\sigma, \kappa_k^\sigma) + \frac{r(S_{k+1}^\sigma) - w(S_k^\sigma, \kappa_k^\sigma) - w(\{\sigma(k+1)\}, \kappa_k^\sigma)}{2} & \text{if } k \in \{1, ..., |N| - 1\}, \end{cases}$$

where $r(S_k^\sigma)$ is the *standardized remainder* for coalition S_k^σ: the value left for S_k^σ after allocating surplus to later entrants (earlier leavers) $N \backslash S_k^\sigma$. Note that for notational simplicity we still use the same notation, i.e. $r(S_k^\sigma)$, as that for TU games in characteristic function form.

We construct the *individual standardized remainder vector* $s^\sigma(w)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$

(or leave the game one by one in the order $\sigma(|N|), \sigma(|N|-1), ..., \sigma(1)$) and assign each player $\sigma(k)$, besides her stand-alone payoff $w(\{\sigma(k)\}, \kappa^{\sigma}_{k-1})$, half of the net surplus from the standardized remainder $r(S^{\sigma}_k)$. Formally, it is the vector in $\mathbb{R}^N$ recursively (start with $|N|$) defined by

$$s^{\sigma}_{\sigma(k)}(w) = \begin{cases} w(\{\sigma(k)\}, \kappa^{\sigma}_{k-1}) + \frac{r(S^{\sigma}_k) - w(S^{\sigma}_{k-1}, \kappa^{\sigma}_{k-1}) - w(\{\sigma(k)\}, \kappa^{\sigma}_{k-1})}{2} & \text{if } k \in \{2, ..., |N|\} \\ \\ r(S^{\sigma}_1) & \text{if } k = 1. \end{cases}$$

Definition 3.1 *The* consensus value $\gamma(w)$ *of the partition function form game* (N, w) *is the average of the individual standardized remainder vectors, i.e.*

$$\gamma(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s^{\sigma}(w).$$

Hence, the consensus value can be interpreted as the expected individual standardized remainder that a player can get by participating in coalitions.

Example 3.2 *This game is from Pham Do and Norde (2002). Consider the partition function form game* (N, w) *defined by*

$$\begin{aligned} \bar{w}(1,2,3) &= (0,0,0), \\ \bar{w}(12,3) &= (2,0),\ \bar{w}(13,2) = (2,1),\ \bar{w}(23,1) = (3,2), \\ \bar{w}(123) &= (10). \end{aligned}$$

With $\sigma : \{1,2,3\} \longrightarrow N$ *defined by* $\sigma(1) = 2$, $\sigma(2) = 1$ *and* $\sigma(3) = 3$, *which is shortly denoted by* $\sigma = (2\ 1\ 3)$, *we get*

$$s^{\sigma}_3(w)(= s^{\sigma}_{\sigma(3)}(w)) = w(3, \{12,3\}) + \frac{w(123, \{123\}) - w(12, \{12,3\}) - w(3, \{12,3\})}{2} = 4.$$

And one can readily calculate

$$s^{\sigma}_1(w)(= s^{\sigma}_{\sigma(2)}(w)) = w(1, \{1,2,3\}) + \frac{r(\{2,1\}) - w(2, \{1,2,3\}) - w(1, \{1,2,3\})}{2} = 3,$$

and $s^{\sigma}_2(w)(= s^{\sigma}_{\sigma(1)}(w)) = r(\{2\}) = 3$. *All individual standardized remainder vectors are given by*

σ	$s^{\sigma}_1(w)$	$s^{\sigma}_2(w)$	$s^{\sigma}_3(w)$
(123)	3	3	4
(132)	$2\frac{3}{4}$	$4\frac{1}{2}$	$2\frac{3}{4}$
(213)	3	3	4
(231)	$4\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{3}{4}$
(312)	$2\frac{3}{4}$	$4\frac{1}{2}$	$2\frac{3}{4}$
(321)	$4\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{3}{4}$

Then, we get $\gamma(w) = \left(3\frac{5}{12}, 3\frac{5}{12}, 3\frac{1}{6}\right)$ *whereas the Shapley value of this game (Pham Do and Norde (2002)) is* $\Phi(w) = \left(3, 3\frac{1}{2}, 3\frac{1}{2}\right)$*. One can verify that the value introduced by Potter (2000) as well as the value introduced by Bolger (1989) yield the same vector* $(3\frac{1}{4}, 3\frac{1}{2}, 3\frac{1}{4})$ *for this game as when* $|N| = 3$ *the value introduced by Potter coincides with Bolger's value (Potter (2000)). The difference between the consensus value and the others stems from the way to share joint surpluses and the fact that the externalities of players are taken into account. The Shapley value still focuses on marginal vectors and rules out externality effects. As for Bolger's value, it considers a different collection of marginal vectors. Potter's value is obtained by considering the sum of an "average worth" of coalitions.*

Similar to the stand-alone recursion of the consensus value for TU games in characteristic function form, we can reformulate the consensus value for partition function form games by modifying the stand-alone reduced game and defining a corresponding recursive formula.

Formally, let $f : PG^N \longrightarrow \mathbb{R}^N$ be a solution concept. For any $w \in PG^N$ and $i \in N$, we introduce the game $(N\backslash\{i\}, w^{-i})$ defined by for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and for all $S \in \kappa_{N\backslash\{i\}}$

$$w^{-i}(S, \kappa_{N\backslash\{i\}}) = \begin{cases} w(S, \kappa_{N\backslash\{i\}} \cup \{i\}) & \text{if } S \subsetneqq N\backslash\{i\} \\ w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) \\ +\frac{w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) - w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2} & \text{if } S = N\backslash\{i\} \end{cases}$$

and call w^{-i} the *stand-alone reduced game* of w with respect to player i.

We say that a solution concept f satisfies the *stand-alone recursion* if and only if for any game $w \in PG^N$ with $|N| \geq 3$ we have

$$\begin{aligned} f_i(N, w) = \quad & \frac{1}{|N|} \sum_{j \in N\backslash\{i\}} f_i\left(N\backslash\{j\}, w^{-j}\right) \\ + \quad & \frac{1}{|N|} \cdot \frac{w(N, \{N\}) - w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) + w(\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\})}{2} \end{aligned}$$

for all $i \in N$.

One can readily check that the consensus value is the unique one-point solution concept on the class of all n-person partition function form games with $n \geq 2$ which is standard for 2-person partition function form games and satisfies stand-alone recursion.

4 Characterizations

This section characterizes the consensus value for partition function form games in an axiomatic way.

Definition 4.1 *In a partition function form game $w \in PG^N$, two players i and j are* completely symmetric *if for all $\kappa_{N\backslash\{i,j\}} \in \mathbb{P}(N\backslash\{i,j\})$ and $S \in \kappa_{N\backslash\{i,j\}}$,*

$$w(S\cup\{i\},(\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{j\}\}\cup\{S\cup\{i\}\}) = w(S\cup\{j\},(\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{i\}\}\cup\{S\cup\{j\}\})$$

and

$$w(\{i\},(\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{i\}\}\cup\{S\cup\{j\}\}) = w(\{j\},(\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{j\}\}\cup\{S\cup\{i\}\}).$$

Definition 4.2 *In a partition function form game $w \in PG^N$, player i is a* null player *if for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and $S \in \kappa_{N\backslash\{i\}}$,*

$$w(S,\kappa_{N\backslash\{i\}}\cup\{\{i\}\}) = w(S\cup\{i\},(\kappa_{N\backslash\{i\}}\backslash S)\cup\{S\cup\{i\}\}).$$

So, a null player always makes zero marginal contributions to any coalition and obtains zero payoff when standing alone. Moreover, we define a quasi-null player as follows.

Definition 4.3 *In a game $w \in PG^N$, player i is a* quasi-null player *if for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and $S \in \kappa_{N\backslash\{i\}}$ such that $S \neq \emptyset$,*

$$w(S,\kappa_{N\backslash\{i\}}\cup\{\{i\}\}) = w(S\cup\{i\},(\kappa_{N\backslash\{i\}}\backslash S)\cup\{S\cup\{i\}\})$$

and

$$w(\{i\},[N]) = 0.$$

Thus, a quasi-null player i will be a null player if $w(\{i\},\kappa_{N\backslash\{i\}}\cup\{\{i\}\}) = 0$ for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$.

By Definition 2.3, one can find that $\Phi_i(w) = 0$ for all $w \in PG^N$ and for any quasi-null player i in (N,w), which implies that the Shapley value is not so convincing: If a quasi-null player can get positive payoffs due to positive externalities, i.e., $w(\{i\},\kappa_{N\backslash\{i\}}\cup\{\{i\}\}) > 0$ for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$, why would she join the others to form the grand coalition and obtain zero payoff finally?

In order to find how much a quasi-null player should obtain, we first introduce the concept of expected stand-alone value. For a partition function form game $w \in PG^N$ and a player $i \in N$, we define player i's *expected stand-alone value* as

$$\begin{aligned} e_i(w) = & \frac{w(N, \{N\})}{|N|} \\ & + \sum_{S \subset N \setminus \{i\}: S \neq \emptyset} \frac{|S|!(|N| - |S| - 1)!}{|N|!} w(\{i\}, \{S\} \cup [N \setminus (S \cup \{i\})] \cup \{\{i\}\}) \\ & - \sum_{j \in N \setminus \{i\}} \sum_{S \subset N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{|N|!} w(\{j\}, [N \setminus (S \cup \{i\})] \cup \{S \cup \{i\}\}). \end{aligned}$$

The expected stand-alone value tells us how much a player may obtain in a partition function form game (N, w) when we focus on the stand-alone side of the game[3]. Since we rule out the consideration on coalition values, immediately, a reference point could be that the value of the grand coalition is equally shared among players, i.e. $\frac{w(N,\{N\})}{|N|}$. Focusing on stand-alone situations implies that we take externality as the only determinant. Given a player $i \in N$, she has two choices concerning externalities, either choosing stand-alone and enjoying the externalities from coalitions consisting of other players or joining some coalitions generating externalities to the players standing alone. Thus, the second term in the above expression corresponds to the first choice and can be understood as player i's *expected gain* from the externalities of all possible coalitions without containing i, where the distribution of coalitions is such that any ordering of the players is equally likely. The last term, corresponding to the second choice, is player i's *expected loss* due to joining coalitions, which is expressed as the other players' gain from the externalities of coalitions containing i.

One can find that in the case that any player has identical stand-alone payoffs in a partition function form game, the expected stand-alone value is comparable to the equal surplus solution for TU games in characteristic function form. Let TU^N denote the set of all TU games in characteristic function form with player set N. The equal solution surplus E is defined as $E_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}$ for all $v \in TU^N$ and for all $i \in N$.

Proposition 4.4 *For a game $w \in PG^N$, if $w(\{i\}, \{S\} \cup [N \setminus (S \cup \{i\})] \cup \{\{i\}\}) = w(\{i\}, [N])$ for all $i \in N$ and for all $S \subset N \setminus \{i\}$, then*

$$e_i(w) = \frac{w(N, \{N\}) - \sum_{j \in N} w(\{j\}, [N])}{|N|} + w(\{i\}, [N])$$

for all $i \in N$.

[3] Or directly in some special situations that people have no information about the values of coalitions but only know players' stand-alone values and the value of the grand coalition, we then could get such a sharing rule, which is actually an equal-surplus-solution style value in partition function form games.

Proof. By the definition of the expected stand-alone value and since $w(\{i\}, \{S\} \cup [N \backslash (S \cup \{i\})] \cup \{\{i\}\}) = w(\{i\}, [N])$ for all $S \subset N \backslash \{i\}$ and for all $i \in N$, it follows that

$$\begin{aligned} & \sum_{S \subset N \backslash \{i\}: S \neq \emptyset} \frac{|S|!(|N|-|S|-1)!}{|N|!} w(\{i\}, \{S\} \cup [N \backslash (S \cup \{i\})] \cup \{\{i\}\}) \\ = & \sum_{|S|=1}^{|N|-1} \frac{|S|!(|N|-|S|-1)!}{|N|!} \binom{|N|-1}{|S|} w(\{i\}, [N]) \\ = & \frac{|N|-1}{|N|} w(\{i\}, [N]). \end{aligned}$$

Similarly, we can show that

$$\sum_{j \in N \backslash \{i\}} \sum_{S \subset N \backslash \{i,j\}} \frac{|S|!(|N|-|S|-2)!}{|N|!} w(\{j\}, [N \backslash (S \cup \{i\})] \cup \{S \cup \{i\}\}) = \frac{\sum_{j \in N \backslash \{i\}} w(\{j\}, [N])}{|N|}.$$

Hence, what remains is obvious. ■

Therefore, the equal surplus solution E for TU games in characteristic function form is actually a special case of the expected stand-alone value. It can be expressed as

$$E_i(v) = \frac{v(N)}{|N|} + \frac{|N|-1}{|N|} v(\{i\}) - \frac{1}{|N|} \sum_{j \in N \backslash \{i\}} v(\{j\})$$

for all $v \in TU^N$ and for all $i \in N$. Here, the second term is the expected gain as being a stand-alone player while the last term is the expected loss due to joining coalitions.

Let $f : PG^N \longrightarrow \mathbb{R}^N$ be a one-point solution concept. We consider the following properties.

- *Efficiency:* $\sum_{i \in N} f_i(w) = w(N, \{N\})$ for all $w \in PG^N$;
- *Complete symmetry:* $f_i(w) = f_j(w)$ for all $w \in PG^N$, and for all completely symmetric players i, j in (N, w);
- *The quasi-null player property:*

  $$f_i(w) = \frac{1}{2} e_i(w)$$

  for all $w \in PG^N$ and for any quasi-null player i in (N, w);
- *Additivity:* $f(w_1 + w_2) = f(w_1) + f(w_2)$ for all $w_1, w_2 \in PG^N$, where $w_1 + w_2$ is defined by $(w_1 + w_2)(S, \kappa) = w_1(S, \kappa) + w_2(S, \kappa)$ for every $(S, \kappa) \in \mathbb{E}(N)$.

The properties of efficiency, complete symmetry, and additivity are clear by themselves. Here, it is necessary to stress the new property: the quasi-null player property.

Let us first discuss the marginal quasi-null player property that assigns zero payoff to a quasi-null player, which is implicitly specified by the Shapley value introduced by Pham Do and Norde (2002). Requiring a solution concept for partition function form games satisfying both efficiency and this marginal quasi-null player property seems inappropriate. For instance, a quasi-null player i who may obtain positive payoff due to the positive externality from coalition $N\backslash\{i\}$ has to accept zero payoff in the game according to this marginal quasi-null player property. Then, it is hard to imagine that player i could have any incentive to join the grand coalition. As a consequence, it is difficult to justify the efficiency axiom. More generally, the players who may enjoy extremely high positive externalities from other coalitions will choose stand-alone as those effects are not well reflected by the solution concepts that adopt marginal contribution approach. So, the externality has to be taken into consideration.

As we know, the marginal quasi-null player property favors coalitions while it biases against the outside individuals. In order to give a fair treatment to both sides, we have to balance the coalition effect and the externality effect. More specifically, to assign a quasi-null player 0 or $e_i(w)$ can be viewed as consequences of two contrastive viewpoints. Concerning the tradeoff between these two extreme opinions[4], an impartial decision could be choosing the average as the gain of a quasi-null player, which results in the so-called quasi-null player property.

In addition, one can see that a null player, as a special quasi-null player, could still get positive worth as long as her expected loss from externalities is less than the average value $\frac{w(N,\{N\})}{|N|}$. This observation implies that the quasi-null player property also has the flavor of egalitarianism or collectivism. The justification is similar to that for the consensus value for TU games in characteristic function form in Ju, Borm and Ruys (2004).

It is shown that the consensus value is the unique function that satisfies these four properties.

Theorem 4.5 *The consensus value satisfies efficiency, complete symmetry, the quasi-null player property and additivity.*

Proof.

(i) Efficiency: Clearly, by construction, $s^\sigma(w)$ is efficient for all $\sigma \in \Pi(N)$.

(ii) Complete symmetry: Let i, j be two completely symmetric players in a partition function form game $w \in PG^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i$,

[4]Cultural and philosophical factors may affect the propensity or choice between the two extreme opinions.

$\sigma(l) = j$, where $i, j \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained by interchanging in σ the positions of i and j, i.e.

$$\bar{\sigma}(m) = \begin{cases} \sigma(m) & \text{if } m \neq k, l \\ i & \text{if } m = l \\ j & \text{if } m = k \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $s_i^{\sigma}(w) = s_j^{\bar{\sigma}}(w)$.

Case 1: $1 < k < l$.

By definition, we know

$$\begin{aligned} s_i^{\sigma}(w) = s_{\sigma(k)}^{\sigma}(w) = \quad & w(\{\sigma(k)\}, \kappa_{k-1}^{\sigma}) + \frac{1}{2}\left(r(S_k^{\sigma}) - w(S_{k-1}^{\sigma}, \kappa_{k-1}^{\sigma}) - w(\{\sigma(k)\}, \kappa_{k-1}^{\sigma})\right), \\ s_j^{\bar{\sigma}}(w) = s_{\bar{\sigma}(k)}^{\bar{\sigma}}(w) = \quad & w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}}) + \frac{1}{2}\left(r(S_k^{\bar{\sigma}}) - w(S_{k-1}^{\bar{\sigma}}, \kappa_{k-1}^{\bar{\sigma}}) - w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}})\right). \end{aligned}$$

Note that, by complete symmetry,

$$w(\{\sigma(k)\}, \kappa_{k-1}^{\sigma}) = w(\{i\}, \kappa_{k-1}^{\sigma}) = w(\{j\}, \kappa_{k-1}^{\bar{\sigma}}) = w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}}),$$

$S_{k-1}^{\sigma} = S_{k-1}^{\bar{\sigma}}$, and apparently $w(S_{k-1}^{\sigma}, \kappa_{k-1}^{\sigma}) = w(S_{k-1}^{\bar{\sigma}}, \kappa_{k-1}^{\bar{\sigma}})$. It remains to show that $r(S_k^{\sigma}) = r(S_k^{\bar{\sigma}})$.

Clearly, $r(S_m^{\sigma}) = r(S_m^{\bar{\sigma}})$ for $m \geq l$. Recursively, we can show that $r(S_{l-t}^{\sigma}) = r(S_{l-t}^{\bar{\sigma}})$ for $t \in \{1, ..., l-k-1\}$ as

$$r(S_{l-t}^{\sigma}) = w(S_{l-t}^{\sigma}, \kappa_{l-t}^{\sigma}) + \frac{1}{2}\left(r(S_{l-t+1}^{\sigma}) - w(S_{l-t}^{\sigma}, \kappa_{l-t}^{\sigma}) - w(\{\sigma(l-t+1)\}, \kappa_{l-t}^{\sigma})\right)$$

and

$$r(S_{l-t}^{\bar{\sigma}}) = w(S_{l-t}^{\bar{\sigma}}, \kappa_{l-t}^{\bar{\sigma}}) + \frac{1}{2}\left(r(S_{l-t+1}^{\bar{\sigma}}) - w(S_{l-t}^{\bar{\sigma}}, \kappa_{l-t}^{\bar{\sigma}}) - w(\{\bar{\sigma}(l-t+1)\}, \kappa_{l-t}^{\bar{\sigma}})\right).$$

Here, since $\sigma(l-t) = \bar{\sigma}(l-t)$ and $S_{l-t}^{\sigma} \backslash \{i\} = S_{l-t}^{\bar{\sigma}} \backslash \{j\}$, by complete symmetry, we know $w(S_{l-t}^{\sigma}, \kappa_{l-t}^{\sigma}) = w(S_{l-t}^{\bar{\sigma}}, \kappa_{l-t}^{\bar{\sigma}})$.

Then, it immediately follows that $r(S_k^{\sigma}) = r(S_k^{\bar{\sigma}})$ as

$$\begin{aligned} r(S_k^{\sigma}) &= w(S_k^{\sigma}, \kappa_k^{\sigma}) + \frac{1}{2}\left(r(S_{k+1}^{\sigma}) - w(S_k^{\sigma}, \kappa_k^{\sigma}) - w(\{\sigma(k+1)\}, \kappa_k^{\sigma})\right) \\ &= w(S_k^{\bar{\sigma}}, \kappa_k^{\bar{\sigma}}) + \frac{1}{2}\left(r(S_{k+1}^{\bar{\sigma}}) - w(S_k^{\bar{\sigma}}, \kappa_k^{\bar{\sigma}}) - w(\{\bar{\sigma}(k+1)\}, \kappa_k^{\bar{\sigma}})\right) \\ &= r(S_k^{\bar{\sigma}}). \end{aligned}$$

Case 2: $1 < l < k$. The proof is analogous to Case 1.

Case 3: $1 = k < l$.
In this case,

$$\begin{aligned} s_i^\sigma(w) = s_{\sigma(1)}^\sigma(w) &= r(S_1^\sigma), \\ s_j^{\bar\sigma}(w) = s_{\bar\sigma(1)}^{\bar\sigma}(w) &= r(S_1^{\bar\sigma}). \end{aligned}$$

What remains is identical to Case 1.
Case 4: $1 = l < k$. The proof is analogous to Case 3.
(iii) Additivity: It is immediate, by definition, to see that $s_{\sigma(k)}^\sigma(w_1 + w_2) = s_{\sigma(k)}^\sigma(w_1) + s_{\sigma(k)}^\sigma(w_2)$ for all $w_1, w_2 \in PG^N$ and for all $k \in \{1, 2, ..., |N|\}$.
(iv) The quasi-null player property: By definition, we know for a partition function form game $w \in PG^N$ and a given ordering $\sigma \in \Pi(N)$,

$$\begin{aligned} r(S_{|N|}^\sigma) = & \quad w(N, \{N\}) \\ r(S_{|N|-1}^\sigma) = & \quad \frac{1}{2}w(N, \{N\}) + \frac{1}{2}w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) - \frac{1}{2}w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) \\ r(S_{|N|-2}^\sigma) = & \quad \frac{1}{4}w(N, \{N\}) + \frac{1}{4}w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) - \frac{1}{4}w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) \\ & + \frac{1}{2}w(S_{|N|-2}^\sigma, \kappa_{|N|-2}^\sigma) - \frac{1}{2}w(\{\sigma(|N|-1)\}, \kappa_{|N|-2}^\sigma) \\ & \dots \\ r(S_2^\sigma) = & \quad \frac{1}{2}r(S_3^\sigma) + \frac{1}{2}w(S_2^\sigma, \kappa_2^\sigma) - \frac{1}{2}w(\{\sigma(3)\}, \kappa_2^\sigma) \\ r(S_1^\sigma) = & \quad \frac{1}{2}r(S_2^\sigma) + \frac{1}{2}w(S_1^\sigma, \kappa_1^\sigma) - \frac{1}{2}w(\{\sigma(2)\}, \kappa_1^\sigma). \end{aligned}$$

Hence, a general expression is provided as follows.

$$r(S_k^\sigma) = \begin{cases} w(N, \{N\}) & \text{if } k = |N| \\ & \\ (\frac{1}{2})^{|N|-k} w(N, \{N\}) & \\ + \sum_{l=k}^{|N|-1} (\frac{1}{2})^{l-k+1} \left(w(S_l^\sigma, \kappa_l^\sigma) - w(\{\sigma(l+1)\}, \kappa_l^\sigma) \right) & \text{if } 1 \leq k \leq |N| - 1. \end{cases}$$

Let player $i \in N$ be a quasi-null player in game w. Let $\sigma(k) = i$. Then, by definition, this quasi-null player's individual standardized remainders in σ, $s_i^\sigma(w) = s_{\sigma(k)}^\sigma(w)$, are explicitly given as

$$\begin{aligned}
s^{\sigma}_{\sigma(|N|)}(w) = &\quad \frac{1}{2}w(\{i\},\kappa^{\sigma}_{|N|-1})\\
s^{\sigma}_{\sigma(|N|-1)}(w) = &\quad \frac{1}{4}w(N,\{N\}) + \frac{1}{4}\left(w(S^{\sigma}_{|N|-1},\kappa^{\sigma}_{|N|-1}) - w(\{\sigma(|N|)\},\kappa^{\sigma}_{|N|-1})\right)\\
- &\quad \frac{1}{2}w(S^{\sigma}_{|N|-2},\kappa^{\sigma}_{|N|-2}) + \frac{1}{2}w(\{i\},\kappa^{\sigma}_{|N|-2})\\
s^{\sigma}_{\sigma(|N|-2)}(w) = &\quad \frac{1}{8}w(N,\{N\}) + \frac{1}{8}\left(w(S^{\sigma}_{|N|-1},\kappa^{\sigma}_{|N|-1}) - w(\{\sigma(|N|)\},\kappa^{\sigma}_{|N|-1})\right)\\
+ &\quad \frac{1}{4}\left(w(S^{\sigma}_{|N|-2},\kappa^{\sigma}_{|N|-2}) - w(\{\sigma(|N|-1)\},\kappa^{\sigma}_{|N|-2})\right)\\
- &\quad \frac{1}{2}w(S^{\sigma}_{|N|-3},\kappa^{\sigma}_{|N|-3}) + \frac{1}{2}w(\{i\},\kappa^{\sigma}_{|N|-3})\\
\ldots &\\
s^{\sigma}_{\sigma(2)}(w) = &\quad \frac{1}{2^{|N|-1}}w(N,\{N\}) + \frac{1}{2^{|N|-1}}w(S^{\sigma}_{|N|-1},\kappa^{\sigma}_{|N|-1})\\
- &\quad \frac{1}{2^{|N|-1}}w(\{\sigma(|N|)\},\kappa^{\sigma}_{|N|-1}) + \ldots + \frac{1}{4}w(S^{\sigma}_{2},\kappa^{\sigma}_{2})\\
- &\quad \frac{1}{4}w(\{\sigma(3)\},\kappa^{\sigma}_{2}) - \frac{1}{2}w(S^{\sigma}_{1},\kappa^{\sigma}_{1}) + \frac{1}{2}w(\{i\},\kappa^{\sigma}_{1})\\
s^{\sigma}_{\sigma(1)}(w) = &\quad \frac{1}{2^{|N|-1}}w(N,\{N\}) + \frac{1}{2^{|N|-1}}w(S^{\sigma}_{|N|-1},\kappa^{\sigma}_{|N|-1})\\
- &\quad \frac{1}{2^{|N|-1}}w(\{\sigma(|N|)\},\kappa^{\sigma}_{|N|-1}) + \ldots + \frac{1}{4}w(S^{\sigma}_{2},\kappa^{\sigma}_{2})\\
- &\quad \frac{1}{4}w(\{\sigma(3)\},\kappa^{\sigma}_{2}) - \frac{1}{2}w(\{\sigma(2)\},\kappa^{\sigma}_{1}) + \frac{1}{2}w(\{i\},\kappa^{\sigma}_{1}).
\end{aligned}$$

A general expression is

$$s^{\sigma}_{\sigma(k)} = \begin{cases} \frac{1}{2}w(\{i\},\kappa^{\sigma}_{|N|-1}) & \text{if } k = |N| \\ \\ (\frac{1}{2})^{|N|-k+1}w(N,\{N\}) & \\ +\sum_{l=k}^{|N|-1}(\frac{1}{2})^{l-k+2}\left(w(S^{\sigma}_{l},\kappa^{\sigma}_{l}) - w(\{\sigma(l+1)\},\kappa^{\sigma}_{l})\right) & \\ -\frac{1}{2}w(S^{\sigma}_{k-1},\kappa^{\sigma}_{k-1}) + \frac{1}{2}w(\{i\},\kappa^{\sigma}_{k-1}) & \text{if } 2 \leq k \leq |N|-1 \\ \\ r(S^{\sigma}_{1}) & \text{if } k = 1. \end{cases}$$

Consider a class P of $|N|$ permutations $\sigma \in \Pi(N)$ such that for $\sigma, \tau \in P$ it holds that for all $j_1, j_2 \in N \backslash \{i\}$

$$\sigma^{-1}(j_1) < \sigma^{-1}(j_2) \Leftrightarrow \tau^{-1}(j_1) < \tau^{-1}(j_2).$$

That is, given an ordering of the players $N\backslash\{i\}$, let quasi-null player i move from the end to the beginning without changing the other players' relative positions. Summing over the above equations, we get

$$\begin{aligned}
\sum_{\sigma\in P} s_i^\sigma(w) = & \sum_{\sigma\in P} s_{\sigma(k)}^\sigma(w) \\
= & \frac{1}{2}w(N,\{N\}) + \frac{1}{2}\sum_{k=1}^{|N|}\left(w(S_k^\sigma,\kappa_k^\sigma) - w(S_{k-1}^\sigma,\kappa_{k-1}^\sigma)\right) \\
+ & \frac{1}{2}\sum_{k=2}^{|N|} w(\{\sigma(k)\},\kappa_{k-1}^\sigma) - \frac{1}{2}\sum_{k=1}^{|N|-1} w(\{\sigma(k+1)\},\kappa_k^\sigma).
\end{aligned}$$

Since i is a quasi-null player, $w(S_k^\sigma,\kappa_k^\sigma)-w(S_{k-1}^\sigma,\kappa_{k-1}^\sigma)=0$ for all $k\in\{1,2,...,|N|\}$. Then,

$$\sum_{\sigma\in P} s_i^\sigma(w) = \frac{1}{2}\left(w(N,\{N\}) + \sum_{k=2}^{|N|} w(\{\sigma(k)\},\kappa_{k-1}^\sigma) - \sum_{k=1}^{|N|-1} w(\{\sigma(k+1)\},\kappa_k^\sigma)\right).$$

Taking all orderings of players into account, we then get

$$\begin{aligned}
\gamma_i(w) = & \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} s_i^\sigma(w) \\
= & \frac{1}{2}\frac{w(N,\{N\})}{|N|} \\
+ & \frac{1}{2}\sum_{S\subset N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-1)!}{|N|!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}) \\
- & \frac{1}{2}\sum_{j\in N\backslash\{i\}}\sum_{S\subset N\backslash\{i,j\}} \frac{|S|!(|N|-|S|-2)!}{|N|!} w(\{j\},[N\backslash(S\cup\{i\})]\cup\{S\cup\{i\}\}) \\
= & \frac{1}{2}e_i(w).
\end{aligned}$$

■

Before proving the uniqueness of the consensus value, we first check the relationship between the consensus value and the Shapley value for partition function form games defined by Pham Do and Norde (2002). Since the Shapley value assigns zero worth to a quasi-null player, one can see that the quasi-null player property can be reformulated as $f_i(w) = \frac{1}{2}\Phi_i(w) + \frac{1}{2}e_i(w)$ for all $w \in PG^N$ and quasi-null player i in (N,w). In fact, interestingly, introducing this property influences all the players in the same way: Each player finally gets an average of her Shapley value and the expected stand-alone value. Formally, we have the following theorem.

Theorem 4.6 *The consensus value is the average of the Shapley value and the expected stand-alone value. That is, for every w in PG^N it holds that*

$$\gamma(w) = \frac{1}{2}\Phi(w) + \frac{1}{2}e(w).$$

Proof. Similar to part (iv) in the proof for Theorem 4.5, one can show that for every $w \in PG^N$,

$$\gamma_i(w) = \frac{1}{2}\left(\frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} m_i^\sigma(w) + e_i(w)\right)$$

for all $i \in N$. ■

In order to prove[5] that the consensus value is the unique solution that satisfies efficiency, complete symmetry, the quasi-null player property and additivity, one needs to consider the "standard" basis of partition function form games. Let $\mathbb{E}'(N)$ be the set of all $(S,\kappa) \in \mathbb{E}(N)$ such that $S \neq \emptyset$. That is, $\mathbb{E}'(N) = \{(S,\kappa) \in \mathbb{E}(N) : S \neq \emptyset\}$. For any $(S,\kappa) \in \mathbb{E}'(N)$, define the partition function

$$\delta_{(S,\kappa)}(S',\kappa') = \begin{cases} 1 & \text{if } (S',\kappa') = (S,\kappa) \\ 0 & \text{otherwise.} \end{cases}$$

We call $\delta_{(S,\kappa)}$ the Dirac game with respect to (S,κ). One can see that the set of all Dirac games, $\left\{\delta_{(S,\kappa)} : (S,\kappa) \in \mathbb{E}'(N)\right\}$, forms a basis of partition function form games. Each $w \in PG^N$ can be uniquely written as

$$w = \sum_{(S,\kappa)\in\mathbb{E}'(N)} w(S,\kappa)\delta_{(S,\kappa)}.$$

If f is a solution on PG^N satisfying additivity, then for all $w \in PG^N$,

$$f(w) = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}).$$

Lemma 4.7 *Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$, and f be a solution on PG^N satisfying additivity and the quasi-null player property. We have*

$$f_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } S(\kappa,i) \neq \{i\} \\ \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) & \text{if } S(\kappa,i) = \{i\}, \end{cases}$$

where w is the partition function such that

$$w(S',\kappa') = \begin{cases} c & \text{if } (S',\kappa') = (S,\kappa) \\ c & \text{if } (S',\kappa') = (S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\}) \\ 0 & \text{otherwise.} \end{cases}$$

[5] We want to note that the proof for the uniqueness is in the same line as Fujinaka (2004).

Proof. Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$.
Case 1: $S(\kappa,i) \neq \{i\}$. Here, one can readily verify that i is a quasi-null player of game $c\delta_{(S,\kappa)}$. Hence, by the quasi-null player property, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$.
Case 2: $S(\kappa,i) = \{i\}$. Since we can write $w = c\delta_{(S,\kappa)} + c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}$, and i is a quasi-null player in w, by additivity and the quasi-null player property, we have $f_i(w) = \frac{1}{2}e_i(w)$ and $f_i(w) = f_i\left(c\delta_{(S,\kappa)}\right) + f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right)$. Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right)$. ■

Theorem 4.8 *There is a unique solution on PG^N satisfying efficiency, complete symmetry, the quasi-null player property and additivity. This solution is the consensus value.*

Proof. From Theorem 4.5, it follows that the consensus value γ satisfies efficiency, complete symmetry, the quasi-null player property and additivity.

Conversely, suppose a solution concept f satisfies these four properties. We have to show that $f = \gamma$. By additivity, it suffices to show that for any $c \in \mathbb{R}$ and any $(S,\kappa) \in \mathbb{E}'(N)$, $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

For any $c \in \mathbb{R}$ and $(S,\kappa) \in \mathbb{E}'(N)$, $\gamma\left(c\delta_{(S,\kappa)}\right)$ is defined as follows. If $S \neq N$ and $\kappa \neq \{S\} \cup [N\backslash S]$, then

$$\gamma_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } |S| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$ because all players are quasi-null players; if $\kappa = \{S\} \cup [N\backslash S]$, by Theorem 4.6, we have

$$\gamma_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) + \frac{1}{2}\cdot c \cdot \frac{(|S|-1)!(|N|-|S|)!}{|N|!} & \text{for all } i \in S \\ \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) + \frac{1}{2}\cdot c \cdot \left(-\frac{|S|!(|N|-|S|-1)!}{|N|!}\right) & \text{otherwise.} \end{cases}$$

For any $c \in \mathbb{R}$ and $(S,\kappa) \in \mathbb{E}'(N)$, let $I\left(c\delta_{(S,\kappa)}\right) = |S|$. In order to prove that $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$, we use a (converse-)induction argument on the number $I\left(c\delta_{(S,\kappa)}\right)$.

If $I\left(c\delta_{(S,\kappa)}\right) = |N|$, then $c\delta_{(S,\kappa)} = c\delta_{(N,\{N,\{N\}\})}$. One can readily check that for all $i \in N$, $\gamma_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$ because $e_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$. Efficiency and complete symmetry imply that for all $i \in N$, $f_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$. Thus, $f\left(c\delta_{(N,\{N,\{N\}\})}\right) = \gamma\left(c\delta_{(N,\{N,\{N\}\})}\right)$. We then complete the first step for the induction argument.

Next, as an induction hypothesis, suppose that for each $k' \geq k+1$, if $I\left(c\delta_{(S,\kappa)}\right) = k'$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$. We need to show that if $I\left(c\delta_{(S,\kappa)}\right) = k$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$. Assume that $I\left(c\delta_{(S,\kappa)}\right) = k$.

Claim 1: If $\kappa \neq \{S\} \cup [N \backslash S]$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

First, we shall show that for each $i \notin S$, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. Let $i \notin S$, there are two cases:

Case 1: $S(\kappa, i) \neq \{i\}$. By Lemma 4.7, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$. Moreover, if $|S| \neq 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right)$$

and if $|S| = 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Case 2: $S(\kappa, i) = \{i\}$. By Lemma 4.7, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right)$. Since $I\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) = k+1$ and $(\kappa\backslash S)\cup\{S\cup\{i\}\} \neq \kappa_{S\cup\{i\}}$, by the induction hypothesis,

$$f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) = \gamma_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) = 0.$$

Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w)$. Hence, if $|S| \neq 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right);$$

if $|S| = 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) = 0 = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Now we need to show that for each $i \in S$, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. Let $i \in S$. If $S \neq \{i\}$, since for all $j \notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = 0$ and all players in S are completely symmetric in $c\delta_{(S,\kappa)}$, by efficiency and complete symmetry, we have for all $i \in S$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

If $S = \{i\}$, since for all $j \notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = \gamma_j\left(c\delta_{(S,\kappa)}\right)$, by efficiency, obviously, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$.

Claim 2: If $\kappa = \{S\} \cup [N \backslash S]$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

Let $i \notin S$. Since $S(\kappa, i) = \{i\}$, by Lemma 4.7,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right).$$

Since $I\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) = k+1$ and $(\kappa\backslash S)\cup\{S\cup\{i\}\} = \kappa_{S\cup\{i\}}$, by the induction hypothesis,

$$\begin{aligned} f_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) &= \gamma_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) \\ &= \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\backslash S)\cup\{S\cup\{i\}\})}\right) + \frac{1}{2}\cdot c \cdot \frac{k!(|N|-k-1)!}{|N|!}. \end{aligned}$$

Therefore,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!}.$$

Thus, if $|S|>1$, $\frac{1}{2}e_i(w) - \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = 0 = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$, so

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

If $|S|=1$, $\frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = 0$ and $\frac{1}{2}e_i(w) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$, so we also have

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Let $i\in S$. If $S\neq\{i\}$, since for all $j\notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = \gamma_j\left(c\delta_{(S,\kappa)}\right) = -\frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!}$ and all players in S are completely symmetric in $c\delta_{(S,\kappa)}$, by efficiency and complete symmetry, we have for all $i\in S$,

$$\begin{aligned} f_i\left(c\delta_{(S,\kappa)}\right) &= \frac{|N|-k}{k}\cdot\frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} \\ &= \frac{1}{2}\cdot c\cdot\frac{(k-1)!(|N|-k)!}{|N|!} \\ &= \gamma_i\left(c\delta_{(S,\kappa)}\right). \end{aligned}$$

If $S=\{i\}$, by efficiency, obviously, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. ■

We now provide an alternative characterization for the consensus value by means of the transfer property, which is in the same spirit as that for the Shapley value for the case of TU games in characteristic function form (cf. Feltkamp (1995)).

The transfer property, introduced by Dubey (1975), that in some sense substitutes for additivity, is defined as follows. For any two partition function form games $w_1, w_2\in PG^N$, we first define the games $(w_1\vee w_2)$ and $(w_1\wedge w_2)$ by $(w_1\vee w_2)(S,\kappa) = \max\{w_1(S,\kappa), w_2(S,\kappa)\}$ and $(w_1\wedge w_2)(S,\kappa) = \min\{w_1(S,\kappa), w_2(S,\kappa)\}$ for all $S\in\kappa$ and $\kappa\in\mathbb{P}(N)$. Let $f: PG^N \longrightarrow \mathbb{R}^N$ be a solution concept on the class of partition function form games. Then, f satisfies the transfer property if $f(w_1\vee w_2) + f(w_1\wedge w_2) = f(w_1) + f(w_2)$ for all $w_1, w_2\in PG^N$.

In order to characterize the consensus value on the class of all partition function form games by the transfer property, we need the following lemma. Here, the zero game in PG^N that is defined by $w(S,\kappa)=0$ for all $(S,\kappa)\in\mathbb{E}(N)$ is denoted by $\underline{0}$.

Lemma 4.9 *Let f be a solution on PG^N satisfying the transfer property, with*[6] $f(\underline{0}) = 0$. *Then, for all games $w \in PG^N$,*

$$f(w) = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}). \tag{1}$$

Proof. We prove in three steps that equation (1) holds.
Step 1: For the class of all non-negative games w the proof is by induction on

$$k(w) := |\{S|(S,\kappa) \in \mathbb{E}(N) \text{ and } w(S,\kappa) > 0\}|.$$

Here, a game w is non-negative if $w(S,\kappa) \geq 0$ for all $(S,\kappa) \in \mathbb{E}(N)$.
If $k(w) = 0$, then $w = \underline{0}$, so $f(w) = 0 = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)})$.
Take $k > 0$ and suppose equation (1) holds for all non-negative games w with $k(w) < k$. For a non-negative game w with $k(w) = k$, choose an embedded coalition $(S',\kappa') \in \mathbb{E}(N)$ such that $w(S',\kappa') > 0$. Then $k(w - w(S',\kappa')\delta_{(S',\kappa')}) = k - 1$, $(w - w(S',\kappa')\delta_{(S',\kappa')}) \vee (w(S',\kappa')\delta_{(S',\kappa')}) = w$ and $(w - w(S',\kappa')\delta_{(S',\kappa')}) \wedge (w(S',\kappa')\delta_{(S',\kappa')}) = 0$. Hence, using the induction hypothesis and the transfer property, we obtain

$$\begin{aligned} f(w) = & \quad f(w - w(S',\kappa')\delta_{(S',\kappa')}) + f(w(S',\kappa')\delta_{(S',\kappa')}) \\ & - \quad f((w - w(S',\kappa')\delta_{(S',\kappa')}) \wedge (w(S',\kappa')\delta_{(S',\kappa')})) \\ = & \sum_{(S,\kappa)\in\mathbb{E}'(N)} f[(w - w(S',\kappa')\delta_{(S',\kappa')})(S,\kappa)\delta_{(S,\kappa)}] + f(w(S',\kappa')\delta_{(S',\kappa')}) - f(\underline{0}) \\ = & \sum_{(S,\kappa)\in\mathbb{E}'(N);\ (S,\kappa)\neq(S',\kappa')} f(w(S,\kappa)\delta_{(S,\kappa)}) + f(w(S',\kappa')\delta_{(S',\kappa')}) - f(\underline{0}) \\ = & \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}). \end{aligned}$$

Step 2: For non-positive games one proves analogously (interchanging the operations $\wedge$ and $\vee$) that equation (1) holds.
Step 3: For an arbitrary game w, split the game into its non-negative part $v \vee \underline{0}$ and its non-positive part $v \wedge \underline{0}$. The transfer property and steps 1 and 2 imply

$$\begin{aligned} f(w) &= f(w) + f(\underline{0}) \\ &= f(w \vee \underline{0}) + f(w \wedge \underline{0}) \\ &= \sum_{(S,\kappa)\in\mathbb{E}'(N)} [f((w \vee \underline{0})(S,\kappa)\delta_{(S,\kappa)}) + f((w \wedge \underline{0})(S,\kappa)\delta_{(S,\kappa)})] \\ &= \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}). \end{aligned}$$

[6]Note that $f(\underline{0}) = 0$ is a weak requirement because any solution concept satisfying efficiency and (complete) symmetry yields this outcome.

Hence equation (1) holds for all partition function form games. ∎

Note that the converse is also true: If a solution concept f on PG^N satisfies equation (1) for all games $w \in PG^N$, then f satisfies the transfer property and $f(\underline{0}) = 0$.

Below we introduce a lemma which is similar to Lemma 4.7.

Lemma 4.10 *Let $c \in \mathbb{R}$, $(S, \kappa) \in \mathbb{E}'(N)$ and $i \notin S$, and f be a solution on PG^N satisfying the transfer property with $f(\underline{0}) = 0$ and the quasi-null player property. We have*

$$f_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } S(\kappa,i) \neq \{i\} \\ \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) & \text{if } S(\kappa,i) = \{i\}, \end{cases}$$

where w is defined in Lemma 4.7.

Proof. Let $c \in \mathbb{R}$, $(S, \kappa) \in \mathbb{E}'(N)$ and $i \notin S$. The proof is the same as that for Lemma 4.7 except for the case when $S(\kappa, i) = \{i\}$. Since here we can write $w = c\delta_{(S,\kappa)} \vee c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}$, by the transfer property, we have

$$\begin{aligned} & f(c\delta_{(S,\kappa)} \vee c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}) + f(c\delta_{(S,\kappa)} \wedge c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}) \\ = \quad & f(w) + f(\underline{0}) \\ = \quad & f(c\delta_{(S,\kappa)}) + f(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}). \end{aligned}$$

Thus, $f_i(w) = f_i\left(c\delta_{(S,\kappa)}\right) + f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. Moreover, since i is a quasi-null player in w, by the quasi-null player property, we know $f_i(w) = \frac{1}{2}e_i(w)$. Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. ∎

Using Lemma 4.9 and Lemma 4.10, we now prove the following.

Theorem 4.11 *The consensus value is the only one-point solution on the class of partition function form games that satisfies efficiency, symmetry, the quasi-null player property and the transfer property.*

Proof. First of all, we claim that a solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ satisfying additivity on PG^N also satisfies the transfer property on PG^N. To prove this, take $w_1, w_2 \in PG^N$. Then, using additivity,

$$\begin{aligned} f(w_1 \vee w_2) + f(w_1 \wedge w_2) &= f(w_1 \vee w_2 + w_1 \wedge w_2) \\ &= f(w_1 + w_2) \\ &= f(w_1) + f(w_2). \end{aligned}$$

Therefore, the consensus value satisfies the transfer property.

By Lemma 4.10 and using the same technique in the proof for Theorem 4.8, one can readily see that requiring a solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ to satisfy efficiency, complete symmetry, and the quasi-null player property, it easily follows that f is uniquely determined for (multiples of) Dirac games. Moreover, based on Lemma 4.9, we know that a solution f satisfying the transfer property is uniquely determined for any game in PG^N, since the class of Dirac games forms a basis of PG^N. ■

Similarly, we can characterize the Shapley value for partition function form games by means of this transfer property.

Theorem 4.12 *The Shapley value is the only one-point solution on the class of partition function form games that satisfies efficiency, symmetry, the (quasi-)null player property and the transfer property.*

One may notice that a nice feature of the consensus value for partition function form games lies in the individual rationality.

First, we define superadditivity for partition function form games. A partition function form game $w \in PG^N$ is called *superadditive* if it satisfies

$$w(S \cup T, \{S \cup T\} \cup \kappa_{N \setminus (S \cup T)}) \geq w(S, \{S\} \cup \{T\} \cup \kappa_{N \setminus (S \cup T)}) + w(T, \{S\} \cup \{T\} \cup \kappa_{N \setminus (S \cup T)})$$

for all $S, T \subset N$ and $\kappa_{N \setminus (S \cup T)} \in \mathbb{P}(N \setminus (S \cup T))$ with $S \cap T = \emptyset$.

Theorem 4.13 *If a partition function form game $w \in PG^N$ is superadditive and with nonnegative externalities on individual players, that is, $w(\{i\}, \kappa_{N \setminus \{i\}}) \geq w(\{i\}, [N])$ for all $i \in N$ and $\kappa_{N \setminus \{i\}} \in \mathbb{P}(N \setminus \{i\})$, then the consensus value satisfies* individual rationality, *that is, $\gamma_i(w) \geq w(\{i\}, [N])$ for all $i \in N$.*

Proof. By Definition 3.1, it is easy to see that in any superadditive game with nonnegative externalities on individual players, the individual standardized remainder $s_i^\sigma(w)$ is greater than or equal to the stand-alone value $w(\{i\}, [N])$ for all $i \in N$ and $\sigma \in \Pi(N)$. ■

This is a very reasonable property. However, not all solution concepts satisfy it. See the following example where the game is taken from Cornet (1998).

Example 4.14 *Let the game (N, w) be given by $N = \{1, 2, 3\}$ and*

$$\begin{aligned} \bar{w}(1,2,3) &= (0,0,0), \\ \bar{w}(12,3) &= (0,3), \ \bar{w}(13,2) = (0,3), \ \bar{w}(23,1) = (3,0), \\ \bar{w}(123) &= (4). \end{aligned}$$

As the game is superadditive and with nonnegative externalities on individual players, the consensus value satisfies individual rationality. Indeed, one can check that the consensus value of this game $\gamma(w) = (\frac{1}{3}, \frac{11}{6}, \frac{11}{6})$, *which coincides with the Shapley value defined by Pham Do and Norde (2002) in this game, is greater than* $(0, 0, 0)$. *However, the Myerson value is* $(-\frac{5}{3}, \frac{17}{6}, \frac{17}{6})$; *the Feldman value as well as the Bolger's and Potter's value are* $(-\frac{1}{6}, \frac{25}{12}, \frac{25}{12})$, *the Shapley value defined by Feldman (1994) is* $(-\frac{2}{3}, \frac{7}{3}, \frac{7}{3})$.

5 A generalization of the consensus value

By relaxing the way of sharing remainders, we get a generalization of the consensus value: the *generalized consensus value.*

We define the generalized remainder, with respect to an order $\sigma \in \Pi(N)$, for given $\theta \in [0, 1]$, recursively by

$$r_\theta(S_k^\sigma) = \begin{cases} w(N, \{N\}) & \text{if } k = |N| \\ \\ w(S_k^\sigma, \kappa_k^\sigma) \\ +(1-\theta)\left(r_\theta(S_{k+1}^\sigma) - w(S_k^\sigma, \kappa_k^\sigma) - w(\{\sigma(k+1)\}, \kappa_k^\sigma)\right) & \text{if } k \in \{1, ..., |N|-1\}. \end{cases}$$

The generalized remainder is the value left for S_k^σ after allocating surplus to later entrants $N \backslash S_k^\sigma$ according to share parameter θ.

Correspondingly, the individual generalized remainder vector $s_\theta^\sigma(w)$ is the vector in $\mathbb{R}^N$ defined by

$$(s_\theta^\sigma)_{\sigma(k)}(w) = \begin{cases} w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) \\ +\theta\left(r_\theta(S_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma) - w(\{\sigma(k)\}, \kappa_{k-1}^\sigma)\right) & \text{if } k \in \{2, ..., |N|\} \\ \\ r_\theta(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 5.1 *For any* $w \in PG^N$, *the generalized consensus value,* $\gamma_\theta(w)$, $\theta \in [0, 1]$, *is the average of the individual generalized remainder vectors, i.e.*

$$\gamma_\theta(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s_\theta^\sigma(w).$$

Note that the consensus value corresponds to the case $\theta = \frac{1}{2}$.

As mentioned in Section 4, dependent on the degree to which that the coalition effect or externality effect is preferred by a society, the quasi-null player property can be generalized. Defining the *θ-quasi-null player property* of a one-point solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ by $f_i(w) = (1-\theta)e_i(w)$ for all $w \in PG^N$ and any quasi-null player $i \in N$ for w, we obtain the following theorem.

Theorem 5.2 *For $\theta \in [0, 1]$:*

(a) The generalized consensus value γ_θ is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the θ-quasi-null player property and additivity.

(b) For any $w \in PG^N$, it holds that

$$\gamma_\theta(w) = \theta\Phi(w) + (1-\theta)e(w)$$

(c) The generalized consensus value γ_θ is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the θ-quasi-null player property and the transfer property.

Proof. Following the same way to prove Theorem 4.5, Theorem 4.6, Theorem 4.8, and Theorem 4.11, it is easily established. ■

In particular, for $\theta = 1$, the generalized consensus value is the Shapley value; for $\theta = 0$, the generalized consensus value equals the expected stand-alone value.

Corollary 5.3 *(a) The expected stand-alone value is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the 0-quasi-null player property and additivity.*

(b) The expected stand-alone value is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the 0-quasi-null player property and the transfer property.

The proof is omitted as it is obvious.

The idea of defining the consensus value can also be extended. If taking the size of the incumbent party S into consideration, we can argue on a basis of a proportional principle that given an ordering of players the entrant should get $\frac{1}{|S|+1}$ of the joint surplus while the incumbents get a share of $\frac{|S|}{|S|+1}$, which results in another solution concept, namely, *the coalition-size-based consensus value* for partition function form games.

6 Some applications of the consensus value

6.1 Application to oligopoly games

Along the same line as Pham Do and Norde (2002), this section first applies the consensus value to oligopoly games in partition function form.

Let us focus on a linear oligopoly market of a homogeneous good with asymmetric costs, no fixed costs and no capacity constraints. Such an oligopoly is defined by the vector $(b; c) \in \mathbb{R}_{+}^{n+1}$, where $b > 0$ is the intercept of the inverse demand function, $c = (c_1, c_2, ..., c_n) \geq 0$ is the marginal cost vector. Without loss of generality, assume $c_1 \leq c_2 \leq ... \leq c_n$. We also assume that an equilibrium price always exceeds the largest marginal cost, i.e. $\frac{b+\sum_{j=1}^{n} c_j}{n+1} > c_n$. Note that this assumption is equivalent to the requirement of positive market shares at the equilibrium for all players (Zhao (2001)). For each supply (input) vector $x = (x_1, x_2, ..., x_n)$, the price is $p(x) = b - \sum_{i=1}^{n} x_i$, whereas player i's cost and profit (payoff) are $C_i(x_i) = c_i x_i$ and

$$\pi_i(x) = p(x)x_i - C_i(x_i) = \left(b - \sum_{i=1}^{n} x_i\right) x_i - c_i x_i,$$

respectively. Player i's reaction curve is implicitly defined by the first order condition:

$$\frac{\partial \pi_i(x)}{\partial x_i} = p(x) - x_i - c_i = 0, \text{ or } x_i = \frac{b - c_i - \sum_{j \neq i} x_j}{2}. \tag{2}$$

A Cournot-Nash equilibrium is a vector such that each player's action x_i is a best response to the complementary choice $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. This equilibrium is graphically the intersection point of all reaction curves and algebraically the solution of the system of the equations (2). The unique equilibrium, $x^* = (x_1^*, x_2^*, ..., x_n^*)$, is determined by

$$x_i^* = \frac{b - n \cdot c_i + \sum_{j \neq i} c_j}{n+1}$$

and the payoff of player i at this equilibrium is

$$\pi_i(x^*) = (x_i^*)^2 = \frac{(b - n \cdot c_i + \sum_{j \neq i} c_j)^2}{(n+1)^2}.$$

Now suppose that after sufficient communication, some players may agree to cooperate (for example, players intend to adjust negative externalities which are caused by decreasing returns to inputs). In such a situation a coalition structure might form, in which, however, the payoff of coalition S depends on the behavior of the players outside S, and visa versa. Notice that the payoff for coalition S under one coalition structure is different from that under another coalition structure if the number of coalitions is different. Assume that the marginal cost of coalition S is $c_S = min_{i \in S} c_i$, that is, the most efficient technology in a coalition can be costlessly adopted by all players in that coalition. Moreover, if a coalition structure $\kappa = \{S_1, S_2, ..., S_k\}$ is formed, then, in equilibrium each coalition S in κ will choose the total (input) quantity levels to maximize the sum of its members' profits, given the total inputs of the other coalitions in κ.

Let $x_{S_j} = \sum_{i \in S_j} x_i$ denote the total input level for a coalition S_j and $\pi_{S_j}(x)$ denote the profit of coalition S_j under coalition structure κ,

$$\pi_{S_j}(x) = p(x)x_{S_j} - C_S(x_{S_j}) = \left(b - \sum_{i=1}^{k} x_{S_i}\right) x_{s_j} - c_{S_j} x_{S_j}.$$

Coalition S_j's reaction curve under coalition structure κ is also implicitly defined by the first order condition:

$$\frac{\partial \pi_{S_j}(x)}{\partial x_{S_j}} = p(x) - x_{S_j} - c_{S_j} = 0, \text{ or } x_{S_j} = \frac{b - c_{S_j} - \sum_{i \neq j} x_{S_i}}{2}.$$

The unique equilibrium under coalition structure κ with quantities $x^* = (x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})$, and profit $\pi_{S_j}(x^*)$ of coalition S_j, is given by

$$x^*_{S_j} = \frac{b - k \cdot c_{S_j} + \sum_{i \neq j} c_{S_i}}{k+1}$$

and

$$\pi_{S_j}(x^*) = \frac{(b - k \cdot c_{S_j} + \sum_{i \neq j} c_{S_i})^2}{(k+1)^2}.$$

The oligopoly game in partition function form (N, w) is determined for every (S_j, κ) by $w(S_j, \kappa) = \pi_{S_j}(x^*)$, where x^* is the equilibrium vector under coalition structure κ.

To get further illustration of how the consensus value can be used we specify the 3-person oligopoly game in partition function form (N, w). The partition function form game is given by $\bar{w}(1,2,3) = (a_1, a_2, a_3)$, $\bar{w}(12,3) = (a_{12}, b_3)$, $\bar{w}(13,2) = (a_{13}, b_2)$, $\bar{w}(23,1) =$

(a_{23}, b_1), $\bar{w}(123) = (a_{123})$, where

$$\begin{aligned}
a_1 &= \frac{1}{16}(b - 3c_1 + c_2 + c_3)^2, \\
a_2 &= \frac{1}{16}(b - 3c_2 + c_1 + c_3)^2, \\
a_3 &= \frac{1}{16}(b - 3c_3 + c_1 + c_2)^2, \\
a_{12} &= \frac{1}{9}(b - 2c_1 + c_3)^2,\ b_3 = \frac{1}{9}(b - 2c_3 + c_1)^2 \\
a_{13} &= \frac{1}{9}(b - 2c_1 + c_2)^2,\ b_2 = \frac{1}{9}(b - 2c_2 + c_1)^2 \\
a_{23} &= \frac{1}{9}(b - 2c_2 + c_1)^2,\ b_1 = \frac{1}{9}(b - 2c_1 + c_2)^2 \\
a_{123} &= \frac{1}{9}(b - c_1)^2.
\end{aligned}$$

Given the ordering of marginal costs, one can easily see that $a_1 \geq a_2 \geq a_3$, and $a_{12} \geq a_{13} = b_1 \geq a_{23} = b_2 \geq b_3$.

The consensus value of this game, $\gamma(w) = (\gamma_i(w))_{i=1,2,3}$, can be computed as follows:

$$\begin{aligned}
\gamma_1(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_1 - a_2 - a_3 + \frac{a_{12} + 3a_{13} - 3a_{23} - b_3}{2}\right) \\
\gamma_2(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_2 - a_1 - a_3 + \frac{a_{12} + 3a_{23} - 3a_{13} - b_3}{2}\right) \\
\gamma_3(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_3 - a_1 - a_2 + b_3 - a_{12}\right).
\end{aligned}$$

Note that if players have identical costs, then $a_1 = a_2 = a_3$ and $a_{12} = a_{13} = a_{23} = b_1 = b_2 = b_3$, and obviously, the consensus value yields an equal payoff to all players, i.e. $\gamma_i(w) = \frac{a_{123}}{3}$.

Consider the following example for further illustration.

Example 6.1 *The game (cf. Pham Do and Norde (2002)) in partition function form (N, w) associated with a linear oligopoly market $(b; c)$, where $b = 20$, $c = (1, 3, 4)$, is given by*

$$\begin{aligned}
\bar{w}(1,2,3) &= (36, 16, 9), \\
\bar{w}(12,3) &= (53.78, 18.78),\ \bar{w}(13,2) = (49, 25),\ \bar{w}(23,1) = (25, 49), \\
\bar{w}(123) &= (90.25).
\end{aligned}$$

The consensus value for this game is $\gamma(w) = (46.833, 24.833, 18.583)$, whereas the Shapley value is $\Phi(w) = (46.70, 24.71, 18.83)$.

6.2 Free-rider, sharing rule and participation incentive

Since the partition function form games can well capture externalities, they provide a suitable framework to analyze the associated issues such as free-rider problem. Below we will investigate the effects of different solution concepts on the participation incentives of the players who may free-ride in a game.

Consider the following partition function form game (N, w) (we may call it a free-rider game) defined by $\bar{w}(1, 2, 3) = (0, 0, 0)$, $\bar{w}(12, 3) = (1, 1)$, $\bar{w}(13, 2) = (1, 1)$, $\bar{w}(23, 1) = (0, 0)$, $\bar{w}(123) = 1$. This game can be interpreted as follows: Three players are considering to set up a joint project. Each player has two choices: *participate* or *stand by*. The success of the project depends on the players' participation. Here, obviously, player 2 and 3 are possible free-riders.

Since both player 2 and 3 are prone to standing by, it is very likely that the project will fail in the end. Thus, everybody becomes a loser due to their "selfish rationality". Given the different sharing rules, which one is better for increasing the possible free-riders' incentive to contribute instead of standing idle? We now check the following solution concepts and compare their influences on players' choices.

	1	2	3
the Shapley value $\Phi(w)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
Bolger, Feldman or Potter's value	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
the consensus value $\gamma(w)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Now we first discuss the effect of the Shapley value on the free-riders's participation incentive. Given the Shapley value as the solution concept for the above game, we know the three players will play the following strategic game.

If player 1 chooses participating, the payoff matrix is provided as below. (Here, the payoffs in each cell are listed in the order of player 1, 2 and 3.)

	3 participate	3 stand by
2 participate	$\frac{2}{3}, \frac{1}{6}, \frac{1}{6}$	$\frac{1}{2}, \frac{1}{2}, 1$
2 stand by	$\frac{1}{2}, 1, \frac{1}{2}$	$0, 0, 0$

While if player 1 chooses standing by, all of them will get zero payoff no matter what strategies player 2 and 3 will choose.

	3 participate	3 stand by
2 participate	$0, 0, 0$	$0, 0, 0$
2 stand by	$0, 0, 0$	$0, 0, 0$

So, obviously, choosing participating is the dominant strategy for player 1. One can easily check this game has three pure-strategy Nash equilibria: (1 stands by, 2 stands by, 3 stands by), (1 participates, 2 participates, 3 stands by) and (1 participates, 2 stands by, 3 participates); and another equilibrium which involves mixed strategies of players 2 and 3: (1 participates, 2 participates with probability $\frac{3}{8}$ and stands by with probability $\frac{5}{8}$, 3 participates with probability $\frac{3}{8}$ and stands by with probability $\frac{5}{8}$). We may call such an equilibrium semi mixed-strategy equilibrium as player 1 still plays a pure strategy.

Similarly, one can check the results due to the implementation of the values by Bolger, Feldman or Potter. The corresponding three pure-strategy equilibria are the same as above, while the third equilibrium is different: (1 participates, 2 participates with probability $\frac{2}{5}$ and stands by with probability $\frac{3}{5}$, 3 participates with probability $\frac{2}{5}$ and stands by with probability $\frac{3}{5}$).

Now we check the consensus value in this game. Despite the fact that the three pure-strategy equilibria are the same as above, the semi mixed-strategy equilibrium is different: (1 participates, 2 participates with probability $\frac{3}{7}$ and stands by with probability $\frac{4}{7}$, 3 participates with probability $\frac{3}{7}$ and stands by with probability $\frac{4}{7}$).

Apparently, $\frac{3}{7} > \frac{2}{5} > \frac{3}{8}$; and the corresponding expected payoff due to the consensus value is also greater than the others: $1\frac{8}{49} > 1\frac{3}{25} > 1\frac{5}{64}$.

Therefore, from the semi mixed-strategy equilibrium, we can see that the consensus value helps to increase the participation incentives of the possible free-riders.

References

[1] Cornet, M. (1998), *Game-theoretic Models of Bargaining and Externalities*, Ph.D. Dissertation, Vrije Universiteit Amsterdam, the Netherlands.

[2] Bolger, E.M. (1989), A Set of Axioms for a Value for Partition Function Games, *International Journal of Game Theory*, 18, 37-44.

[3] Dubey, P. (1975), On the Uniqueness of the Shapley Value, *International Journal of Game Theory*, 4, 229-247.

[4] Feldman, B. (1994), *Value with Externalities: the Value of the Coalitional Games in Partition Function Form*, Ph.D. Thesis, Department of Economics, State University of New York, Stony Brook, NY, USA.

[5] Feltkamp, V. (1995), *Cooperation in Controlled Network Structures*, Ph.D. Dissertation, Tilburg University, the Netherlands.

[6] Fujinaka, Y. (2004), Axiomatizations of the Shapley Value for Partition Function Form Games, *mimeo*, Kobe University, Japan.

[7] Ju, Y., P.E.M. Borm and P.H.M. Ruys (2004), The Consensus Value: a New Solution Concept for Cooperative Games, *CentER Discussion Paper*, 2004-50, Tilburg University.

[8] Luce, R.D., and H. Raiffa (1957), *Games and Decisions: Introduction and Critical Survey*, New York: John Wiley.

[9] Maskin, E. (2003), Bargaining, Coalitions, and Externalities, *Presidential Address to the Econometric Society.*

[10] Myerson, R.B. (1977), Values for Games in Partition Function Form, *International Journal of Game Theory*, 6, 23-31.

[11] Pham Do, K.H., and H. Norde (2002), The Shapley Value for Partition Function Form Games, *CentER Discussion Paper*, No. 2002-04, Tilburg University, The Netherlands.

[12] Potter, A. (2000), A Value for Partition Function Form Games, *Working Paper*, Dept. of Mathematics, Hardin-Simmons University Abilene, Texas, USA.

[13] Roth, A.E. (1988), *The Shapley Value: Essays in Honor of L. S. Shapley*, Cambridge University Press, Cambridge, UK.

[14] Shapley, L.S. (1953), A Value for n-Person Games, In: Kuhn, H., Tucker, A.W. (Eds.), *Contributions to the Theory of Games*, Vol. II. Princeton University Press, Princeton, NJ.

[15] Thrall, R.M., and W.F. Lucas (1963), n-Person Games in Partition Function Form, *Naval Research Logistics Quarterly*, 10, 281-293.

[16] Zhao, J. (2001), A Characterization for the Negative Welfare Effects of Cost Reduction in Cournot Oligopoly, *International Journal of Industrial Organization*, 19, 455-469.