Bayesian Nash Equilibrium in "Linear" Cournot Models with Private Information About Costs

Sjaak Hurkens
Universitat Pompeu Fabra and CREA
Department of Economics and Business
Ramon Trias Fargas 25-27
Barcelona, Spain

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Abstract

Calculating explicit closed form solutions of Cournot models where firms have private information about their costs is, in general, very cumbersome. Most authors consider therefore linear demands and constant marginal costs. However, even within this framework, some details have been overlooked and the correct calculation of all Bayesian Nash equilibria is more complicated than expected. Moreover, multiple symmetric and interior Bayesian equilibria may exist for an open set of parameters. The reason for this is that linear demand is not really linear. The general "linear" inverse demand function is $P(Q) = \max\{a - bQ, 0\}$ rather than $P(Q) = a - bQ$. In particular, price must be nonnegative.

Key Words: Cournot, Private Information.

1 Introduction.

Cournot models with asymmetric information about demand or costs have been successfully used to study the incentives and consequences of acquiring and sharing of information and of the formation of cartels.[1] Calculating explicit closed form solutions of Cournot models where firms have private information is, in general, very cumbersome. Most authors consider therefore linear demands, linear or quadratic costs and information structures that yield linear conditional expectations. An example of such an information structure is where both the (demand or cost) parameter and the signals received are normally distributed. Within this class of models equilibrium strategies are shown to be linear in the private signal and unique. A problem with this approach is that the linearity of strategies and the unboundedness of the support of the parameters and signals implies that demands, costs, quantities and prices may be negative. This is recognized for example by Vives (1999) when he writes

> The assumption of normality is very convenient analytically but has the drawback that prices and quantities may take negative values.

but is then immediately excused when he goes on to state

> However, the probability of this phenomena can be controlled by controlling the variances of the random variables. Furthermore, [...] there are pairs of prior-likelihood that yield the convenient linear conditional expectation property and avoid the mentioned drawback.

Kirby (1988) makes almost the same observation. Examples of the prior-likelihood pairs Vives and Kirby refer to are beta-binomial and gamma-Poisson. Li (1985) emphasizes the importance of the fact that his results do not require the normality assumption as it allows distributions that

> ... are especially appropriate here, because they may obey the non-negativity constraints on the inverse demand or marginal costs.

It seems these arguments settle the case and the assumption of triple linearity (linear demand, costs and conditional expectations) is both analytically convenient and conceptually satisfying (no negative quantities or prices) when the right assumptions are made about the distributions.

In this paper I consider the Cournot model where each firm has perfect but private information about its own constant marginal cost. Inverse demand is known and linear, except for the kink at zero price: $P(Q) = \max\{1 - Q, 0\}$. That is, I take the nonnegativity of price (as well as quantity) seriously. In this framework one would expect to have the least chance of encountering problems as the one described before, as each firm has perfect information about demand

[1] See for example Clarke (1983), Cramton and Palfrey (1990), Gal-Or (1985, 1986), Hauk and Hurkens (2001), Hurkens and Vulkan (2001), Hwang (1993, 1995), Kirby (1988), Li (1985), Li, McKelvey and Page (1987), Novshek and Sonnenschein (1982), Palfrey (1985), Sakai and Yamamoto (1989), Raith (1996), Shapiro (1986), and Vives (1984, 1988, 1990, 2002).

and its own cost (and therefore, about its own preference). The only uncertainty relevant for a firm concerns the output of its rivals, which is the essence of the classic Cournot model. Even within this framework some details have been overlooked and the correct calculation of all Bayesian Nash equilibria is more complicated than expected. In particular, multiple symmetric and interior Bayesian equilibria may exist for an open set of parameters. The assumptions commonly made in the literature to guarantee interior solutions (and thereby taking seriously that quantities must be nonnegative) do not suffice to guarantee uniqueness. In fact, they do not even suffice to prove that their proposed strategies are equilibrium strategies at all.

Let me illustrate this by means of a simple tri-opoly example. Suppose inverse demand is $P(Q) = \max\{1 - Q, 0\}$. Each firm has marginal cost either equal to zero or to one. The probability of any firm having zero cost is 3/5 and is independent of the cost realization of its rivals. Obviously, any firm that has marginal cost of one must produce zero in any equilibrium. Hence, any symmetric equilibrium will be characterized by the quantity $\underline{x}$ produced by a low cost firm. I claim that both $\underline{x} = 5/16$ and $\underline{x}' = 4/11$ characterize a Bayesian equilibrium. I will prove this claim below. Note that in the first candidate equilibrium total output is at most 15/16 so that price is at least 1/16 and thus strictly positive. In the second candidate equilibrium total output could reach 12/11 so that price may hit the boundary of zero.

Given the strategy $\underline{x}$ for the other two firms, firm 1 will either choose a quantity $x \leq 1 - 2 \times (5/16) = 3/8$ or he will choose a quantity $x \in [3/8, 1/2]$. (Obviously, he will never want to produce more than the monopoly quantity 1/2.) In the first case his payoff equals $x(1 - x - 2\mu)$ where $\mu = (3/5)(5/16)$ is the expected quantity produced per firm. The optimal quantity is thus $x = (1 - 6/16)/2 = 5/16 < 3/8$. His payoff will be equal to $(5/16)^2$. If the firm chooses a quantity $x \geq 3/8$ then his expected profit equals

$$2 \times \frac{3}{5} \times \frac{2}{5} \left(x(1 - x - \frac{5}{16}) \right) + (\frac{2}{5})^2 \left(x(1 - x) \right),$$

which takes into account the zero price and profit in case both rivals happen to have low cost as well. Hence, this profit equals $(-64x^2 + 49x)/100$ which can be shown to be at most equal to $2401/25600 < (5/16)^2$. This completes the proof that $\underline{x}$ constitutes an equilibrium.

Now consider the strategy $\underline{x}'$. Given this strategy for the other two firms, firm 1 will either choose a quantity below $1 - 2 \times (4/11) = 3/11$ or he will choose a quantity $x \in [3/11, 1/2]$. In the first case his payoff equals $x(1 - x - 2\mu')$ where $\mu' = (3/5)(4/11)$ is the expected quantity produced per firm. This profit is strictly increasing for $x \leq 3/11$. Hence, it is optimal for him to choose a quantity $x \geq 3/11$ and his expected profit equals

$$2 \times \frac{3}{5} \times \frac{2}{5} \left(x(1 - x - \frac{4}{11}) \right)) + (\frac{2}{5})^2 \left(x(1 - x) \right),$$

which takes into account the zero price and profit in case both rivals happen to have low cost as well. Hence, this profit equals $(-176x^2 + 128x)/275$ which attains its maximum at $4/11$. This completes the proof that $\underline{x}'$ constitutes an equilibrium.

It may come as a surprise to the reader that multiple symmetric Bayesian equilibria in pure strategies exist in this model. Symmetry often seemed to imply uniqueness in these kind of models. The difference between the two equilibria of the example is rather substantial. In the second equilibrium low cost firms produce almost 12% more, profits are thus lower and consumer and total welfare are higher. Hence, policy analysis with respect to entry or collusion, for example, based on the first equilibrium would be inadequate if the second equilibrium were to be played and vice-versa.

The standard approach in the literature has been to ignore the maximum operator in the inverse demand function and find the equilibrium of the pseudo-game where negative prices are allowed. (In this way one would have found only the first equilibrium in the above example.) If it turns out that all prices are nonnegative given these strategies, one would think that one was fully justified to ignore the nonnegativity constraint on the price.[2] This method is quite similar to the one of ignoring the constraints in maximization (or minimization) problems that are expected to be non-binding and then afterwards verifying that one was justified to do so. However, the example in this section demonstrates that ignoring the nonnegativity constraint on the price is of a different nature, as it affects the objective function and not the constraints on the endogenous quantity variables. There are several problems with the standard approach.

First of all, in the best of cases, the standard approach will give us one equilibrium, but no guarantee of uniqueness. Additional assumptions and further analysis would be required to guarantee uniqueness. Uniqueness of equilibrium has been claimed, among others, by Li (1985, Prop. 6), Shapiro (1986, p. 438) and Vives, (2002, Prop. 1).[3]

Second, the standard approach may yield strategies that are such that for some realization of costs total output is higher than the intercept of demand. In the calculation of the profits in the standard approach negative price results from such outputs. If a firm would realize that price cannot drop below zero, it would in fact prefer to be more aggressive in its output choice when it has low cost. Hence, the strategies found by means of the standard approach are then certainly not equilibrium strategies in the model where the nonnegativity constraint is taken seriously.

Finally, a third problem arises in case the standard approach yields strategies that are such that prices are in fact always strictly positive. It is then clear

[2] Of course, most of us would probably not even check this.

[3] Of course, these claims are correct when the nonnegativity constraint on the price (and quantity) are ignored. However, the assumptions that Li (1985) makes about the distributions and the assumptions that Vives (2002) and Shapiro (1986) make to "guarantee" interior solutions, at the very least, suggest that their claims are valid even when the nonnegativity constraints are taken into account.

that no firm would like to deviate from the strategy by a marginal increase or decrease. However, a firm that realizes that price will not drop below zero may want to increase by a relatively big amount its output. Taking seriously the nonnegativity constraint may render the profit function non-concave, so that first order conditions are not sufficient. One should check whether there are profitable deviations of this type.

It is therefore important to be aware of the fact that the nonnegativity constraint of the price may be binding. It is also important to know whether the existing literature that employs the linear Cournot model with private information has made the mistake of ignoring this constraint, and whether the conclusions they obtained are valid or not.

It should be noted that the example in this section is somewhat special in the sense that firms with high cost do not produce at all. That is, for these firms the nonnegativity constraint on their quantity is binding. Many authors make assumptions that (are supposed to) guarantee interior solutions. Now it could be the case, hypothetically, that the assumptions that guarantee nonnegative quantities for all firms, at the same time guarantee strictly positive prices. If that is the case, then of course nothing is lost by ignoring the possibility of prices that hit the boundary and the established results remain valid. In the remainder of the paper I will investigate this issue.

I will show that with two or three firms, a common assumption that guarantees positive quantities also guarantees strictly positive prices. However, I also show by means of an example with four firms that this does not hold in general. Finally, it is shown that in models with a large number of firms both quantities and prices must hit the boundary for a set of cost realizations that has strictly positive probability. This is relevant when studying the asymptotic properties of the equilibrium when the number of firms tends to infinity. For example, Li (1985) and Vives (1988, 2002) are interested in those limit properties. It is also important in the study of free entry, where the number of firms is determined endogenously by a zero profit condition and entry cost is low. (See again Vives, 2002). For example, suppose parameters are such that Cournot equilibria in which quantities and prices are always strictly positive exist only in case of $n \leq 10$ firms. Suppose moreover that with 10 firms, each firm still makes strictly positive net profit. Then one is forced to consider a Cournot oligopoly with 11 firms, and thus one has to take the nonnegativity constraints seriously.

2 The Model: Standard analysis.

There are $n > 1$ firms competing in quantities in a market for a homogeneous good with inverse demand function $P(Q) = \max\{1 - Q, 0\}$. Each firm i draws a constant marginal cost c_i from a distribution F over the interval $[\underline{c}, \bar{c}] \subseteq [0, 1)$ where $\underline{c} < \bar{c}$. The draws for different firms are assumed to be independent. Each firm is only informed about its own marginal cost and the distribution function F, and this is common knowledge. A pure strategy for firm i is a mapping

$x_i : [\underline{c}, \bar{c}] \to [0, \infty)$, that is, for each realization of its cost, the firm determines a nonnegative output. Given strategies $x_j(\cdot)$ and cost realizations c_j, firm i's profit equals $x_i(c_i)[P(\sum_j x_j) - c_i]$. Firms maximize expected profit.

In the remainder of this section I will first replicate the standard analysis of the "quasi-model" where the nonnegativity constraint on the inverse demand function is ignored. Then I will introduce an obvious necessary (but not yet sufficient) condition for the equilibrium of the quasi-model to be an equilibrium of the true model. I will show that the common explicit assumption made in the literature that firms produce strictly positive quantities for any realization of costs, does *not* imply this necessary condition. I will then show that in the case of duopoly the solution of the quasi-model is the unique solution of the true model under a mild assumption.

The standard analysis of the model goes as follows. First ignore the maximum operator and solve the model in which payoffs are given by

$$\pi_i(x) = x_i(1 - x_i - \sum_{j \neq i} x_j - c_i), \tag{1}$$

where x_k denotes the output of firm k and x denotes the vector of outputs. This yields for all i

$$x_i(c_i) = \max\{(1 - \sum_{j \neq i} \mu_j - c_i)/2, 0\}, \tag{2}$$

where $\mu_j = E[x_j(c_j)] = \int_{\underline{c}}^{\bar{c}} x_j(c_j) dF(c_j)$.

To proceed one needs to focus on either interior solutions or on symmetric solutions. I will start with the interior solutions.

Assuming an interior solution it follows that

$$\mu_i = (1 - \sum_{j \neq i} \mu_j - \tilde{c})/2 \tag{3}$$

where $\tilde{c} = \int_{\underline{c}}^{\bar{c}} c_j dF(c_j)$ denotes average marginal cost. It follows immediately that for all i, $\mu_i = 1 - \sum_{j=1}^{n} \mu_j - \tilde{c}$ and thus

$$\mu_i = \frac{1 - \tilde{c}}{n + 1}. \tag{4}$$

Substituting (4) back into (2) yields finally

$$x_i(c_i) = \frac{2 + (n-1)\tilde{c} - (n+1)c_i}{2(n+1)}. \tag{5}$$

Note that the symmetry of the solution follows from the symmetry of the firms and the assumption of an interior solution, it does not have to be assumed. To ensure existence of an interior solution one needs that $x_i(\bar{c}) > 0$, that is, one needs to assume that

$$2 + (n-1)\tilde{c} - (n+1)\bar{c} > 0. \tag{6}$$

Most authors recognize this and make the adequate assumptions to guarantee the nonnegativity of equilibrium quantities. (See for example Shapiro (1986) and Vives (2002).)

Others do not assume an interior solution and allow for firms to produce zero. (See for example Cramton and Palfrey, 1990). In this case, one can solve for a solution by assuming symmetry and differentiability of the distribution function F.[4] Let $f = F'$ denote the density function. In this case there must exist a cutoff value $\hat{c}$ such that $x_i(c) = 0$ if $c \geq \hat{c}$ and $x_i(c) = (\hat{c} - c)/2$ otherwise. By equation (2), $\hat{c} = 1 - \sum_{j \neq i} \mu_j$ must hold. By integration of (2) one finds

$$\begin{aligned} \mu_i &= \int_{\underline{c}}^{\hat{c}} ((\hat{c} - c)/2) dF(c) \\ &= \int_{\underline{c}}^{\hat{c}} ((\hat{c} - c)/2) f(c) dc \\ &= \left[\frac{\hat{c} - c}{2} F(c) \right]_{\underline{c}}^{\hat{c}} + \frac{1}{2} \int_{\underline{c}}^{\hat{c}} F(c) dc = \frac{1}{2} \int_{\underline{c}}^{\hat{c}} F(c) dc. \end{aligned}$$

The cutoff value $\hat{c}$ is thus uniquely defined by

$$2(1 - \hat{c}) = (n - 1) \int_{\underline{c}}^{\hat{c}} F(c) dc, \tag{7}$$

whenever $2(1 - \bar{c}) \leq (n - 1) \int_{\underline{c}}^{\bar{c}} F(c) dc = (n - 1) \left([cF(c)]_{\underline{c}}^{\bar{c}} - \int_{\underline{c}}^{\bar{c}} cf(c) dc \right) = (n - 1)[\bar{c} - \tilde{c}]$. That is, whenever there is no interior solution there exists a symmetric solution in which firms do not produce for high enough costs.

We summarize the findings in a Lemma.

Lemma 1 *(i) If (6) is satisfied, the quasi-model has a unique equilibrium, which is symmetric and linear, and is given by*

$$x_i(c_i) = \frac{2 + (n - 1)\tilde{c} - (n + 1)c_i}{2(n + 1)}.$$

(ii) If (6) is not satisfied, the quasi-model has a unique symmetric equilibrium, which is given by

$$x_i(c_i) = \max\{(\hat{c} - c_i)/2, 0\}$$

where $\hat{c}$ is implicitly defined by (7).

We conclude that the quasi-model has in any case a unique symmetric equilibrium. The question is now whether this same strategy profile does also constitute an equilibrium of the true model, and if so, whether it will be the unique symmetric equilibrium of the true model.

A necessary condition for an affirmative answer to the first question is that price is always nonnegative when these strategies are used, that is, $\sum_{j=1}^{n} x_j(\underline{c}) \leq$

[4]For some distributions of costs there may exist additional, asymmetric equilibria.

1. Namely, suppose on the contrary that $\sum_{j=1}^{n} x_j(\underline{c}) > 1$, that is, there is a positive probability that price is negative in the quasi-model when a firm has the lowest possible cost. Since the strategy profile is an equilibrium in the quasi-model, it means that an infinitesimal increase of one firm's output above $x_i(\underline{c})$ has no net effect on the payoff. That is, such output expansion would cause a decrease in payoff in the cases where price is negative (as output increases and price becomes even more negative) which is exactly offset by the increase in payoff when price is positive. However, in the true model, the first negative effect is not present (as price cannot drop below zero) and thus there is a strictly positive net gain from increasing the output.

Hence, we obtain immediately

Lemma 2 *(i) A necessary condition for the unique interior solution of the quasi-model to be an equilibrium of the true model is that*

$$(n-1)\tilde{c} - (n+1)\underline{c} \leq 2/n, \tag{8}$$

(ii) A necessary condition for the unique symmetric boundary solution of the quasi-model to be an equilibrium of the true model is that

$$n(\hat{c} - \underline{c}) \leq 2. \tag{9}$$

Note that in the case of a duopoly these conditions are in fact always satisfied since $\tilde{c} \leq \bar{c} < 1$ and also $\hat{c} \leq \bar{c} \leq 1$. Of course, in a duopoly situation none of the firms will produce more than 1/2, the monopoly quantity of a firm with zero cost, so that total output never exceeds one and price will always be strictly positive. This then also implies that in a linear duopoly with private information about constant marginal costs that satisfy (6) the equilibrium is unique. It can also be shown that (in the case of duopoly) when (6) is not satisfied, and thus no interior equilibria exist, that then the equilibrium is unique (as long as the distribution function F is differentiable).

Lemma 3 *In case of a duopoly the (true) model has a unique equilibrium. whenever (6) is satisfied or F is differentiable with $F'(c) > 0$ for all $c \in [\underline{c}, \bar{c}]$.*

Proof. As no firm will produce more than $(1-\underline{c})/2$ price will never drop below zero in the quasi-model and therefore both models have the same equilibria. In case (6) is satisfied, uniqueness of the equilibrium has already been established. If it is not and F is differentiable, then it has been established before that equilibrium strategies must be of the cutoff type, but with possibly different cutoff values $\hat{c}_1 \leq \hat{c}_2$. Hence $x_1(c) = \max\{(\hat{c}_1 - c)/2, 0\}$ and $x_2(c) = \max\{(\hat{c}_2 - c)/2, 0\}$ where $\hat{c}_i = 1 - \int_{\underline{c}}^{\hat{c}_j} (\hat{c}_j - c)/2dF(c)$ and $j \neq i$. Suppose that $\hat{c}_1 < \hat{c}_2$. Then

$$0 > \hat{c}_1 - \hat{c}_2 > \int_{\underline{c}}^{\hat{c}_1} (\hat{c}_2 - \hat{c}_1)/2dF(c) = F(\hat{c}_1)(\hat{c}_2 - \hat{c}_1)/2 > 0.$$

The contradiction shows that only symmetric equilibria will exist and it has already been established that there is only one symmetric equilibrium. ■

However, for oligopolies with more than 2 firms, (8) is not always true. For example, with $n = 3$ the condition reads $\tilde{c} \leq 2\underline{c} + \frac{1}{3}$ which is not satisfied in case $\underline{c} = 0$, $\bar{c} = 1/2$ and $\tilde{c} = 3/8$. Notice that in this case the standard assumption (6) is satisfied. Hence the interior equilibrium of the quasi-model is in this case not an equilibrium of the true model.

Similarly, if cost is uniformly distributed on $[0, 3/4]$, (8) is not satisfied and the quasi-model has only an equilibrium with cutoff strategies. However, in this case one can calculate the cutoff value $\hat{c}$ from (7) to be equal to the positive root of $2y^2 + 3y - 2 = 0$ which is strictly larger than $2/3$. Hence, (9) is violated and the strategies $x_i(c) = \max\{(\hat{c} - c)/2, 0\}$ are not optimal for small values of c in the true model.

A common assumption explicitly made in the literature is that firms would always be willing to produce a positive output.[5] This means that if one firm has the highest possible cost $\bar{c}$ and the remaining $n-1$ firms all have the lowest cost $\underline{c}$, and all of this is known, the high cost firm will still produce. This assumption is equivalent to

$$1 + (n-1)\underline{c} - n\bar{c} > 0, \tag{10}$$

which is easily seen to imply (6). For $n = 3$ it also implies (8) (as $\tilde{c} \leq \bar{c} < (1 + 2\underline{c})/3$ so that $2\tilde{c} - 4\underline{c} < (2 - 8\underline{c})/3 \leq 2/3$). This means that there is some hope that the equilibrium strategies calculated for the quasi-model are also equilibrium strategies for the true model. It is not guaranteed, though, as one would have to check whether a firm would prefer to deviate to a much higher quantity that would provoke price to fall to zero in some circumstances. I will not pursue this investigation here. Namely, for $n > 3$ assumption (10) does certainly not guarantee (8). Consider, for example, the case with $n = 4$, $\underline{c} = 0$, $\bar{c} = 1/5$ and $\tilde{c} > 1/6$. Assumption (10) holds but necessary condition (8) does not.

Hence, the assumptions usually made in the literature are not yet sufficient to guarantee that the strategies calculated above are really equilibrium strategies. In fact, when (8) is not satisfied, these strategies are certainly not equilibrium strategies and some other equilibrium must exist (taking existence for granted for the moment). And even when (8) is satisfied, one cannot yet conclude that the strategies are equilibrium strategies as one has to check for deviations that yield the price equal to zero in some circumstances. That is, condition (8) is necessary but not sufficient. But even if the parameters of the model are such that the strategies calculated are indeed equilibrium strategies, it is feasible (under some parameter conditions) that additional symmetric equilibria exist. Hence the claimed uniqueness is not achieved! I will demonstrate this by means of an example in the next section.

Before I come to the example, let me notice that for large enough n, neither (6) nor (9) will hold (whenever $F(\underline{c}) = 0$).[6] This means that for large n, the

[5] For example, Shapiro (1986, p. 436) and Vives (2002, p. 364) make this assumption.

[6] It is clear that (6) does not hold for large n as the left-hand side of the inequality is strictly

equilibrium cannot be interior and the nonnegativity constraint of the quantities will necessarily bind for some (high) cost parameters. Also, the nonnegativity constraint on prices will necessarily bind for some (low) cost parameters. This implies that a thorough analysis of free entry equilibria must either assume high fixed costs of entry (to keep the number of firms that enter down) or is forced to calculate and use the correct equilibrium strategies taking into account both nonnegativity constraints on prices and quantities. The latter also applies for studies of the competitive limit of oligopoly markets.

3 Leading example

Let $n = 4$ and assume the marginal cost can take only two values, $\underline{c} = 0$ and $\bar{c} = 1/5$. Let p denote the probability of zero marginal costs. For any p, condition (10) is satisfied, and thus also the necessary condition for positive quantities (6) is satisfied. The necessary condition for nonnegative prices (8) is satisfied if and only if $p \geq 1/6$.

This does not mean that for $p \geq 1/6$ the strategies calculated in (5) are in fact equilibrium strategies. This is most easily seen for the case $p = 1/6$. For this parameter value equation (5) yields the strategies $x_i(0) = 0.25$ and $x_i(1/5) = 0.15$. As guaranteed by the conditions, all quantities and prices are nonnegative. In particular, when all four firms have zero cost the prevailing price will be exactly equal to 0. But it is clear that it is not optimal to produce 0.25 when having low cost when we take into account that price cannot become negative. Namely, 0.25 would be optimal when payoffs are really given by equation (1). An infinitesimal increase of this quantity would yield a negative price (and payoff) in the case that all rivals have low cost which would be exactly offset by the increase of payoff in the other cases. When we take into account that price cannot drop below zero, the negative effect disappears while the positive effect remains. This means that a firm would in fact prefer to produce a bit more than 0.25 when it has low cost. It is shown below that the unique symmetric Bayesian equilibrium in this case is given by $x_i(0) = \frac{119}{475} \approx 0.250526$ and $x_i(1/5) = \frac{641}{4275} \approx 0.149942$.

I will now characterize all symmetric Bayesian equilibria in pure strategies. Let $\underline{x}(p)$ denote the quantity of the low cost firm and let $\bar{x}(p)$ denote the quantity of the high cost firm in a symmetric Bayesian equilibrium. For brevity I will sometimes omit the argument p when no confusion can result. The low cost firm will produce more than the high cost firm so $\underline{x}(p) > \bar{x}(p)$. It will be convenient to distinguish six different cases.

Case I $0 < 1 - 4\underline{x}$

Case II $1 - 4\underline{x} \leq 0 < 1 - 3\underline{x} - \bar{x}$

decreasing in n. To see that (9) does not hold for large n is more involved, as the cutoff value $\hat{c}$ is determined implicitly by (7) and depends on n. However, it is easy to see that (9) holds if and only if $2(1 - (\frac{2}{n} + \underline{c})) \leq (n-1) \int_{\underline{c}}^{\underline{c}+2/n} F(c)dc$. In the limit as n tends to infinity the left-hand side converges to $2(1 - \underline{c}) > 0$, while the right-hand side converges to $2F(\underline{c})$.

Case III $1 - 3\underline{x} - \bar{x} \leq 0 < 1 - 2\underline{x} - 2\bar{x}$

Case IV $1 - 2\underline{x} - 2\bar{x} \leq 0 < 1 - \underline{x} - 3\bar{x}$

Case V $1 - \underline{x} - 3\bar{x} \leq 0 < 1 - 4\bar{x}$

Case VI $1 - 4\bar{x} \leq 0$

Clearly, cases V and VI cannot occur as it would imply that the low cost firm always faces a price of zero in equilibrium. Case IV cannot occur either. Namely, in this case the low cost firm would only face a positive price in the case all other firms have high cost and choose $\bar{x}$. The optimal quantity for the low cost firm would therefore be to produce $(1 - 3\bar{x})/2$. Hence, $\underline{x} = (1 - 3\bar{x})/2$. But then $1 - 2\underline{x} - 2\bar{x} = \bar{x} > 0$, which is impossible in case IV. The other cases are more involved and I will study them in turn. Note that in case I price will always be positive, whereas in case II price is positive only if at least one of the four firms has high cost. In case III price is positive only if at east two firms have high cost.

CASE I: $1 - 4\underline{x}(p) > 0$. In this case the equilibrium candidate strategies must be given by (5). That is

$$\underline{x}(p) = \frac{13 - 3p}{50}, \bar{x}(p) = \frac{8 - 3p}{50}. \tag{11}$$

Clearly, one necessary condition is that $p > 1/6$ so that $\underline{x}(p) < 0.25$ and $\bar{x}(p) < 0.15$. Note that the expected payoff to a low cost firm will be equal to $\underline{x}(p)^2$.

I need to verify that the low cost firm will not want to deviate to a quantity $x \in (1 - 3\underline{x}, 1 - 2\underline{x} - \bar{x})$. The payoff function for the low cost firm for quantities x in this interval equals

$$\begin{aligned}\pi_1(x) \quad = \quad & 3p^2(1-p)x(1 - x - 2\underline{x} - \bar{x}) + 3p(1-p)^2 x(1 - x - \underline{x} - 2\bar{x}) \\ & +(1-p)^3 x(1 - x - 3\bar{x}).\end{aligned} \tag{12}$$

This function is concave in x and attains its maximum at

$$\begin{aligned} x_1 \quad &= \quad \frac{3p^2(1 - 2\underline{x} - \bar{x}) + 3p(1-p)(1 - \underline{x} - 2\bar{x}) + (1-p)^2(1 - 3\bar{x})}{2(3p^2 + 3p(1-p) + (1-p)^2)} \\ &= \quad \frac{1 + p + p^2 - 3\bar{x} - 3p(1+p)\underline{x}}{2(1 + p + p^2)} \\ &= \quad \frac{26 + 20p + 20p^2 + 9p^3}{100(1 + p + p^2)} \end{aligned} \tag{13}$$

Note that $x_1 > 1 - 3\underline{x}(p)$ if and only if $p < 0.169194$. The payoff obtained from using this quantity equals $(1 - p^3)(x_1)^2$ which is less than or equal to $\underline{x}(p)^2$ if and only if $p \geq \hat{p} \approx 0.167902$ where $\hat{p}$ is the real root in $(0, 1)$ of $(1 - p^3)x_1^2 = \underline{x}(p)^2$. Hence, for $p \geq \hat{p}$ the low cost firm will not want to deviate to any $x \in (1 - 3\underline{x}, 1 - 2\underline{x} - \bar{x})$.

Next I need to verify that the low cost firm will not want to deviate to a quantity $x \in (1 - 2\underline{x} - \bar{x}, 1 - \underline{x} - 2\bar{x})$. The payoff function for the low cost firm for quantities x in this interval equals

$$\pi_2(x) = 3p(1-p)^2 x(1 - x - \underline{x} - 2\bar{x}) + (1-p)^3 x(1 - x - 3\bar{x}). \quad (14)$$

This function is concave in x and attains its maximum at

$$\begin{aligned} x_2 &= \frac{3p(1-p)^2(1 - \underline{x} - 2\bar{x}) + (1-p)^3(1 - 3\bar{x})}{2(3p(1-p)^2 + (1-p)^3)} \\ &= \frac{1 + 2p - 3(p+1)\bar{x} + 3\underline{x}}{2(1+2p)} \quad (15) \\ &= \frac{13 + 23p + 9p^2}{50(1+2p)}. \end{aligned}$$

However, it can be verified that $x_2 < 1 - 2\underline{x}(p) - \bar{x}(p)$ for any value of p. Hence, such a deviation will not be optimal.

It is then also not optimal for the low cost firm to deviate to an even higher quantity above $1 - \underline{x} - 2\bar{x}$.

It remains to be verified that the high cost firm will not want to deviate to a quantity above $1 - 3\underline{x}(p)$. But this is clear as $1 - 3\underline{x}(p) \geq 2/5$ for all p and $2/5$ is in fact the monopoly quantity of the high cost firm.

I conclude that the strategies in (11) constitute an equilibrium if and only if $p \geq \hat{p} \approx 0.167902$.

CASE II: $1 - 4\underline{x}(p) \leq 0$ and $1 - 3\underline{x}(p) - \bar{x}(p) > 0$.

In this case the low cost firm's best reply $\underline{x}$ must be in the interval $(1 - 3\underline{x}, 1 - 2\underline{x} - \bar{x})$. As seen before, for quantities x within this interval the firm's profit function is given by $\pi_1(x)$ in (12). Thus, by (13) it follows that

$$\underline{x} = \frac{1 + p + p^2 - 3\bar{x} - 3p(1+p)\underline{x}}{2(1 + p + p^2)}. \quad (16)$$

The high cost firm will have the reaction function derived in equation (2) and thus

$$\bar{x} = (1 - 3p\underline{x} - 3(1-p)\bar{x} - 1/5)/2. \quad (17)$$

The candidate equilibrium is thus the solution of the system of equations (16) and (17) which gives

$$\bar{x}(p) = \frac{8 + 5p + 5p^2 - 15p^3}{25(2 + 2p + 2p^2 - 3p^3)} \quad (18)$$

$$\underline{x}(p) = \frac{13 + 10p + 10p^2 - 15p^3}{25(2 + 2p + 2p^2 - 3p^3)}. \quad (19)$$

A necessary condition for $\underline{x} \geq 1/4$ is $p \leq 0.176991$. For any p we have that $1 - 3\underline{x}(p) - \bar{x}(p) > 0$.

I have to check that the low cost firm does not want to deviate to a quantity below $1 - 3\underline{x}(p)$. The best of such low quantities would be equal to $x' = (1 - 3(p\underline{x}(p) + (1-p)\bar{x}(p))/2$. This is less than $1 - 3\underline{x}(p)$ only for $p > 0.173746$. So for $p < 0.173746$ the low cost firm will certainly not deviate in this fashion. For intermediate values of p I have to compare the payoffs. Deviating yields no more than sticking to $\underline{x}(p)$ if $(x')^2 \leq (1-p^3)\underline{x}(p)^2$. That is, for $p \leq \tilde{p} \approx 0.175322$ where $\tilde{p}$ is the real root in $(0,1)$ of $(x')^2 = (1-p^3)\underline{x}(p)^2$.

I also have to check that the low cost firm does not want to deviate to a quantity $x \in (1-2\underline{x}-\bar{x}, 1-\underline{x}-2\bar{x})$. For such quantities the firm's profit function is equal to $\pi_2(x)$ as defined in (14). The maximum of this function is obtained by x_2 as defined in (15). Substituting the candidate equilibrium strategies for this case II yields

$$x_2 = \frac{26 + 72p + 90p^2 + 25p^3 - 60p^4}{50(2 + 6p + 6p^2 + p^3 - 6p^4)}.$$

It can be verified that $x_2 < 1 - 2\underline{x}(p) - \bar{x}(p)$ for all p so that such deviations are not optimal. It also follows that deviating to an even higher quantity is not optimal either.

It remains to be shown that the high cost firm will not want to deviate to a quantity above $1 - 3\underline{x}$. It is clear that no quantity above $(1 - 3\bar{x} - 1/5)/2$ will be chosen as expected total output of the other three firms is certainly above $3\bar{x}$. For $p \leq 0.2$, $\bar{x}(p) > 0.147$ so that $(1 - 3\bar{x} - 1/5)/2 < 0.1795$. On the other hand, for the same parameter range $\underline{x}(p) < 0.26$ so that $1 - 3\underline{x} > 0.22$. Hence, the high cost firm will certainly not deviate.

I conclude that the strategies in (16) and (17) constitute an equilibrium if and only if $p \leq \tilde{p} \approx 0.175322$.

CASE III: $1 - 3\underline{x} - \bar{x} \leq 0 < 1 - 2\underline{x} - 2\bar{x}$

Around the equilibrium strategies profit for the low cost firm is given by $\pi_2(x)$ while the payoff for the high cost firm is given by $\pi_1(x) - x/5$. It follows that an equilibrium in this case is characterized by

$$\begin{aligned} \underline{x} &= \frac{1 + 2p - 3(p+1)\bar{x} + 3\underline{x}}{2(1+2p)} \\ \bar{x} &= \frac{4 - 5p^3 - 15(1-p)\bar{x} - 15(1-p^3)\underline{x}}{10(1-p^3)} \end{aligned}$$

from which I find

$$\begin{aligned} \underline{x}(p) &= \frac{-13 - 23p + 30p^2 - 5p^3 + 5p^4}{5(1-p)(5 - 9p + 12p^2 + p^3)} \\ \bar{x}(p) &= \frac{4 - p + 30p^2 - 20p^3 - 10p^4}{5(1-p)(5 - 9p + 12p^2 + p^3)} \end{aligned}$$

However, this gives negative values of $\underline{x}(p)$ for any value of p. Hence, there exists no equilibrium in this case.

4 Conclusion

The linear Cournot model with private information about costs is widely used in the literature. Usually boundary conditions are either ignored or assumed away. However, the assumptions made only concern the nonnegativity constraints for quantities. In this note I have shown that also the nonnegativity constraint for price has to be taken into account. The main example shows that multiple pure, symmetric and interior Bayesian Nash equilibria exist for some parameters. Moreover, it has been shown that in the case of many firms, the assumption that all firms will produce strictly positive quantities cannot hold in any equilibrium. This is especially relevant for studies of free entry, of information aggregation, information acquisition and information sharing, and of the study of collusion.

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