

Dynamic Accumulation in Bargaining Games

Francesca Flamini ^(*, ‡)

Department of Economics

University of Glasgow

Abstract: In many bargaining situations the decisions that parties take at one point in time affect their future bargaining opportunities. We consider an ultimatum bargaining game in which parties can decide not only how to share a current surplus but also how much to invest in order to generate future surpluses. We show that there is a unique Markov perfect equilibrium (MPE) in which a proposer consumes the whole surplus not invested. Moreover, when the proposer has a sufficiently high discount factor, his MPE investment level is higher than his opponent's, for a given capital stock. Finally, we show that bargaining can lead to underinvestment.

JEL classification: C61, C72, C73, C78.

Keywords: dynamic accumulation, bargaining, recursive optimisation.

* Francesca Flamini, Department of Economics, University of Glasgow, Adam Smith Building, G12 8RT, Glasgow UK, email: f.flamini@socsci.gla.ac.uk, Tel no. +44 (0)141 330 4660, Fax no. +44 (0)141 330 4940.

‡ I would like to thank Robert Evans, Campbell Leith, Chol-Won Li, Marco Mariotti and the participants of the seminars at Edinburgh University, Glasgow University and the participants of the 10th International Symposium on Dynamic Games and for very useful suggestions. All errors remain mine.

1 Introduction

In many bargaining situations, the decisions that parties take at one point in time affect the size of future surpluses. For instance, members of a household decide not only how to allocate current consumption among themselves but also how much to save for tomorrow's consumption. Partners in a business need to agree how to share current profits among themselves, and also whether and how any remaining profit should be re-invested. Political parties attempt to find agreement over an issue by taking into account the fact that their current decisions can increase goodwill and facilitate agreement over issues in the future. Colluding firms can negotiate over how to share current profits between themselves, while also investing in order to generate profits in the future.

The aim of this paper is to investigate bargaining games characterised by a dynamic accumulation problem, that is, players attempt to agree not only on how to share a surplus but also on the level of investment, which will then affect the capital stock and the amount of future surpluses. The capital stock we have in mind does not need to be physical, but can simply reflect the value of the ongoing relationship itself (e.g., it can represent goodwill generated by the agreement between political parties on some issues). This implies that our framework can be applied to several bargaining situations.

A dynamic accumulation problem combined with a bargaining process is almost unexplored in economics. As far as we know, only Muthoo (1999, section 10.3) has considered the problem of an alternating bargaining process combined with a dynamic accumulation game. However, he analyses it as a possible application of an infinitely repeated game in which two parties share an infinite series of cakes of constant size (Muthoo, 1995). In particular, the investment problem is strongly simplified since the focus is on steady-state stationary subgame perfect equilibria and therefore, players

should invest so as to have the same surplus in each period. For this reason, the problem of identifying investment paths in bargaining games remains open.

We attempt to address this question, by analysing an ultimatum bargaining framework in which players are able to invest part of the surplus. By introducing the possibility of accumulation, the game is modified in a non-trivial way, in particular, it can be non-stationary (by stationary game we mean a game characterised by the same subgame at specific nodes). This implies that the computation of an equilibrium is not as simple as in the case of a stationary game. Moreover, there can be multiple equilibria (indeterminacy of equilibria due to non-stationarity has been shown by, for instance, Binmore, 1987).

The equilibrium concept on which we focus is the Markov subgame perfect equilibrium (MPE). Following Fudenberg and Tirole (1990), an MPE is a subgame perfect equilibrium in which players' strategies are restricted to depend on the past history of play through the state variable (i.e., Markov strategies). In our case, the state variable is the capital stock, k_t (plus the rules of the game that define the identity of the proposer). As Maskin and Tirole (2001) point out, this means that only those aspects of the past that are significant should have an appreciable influence on behaviour. Moreover, Markov strategies represent the simplest form of behaviour that is consistent with rationality (Maskin and Tirole, 2001).

The main result of our analysis is that a unique MPE exists. This is characterised by demands in which the proposer obtains the entire surplus not invested and invests more than his opponent would have done provided that the proposer is sufficiently patient. Within an ultimatum framework a proposer can focus on his intertemporal optimisation by recognising that in the future he can propose again (with a positive probability and his opponent can propose with the complementary probability). Indeed, an acceptance is

always obtained, when a rejection implies the end of the game. However, such an equilibrium is not specific to ultimatum bargaining, and it can also be sustained under alternative bargaining procedures (even, a potentially infinite bargaining stage, although in this case other MPE can exist). The assumption of an ultimatum bargaining procedure (in which a rejection implies the end of the game) is restrictive but it allows us to fully derive the results. This is a first step to the solution of more interesting bargaining procedures.

Moreover, we show that bargaining leads to underinvestment. The intuition is as in the classic hold-up problem although in this case we have a dynamic set-up. In particular, since a proposer can extract the entire surplus not invested, he can fund the current level of investment by reducing his own consumption. In this sense he bears all the cost of investing. Additionally, the proposer will take into account the fact that in the future his opponent enjoys a positive probability of being a proposer and therefore today's proposer needs to take into account that he will share the benefits of his initial investment with the other player. Within the context of the social planner's problem the incentives are different for two reasons. First, both players share the cost of the investment, second, they equally share the benefits of this investment (under the assumption of symmetric players). This is why the social planner's investment path is higher.

In the following section the model is presented and solved. We then consider the optimum for the social planner (in section 3). We discuss alternative bargaining procedures in section 4. Section 5 concludes this paper.

2 The Game

Two players (for instance, firms, political parties, members of a household, etc.), named 1 and 2, engage in the production of a surplus and its division between themselves. In particular, having produced a surplus from a given capital stock, the two players bargain over how much to invest and how to share the consumption of the remaining surplus between themselves. The level of investment affects the future capital stock and, consequently, the surplus available in the following bargaining stage.

The game consists of two distinct phases: a production and a bargaining stage. Each phase can only start when the other has finished. We assume that the bargaining stage is characterised by an ultimatum procedure (only one round). Therefore, while the bargaining stage is ‘very quick’ (indeed, to simplify, is assumed to be timeless), production takes time. The parameter τ indicates the interval of time required to generate surplus. A time period is indicated by t with $t = 0, 1, \dots, \infty$ (in other words, t indicates the number of production stages which have taken place). A surplus is generated according to the production function $F(k_t) = k_t^\rho$, with $0 < \rho \leq 1$, where k_t is the capital stock at period t . Once the output is generated, $F(k_t)$, the bargaining stage begins, in which players attempt to divide $F(k_t)$. Player i 's time preference is represented by a *between-cake* discount factor $\alpha_i = \exp(-r_i\tau)$, where r_i is his discount rate. In the first period, $t = 0$, a bargaining stage starts. The surplus available is 1, by assumption. In general, in period t player 1 (2, respectively) can become a proposer with probability¹ p ($1-p$, respectively), with $0 < p < 1$. A proposal by player i is a pair $({}_i x_t, {}_i I_t)$, where ${}_i I_t$ is the investment level proposed by i at time t and ${}_i x_t$ is the share demanded by i over the remaining surplus at time t . The proposal ${}_i x_t, {}_i I_t$ depends on capital, denoted by

k_t . Our notation is simplified, in the sense that the subscript t in the proposal indicates that this is conditional on the capital stock at t , k_t .

On the one hand, if the proposal is accepted, the bargaining stage ends and the proposer's current per-period utility is,

$$u_i(x_t, iI_t, k_t) = [x_t(F(k_t) - iI_t)]^{1-\eta}/(1-\eta), \quad (1)$$

with $0 < \eta < 1$. This is a general utility function: for η that goes to one, the per-period utility function tends to the logarithmic case, while for η that tends to zero, the utility becomes linear in consumption (the assumption of linear utility, $\eta=0$, common in bargaining games, must be excluded for the existence of a solution of the intertemporal optimisation problem, see Ljungqvist and Sargent, 2000). Once players agree over the division of a surplus, production takes place. The output available at the next bargaining stage (at $t + 1$) is $F(k_{t+1})$, where k_{t+1} is the capital stock in the next period and it is given by the agreed level of investment iI_t and the capital remaining after depreciation, $k_{t+1} = iI_t + (1-\lambda) k_t$, where λ is the depreciation rate ($0 < \lambda < 1$).

On the other hand, if there is a rejection, the game finishes and the players get (almost²) zero payoffs. Therefore, in the case of rejection it is not only the surplus $F(k_t)$ that disappears but also the capital stock k_t . For instance, suppose that two colluding firms get zero profits if they cease to collaborate. One reason could be that they compete à la Bertrand and there is no second-hand market for k_t . Then, if the firms do not have frequent contacts, tacit collusion might be difficult to sustain. We can assume that as soon as one of the colluding firms rejects a proposal, the collaborative relationship is

¹ Similar results can be established if the bargaining stage is characterised by an alternating-offer rather than a random-proposer procedure. These results are available from the author upon request.

compromised. We can also think of the capital stock k_t as not being physical but simply reflecting the investment in the relationship itself. For instance, suppose a country is enjoying a recently established democracy. Failure of the new political parties to reach agreement on political issues may threaten the viability of the democracy itself.

The case in which, after a rejection, only the surplus at that stage disappears ($F(k_t)$), but not the capital (k_t) is interesting but technically more demanding and we discuss it in section 3. We now turn to discuss the MPE's in the case in which a rejection implies the end of the game.

2.1 *The Equilibrium*

First of all, if an MPE exists in this framework, it must be without delay (if an MPE with delay is assumed to exist, then it can be shown that a profitable deviation exists, so that the strategies which defined such a delay cannot sustain an MPE). To define an MPE, we need to solve the following problem. The proposer at time t , say i (with $i = 1, 2$), will maximise his expected discounted utility, $v_i(k_t)$, with respect to the share of the surplus he demands for his consumption, ${}_i x_t$, and the amount of the surplus to be invested, ${}_i I_t$, with $0 \leq {}_i x_t \leq 1$ and $0 \leq {}_i I_t \leq F(k_t)$. Player i 's expected discounted utility at time t is given by his current per-period utility, $u_i({}_i x_t, {}_i I_t, k_t)$, plus the future expected utility, $E_s \sum_s \alpha_i^{t+s} u_i({}_z x_{t+s}, {}_z I_{t+s}, k_{t+s})$, where the per-period utility $u_i({}_z x_{t+s}, {}_z I_{t+s}, k_{t+s})$ is defined in (1), with z proposing at $t+s$, $z = 1, 2$ and $s = 1, 2, \dots$. The expectation is taken with respect to the probabilities of becoming a proposer at $t+s$. Since the identity of a proposer at one point in time affects the future investment strategies via the agreed level

² As long as players outside option is sufficiently small, and therefore not binding, the analysis below still holds.

of investment (and therefore the capital stock for next period), the explicit form of $v_i(k_t)$ is as follows,

$$u_i(x_t, I_t, k_t^{h^t}) + \sum_{p=1}^{\infty} \sum_{s=2^{p-1}}^{2^p-1} \left(\prod_{h \in h^{t+s} \setminus h^t} h \right) \alpha_i^{t+s} [p_i u_i(x_{t+s}, I_{t+s}, k_{t+s}^{h^{t+s}}) + (1-p_i) u_i(x_{t+s}, I_{t+s}, k_{t+s}^{h^{t+s}})] \quad (2)$$

where each element of a potential history h^t , h , is equal to p_i assuming player i proposes and $(1-p_i)$ when player j is assumed to propose (except for the unique element of $h^{t+1} \setminus h^t$, that is 1 by assumption) for $t = 1, 2, \dots, \infty$. The potential history h^t uniquely indicates the sequence of proposers to reach the node t from 1, where the nodes considered are only the ones in which an offer is to be made. These are numbered sequentially from 1. At each node there are two possibilities either i or j will propose next, in each period the lowest number is given to the node where i will propose next. Accordingly, the product of the elements of h^t ($\prod h$) gives the probability of reaching node t from node 1, while h^t as a superscript of k_t indicates the actual history of proposals which defines the capital stock at time t .

Similarly, the expected discounted utility of a responder who accepts player i 's proposal (x_t, I_t) , i.e., $w_i(k_t)$, can be written in the following form,

$$u_j(x_t, I_t, k_t^{h^t}) + \sum_{p=1}^{\infty} \sum_{s=2^{p-1}}^{2^p-1} \left(\prod_{h \in h^{t+s} \setminus h^t} h \right) \alpha_j^{t+s} [p_i u_j(x_{t+s}, I_{t+s}, k_{t+s}^{h^{t+s}}) + (1-p_i) u_j(x_{t+s}, I_{t+s}, k_{t+s}^{h^{t+s}})] \quad (3)$$

Player i 's proposal is accepted immediately if and only if the responder obtains at least as much as he would get in the case of rejection, which is zero in this case (i.e., $w_j(k_t) \geq 0$). Then, a proposer's maximisation problem is given by,

$$\begin{aligned}
& \max_{\substack{0 \leq x_t \leq 1, \\ 0 \leq I_t \leq f(k_t)}} \frac{[{}_i x_t (F(k_t) - {}_i I_t)]^{1-\eta}}{1-\eta} + \alpha_i ((1-p_i)w_i(k_{t+1}) + p_i v_i(k_{t+1})) \\
& \text{s.t. } w_j(k_t) \geq 0 \\
& \text{with } k_{t+1} = \begin{cases} k_t(1-\lambda) + {}_i I_t & \text{in case of acceptance} \\ 0 & \text{otherwise} \end{cases} \\
& \text{and } p_i = 1-p_j, p_1 = p \text{ for } i \neq j, \text{ and } i, j = 1, 2.
\end{aligned}$$

Since an MPE is characterised by a time invariant rule mapping the state variable k_t into the decision variables ${}_i x_t$ and ${}_i I_t$, then the problem can be written in a recursive form (the Bellman equation). A proposer's optimisation problem becomes,

$$\begin{aligned}
V_i(k_t) &= \max_{\substack{0 \leq x_t \leq 1, \\ 0 \leq I_t \leq k_t}} \frac{[{}_i x_t (F(k_t) - {}_i I_t)]^{1-\eta}}{1-\eta} + \alpha_i ((1-p_i)W_i(k_{t+1}) + p_i V_i(k_{t+1})) \quad (4) \\
k_{t+1} &= \begin{cases} k_t(1-\lambda) + {}_i I_t & \text{in case of acceptance} \\ 0 & \text{otherwise} \end{cases} \\
& \text{and } p_i = 1-p_j, p_1 = p \text{ for } i \neq j, \text{ and } i, j = 1, 2; \\
\text{s.t. } W_j(k_t) &\geq 0 \quad (5)
\end{aligned}$$

where $V_i(k_t)$ is the optimum $v_i(k_t)$ or *value function*, and $W_i(k_t)$ is the optimal expected utility when player i is a responder, $w_i(k_t)$. Condition (5) guarantees that the proposal at time t is accepted immediately. However, this is always satisfied since ${}_i x_t$ belongs to $[0,1]$ and ${}_i I_t$ to $[0, k_t]$. The indifference conditions are important instruments in deriving the solution of bargaining problems with a stationary structure. In our (non-stationary) game, the indifference conditions, which are (5) as an equality, cannot hold (unless the between cake discount factors α_i are zero). When there is accumulation a responder is

able to obtain a positive surplus at some point in the future, and therefore his optimal expected utility $W_i(k_t)$ can be strictly positive³.

Since any investment decision made by a player at time t , affects the whole stream of future profits (by the equation of motion of the capital stock), the bargaining stages are strongly interconnected even within a simplified bargaining structure such as the ultimatum framework. Given the ultimatum structure, the focus is on an MPE in which a proposer is able to consume the whole portion of the surplus not invested, in other words a proposer's optimisation problem is as follows,

$$V_i(k_t) = \max_{0 \leq I_t \leq k_t} \frac{(F(k_t) - I_t)^{1-\eta}}{1-\eta} + \alpha_i((1-p_i)W_i(k_{t+1}) + p_i V_i(k_{t+1})) \text{ with} \quad (6)$$

$$k_{t+1} = k_t(1-\lambda) + I_t \text{ in case of acceptance and } 0 \text{ otherwise} \quad (7)$$

To solve the problem we use the *guess and verify* method. This consists in 'guessing' the form of the value function but leaving the coefficients undetermined, and then 'verifying the guess' by showing that there are values of the coefficients that make the guess correct. The *guess and verify* method relies on the uniqueness of the solution. This is ensured by the assumption of concave utility functions, a linear production function ($F(k_t) = k_t$) and a linear equation of motion (see Stokey, Lucas and Prescott, 1993 or Levhari and Srinivasant, 1969). Our 'guess' is that the value function is a function of the capital stock of the same form as the utility function. Then the players' optimisation problem can be written as follows,

³ In our framework, the equilibrium shares x_t cannot be larger than one. In other words, we exclude the case in which a proposer can extract all his opponent's expected discounted utility in the case of acceptance, $W_j(k_t)$. This case, even though interesting, complicates the analysis, because it requires

$$\phi_i k_t^{1-\eta} / (1-\eta) = \max_{c_t} c_t^{1-\eta} / (1-\eta) + \alpha_i \beta_i k_{t+1}^{1-\eta} / (1-\eta) \text{ w.r.t. } c_t, \quad (8)$$

$$\text{where } k_{t+1} = k_t(2 - \lambda) - c_t \text{ if there is an acceptance, } 0 \text{ otherwise.} \quad (9)$$

$$\beta_i = p_i \phi_i + (1 - p_i) \mu_i \quad (10)$$

with c_t equal to the consumption level proposed by player i (i.e., $k_t - c_t$) and $p_j = 1 - p_i$, p_i in $[0,1]$, with $i = 1, 2$. The coefficients ϕ_i and μ_i , and consequently β_i with $i = 1, 2$, are undefined at this point. The expected maximum discounted utility of a responder at time t , $\mu_i k_t^{1-\eta} / (1-\eta)$ is given by,

$$\mu_i k_t^{1-\eta} / (1-\eta) = 0 + \alpha_i [(1 - p_i) \mu_i / (1-\eta) + p_i \phi_i / (1 - \eta)] (\varphi_j k_t)^{1-\eta} \quad (11)$$

where $\varphi_j k_t$ is the capital stock at time $t + 1$, after player j has been a proposer at time t .

The FOC for the problem (8) – (10) is the following,

$$c_t^{-\eta} - \alpha_i \beta_i (k_t(2-\lambda) - c_t)^{-\eta} = 0, \quad (12)$$

which implies,

$$c_t = (2-\lambda) k_t / (1 + \alpha_i^{1/\eta} \beta_i^{1/\eta}). \quad (13)$$

Optimal consumption is a linear function of the capital stock. Moreover, if the expected utility in the continuation game ($\alpha_i \beta_i$) is higher for a given level of capital stock, a proposer consumes less and invests more. Using (13), the equation of motion can be written as,

another state variable apart from the capital stock to keep track of the responder's expected payoff before making an offer.

$$k_{t+1} = \varphi_i k_t, \text{ where } \varphi_i = (2-\lambda) \alpha_i^{1/\eta} \beta_i^{1/\eta} / (1 + \alpha_i^{1/\eta} \beta_i^{1/\eta}). \quad (14)$$

After the *guess*, it is necessary to verify that there is a solution for the coefficients, ϕ_i, μ_i such that they are well defined. By using (13), we obtain two equations, (10) and (11), in the two undefined coefficients, ϕ_i and μ_i . Then, if there is a solution to this system, the guess was right and the verification phase ends. The solution of the system (10) and (11) also gives the optimal investment path. Moreover, this solution is unique. We show the conditions under which the solution exists. This is sufficient to complete the verification.

The solution to the problem is completed with another constraint, the so-called *transversality* condition, which imposes that at the limit as t tends to zero, the utility value of the discounted capital stock ($\alpha_i^t k_t dV_i(k_t)/dk_t$) goes to zero. This is as follows,

$$\lim (\alpha_i)^t \phi_i k_t^{1-\eta} = 0 \text{ as } t \rightarrow \infty \text{ for any } i = 1, 2. \quad (15)$$

The solution to the problem (8) – (10) given the constraint (15) is characterised by the following remark and proposition.

Remark 1: *In this game, a unique MPE can be established under the conditions identified in Appendix A.*

Proposition 1: *In the unique MPE, there is a player who invests more, for a given capital stock k_t . This player is characterised by possessing a sufficiently high between-cake discount factor.*

Proof: From Appendix A, in general in equilibrium there is a player, say i , such that $\alpha_i > \alpha_j \psi_j / \psi_i$. From (A.5) in Appendix A, this implies that $\phi_i > \phi_j$. Then, from the FOC, player i has a smaller consumption level than j , for a given capital stock k_t , if $\alpha_i \beta_i > \alpha_j \beta_j$. But this inequality holds, since $\beta_i = \psi_i \phi_i$ and $\alpha_i \psi_i \phi_i > \alpha_j \psi_j \phi_j$. Then player i successfully proposes to invest more for a given capital stock, when $\alpha_i > \alpha_j \psi_j / \psi_i$.

□

Given the ultimatum procedure, a proposer can optimise his expected utility without fearing a rejection in equilibrium. Since in general there is a player who minds relatively more about the future, such a player invests more than his opponent, given the capital stock k_t . As a result the growth path is higher and when at one point in the future this player will propose again he will be able to extract a larger surplus.

In spite of the simplicity of the bargaining structure, an explicit solution of the accumulation problem is not straightforward. However, the existence, uniqueness and characterisation of the MPE can be established. Moreover, for the special case of $\eta = 1/2$ a more detailed analysis can be done. We conclude this section with the results of the model when players still have concave utility function but in a specific form, $\eta = 1/2$.

Lemma 1: *In the case of symmetry, that is $\alpha_i = \alpha$, $\delta_i = \delta$ and $p_i = 1/2$ for any i , with $\eta = 1/2$, for $\alpha(2-\lambda)^{1/2} < 1$, there is a unique MPE. This is characterised by the investment plan (13) with coefficients defined as follows,*

$$\phi_i = [(2 - \lambda)/(1 - (2-\lambda)\psi_i^2 \alpha^2)]^{1/2}, \quad (16)$$

$$\beta_i = \psi_i \phi_i \text{ and } \mu_i = \alpha^2 \phi_i (2-\lambda) \psi_i^2, \quad (17)$$

where $\psi_1 = \psi_2 = [1 - (1 - \alpha^2(2-\lambda))^{1/2}]/\alpha^2(2-\lambda) < 1$.

Proof: Appendix B.

□

In the case of symmetry, the unique MPE is symmetric. Moreover, when production becomes quicker (α increases) a proposer consumes less and invests more (by using (13), (16) and (17), since the solution (ψ_1, ψ_2) is an increasing function of the between-cake discount factor α). Indeed, in this case the future becomes more important to players and therefore the investment path is higher so that in the future a proposer can extract larger surpluses.

When the symmetry assumption is relaxed the computation of the equilibrium (in particular ψ^*_j for $j=1,2$) is less straightforward (in particular, if a component of the pair (ψ^*_1, ψ^*_2) belongs to T_i , the other component may not belong to T_j , with $i, j = 1,2, i \neq j$, and then the equilibrium is undefined). However, even in the case of an extreme asymmetry between players, a simple example in which there is still a solution, is when at the limit, one of the players, say 1, has a very high probability to propose, p_i tends to 1. In this case, the unique real solution is defined by $\psi_i = p_i$ with $i = 1,2$, where the relevant interval for ψ^*_i is T_i equal to $[p_i, \min\{((2-\lambda)^{1/2}\alpha_i)^{-1}, ((2-\lambda)\alpha_j\alpha_i)^{-1}\})$, with λ sufficiently high, $(2-\lambda)^{1/2}\alpha_i < 1$, for $i = 1,2$. Indeed, the overall payoff to player 1 as a proposer is as the payoff in the continuation game, for a given capital stock (i.e., $\beta_i = \phi_i$). If player 2 becomes a responder (even though with a probability that tends to zero) his expected payoff is zero ($\mu_2 = 0$).

2.2 A Note on the Linear-Quadratic Form

The most studied recursive optimisation problems are linear quadratic, (i.e., the constraint is a linear function of the state variable and the per-period objective function

is quadratic), since such problems are characterised by a simple solution (the ‘guess’ is not required, as the solution is known to be linear, see Ljungqvist and Sargent, 2000). Our bargaining problem with dynamic accumulation can also be transformed into the linear quadratic form. Indeed, the model can be expressed in terms of differences between the actual and an unreachable target and players minimise a quadratic loss function with respect to the difference between the actual level of consumption and a target level. It can be shown that this transformation does not change the qualitative results established in Remark 1 and Proposition 1 (see Flamini, 2002). The only simplification is in the analytical derivation of the conditions for the existence of a solution. However, since in classic bargaining theory players have concave/linear utility functions rather than quadratic loss functions, we prefer to maintain the standard framework.

3 The Social Planner’s Accumulation Plan

In this section, we show that if there is a social planner who is able to choose the consumption and investment levels for the two players, he would always invest more than the non-cooperative players. Moreover, instead of leaving the whole consumable surplus to a player, the social planner would divide it equally between players (given the symmetry of players’ per-period utility).

To prove this we first solve the social planner’s optimisation problem. We then compare the social optimum with the MPE investment level. To simplify the problem players are symmetric. The social planner’s optimisation problem is a classic growth problem, where the optimal consumption plan is to divide the surplus equally between

the players (since the players are symmetric). The optimum investment plan is given by the solution of the following recursive problem,

$$\phi_s \frac{k_t^{1-\eta}}{1-\eta} = \max_{I_t} \frac{(k_t - I_t)^{1-\eta} 2^\eta}{1-\eta} + \alpha \phi_s \frac{k_{t+1}^{1-\eta}}{1-\eta} \quad (18)$$

$$k_{t+1} = k_t(1-\lambda) + I_t \quad \text{with } i \neq j \text{ and } i, j = 1, 2$$

where, ϕ_s is the undefined coefficient of the value function related to the social planner's problem. This is a standard recursive accumulation problem. It can be shown (e.g., in Ljungqvist and Sargent, 2000) that the optimal investment plan for the social planner, I_t , is given by,

$$I_t = k_t [\alpha^{1/\eta} (2-\lambda)^{(1-\eta)/\eta} - (1-\lambda)] \quad \text{with } \alpha(2-\lambda)^{1-\eta} < 1. \quad (19)$$

Therefore, the social planner invests more if the depreciation rate increases and/or production becomes quicker (α increases).

To facilitate the comparison between the growth path with and without bargaining, we assume that $\eta = 1/2$. Then, the social optimum in (19) is as follows,

$$I_t = k_t [\alpha^2(2-\lambda) - (1-\lambda)] \quad \text{with } \alpha(2-\lambda)^{1/2} < 1 \quad (20)$$

While using (13), the optimum non-cooperative investment level in the case of symmetry with $\eta = 1/2$ is given by,

$$iI_t = k_t (2-\lambda)(\alpha^2\beta^2-(1-\lambda))/(1 + \alpha^2\beta^2), \quad \text{with } \alpha(2-\lambda)^{1/2} < 1 \quad (21)$$

where $\alpha = \alpha_i$ and $\beta_i = \beta$ for any i . The social planner's level of investment is at least as large as the non-cooperative level if the following inequality holds,

$$[\alpha^2(2-\lambda)-(1-\lambda)] \geq (2-\lambda)(\alpha^2\beta^2-(1-\lambda))/(1 + \alpha^2\beta^2), \quad (22)$$

Then, it is sufficient to show that $\beta^2(1-\alpha^2(2-\lambda))$ is not larger than $(2-\lambda)$. Given Lemma 1, $\beta^2 = \psi^2\phi^2 = (2-\lambda)\psi^2/(1-(2-\lambda)\psi^2\alpha^2)$, with $\psi < 1$. After some manipulation (22) can be written as $\psi^2 \leq 1$, which is always true. Then, the social planner always invests more than the non-cooperative players. The intuition is that since a proposer does not fear a rejection, he can fund the current level of investment by reducing his own level of consumption. Additionally, the proposer's intertemporal optimisation will also take into account the fact that in the future his opponent can propose with a positive probability and therefore his consumption will be very different. In the context of the social planner's problem the incentives are different for two reasons. First, both players share the cost of the investment; second, they equally share the benefits of this investment.

This result is related the conventional hold up problem, in which there is underinvestment since a player incurs all the costs of investment but cannot appropriate all the benefits from the bargaining process. An important difference between the conventional hold-up problem and our game is that in the latter the accumulation problem is dynamic in that it is repeated more than once⁴. However, the main incentive to under-invest still holds.

⁴ In the literature the hold-up problem has been solved in a number of ways, for instance, when players have a matching problem (see, Cole et al, 2001), when the investment cost is not sunk (Muthoo, 1996) or when the investment decision is unobservable and there are repeated-offers (Gul, 2001).

4 The Bargaining Procedure

The bargaining procedure in our model is relatively simple. The assumption that the game ends in the case of rejection allowed us to tackle the accumulation problem that the players face. Consider the two following alternatives. In the first one, we assume that only the surplus disappears after a rejection, but a new production stage can take place. Then, capital depreciates and the new capital stock is $k_{t+1} = (1-\lambda)k_t$. In the second alternative, players play the bargaining stage (potentially forever) until an agreement is reached - either as in Rubinstein (1982) with an alternating-offer structure or with random proposers. In this case there is no production (and therefore no depreciation) after a rejection. As a result the capital stock in the next bargaining round is simply⁵ k_t . The two cases are very similar in terms of the continuation game, since there is still a positive capital stock after a rejection. This feature makes the problem technically very demanding, since the optimisation for a proposer who attempts to make an acceptable offer is now constrained by the acceptance condition. In other words, a proposer's problem has a recursive structure which includes a constraint, the acceptance condition, which in turn embodies another recursive problem (i.e., the responder's optimisation problem in the case of rejection). The solution of a recursive problem constrained by another recursive problem is unsolved as far as we know. It is possible to show that there is an MPE like the one defined for the ultimatum structure if the constraint is not binding (at all points in time and for each player)⁶. However, many MPE (with and without delay) can also be sustained, given the non-stationary structure of the game. In an ultimatum bargaining framework this feature does not hold, since a proposer can focus on his intertemporal optimisation without fearing a rejection.

In conclusion, a dynamic accumulation problem within a bargaining game is often intractable. The problem becomes tractable, when either the bargaining stage is simple (as the one considered in section 2) or the accumulation problem is simple (as in Muthoo, 1999, where the level of investment is such that the surplus has always the same size).

6 Conclusion

As far as we know our model is the first attempt to solve an accumulation problem within a bargaining model. This is characterised by parties who need to agree not only on how divide a surplus for their consumption, but also on how much to invest, which affects the size of surpluses available in the future. When a rejection of a proposal induces the end of the game, we showed that there is a unique Markov perfect equilibrium (MPE) in which a proposer consumes the whole surplus not invested. This equilibrium can also be sustained under more complicated bargaining procedures (for instance in a potentially infinite bargaining game, or when a production stage follows the rejection of a proposal). However, in these cases the analytical solution of the recursive optimisation problem is technically very demanding, since the maximisation problem of a proposer embodies another recursive problem (via a constraint). Only when either the bargaining or the investment problem is simplified can a full characterisation of the solution be derived.

With a focus on the former, it was shown that when the proposer has a sufficiently high discount factor, his MPE investment level is higher than his opponent's, for a given

⁵ In this case, a within-cake discount factor applies after a rejection, as in the classic bargaining problem (Rubinstein, 1982 see also Muthoo, 1995, where there are between- and within-cake discount factors).

capital stock. Moreover, it is smaller than the optimal level of investment chosen by a social planner for the same mechanism at work in the standard hold-up problem.

In general, the capital stock in our framework does not need to be physical, but can simply derive from the ongoing relationship itself. Therefore, this framework can represent an important first step for further investigation of the dynamic accumulation problem in many bargaining situations.

Appendix A

Proof of Remark 1. The proof consists in two parts. In part (i) we show that the unknown parameters ϕ_i , β_i and μ_i can be written as functions of interdependent variables ψ_i and ψ_j . In part (ii) we show under which conditions the two auxiliary variable ψ_i and ψ_j are uniquely defined. This proof extends results contained in Lockwood *et al.* (1996) in several respects. Lockwood *et al.* do not deal with a bargaining problem. Moreover, their players possess the same rate of time preference and face a linear-quadratic problem which is the simplest form that the optimisation problem can assume.

(i) Using (13) and the equation of motion, after some manipulations, problem (8) can be written as,

$$\phi_i = (2 - \lambda)^{1-\eta} (1 + \alpha_i^{1/\eta} \beta_i^{1/\eta})^\eta. \quad (\text{A.1})$$

Moreover, (11) can be written explicitly for μ_i for any $i = 1, 2$,

⁶ See Flamini (2002).

$$\mu_i = \alpha_i p_i \phi_i \varphi_j^{1-\eta} / (1 - \alpha_i(1-p_i) \varphi_j^{1-\eta}). \quad (\text{A.2})$$

By using (A.2), (10) can be solved for β_i ,

$$\beta_i = \psi_i \phi_i, \text{ where } \psi_i = p_i / (1 - \alpha_i(1-p_i)(\varphi_j)^{1-\eta}). \quad (\text{A.3})$$

By using (A.3), (A.1) becomes an equation in ϕ_i ,

$$\phi_i = (2 - \lambda)^{1-\eta} (1 + \alpha_i^{1/\eta} \psi_i^{1/\eta} \phi_i^{1/\eta})^\eta. \quad (\text{A.4})$$

The solution to (A.4) is the following,

$$\phi_i = (2 - \lambda)^{1-\eta} / (1 - (2-\lambda)^{(1-\eta)/\eta} \psi_i^{1/\eta} \alpha_i^{1/\eta})^\eta. \quad (\text{A.5})$$

Using (A.5) and $\beta_i = \psi_i \phi_i$, φ_i as defined in (14), can be written as follows,

$$\varphi_i = (2 - \lambda)^{1/\eta} \psi_i^{1/\eta} \alpha_i^{1/\eta}. \quad (\text{A.6})$$

This implies that ψ_i in (A.3) can be written as a function of ψ_j ,

$$\psi_i = p_i / (1 - \alpha_i^{1/\eta} (1-p_i) (2 - \lambda)^{(1-\eta)/\eta} \psi_j^{(1-\eta)/\eta}). \quad (\text{A.7})$$

Since by definition, $\beta_i = p_i \phi_i + (1-p_i) \mu_i$, by using (A.3),

$$\mu_i = (\psi_i - p_i) \phi_i / (1-p_i). \quad (\text{A.8})$$

System (A.7) consists of two equations in two unknowns ψ_i and ψ_j . If there is a solution (ψ^*_1, ψ^*_2) to (A.7), then this implies a solution to (A.5), (A.3) and (A.8). This defines a ‘non-negative equilibrium’, if the transversality condition and the following non-negative constraints hold: $\phi_i \geq 0$, $\beta_i \geq 0$ and $\mu_i \geq 0$. Note that the condition of immediate acceptance is equal to the non-negative constraints $\mu_i \geq 0$ for any i .

ii) System (A.7) can be written as a pair of equations where ψ_i is the explicit variable.

That is to say,

$$\psi_i = f_i(\psi_j) = p_i / (1 - a_i \psi_j^{(1-\eta)/\eta}) \quad (\text{A.9})$$

$$\psi_i = f_j(\psi_j) = (\psi_j - p_j)^{\eta/(1-\eta)} / (a_j \psi_j^{\eta/(1-\eta)}) \quad (\text{A.10})$$

with $a_i = \alpha_i^{1/\eta}(1-p_i)(2-\lambda)^{(1-\eta)/\eta}$ and $a_j = \alpha_j^{1/(1-\eta)}(1-p_j)^{\eta/(1-\eta)}(2-\lambda)$, with $i \neq j$ and $i, j = 1, 2$.

This implies,

$$p_i a_j \psi_j^{\eta/(1-\eta)} = (\psi_j - p_j)^{\eta/(1-\eta)} (1 - a_i \psi_j^{(1-\eta)/\eta}). \quad (\text{A.11})$$

We now investigate the properties of the two functions f and define under which conditions there is a unique non-negative equilibrium. First of all, note that the function f_i is increasing in ψ_j , with a discontinuity at ${}_d\psi_j = 1/a_i$ (see figure 1). Moreover, the function f_i is negative for ψ_j larger than ${}_d\psi_j$. However, for the non-negative constraints and the requirement of a finite value function, it must be that the upper bound for ψ_j is ${}_d\psi_j$. On the other hand, a lower bound for ψ_i is p_i , since $f_i(0) = p_i$. Regarding function f_j , this is increasing in ψ_j , with an upper bound in ${}_d\psi_i = 1/a_j$. Moreover, $f_j(p_j) = 0$. This implies that the function f_i must be convex (its second derivative is positive), while f_j must be concave (its second derivative is negative) between p_j and ${}_d\psi_j$. Given the

properties of the functions f_i , with $i = 1, 2$, these intersect at most twice in the space $[p_i, {}_d\psi_i] \times [p_j, {}_d\psi_j]$, although the equation (A.10) is of degree not smaller than $((\eta/(1-\eta))^2 + 1$ if $\eta > 1/2$ and $((1-\eta)/\eta)^2 + 1$ otherwise). An intersection point is indicated by $\psi^* = (\psi_1^*, \psi_2^*)$.

We now refine the space of interest for the solution (ψ_1^*, ψ_2^*) to take into account the constraints that a non-negative equilibrium must satisfy, namely, condition (15), and the non negative-constraints. By using (A.5) and the non-negative condition for β_j , ψ_j must be smaller than ${}_u\psi_j = 1/[(2-\lambda)^{1-\eta}\alpha_j]$, with $j=1,2$. Moreover, the transversality condition is satisfied if $\alpha_i\varphi_j^{(1-\eta)} < 1$, with⁷ $\varphi_j = \max \{\varphi_1, \varphi_2\}$, for any $i, j = 1, 2$. This implies $\psi_j < {}_{TC}\psi_j = 1/[\alpha_j(2-\lambda)\alpha_i^{\eta/(1-\eta)}]$ with $i, j = 1, 2$. Therefore, the upper bound for ψ_j is ${}_{UB}\psi_j = \min \{ {}_d\psi_j, {}_u\psi_j, {}_{TC}\psi_j \}$, while the lower bound is simply ${}_{LB}\psi_j = p_j$. A solution (ψ_1^*, ψ_2^*) implies a non-negative equilibrium, if it lies in the space $T = T_1 \times T_2$, with $T_i = [{}_{LB}\psi_i, {}_{UB}\psi_i]$ and $i = 1, 2$.

A necessary condition for the existence of an equilibrium is that ${}_{LB}\psi_i < {}_{UB}\psi_i$. Additionally, we require that the two functions, f_i for any i , are sufficiently curved so that they cross at least once. An explicit solution for a general η is not straightforward. We investigate the existence of a non-negative equilibrium in more detail when η assumes a specific value, $\eta = 1/2$ (see Appendix B). For the general case, we can conclude that if the two functions, f_i for any i , cross once, at ψ^* , then there is a unique solution if ψ^* is in T . If the functions f_i , for any i , cross twice, at ${}^-\psi^* = ({}^-\psi_1^*, {}^-\psi_2^*)$ and ${}^+\psi^* = ({}^+\psi_1^*, {}^+\psi_2^*)$, then there is a unique solution if either ${}^-\psi^*$ or ${}^+\psi^*$ is in T . If neither or both the solutions ${}^-\psi^*$ and ${}^+\psi^*$ are in T , then the *guess and verify* method has failed to yield the solution, since this is based on the uniqueness of the solution.

⁷ This is a stronger condition, it is sufficient $\alpha_i(p_j\varphi_j + p_i\varphi_i)^{(1-\eta)} < 1$.

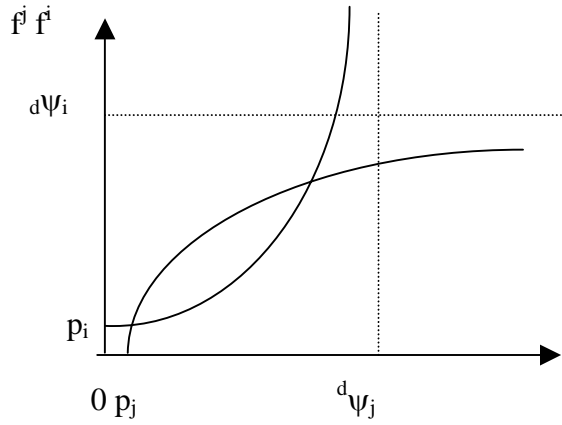


Figure 1: Representation of system (A.9) and (A.10).

Appendix B

Proof of Lemma 1. When $\eta = 1/2$, the equation (A.11) can be written as follows,

$$a_i \psi_j^2 - (1 - a_j p_i + a_i p_j) \psi_j + p_j = 0. \quad (\text{B.1})$$

There are at most two positive solutions to (B.1) if (i) $(1 - a_j p_i + a_i p_j) > 0$ (following the Descartes' rules of signs) and (ii) the discriminant of (2.1), Δ , is non-negative. Condition (i) can also be written as $b > 0$, where

$$b = 1 + (2-\lambda)[(1 - 2p_i) \alpha_i^2 + (\alpha_i^2 - \alpha_j^2) p_i^2] > 0$$

Then (i) is guaranteed by $p_i \leq 1/2$ and $\alpha_i \geq \alpha_j$. Regarding (ii), after some manipulation the discriminant can be written as

$$\Delta = (2-\lambda)(\alpha_i^2 - \alpha_j^2) p_i^2 - 2\alpha_i (2-\lambda)^{1/2} (\alpha_i(2-\lambda)^{1/2} - 1)p_i + (\alpha_i(2-\lambda)^{1/2} - 1)^2.$$

In addition to the condition for (i), Δ is positive for any p , if $\alpha_i \leq (2-\lambda)^{-1/2}$. Since if $\alpha_i > (2-\lambda)^{-1/2}$ then (ii) requires that p must be sufficiently small⁸, to simplify let α_i be smaller than $(2-\lambda)^{-1/2}$. To sum up, a necessary solution for a positive solution to , is $p_i \leq 1/2$ and $\alpha_j \leq \alpha_i \leq (2-\lambda)^{-1/2}$.

The solutions to are as follows,

$$\psi_{*j}^* = [b \pm \sqrt{\Delta}]/2a_i \text{ with } i, j = 1, 2 \text{ and } i \neq j. \quad (\text{B.2})$$

If there are two positive solutions, they are named $\bar{\psi}_{*j}^*$ and $^+\psi_{*j}^*$, (with $\bar{\psi}_{*j}^* < ^+\psi_{*j}^*$, and $j = 1, 2$). Under symmetry (i.e., $\alpha_i = \alpha_j = \alpha$ and $p_i = 1/2$), the solutions are as follows,

$$\psi_{*j}^* = [1 \pm \sqrt{(1 - \alpha^2(2-\lambda))}]/\alpha^2(2-\lambda). \quad (\text{B.3})$$

These are real if $\alpha^2(2-\lambda) < 1$. Then, the non-negative equilibrium is given by $\bar{\psi}_{*j}^*$ since this belongs to T_j , while $^+\psi_{*j}^*$ does not, where $T_j = (1/2, 1/(\alpha(2-\lambda)^{1/2}))$, with $j = 1, 2$. Then the payoff to a proposer is higher than the payoff to a responder for a given level of capital (i.e., $\phi_i > \beta_i$, then $\phi_i > \mu_i$ for any i), since $\bar{\psi}_{*j}^* < 1$.

⁸ It must be $p < (2-\lambda)^{1/2} (\alpha_i(2-\lambda)^{1/2} - 1)(\alpha_i - \alpha_j)/(2-\lambda)(\alpha_i^2 - \alpha_j^2) < 1/2$.

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