Lost in Translation: Basis Utility and Proportionality in Games[1]

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abstract

A player's basis utility is the utility of no payoff. Basis utility is necessary for the coherent representation of the equal split bargaining solution. Standard axioms for the Nash (1950) bargaining solution do not imply independence from basis utility. Proportional bargaining is the unique solution satisfying efficiency, symmetry, affine transformation invariance and monotonicity in pure bargaining games with basis utility. All existing cooperative solutions become translation invariant once account is taken of basis utility. The noncooperative rationality of these results is demonstrated though an implementation of proportional bargaining based on Gul (1988). Quantal response equilibria with multiplicative error structures (Goeree, Holt and Palfrey (2004)) become translation invariant with specification of basis utility. Equal split and proportional bargaining join the Kalai-Smorodinsky (1975) solution in a family of endogenously proportional monotonic pure bargaining solutions.

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1 Introduction

There is no representation of *basis utility* in games. Basis utility is a natural reference point on a player's utility scale that enables determination of the marginal utilities of outcomes. The *basis point* in a game represents all players's basis utilities.

The immediate context of this work is the formal representation of bargaining games. The value to a player of a particular outcome is typically represented by an expected utility function. The expected utility of a lottery over outcomes is the utility of the outcomes weighted by their probability. Expected utility preferences are (*affine*) *transformation invariant*, they do not change under *translation* by adding a constant or *rescaling* from multiplication by a positive number.

A game theoretic solution concept is transformation invariant if affine transformation of a player's utility function leads to the same transformation of her payoff or allocation. Transformation invariance is necessary if a player's real payoff is to be independent of the representation of her preferences. Myerson (1991: 18) advises that "we should be suspicious any theory of economic behavior that requires distinguishing between such equivalent representations."

Omission of a game's basis point leads *basis dependent* solutions to appear *translation dependent*: A player's implied basis utility is then the zero point of any expected utility representation of his preferences both before and after translation. Section 2 of this paper defines basis utility and shows that it is part of the common knowledge of a complete information game. Minimax equilibria, stable sets, Nash equilibrium and core-related solutions are based on inequalities provide little role for basis utility.[2] It is Nash's (1950) axiomatic approach that allows basis utility an easy entry point into game theory as a reference point in bargaining models.

Cooperative solutions are first taken up in section 3, which develops results for three specific cases: (i) Representation of basis utility is necessary to properly characterize the equal split bargaining outcome. (ii) Nash's (1950) axioms are not sufficient to identify a solution in games with basis utility. (iii) Proportional pure bargaining is uniquely characterized by efficiency, symmetry, transformation invariance and monotonicity in games with basis utility. Proportional allocations of this type have been considered translation dependent.[3] Conditions are then developed under which all solutions for the standard characteristic function game are translation invariant.

Section 4 concerns noncooperative games. An implementation of proportional bargaining based on Gul (1988) affirms the noncooperative rationality of basis utility dependent cooperative solutions. In the limiting case, a player's probability of selection to propose is proportional to her expected payoff. The equilibrium is translation invariant when selection probabilities are based on marginal utilities. Goeree,

[2] von Neumann and Morgenstern (1944) clearly considered that marginal utilities, and thus, basis utilities, could be determined. See the first paragraph of section 3.8.2, p. 30 in the 1947 edition.

[3] E.g., Aumann and Maschler (1985: 210), fn. 26 and Hart and Mas-Colell (1996: 595), fn. 10.

Holt and Palfrey (2004) find that quantal response equilibria with multiplicative error structures better model experimental data than additive error models. These models are shown to be basis dependent and not translation dependent.

Section 5 returns to cooperative pure bargaining. Equal split is characterized in general pure bargaining games. The similarity of equal split, Kalai-Smorodinsky (1975) and proportional bargaining and their relationship to the proportional solutions of Kali (1977) and Roth (1979) are considered. Several approaches are offered to structure the expanded universe of pure bargaining solutions.

2 Basis Utility

Definition 2.1 *The basis utility of a player in a game is the utility of no payoff.*

Basis utility enables determination of the marginal utility of all potential payoffs.[4] Nash (1950) writes that players in a game have "full knowledge of the tastes and preferences" of other players. Knowledge of the utility of a probabilistic payoff must be considered common knowledge (up to transformation invariance) in cooperative and noncooperative games. This implies knowledge of basis utility. Assume player i assigns $U^i_j(A)$ as the utility to j of receiving real payoff A and $U^i_j(A|p)$ to receiving A with probability p. Represent i's measure of j's basis utility by $U^i_j(\varnothing)$. Since these are expected utilities, i can infer that $U^i_j(\varnothing) = 1/(1-p)\, U^i_j(A) - p/(1-p)\, U^i_j(A|p)$.

Proposition 2.1 *Basis utility is part of the common knowledge of a complete information game.*

Utilities and marginal utilities constructed for a player i are, of course, not comparable with those for a player j. However, knowledge of basis utility makes it possible make valid comparisons between players regarding ratios of the marginal utility of outcomes to the marginal utility of individual worths. This requires that all players (and observers) calculate the same ratios. Consider a cooperative game w where player j's individually rational payoff is strictly greater than her basis utility. Assume that the game w is an observer's representation of the game and that w^i is representation of any player i.

Proposition 2.2 *The ratio of the marginal utility of an outcome Z to any player j divided by the marginal utility of j's individual payoff is the same according to the utility function constructed by any player in the game w or any observer, provided that $w(\overline{j}) \neq U_j(\varnothing)$.*

[4]Other definitions might be appropriate under specific circumstances. For example, in an arbitration, parties might agree that certain funds received by one party should not be taken into account in determining its current economic status. Specific types of games might also warrant different definitions of basis utility. In market games players's individual worths are generally the utility of their endowments, which might be thought the proper measure of basis utility.

Proof: Let the observer's utility scale for player j be U_j. For any player i there must be constants a^i and $b^i > 0$ such that $U_j = a^i + b^i U_j^i$. Then

$$\frac{U_j^i(Z) - U_j^i(\varnothing)}{w^i(\bar{j}) - U_j^i(\varnothing)} = \frac{(a^i + b^i U_j(Z)) - (a^i + b^i U_j(\varnothing))}{(a^i + b^i w(\bar{j})) - (a^i + b^i U_j(\varnothing))} = \frac{U_j(Z) - U_j(\varnothing)}{w(\bar{j}) - U_j(\varnothing)}.$$

□

3 Cooperative Games

3.1 Soft bargaining and the equal split solution

In the equal split solution players split the payoff equally, irrespective of disagreement payoffs. This *soft bargaining* – in contrast the commonly expected *hard bargaining* – might be justified by fairness considerations that are rational in a larger context.[5] Soft bargaining is observed in a variety of experiments, most dramatically in the ultimatum game.[6] This example demonstrates the basis dependence of the equal split solution and defines key elements of the formal bargaining framework employed in this paper.

Assume players 1 and 2 can share \$100 if they can agree on its division. Otherwise they receive \$10 and \$30, respectively. Start with players's utilities defined as equal to the dollars received. The bargaining game is then $B = (d, S)$, where $d = (10, 30)$ is the disagreement point and $S = \{\,(x, y) \in \mathbb{R}^2 |\, x + y \leq 100\}$ is the set of feasible bargaining outcomes. Refer to d and S as *elements* of the bargaining game B.

Definition 3.1 *A bargaining game is* symmetric *if all comparable elements of the game are symmetric. All elements are* comparable *unless specifically excluded from comparison. An element Z is symmetric if and only if for any players i and j in the game and any vector $x \in Z$, $y \in Z$ as well, where $y_i = x_j$ and $y_j = x_i$ and $y_k = x_k$ for all $k \neq i, j$.*

Definition 3.2 *If a cooperative solution is* symmetric *then all players receive equal allocations in a symmetric game.*

Definition 3.3 *A cooperative solution is* efficient *if, for $x = F(d, S)$, there is no $y \in S$ such that $y \geq x$ and $y \neq x$.*

[5] See, e.g., Huck and Oechssler (1999) and Lopomo and Ok (2001).

[6] In the ultimatum game one player makes a take-it or leave-it offer to another, and each player receives zero if it is rejected. Equal split does not describe the structure of an ultimatum game, but the choice between equal split and, say, Nash bargaining provides a useful cooperative perspective.

Proposition 3.1 *The TU equal split solution identified by efficiency, symmetry and the noncomparability of disagreement payoffs is basis dependent.*

Proof: Efficiency, symmetry and the noncomparability of disagreement payoffs determine $ES(B) = (50, 50)$. Define B^* by adding 100 to player 1's utility function: $B^* = (d^*, S^*)$, where $d^* = (110, 30)$ and $S^* = \{\,(x, y) \in \mathbb{R}^2 |\, x + y \leq 200\}$. The axioms again imply a symmetric outcome: $ES(B^*) = (100, 100)$. But this corresponds to $100 - 100 = \$0$ for player 1. Translation invariance implies the outcome $(150, 50) = ES(B) + (100, 0) = (150, 50)$. The basis points of B and B^* are clearly $(0, 0)$ and $(100, 0)$. B^* is no longer symmetric if basis utility is made a comparable element. Translation of B^* to make it symmetric restores the proper allocations. □

Representation of the basis point allows for a natural transformation invariant representation of equal split. This is done now for the case of linear utility functions. Generalization to general pure bargaining games is deferred to theorem 5.1.

Definition 3.4 *A* proper n-player pure bargaining game *is represented by the triple* $B = (\xi, d, S)$, *where* $\xi \in \mathbb{R}^N$ *is the basis point,* $d \in \mathbb{R}^N$ *is the disagreement point,* $S \subset \mathbb{R}^N$ *is the set of feasible alternatives and* $\mathbb{R}^N$ *is the* n*-dimensional space indexed by the set* N *of the* n *players* $i = 1, \ldots, n$.

Definition 3.5 *Direct or Hadamard multiplication is represented by the symbol* $\odot$. *If* a *and* b *are both* n*-vectors, then* $a \odot b = (a_1\, b_1, a_2\, b_2, \ldots, a_n\, b_n)$. *If* $a \in \mathbb{R}^N$ *and* B *is a subset of* $\mathbb{R}^n$, *then* $a \odot B = \{x \in \mathbb{R}^n |\; y \in B$ *and* $x = a \odot y\}$.

Definition 3.6 *Direct addition and subtraction are represented by the symbols* $\oplus$ *and* $\ominus$. *If* $a \in \mathbb{R}^N$ *and* B *is a subset of* $\mathbb{R}^n$, *then* $a \oplus B = \{x \in \mathbb{R}^n |\; y \in B$ *and* $x = a + y\}$ *and direct subtraction is defined analogously. Direct multiplication has precedence.*

Definition 3.7 *A solution* F *for a proper* n*-player pure bargaining game* $B = (\xi, d, S)$ *is* affine transformation invariant *if and only if* $F(s \odot \xi + c, s \odot d + c, s \odot S \oplus c) = s \odot F(\xi, d, S) + c$ *for any* $s > 0^N \in \mathbb{R}^N$ *and* $c \in \mathbb{R}^N$.

Proposition 3.2 *Assume, in a two-player bargaining game* $B = (\xi, d, S)$, *that the efficient surface of* S *is linear and that the bargaining solution is efficient, symmetric, independent of individual payoffs, and affine transformation invariant. Let the maximum utility obtainable by player 1 while player 2 obtains at least* ξ_2 *be* M_1 *and the similar maximum for player 2 be* M_2. *The solution is then the equal split solution and the payoffs are*

$$ES(\xi, d, S) = \left(\frac{1}{2}\,(M_1 + \xi_1), \frac{1}{2}\,(M_2 + \xi_2)\right).$$

Proof: First translate utilities so that the basis point is zero. Then rescale so that players's utility is one-to-one transferrable by multiplication with constants $x = (1, (M_1 - \xi_1)/(M_2 - \xi_2))$. Symmetry gives the outcome $(1/2(M_1 - \xi_1), 1/2(M_1 - \xi_1))$. Reverse transformation completes the proof. □

3.2 The Nash bargaining solution

Changes in basis utility have no effect on the Nash (1950) bargaining solution. However, in proper pure bargaining games this independence must be assumed since it cannot be inferred from Nash's axioms of efficiency, symmetry, transformation invariance and independence from irrelevant alternatives[7]

Let x^N be the $1 \times n$ vector with $x_i^N = x$ for all $i \in N$. The game normalized so that the disagreement point is 0^N and the Nash solution is 1^N can no longer be made symmetric by IIA in the sense of definition 3.1 because it cannot be guaranteed that basis utility will be symmetric as well.

Proposition 3.3 *Characterization of the Nash bargaining solution in a proper pure bargaining game $B = (\xi, d, S)$ requires that basis utility be declared a noncomparable element of the game.*

Basis utility is readily identified in Nash's detailed bargaining example. Two players, Bill and Jack, bargain over personal items. Table 1 reports the utility of these items to each player. Nash assumes utilities are additive. This requires that these are marginal utilities and that the basis point in the example is $\xi = (0, 0)$. When the game is normalized $\xi^* = (-1, -1.2)$. This is illustrated in Figure 1.

If the reported utilities need not be marginal utilities, then adding one to Bill's utilities would have no effect on the bargain reached. The Nash solution of the original game is $(24, 11)$ for Bill and Jack. In the transformed game the outcome is $(30, 10.33)$. Translation invariance is clearly violated: Jack's utility changes.[8]

3.3 Proportional bargaining

Kalai (1977) and Roth (1979) study a type of proportional solution that is independent of basis utility. Individual worths are normalized to zero so that the feasibility set completely represents the standard pure bargaining game. Kalai's principal result is that a solution that is weakly Pareto optimal, homogeneous (i.e., $c\,F(S) = F(c\,S)$

[7] Independence of irrelevant alternatives in proper pure bargaining games requires that if $B = (\xi, d, S)$, $B^* = (\xi, d, S)$, $S \subset T$, $x = F(B^*)$ and $x \in S$, then $x = F(B)$.

[8] In the original game Bill gives Jack the book, whip, ball and bat. Jack gives Bill the pen, toy and knife. In the transformed game, with 1/3 probability the original solution obtains and with 2/3 probability the exchange is the same except that Bill keeps his ball.

for $c > 0$), strongly individually rational and monotonic must be proportional. Strikingly, however, the proportions are exogenously determined.

Social choice models of *endogenous proportional* allocation where individual worths determine relative shares are studied by O'Neill (1980), Moulin (1987) and Young (1988). There have been no endogenous proportional pure bargaining solutions similar to these models because the basis point was not represented in the pure bargaining game. Its inclusion allows the disagreement point to determine the vector of proportionality. Since individual worths are the only measure of the strength of players in a pure bargaining game, this result has an obvious natural interpretation.

The set S of feasible bargaining outcomes is required to be convex, compact, comprehensive and nonlevel. Comprehensive means that if $y \in S$ and $x < y$, then $x \in S$ as well. Nonlevel means that if x is a weakly efficient allocation, then it must be (strongly) efficient as well. That is, if $x \in S$ and there is no $y \in S$ such that $y > x$, then there is no $y \in S$ such that $y \geq x$ and $y \neq x$.

Definition 3.8 *The proportional solution for the bargaining game $B = (\xi, d, S)$ with $d > \xi$ is the unique point z on the efficient surface of S such that, for some $c > 0$,*

$$c = \frac{z_i - \xi_i}{d_i - \xi_i}, \qquad i = 1, 2, \ldots, n. \tag{1}$$

Definition 3.9 *A bargaining solution F is* monotonic *if and only if given any two bargaining games $B = (\xi, d, S)$ and $B^* = (\xi, d, T)$ where $S \subset T$, $F(B^*) \geq F(B)$.*

Theorem 3.1 *The proportional bargaining solution is the unique pure bargaining solution that is efficient (def. 3.3), symmetric (def. 3.2), affine transformation invariant (def. 3.7) and monotonic in all proper pure bargaining games with $d > \xi$.*

Proof: Create a normalized game $B^* = (\xi^*, d^*, S^*)$ from $B = (\xi, d, S)$ where $d > \xi$ with $\xi^* = 0^N$ and $d^* = 1^N$ as follows. Let $S^0 = S \ominus \xi$ (def. 3.6). Then define $\delta = d - \xi$ and let $S^* = \delta^{-1} \odot S^0$, where $\delta^{-1} = (1/\delta_1, 1/\delta_2, \ldots, 1/\delta_n)$.

Define $c = \max\{c \mid c\,1^N \in S^*\}$. Let π^i be an n-vector with $\pi^i_i = c + \epsilon$, $\epsilon > 0$, and $\pi^i_j = 0$ for $i \neq j$. Define T to be the comprehensive set based on the convex hull generated by the points $\{c\,1^N, \pi^1, \pi^2, \ldots, \pi^n\}$. Choose ϵ small enough that $T \subset S^*$ and note that $c\,1^N \in T$ and T is symmetric with respect to players. By symmetry $c^N = F(0^N, 1^N, T)$. Monotonicity now requires that $c^N = F(0^N, 1^N, S^*)$. Any other outcome will require at least one player to receive less than c.

Restoring the utility scales of the original game, we find that, for every player i, $F_i(\xi, d, S) = (d_i - \xi_i)c + \xi_i$, or $c = (F_i(\xi, d, S) - \xi_i)/(d_i - \xi_i)$ for every player i. $\square$

3.4 General cooperative games

The worth of the null coalition currently conveys no information. One way to represent basis utility in coalitional games is to let the null coalition represent the basis point. In continuity with the notation already used, let $w(\varnothing) = \xi$.

Definition 3.10 *A game w in* proper characteristic function form *is a set function from the coalitions $S \subset 2^N$, $S \neq \varnothing$, to $\mathbb{R}^S$, with $w(\varnothing) \equiv \xi$.*

Note that affine transformation of the characteristic function requires that the worth of the null coalition must be transformed as well. This definition simplifies for transferrable utility games. Any game in proper form has a marginal representation.

Definition 3.11 *Let w be a proper game with $w(\varnothing) = \xi \in \mathbb{R}^N$. Let w^* be the* marginal *form of w. For any coalition $S \subset N$, let ξ_S be the restriction of ξ to the players in S, with $\xi_\varnothing \equiv 0$. Then $w^*(S) = w(S) \ominus \xi_S$ (def. 3.6).*

In a marginal form game $w^*(\varnothing) = 0^N$. Proposition 2.1 shows that the basis point is part of the common knowledge of the game. Therefore, it seems reasonable to assume that $\xi = 0^N$ unless explicitly stated otherwise, and, hence, that the standard characteristic function is, effectively, a marginal form game.

Definition 3.12 *A solution function is in* marginal form *if it does not reference the basis point.*

Definition 3.13 *Let ϕ^* be a solution function with marginal form and let w be a cooperative game in proper characteristic function form with $w(\varnothing) = \xi$ and marginal representation w^*. Then define*

$$\phi^*(w) = \phi^*(w^*) + \xi.$$

The result of the application of a solution function in marginal form to a proper form game is defined to be the solution on the marginal form game plus the basis point. The marginal form is directly analogous to the 0-normalized standard characteristic function game. The proper form of the solution function can now be inferred. The following proposition trivially follows.

Proposition 3.4 *All marginal form solutions are translation invariant.*

Definition 3.14 *Let ϕ be a solution function defined on proper characteristic function games. Let $v = w$, except that $v(\varnothing) \neq w(\varnothing)$. Basis utilities change in v. If $\phi(v) = \phi(w)$ for any game w and any value $v(\varnothing)$, then ϕ is* basis independent*.*

Proposition 3.5 *The property identified as translation invariance in standard form games without representation of basis utility is actually* basis independence.

Proof: If a solution appears translation invariant in standard characteristic function games, it must be independent of basis utility as the implied basis point is always $(0, 0)$, even after translation. If a solution is basis dependent, it must appear translation dependent because the implied basis point is not subjected to translation. □

4 Noncooperative games

The principal task of this section is to demonstrate the noncooperative rationality of basis utility with a translation invariant model of proportional bargaining. Additionally, quantal response equilibrium is shown to provide a pure noncooperative application of basis utility. There is at least one prior point of contact between basis utility and noncooperative games. A player in the Hart and Mas-Colell (1996) model of the consistent NTU value that is 'removed' due to a breakdown in negotiations receives zero terminal payoff. This payoff must function like the player's basis utility if the game is be translation invariant.

4.1 Noncooperative proportional bargaining

Proportional bargaining (th. 3.1) is modeled using the basic setup developed by Gul (1998) to implement the Shapley value. Two players are endowed with productive assets that yield an income stream. In each time period, a player is selected to bid a constant stream of payments for the other's resources. If the bid is accepted, the bidder receives the assets of the acceptor, the acceptor receives the promised payments and bargaining ends. If the bid is rejected, this stage game repeats.

Let $w(\overline{12})$ be the worth of the joint assets, let $w(\overline{1})$ and $w(\overline{2})$ be the individual worths and let $w(\varnothing) = (\xi_1, \xi_2)$. The game must be superadditive: $w(\overline{12}) > w(\overline{1}) + w(\overline{2})$. The common discount *factor* is δ, with $0 < \delta < 1$. Let c_t^i be the value of the assets owned by player i at time t. The utility provided to player i at time t is defined to be $(1 - \delta)\, c_t^i$. The present discounted utility at time t_0 to player i given an prospective asset holding history $\{c_t^i\}_{t=t_0}^{\infty}$ is then

$$U^i(t_0) = \sum_{t=t_0}^{\infty} \delta^{t-t_0} (1 - \delta)\, c_t^i. \tag{2}$$

The selection of players to propose differs from the Gul model. Both players submit bids before the bidder is selected. The probability of selecting player i to bid is set proportional to player j's bid for i's assets. This selection procedure can be seen as a natural way to reflect the impact of a player's strength on the bargaining process. Let b_t^{ji} be the bid by player j for i's assets at time t. The probability p_t^i of i's selection to bid in period t, conditional on b_t^{ji} and b_t^{ij} is

$$p_t^i = \frac{b_t^{ji} - \xi_i}{(b_t^{ji} - \xi_i) + (b_t^{ij} - \xi_j)}. \tag{3}$$

Computing probabilities based on the marginal utility of bids makes selection probabilities independent of the translation of players's utility scales.

Given that no bid has been accepted, the complete history of the game prior to time t, is $h_{t-1} = (b_k^{ij}, b_k^{ji})_{k=1}^{k=t-1}$. The complete set of all such possible histories prior to time t is H_{t-1}. A strategy for player i at time t given a history $h_{t-1} \in H_{t-1}$ is $\sigma_t^i(w, \delta, h_{t-1}) = (b_t^{ij}, r_t^i)$, where i will accept any bid $b_t^{ji} \geq r_t^i$ if j is selected to bid. Let σ_t^i contain a single strategy for each possible history to time $t-1$. A complete strategy for i is $\Sigma^i = (\sigma_t^i)_{t=1}^{t=\infty}$, the set of all strategy profiles is $\Sigma = \Sigma^1 \times \Sigma^2$ and a complete description of the game is then $\Gamma_1 = (\Sigma, (U^1, U^2), w, \delta)$.

Theorem 4.1 *In the unique stationary subgame perfect equilibrium of* Γ_1 i *offers* j

$$\bar{b}^{ij} = \delta \frac{w(\bar{j}) - \xi_j}{\sum_{i=1}^{i=2} w(\bar{i}) - \xi_i} \left(w(\overline{12}) - \xi_1 - \xi_2\right) + (1-\delta)(w(\bar{j}) - \xi_j) \; + \; \xi_j,$$

and $\bar{r}^j = \bar{b}^{ij}$. *The expected utilities during bargaining and at any time* t *before a bidder is selected are the allocations determined by proportional pure bargaining (eq. 1)*

$$\bar{U}^i = \frac{w(\bar{i}) - \xi_i}{\sum_{j=1}^{j=2} w(\bar{j}) - \xi_j} \left(w(\overline{12}\,) - \xi_1 - \xi_2\right) \; + \; \xi_i, \qquad\qquad i = 1, 2.$$

Proof: Considering stationary strategies, history is irrelevant and each player computes optimal strategies under the assumption that if the current bid is rejected that agreement will be reached in the next time period. Expected utility before selection of a bidder is

$$\bar{U}^i = p^i \Big(w(\overline{12}\,) - \bar{b}^{ij} \Big) + p^j \, \bar{b}^{ji}, \qquad\qquad i = 1, 2; \; j \neq i.$$

Equilibrium bids are player's continuation values and are the solution of the equations

$$\bar{b}^{ij} = \delta \bar{U}^j + (1-\delta)\, w(\bar{j}), \qquad\qquad i = 1, 2; \; j \neq i.$$

□

Proposition 4.1 *The game Γ_1 is translation invariant.*

Proof: Let $w^*(\varnothing) = \xi^* = (\xi_1 + x, \xi_2)$, $w^*(\overline{12}) = w(\overline{12}) + x$, $w^*(\overline{1}) = w(\overline{1}) + x$ and $w^*(\overline{2}) = w(\overline{2})$. Then $\bar{b}^{ji^*} = \bar{b}^{ji} + x$, $\bar{U}^{i^*} = \bar{U}^i + x$, $\bar{b}^{ij^*} = \bar{b}^{ij}$ and $\bar{U}^{j^*} = \bar{U}^j$. □

Remark 4.1 *Theorem 4.1 easily generalizes to n-player pure bargaining games.*

Remark 4.2 *Selection probabilities can be based on the average of both players's proposals,* e.g., $p^i = (b^{ji} + b^{ii})/((b^{ji} + b^{ii}) + (b^{ij} + b^{jj}))$. *The outcome in the limit, as $\delta \to 1$, is the same. However, expected utility for $\delta < 1$ is no longer exactly the proportional solution.*

Remark 4.3 *Complete translation invariance can easily be shown in (NTU) hyperplane games, and with some work, in general NTU games.*

Remark 4.4 *A TU and NTU implementation of proportional pure bargaining based on the game of Hart and Mas-Colell (1996) is included in Feldman (2002).*

4.2 Quantal response equilibria

Quantal response equilibria (QRE) are a refinement introduced by McKelvey and Palfrey (1995). "Trembles" or misperceptions of payoffs cause deviations from best response and are modeled with a statistical response function. Under general conditions a unique equilibrium is selected as the size of trembles goes to zero. QRE is defined with an additive error structure. This guarantees translation invariance. Goeree, Holt and Palfrey (2004) (GHP) introduce regular QRE. One feature of these equilibria is that they allow a multiplicative error structure, which the authors find provides a better fit to experimental data. GHP consider the multiplicative error model to be translation dependent and thus that "[t]ranslation invariance is not plausible in settings where the magnitudes of perception errors or preference shocks depend on the magnitudes of expected payoffs." (2004: 19.)

A regular n-player QRE for may be defined as follows. Let $S_i = (s_{i1}, s_{i2}, \ldots, s_{iJ_i})$, be i's pure strategy set, where J_i is the number of i's pure strategies. Let $\sigma_i \in \Sigma_i$ be a mixed strategy over S_i, let $\sigma \in \Sigma$ be a complete profile of mixed strategies, and let σ_{-i} represent the strategy profile of all players except i. Player i's expected payoff from a strategy profile σ is $\pi_i(\sigma)$.

Represent undisturbed payoffs as a function of strategy choice to any i given σ_{-i} by the function $\bar{\pi}_i(\sigma) = (\pi_i(s_{i1}, \sigma_{-i}), \pi_i(s_{i2}, \sigma_{-i}), \ldots, \pi_i(s_{iJ_i}, \sigma_{-i}))$. Collect the $\bar{\pi}_i$ into the profile $\bar{\pi}(\sigma) = (\pi_1(\sigma), \pi_2(\sigma), \ldots, \pi_n(\sigma))$. Player i's perceived payoff from strategy j, $\hat{\pi}_{ij}(s_{ij}, \sigma_{-i})$, is affected by a privately observed random disturbance that may be a function of her strategy choice: $\hat{\pi}_{ij}(s_{ij}, \sigma_{-i}) = g(\bar{\pi}_i(s_{ij}, \sigma_{-i}), \epsilon_{ij})$.

Let $P_i : \bar{\pi}_i \rightarrow \Sigma_i$ be the regular quantal response function for player i. The regular QRE is a reduced form approach because P_i implies $g(\bar{\pi}_i, \epsilon_i)$ and the distribution of ϵ_i. GHP place restrictions directly on the response functions of regular QRE that ensure representation of boundedly rational choice behavior. A strategy profile σ is a regular QRE if and only if $P_i(\bar{\pi}_i(\sigma_{-i})) = \sigma_i$ for all $i = 1, \ldots, n$.

The canonical quantal response function based on multiplicative error is the power model, under which the probability of i playing strategy j is

$$P_{ij} = \frac{(\pi_{ij})^{\frac{1}{\mu}}}{\sum_{k=1}^{J_i} (\pi_{ik})^{\frac{1}{\mu}}}, \tag{4}$$

where $\mu \geq 0$ is a constant determining players's discrimination ability. As $\mu \rightarrow 0$ the probability of all players playing their best response goes to one. McKelvey and Palfrey prove, for logit response functions, that there is generically a unique branch of the equilibrium correspondence based on the discrimination parameter that contains the unique regular QRE under no discrimination and a perfect discrimination QRE that is also a Nash equilibrium. This branch can be thought of as representing a learning process that leads to a unique Nash equilibrium.

Figure 2 shows a simple three-player coordination game Γ_2. The strategy profiles (U, L, W) and (D, R, E) are both Nash equilibria. The QRE equilibrium using the power response function is (U, L, W) when $x = 0$. However, increasing all of player 3's payoffs by one by setting $x = 1$ leads to the selection of (D, R, E). This apparent translation dependence disappears if response probabilities in eq. 4 are determined by marginal utilities. Basis and not translation dependence appears in the quantal power response function.

Proposition 4.2 *All quantal response functions using marginal utilities, payoffs relative to basis utilities, are translation invariant.*

5 Focal Points, Monotonicity and Pure Bargaining

This section completes the presentation of pure bargaining results and provides some interpretation. Equal split is first characterized in proper pure bargaining games.

5.1 Equal split in general pure bargaining games

Definition 5.1 *Let $B = (\xi, d, S)$ be an n-player pure bargaining game. Consider a set $x^i \in S$, $i = 1 \ldots, n$ and a $y \in \mathbb{R}^N$. For any i, let $x^i \in S$ maximize x_i^i subject to the further restriction that $x_j^i \geq y_j$ for all $j \neq i$. Then the* maximal aspirations point relative to y *is $M_y = M(y, S) = (x^1, \ldots, x^n)$. Define $M_\xi = M(\xi, S)$ as the* ξ-maximal aspirations point *of B and $M_d = M(d, S)$ as the* d-maximal aspirations point *of B.*

The ξ-maximal aspirations point shows the most a player can receive when all other players receive at least their basis utility. The standard d-maximal aspirations point represents the most a player can receive when all others receive at least their disagreement payoffs.

Definition 5.2 *Let $B = (\xi, d, S)$ and $B^* = (\xi, d, T)$ have a common maximal aspirations reference point y. A solution F is* restricted monotonic *if and only if $M_y = M(y, S) = M(y, T)$ and $S \subset T$ imply that $F(B^*) \geq F(B)$.*

Restricted monotonicity weakens the definition of monotonicity (def. 3.9) by requiring that two feasibility sets share the same maximal aspirations point.

Theorem 5.1 *The equal split solution is the unique solution for the game $B = (\xi, d, S)$ that is efficient (def. 3.3), symmetric (def. 3.2), affine transformation invariant (def. 3.7), restrictedly monotonic (def. 5.2), and shows noncomparability of disagreement payoffs and comparability of $M_\xi = M(\xi, S)$ (see def. 3.1).*

Proof: Normalize B so that $\xi^* = 0^N$ and $M_\xi^* = 1^N$ and define $c = \max\{c \mid c\,1^N \in S^*\}$. Let π^i be an n-vector with $\pi_i^i = 1$, and $\pi_j^i = 0$ for $i \neq j$. Define T to be the comprehensive set based on the convex hull generated by the points $\{c\,1^N, \pi^1, \pi^2, \ldots, \pi^n\}$. By symmetry $c^N = F(0^N, 1^N, T)$. Restricted monotonicity then requires that $c^N = F(0^N, 1^N, S^*)$ as well. □

The equal split solution is the point on the line between ξ and M_ξ that intersects the efficient surface of S. The sense of equality in the general equal split solution is in the nature of a proportionality property. Consider the range from any player's maximal expectations to their basis utility. Each player loses relative to maximal aspirations or gains relative to basis utility in equal proportion.

Proposition 5.1 *Let $x = ES(\xi, d, S)$ be the equal split solution, let b be the ξ-maximal aspirations point and let b be strictly greater than ξ. Then there is a k such that*

$$\frac{b_i - x_i}{b_i - \xi_i} = k \quad \textit{and} \quad \frac{x_i - \xi_i}{b_i - \xi_i} = 1 - k, \qquad i = 1, 2, \ldots, n.$$

5.2 Monotonic solutions

As can be seen by the proof, equal split is a direct variation on the Kalai and Smorodinsky (1975) bargaining solution where basis utility replaces the disagreement point and ξ-maximal aspirations replace d-maximal aspirations. There is an analog to proposition 5.1 for Kalai-Smorodinsky bargaining. Thus equal split, Kalai-Smorodinsky and proportional bargaining are all monotonic and have proportional qualities. The relationship between monotonicity and proportionality shown by Kalai (1977) also appears in these endogenously proportional solutions. However, Kalai's (1975) solutions are homogeneous and not translation invariant because the exogenous proportionality vector is translation dependent.

There is an essential similarity between equal split, Kalai-Smorodinsky and proportional bargaining. Given efficiency, symmetry, transformation invariance and the appropriate monotonicity axiom, the salience of any two reference points identifies a solution. These solutions are intuitive. Two points determine a line. The solution is the intersection of this line with the efficient bargaining surface. Monotonicity merely identifies this intersection mathematically. This simplicity can seem like a weakness. There is little subtlety and no sense of marginal equilibrium. However, this simplicity is likely a strength. Schelling writes

> [G]ame characteristics that are relevant to sophisticated mathematical solutions ... might not have the power of focusing expectations and influencing the outcome ... except when the same solution can be reached by an alternative less sophisticated route. (1960: 113, edited)

Indeed, the less sophisticated the nonmathematical route, the greater the power of focusing expectations might reasonably be. The salience of two reference points makes a monotonic solution a focal point.

5.3 Pure bargaining choices

Section 5.2 provides a reference-point based approach focusing expectations in bargaining. Given such a focus, bargaining mechanisms consistent with these expectations might then be favored. If the salience of only the disagreement point is thought to guide or allow expectations to move toward Nash bargaining, this approach becomes more complete.

Figure 4 illustrates another approach to solution selection, one based on the characteristics of bargaining outcomes. The primary choice is between equal and proportional gain. There is no 'soft' variant of proportional pure bargaining because the disagreement point is essential to proportional outcomes. The next choice is then between the soft and hard variants of equal gain bargaining, with equal split being the soft bargaining solution. There are two variants of equality-based hard bargaining. Monotonic Kalai-Smorodinsky bargaining provides gains that are in strict equal proportion relative to the disagreement and maximal aspirations points. IIA-based Nash outcomes deviate from this strict equality when and to the extent that doing so will increase the product of player's payoffs relative to the disagreement point.

The nature of noncooperative implementations provides the last approach to comparing pure bargaining models. Nash bargaining results when players have equal participation in the game (e.g., Binmore, Rubinstein and Wolinsky (1986) and Hart and Mas-Colell (1996)). Theorem 4.1 and Feldman (2002) show that proportional bargaining results when players's probability of proposing is proportional to their expected payoff. Moulin (1984) shows that Kalai-Smorodinsky bargaining can be implemented in a game where players first bid *probabilities* for the right to propose, the player with the highest bid proposes first and the second player has the right to make a last counteroffer with the probability of the winning bid. Finally, Huck and Oechssler (1999) find equal split is the equilibrium outcome in an evolutionary setting and Lopomo and Huck (2001) find equal split in cases of interdependent preferences. None of these models can be considered inherently more rational than the others, but each has aspects that make it more relevant to particular bargaining environments.

6 Conclusion

Recognition of basis utility expands pure bargaining theory with two new endogenous proportional solutions. Equal split provides a model of commonly observed experimental outcomes. Pure proportional bargaining is the pure bargaining version of the TU proportional value of Ortmann (2000) and the NTU proportional value of Feldman (1999, 2002). With the Kalai-Smorodinsky (1975) bargaining solution they form a versatile family of monotonic pure bargaining models. Basis utility also allows moves of nature in noncooperative games, such as the selection of proposers and trembles, to be conditioned on payoffs without creating translation invariant equilibria.

Endogenous proportionality was lost without basis utility, which was obscured in part by the mechanics of translation invariance. Basis utility expands the range of interpersonal comparisons that can be made in the expected utility framework beyond those of Kalai (1977) and Myerson (1977). Thompson's (1998: 197) negotiation text sees consensus interpersonal comparison and proportionality as "the heart of equity theory." Proportionality, here, should be taken in the sense of the ratios of Kalai (1977) and propositions 2.2 and 5.1 and not simply proportional bargaining. This is not a new idea. Moulin (1999) quotes Aristotle in his survey of social choice allocation rules: "Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences."

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Item	Utility to Bill	Utility to Jack
Bill's items:		
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1
Jack's items:		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

Table 1: Bargaining example from Nash (1950).

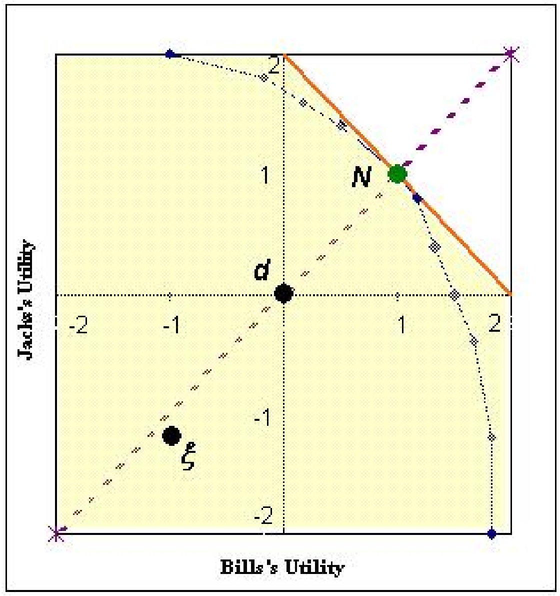

Figure 1: Nash (1950) example normalized and including basis utility point $\xi = (-1, -1.2)$.

	W	
	L	R
U	3, 3, 5 + x	1, 1, 1 + x
D	1, 1, 1 + x	2, 2, 1 + x

	E	
	L	R
U	2, 2, 1 + x	1, 1, 1 + x
D	1, 1, 1 + x	4, 4, 2 + x

Figure 2: Coordination game Γ_2.

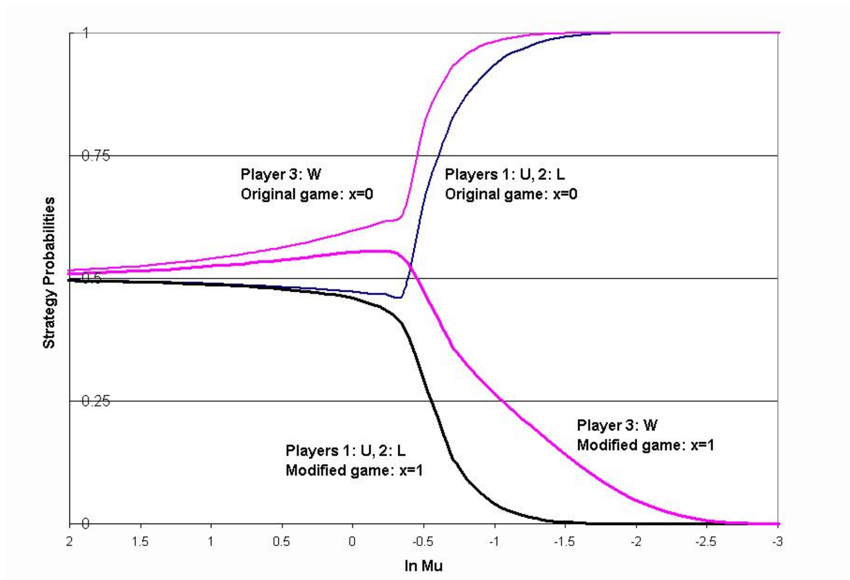

Figure 3: Quantal response graph for Γ_2 showing apparent translation dependence.

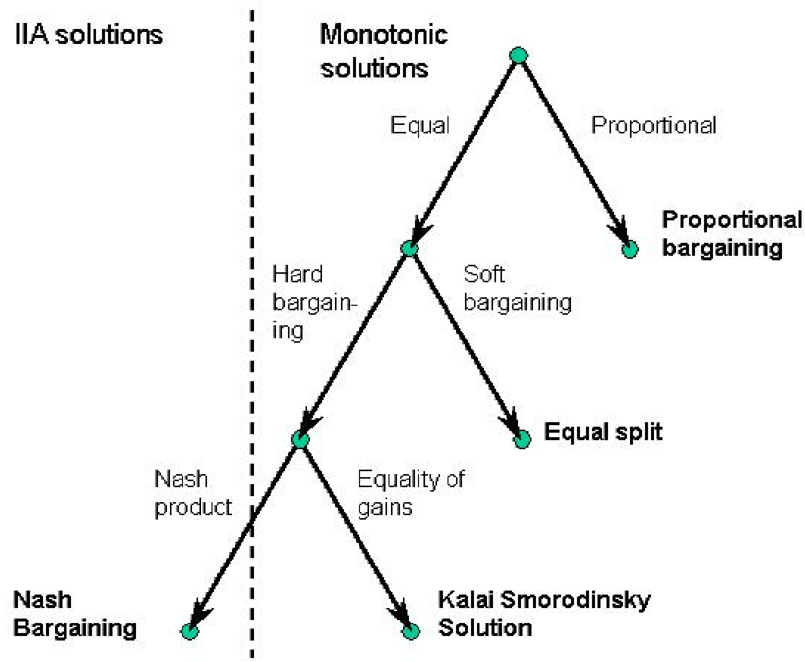

Figure 4: Tree of IIA and monotonic bargaining solutions.