Cutting a Pie Is Not a Piece of Cake

Julius B. Barbanel
Department of Mathematics
Union College
Schenectady, NY 12308
barbanej@union.edu

Steven J. Brams
Department of Politics
New York University
New York, NY 10003
steven.brams@nyu.edu

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Abstract

Gale (1993) posed the question of whether there is necessarily an undominated, envy-free allocation of a pie when it is cut into wedge-shaped pieces or sectors. For two players, we give constructive procedures for obtaining such an allocation, whether the pie is cut into equal-size sectors by a single diameter cut or into two sectors of unequal size. Such an allocation, however, may not be equitable—that is, give the two players exactly the same value from their pieces.

For three players, we give a procedure for obtaining an envy-free allocation, but it may be dominated either by another envy-free allocation or an envy-causing allocation. A counterexample shows that there is not always an undominated envy-free allocation for four or more players if the players' preferences are not absolutely continuous with respect to each other. If we do make this assumption, then the existence question remains open for four or more players. For three players, the question of whether there exists an undominated envy-free allocation is open whether or not the players' preferences are absolutely continuous with respect to each other.

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Introduction

The general problem of fair division, and the specific problem of cutting a cake fairly, have received much attention in recent years (for overviews, see Brams, Taylor, and Zwicker, 1995; Brams and Taylor, 1996; Robertson and Webb, 1998; and Barbanel and Brams, 2004). Cutting a pie into wedge-shaped sectors, by contrast, has received far less attention, though it would seem that the connection between cake-cutting and pie-cutting is close. Mathematically, if a cake is a line segment, it becomes a pie when its endpoints are connected to form a circle.

In this paper, we assume that a pie is a disk, or filled-in circle, and all cuts are made between the center and a point on the circumference (as one would cut a real pie). These cuts divide a pie into sectors, exactly one of which is given to each player.

Gale (1993) asked whether there is necessarily an allocation of the sectors that is envy-free and undominated. An *envy-free allocation* is one in which each player receives a sector that it believes is at least as desirable as that which any other player receives. An *undominated allocation* is one for which there is no other allocation in which at least one player receives a sector it strictly prefers and the other players receive sectors they value at least as much.

We answer Gale's question affirmatively for two players by specifying constructive procedures that yield envy-free and undominated allocations. We do this for both wedge cuts and diameter cuts, when the wedges are 180-degree sectors that divide the pie exactly in half in surface area (or volume). Unless otherwise stated, when we are discussing wedge cuts and say that an allocation is undominated, we mean that it is undominated with respect to wedge-cut allocations. A similar statement holds for diameter cuts.

We begin by specifying a procedure for diameter cuts that produces an equitable allocation. An *equitable allocation* is one in which each player gets exactly the same value from its piece as the other players get from their pieces. We show that envy-free allocations that are equitable need not be undominated, even for two players.

Next, we describe a variation of this procedure that yields an allocation that is envy-free and undominated but not necessarily equitable. Then, for two players, we give a wedge procedure that produces an allocation that is envy-free and undominated.

For three players, we give a procedure for cutting a pie into three sectors such that the resulting allocation is envy-free but not necessarily equitable or undominated. In fact, we give an example of an envy-free allocation produced by this procedure that is dominated by another envy-free allocation as well as an allocation that causes envy, both of which are undominated. For pies in general, however, we do not know whether there always exists a three-player undominated, envy-free allocation.

For four players, surprisingly, we do have an answer: There may not exist such an allocation. While the players' preferences are continuous in our counterexample, they are not absolutely continuous with respect to each other. We will say more about this later.

To summarize, for two players we have a procedure that yields an undominated and envy-free allocation of a pie, and for four players we know that such an allocation may not exist when preferences are not absolutely continuous with respect to each other. For three players, Gale's question remains open.

Pie-Cutting Procedures

To begin the analysis, we make the following assumptions:

1. *Goals*. The goal of the players is to maximize the minimum-value pieces (*maximin pieces*) they can guarantee for themselves, regardless of what the other players do.

Remark. This implies that the players are *risk-averse*: They never choose strategies that might yield them more-valued pieces if these strategies entail the risk of their getting less-valued pieces.

2. *Measures*. Each player has a finitely additive, nonatomic probability measure over the pie.

Remark. Roughly speaking, the value of disjoint pieces of pie can be summed, any piece of pie that has positive value to a player has a subpiece that has smaller positive value to that player, and each player assigns a value of 1 to the whole pie. Notice that the non-atomic nature of the measures guarantees that every player's measure assigns value zero to each radius of the pie. This implies that the preferences of the players, which are based on their measures, are continuous, enabling us to invoke the Intermediate-Value Theorem. Thus, for example, imagine two adjacent wedge-shaped pieces, and some player values one piece more than the other. As the boundary separating these pieces rotates along the circumference from the less-valued piece to the more-valued piece, there will be some intermediate point where the player values the two pieces equally.

3. *Incomplete information*. The players do not know the preferences of other players.

We next specify procedures that lead to an envy-free allocation of a pie using a single diameter cut. One procedure yields an equitable allocation, the other an undominated allocation. As we will see later, the undominated allocation may be dominated by an allocation that uses wedge cuts.

At various points in the arguments that follow, we will claim that a certain graph has a maximum value. In all cases we consider, this is always justified by the Extreme-Value Theorem: A continuous function on a finite closed interval achieves a maximum.

Two-Player Diameter Procedures

We give below two rules (D1 and D2) that give an envy-free and equitable allocation, but this allocation need not be undominated. When we substitute revised rules D1' and D2' for D1 and D2, respectively, we obtain an envy-free and undominated allocation, but this allocation need not be equitable.

D1. *Randomly choose a diameter of the pie, and randomly assign one of the two pieces determined by this diameter to Player A and the other to Player B. Rotate the diameter 360 degrees. As it rotates, draw two graphs. At each point in the rotation, a red graph indicates the value that Player A assigns to its piece, and a blue graph indicates the value that Player B assigns to its piece.*

Theorem 1. *The red graph and the blue graph have at least one point of intersection.*

Proof. Suppose, by way of contradiction, that this is not so. Assumption 2 implies that the red and blue graphs are graphs of continuous functions. If two such graphs do not intersect, then one of the graphs must always be above the other. Assume, without loss of generality, that the red graph is always above the blue graph. Since this is true at the beginning and at the 180-degree point, additivity tells us that Player A assigns greater value to the whole pie than Player B does. But this contradicts our assumption that all measures assign value 1 to the whole pie. Q.E.D.

Theorem 2. *For at least one point of intersection, both players will get a common value of at least 50%.*

Proof. Any point of intersection corresponds to an allocation in which both players get a common value. It is easy to see that intersection points come in pairs, separated by 180 degrees. If both players get less than 50% at some intersection point, then both players must get more than 50% at this intersection's 180-degree pair. Q.E.D.

D2. *Choose an intersection point that maximizes the common value of the players. Make the diameter cut at this point, allocating to each player its preferred half, or either half if the maximum is 50% for both players.*

Theorem 3. *The resulting allocation is envy-free and equitable.*

Proof. Immediate from the construction.

We next extend D1 and revise D2 to give a rule that ensures that an envy-free allocation is undominated.

D1'. *Randomly choose a diameter of the pie, and randomly assign one of the two pieces determined by this diameter to Player A and the other to Player B. Rotate the diameter 360 degrees. As it rotates, draw two graphs. At each point in the rotation, a red graph indicates the value that the Player A assigns to its piece, and a blue graph indicates the value that Player B assigns to its piece. Use these red and blue graphs to draw two new graphs, one yellow and one green. The yellow graph is the graph that gives the minimum of the red and blue values at each point, and the green graph is the graph that gives the maximum of the red and blue values at each point.*

D2'. *Make the diameter cut that corresponds to the maximum value of the yellow graph. If the maximum of the yellow graph occurs at more than one point, choose the point that has the largest (or a tied-for-largest) green value, and make the diameter cut at this point. Allocate to each player its preferred half.*

We note, by assumption 2, that the red and blue graphs, and therefore the yellow and green graphs, are graphs of continuous functions. Hence, the Extreme-Value Theorem applies.

Theorem 4. *The resulting allocation is envy-free and undominated.*

Proof. Assume, without loss of generality, that the resulting allocation occurs at a point where the yellow graph (which is the minimum of the red and blue graphs) agrees with the red graph, and that the value of the piece of pie corresponding to this point is $x\%$. Thus, Player A obtains $x\%$, which must be at least 50% by Theorem 2. Then Player B must obtain $y\% \geq x\%$. Hence, the resulting allocation is envy-free. By D2', whichever player receives the less-valued piece cannot obtain more than $x\%$; and if one player receives *x%*, the other player cannot obtain more than *y%*. Hence, the resulting allocation is undominated. Q.E.D.

The construction given by D1 and D2, which maximizes the common value of the players, does not preclude there being a better allocation for both players if they value their pieces differently. Hence, the resulting allocation need not be undominated. Nor does the construction given by D1' and D2' require that the players receive pieces they value equally, so the resulting allocation need not be equitable.

We next present two examples. The first yields (i) an allocation that is envy-free and equitable but not undominated and (ii) an allocation that is envy-free and undominated but not equitable, which proves the following:

Theorem 5. *An envy-free allocation that is equitable need not be undominated, and one that is undominated need not be equitable.*

Proof. *Example 1.* We associate points on the circumference of the pie with the numbers on a clock, indicating the degrees of each sector in parentheses. Players A and B associate the following values with two different sectors each that comprise 12 hours:

Player A: 11-1 o'clock (60°)—90%; 1-11 o'clock (300°)—10%.

Player B: 3-9 o'clock (180°)—60%; 9-3 o'clock (180°)—40%.

We assume that each player's valuation is uniformly distributed within each sector. The initial assignments, as given by D1 or D1', are the diameter from 12 to 6 o'clock, with Player A taking the 12-6 o'clock piece and Player B taking the 6-12 o'clock piece. In addition, we assume the rotation is clockwise. The corresponding graphs are as in Figure 1 where, as labeled, the solid graph corresponds to Player A's valuation and the dashed graph corresponds to Player B's valuation. (Thus, the solid graph is what we previously called the red graph, and the dashed graph is what we previously called the blue graph.)

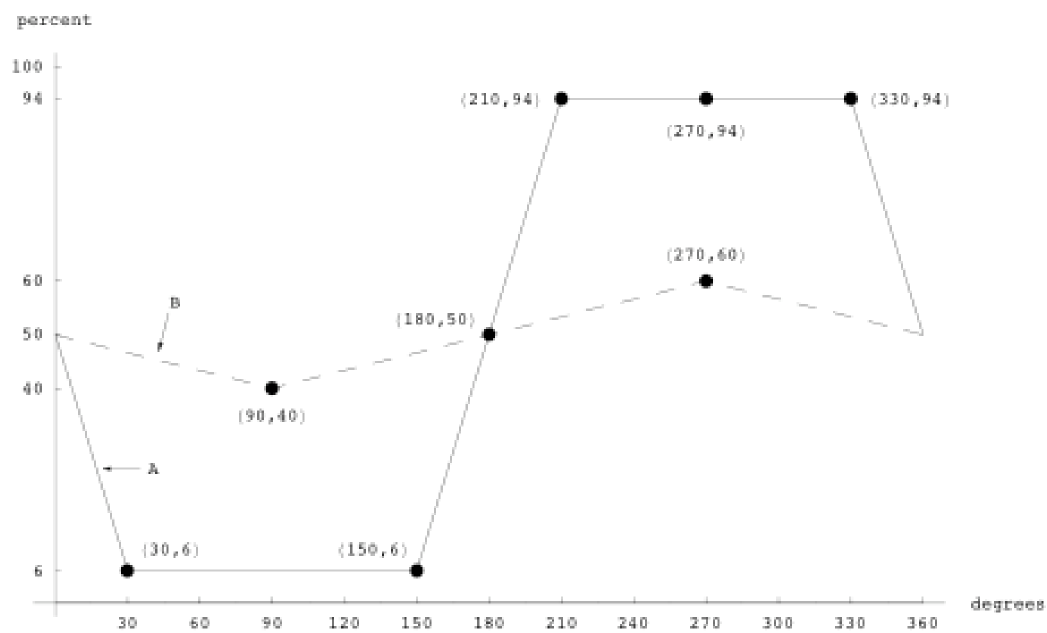

Figure 1

Percent Allocations of Pie to Players A and B in Example 1
as Diameter Cut Rotates Clockwise 360 Degrees from 12-6 O'Clock Position

The two graphs intersect at three points: (0, 50), (180, 50), and (360, 50). Of course, the first and third of these points correspond to the same allocation. Thus, there are two different possible allocations that result when rules D1 and D2 are applied, both of which result in each player's receiving what it sees as 50% of the pie. Thus, each of these allocations is envy-free and equitable. But it is clear from Figure 1 that each of these allocations is dominated: Any rotation greater that 180 degrees and less than 360 degrees results in an allocation that is better for both players.

In particular, note that the point (270, 60) is the maximum point on the minimum graph (i.e., the graph that we previously called yellow). This is the only point at which the maximum is attained. Thus, the application of rules D1' and D2' results in a rotation of 270 degrees, giving an allocation in which Player A receives 60% of the pie and Player B receives 94% of the pie, each according to its own valuation. While this allocation is envy-free and undominated, it is clearly not equitable. Q.E.D.

We next show that an envy-free allocation may satisfy the two properties that it did not simultaneously satisfy in Example 1.

Theorem 6. *An envy-free allocation may be equitable and undominated.*

Proof. *Example 2*. Players A and B associate the following values with two different sectors that comprise 12 hours:

Player A: 12-6 o'clock (180º)—40%; 6-12 o'clock (180º)—60%.

Player B: 4-8 o'clock (30º)—60%; 8-4 o'clock (330º)—40%.

As in Example 1, the initial assignments, as given by D1 and D1', are the diameter from 12 to 6 o'clock, with Player A taking the 12-6 o'clock piece and Player B taking the 6-12 o'clock piece. The rotation is clockwise. The corresponding graphs are as in Figure 2 where, as in Example 1, the solid graph corresponds to Player A's valuation and the dashed graph corresponds to Player B's valuation.

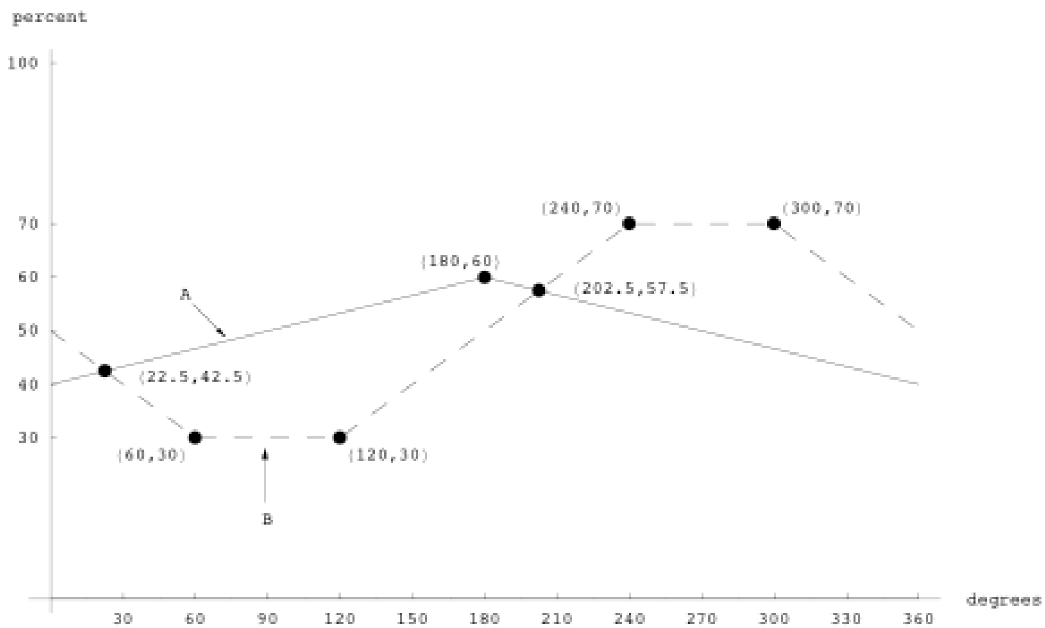

Figure 2

Percent Allocations of Pie to Players A and B in Example 2
as Diameter Cut Rotates Clockwise 360 Degrees from 12-6 O'Clock Position

The two graphs intersect at the points (22.5, 42.5) and (202.5, 57.5). (The allocations corresponding to these points can be viewed as being obtained from each other by having the players switch pieces: They are 180 degrees apart, and the sum of each player's valuation of its piece in these two allocations is 100%.) Clearly, (202.5, 57.5) corresponds to an envy-free allocation and (22.5, 42.5) does not. The former is the envy-free and equitable allocation obtained from rules D1 and D2. In this allocation, the two players receive pieces of pie that they value equally. This common value is 57.5% and occurs at a rotation of 202.5 degrees.

In considering the result of applying rules D1' and D2', we see that something very curious—and different from Example 1—occurs. The point (202.5, 57.5) is the maximum point on the minimum graph; hence, the application of these rules results in a rotation of 202.5 degrees and an allocation in which each player receives 57.5% of the pie, according to each's own valuation. Consequently, rules D1 and D2 result in the same allocation as do rules D1' and D2'. This allocation is envy-free, equitable, and undominated. Q.E.D.

Examples 1 and 2 together show that an envy-free allocation may be (i) equitable but not undominated, (ii) undominated but not equitable, or (iii) both undominated and equitable. An interesting question is whether allocations of wedge-shaped pieces can dominate allocations of diameter pieces that are obtained using rules D1' and D2'. The answer is "yes," as we will show in the next section.

Two-Player Wedge Procedure

In this section, we give four rules, W1, W2, W3, and W4, for obtaining a wedge allocation that is undominated and envy-free. After doing so, we use Example 2 to show that such an allocation can dominate an allocation of the sort obtained in the previous section.

W1. *Player A places two knives at radii such that the pieces of pie so determined are each 50% of the pie, according to Player A. Player A rotates the knives 360 degrees around the pie, maintaining the 50% sizes, in its view.*

W2. *Player B chooses the position of the knives such that one of the pieces so determined is of maximal size in its view.*

It is straightforward to show that the aforementioned piece will have a value of at least 50% to Player B. Thus, if we cut the pie at these knife locations and give Player B its preferred piece and Player A the other piece, then the allocation is envy-free. However, it may be necessary to perform an additional operation to ensure that the allocation is undominated.

W3. *Player B places the knives at the boundary of its maximal piece obtained in W2. Suppose that, in Player B's view, this piece is y% of the pie. Player B rotates the knives 360 degree around the pie, maintaining this y% size in its view.*

W4. *Player A chooses the position of the knives such that the other piece of the pie (i.e., not the one whose size, according to Player B, is being maintained at y%) is of maximal size in its view. At this maximum value for Player A in the rotation, give Player B the sector whose size it was maintaining at y%, and give Player A the other sector.*

Theorem 7. *The resulting allocation is envy-free and undominated.*

Proof. Suppose that Player A views its piece as x% of the pie. By construction, $x \geq 50$ and, as noted above, $y \geq 50$. Thus, the allocation is envy-free. Also, by construction, if Player B obtains y%, then Player A cannot obtain more than x%; and if Player A obtains 50%, then Player B cannot obtain more than y%. It follows that if Player A obtains x%, then Player B cannot obtain more than y% (because $x \geq 50$). Hence, the allocation is undominated. Q.E.D.

It may be that a wedge allocation dominates an allocation obtained using rules D1' and D2' (which need only be undominated with respect to diameter cuts). To illustrate, we return to Example 2. As we saw, rules D1' and D2' yield a diameter cut after a rotation of 202.5 degrees, and this allocation gives Players A and B a common value of 57.5%, according to each's own valuation. This is dominated by the wedge allocation that gives the 8-4 o'clock sector to Player A and the 4-8 o'clock sector to Player B. This allocation gives Player A a value of (2/3)(60%) + (2/3)(40%) = 66 2/3% and Player B a value of 60% of the pie.

Three-Player Wedge Procedure

We next give a procedure, whose rules are stated in the next paragraph, for dividing a pie into three sectors that results in an envy-free allocation. Unlike the two-player procedures, this procedure does not ensure either an equitable or an undominated allocation.

Rules of three-player wedge procedure. Player A rotates three knives around a pie, each along a radius, maintaining a 1/3-1/3-1/3 allocation for itself. Player B calls "stop" when it thinks two of the pieces are tied for largest, which must occur for at least one set of positions in the rotation (see below). The players then choose pieces in the order C first, B second, and A third.

Theorem 8. *The three-player wedge procedure yields an envy-free, but not necessarily equitable or undominated, allocation.*

Proof. To show that there must be at least one set of knife positions in the rotation such that Player B thinks there are two pieces that tie for most-valued, let us call the three pieces determined by the beginning positions of the knives piece *i*, piece *ii*, and piece *iii*. (These pieces will change as Player A rotates the knives.) Let Player B specify its most-valued piece at the start of the rotation. If there is a tie, then we are done. If not, then Player A begins rotating the three radial knives. We assume, without loss of generality, that Player B's most-valued piece at the start of the rotation is piece *i*, and that Player A rotates the three knives in such a way that piece *i* moves toward the original position of piece *ii*. Because, in Player A's view, each of the three pieces is 1/3 of the pie, piece *i* will eventually occupy the position of the original piece *ii*. At this point, piece *iii* occupies the original position of piece *i*, and hence Player B must think that this new piece *iii* is the largest piece. Because, in Player B's view, piece *i* starts out largest and another piece becomes largest as the rotation proceeds, it follows from continuity (assumption 2) that there must be a position in the rotation when Player B views two pieces as tied for largest.

To see that the procedure gives an envy-free allocation, note that the first player to choose, Player C, can take a most-valued piece, so it will not be envious. If Player C takes one of Player B's tied-for-most-valued pieces, Player B can take the other one; otherwise, Player B can choose either of its two tied-for-most-valued pieces. Because Player A values all three pieces equally, it does not matter which piece it gets.

But the existence of such an allocation does not imply that there is not another allocation that dominates it. For example, a rotation of the three knives by Player A could break Player B's tie of two largest pieces in a way that gives some players more-valued pieces and no player a less-valued piece. Finally, there is nothing in the construction that ensures that all three players will value their pieces equally, so the procedure does not, in general, give equitability. Q.E.D.

We next give an example to illustrate the three-player wedge procedure. Not only does it show that that an envy-free allocation may be dominated by another envy-free allocation, but it also shows that it may be dominated by an *envy-causing allocation*, in which at least one player thinks that another player receives a larger portion than it does.

Theorem 9. *If there are three players, an envy-free allocation may be dominated by another envy-free allocation and an envy-causing allocation.*

Proof. *Example 3*. Players A, B, and C associate the following values with two different sectors each:

Player A: 12-4 o'clock (120º)—2/3; 4-12 o'clock (240º)—1/3.

Player B: 12-4 o'clock (120º)—1/2; 4-12 o'clock (240º)—1/2.

Player C: 12-2 o'clock (60º)—1/2; 2-12 o'clock (300º)—1/2.

We start by applying the 3-person wedge procedure. One set of positions of Player A's three knives that cuts the pie into three equal-valued sectors for A is (1, 3, 8) o'clock, because A obtains the following values from each sector:

1-3 o'clock: (1/2)(2/3) = 1/3;

3-8 and 8-1 o'clock: (1/4)(2/3) + (1/2)(1/3) = 1/3 each.

If (1, 3, 8) o'clock are, in fact, the *initial* positions of Player A's three knives, then Player B will call "stop" immediately, because two of the sectors (3-8 and 8-1 o'clock) tie for largest for B [value: (1/4)(1/2) + (1/2)(1/2) = 3/8 each]. By contrast, the 1-3 o'clock sector is worth (1/2)(1/2) = 1/4 to B.

Now Player C will choose the 8-1 o'clock sector, whose value to it is (2/5)(1/2) + (1/2)(1/2) = 9/20, which is more than the 1-3 o'clock sector [value: (1/2)(1/2) + (1/10)(1/2) = 3/10] or the 3-8 o'clock sector [value: (1/2)(1/2) = 1/4]. Next, Player B will choose the 3-8 o'clock section (3/8), which is more than the 1-3 o'clock sector (1/4); and Player A will be left with the 1-3 sector (1/3). In sum, the 3-person wedge procedure yields values of

(1/3, 3/8, 9/20) = (.333…, .375, .450) to (A, B, C) in sectors (1-3, 3-8, 8-1). (i)

This is an envy-free allocation, as we have shown. However, it is dominated by cutting the pie at (2, 6, 12), which yields values of

(5/12, 3/8, 1/2) = (.416…, .375, .500) to (A, B, C) in sectors (2-6, 6-12, 12-2). (ii)

This is not only an envy-free allocation but also an undominated one. (We leave the proof of the latter to the reader, because our main purpose is to illustrate that one envy-free allocation may dominate another one.)

We next show that (i) may be dominated by an envy-causing allocation. Consider the undominated, envy-free allocation (ii), and switch the 2 o'clock cutpoint to 2:30 o'clock. The resulting allocation yields values of

(1/3, 3/8, 21/40) = (.333…, .375, .525) to (A, B, C) in sectors (2:30-6, 6-12, 12-2:30). (iii)

This allocation is not envy-free: A envies C for getting what it thinks is 5/12 in the 12-2:30 o'clock sector, which is more than its 1/3 allocation in the 2:30-6 o'clock sector. But (iii) dominates wedge-procedure allocation (i), which is envy-free. Q.E.D.

Although envy-causing allocation (iii) dominates envy-free allocation (i), it does not dominate envy-free allocation (ii). Indeed, (iii), like (ii), is undominated, which we leave for the reader to show.

We leave open the question of whether there is always a three-player undominated, envy-free allocation of a pie. For cake, the answer is "yes," and there are two procedures, using two cuts, for finding such an allocation (Stromquist, 1980; Barbanel and Brams, 2004). In fact, *every* envy-free allocation of a cake using the minimal number of cuts (*n*-

1 if there are n players) is undominated (Gale, 1993; see also Brams and Taylor, 1996, pp. 150-151), but there is no known procedure for finding such an allocation if $n > 3$.

Four Players: There May Be No Undominated, Envy-Free Allocation

In previous sections, we made no assumption about whether the players' measures are absolutely continuous with respect to each other. Measures on a pie are *absolutely continuous with respect to each other* if, whenever a piece of pie has positive measure to one player, it has positive measure to all players. We will say that preferences are *absolutely continuous with respect to each other* if the underlying measures are absolutely continuous with respect to each other. When this property is not satisfied, there may be no allocation that is both envy-free and undominated.

Theorem 10. *If there are four players, there exists a pie for which there is no allocation that is both envy-free and undominated.*

Proof. *Example 4*. The players associate the following values with two different sectors each, but these sectors do not comprise 12 hours:

- **Players A** & **B**: 12-3 o'clock (90º)—50%; 6-9 o'clock (90º)—50%;
- **Players C** & **D**: 3-6 o'clock (90º)—51%; 9-12 o'clock (90º)—49%.

Suppose, by way of contradiction, that P is an allocation of this pie to Players A, B, C, and D that is envy-free and undominated.

Claim 1. Allocation P cannot give Player A or Player B a piece of both the 12-3 o'clock sector and the 6-9 o'clock sector.

Proof. Assume this is not so. Then allocation *P* must give Player A or Player B either all the 3-6 o'clock sector or all the 9-12 o'clock sector. Assume, without loss of generality, that Player A receives all the 9-12 o'clock sector. Then envy-freeness demands that Players C and D receive equal portions of the 3-6 o'clock sector. But then Players C and D each view their pieces as at most 25 1/2% of the pie and are, therefore, envious of Player A, which receives 49% of the pie in their view. This contradicts our assumption that *P* is envy-free and thus establishes the claim.

Claim 2. Allocation P cannot give both Player A and Player B a piece of either the12-3 o'clock sector or the 6-9 o'clock sector.

Proof. Assume this is not so and suppose, without loss of generality, that allocation *P* gives both Player A and Player B a piece of the 12-3 o'clock sector, and that Player B's piece is clockwise of Player A's piece. By Claim 1, neither Player A nor Player B can receive a piece of the 6-9 o'clock sector.

Assume, without loss of generality, that Player C's piece is the next piece clockwise from Player B's piece. Since *P* is undominated, the knife separating Player B's and Player C's pieces must be at 3 o'clock (else a movement of this knife would produce an allocation that dominates allocation *P*). Likewise, the knife separating Player D's and Player A's pieces must be at 12 o'clock.

Envy-freeness implies that the knife separating Player A's and Player B's pieces must be at 1:30, splitting the 12-3 o'clock sector equally. Thereby, each of these players gets a piece that it views as 25% of the pie.

Where is the knife separating Player C's and Player D's pieces? This knife must be at 7:30, thereby splitting the 6-9 o'clock sector into equal pieces of size 25% of the pie from the perspectives of Players A and B. Any other position of this knife would make Players A and B envy either Player C or Player D. But then Player D will be envious of Player C, because Player D views its piece as 49% of the pie, but it views Player C's piece as 51% of the pie. This contradicts our assumption that *P* is envy-free and thus establishes the claim.

Continuing with the proof of the theorem, it follows from the two claims that we may assume, without loss of generality, that Player A receives a piece from the 12-3 o'clock sector, that Player B receives a piece from the 6-9 o'clock sector, that Player C's piece is the next piece clockwise from Player A's piece, and that Player D's piece is the next piece clockwise from Player B's piece. An argument similar to that used in the proof of Claim 2 shows that because allocation *P* is assumed to be undominated, the knives separating the players' pieces must be at 3, 6, 9, and 12 o'clock. It follows that allocation *P* gives the players pieces that they value as follows:

Players A & B: 50% each; **Player C**: 51%; **Player D**: 49%.

Because Player D views Player C's piece as being 51% of the pie, Player D envies Player C. This contradicts our assumption that *P* is envy-free and so proves the theorem. Q.E.D.

It is trivial to extend the proof of Theorem 10 to more than four players. For example, for five players, simply add a new sector that Player E views as 100% of the value of the pie and the other four players see as valueless.

The measures that underlie the players' preferences in Theorem 10 are not absolutely continuous with respect to each other. By contrast, all our earlier theorems required no assumption about absolute continuity—they held with or without this assumption. This leaves two open questions:

Open Question 1. For three players, does there always exist an undominated, envy-free allocation of a pie (with or without the assumption that the preferences are absolutely continuous with respect to each other)?

Open Question 2. For four or more players, does there always exist an undominated, envy-free allocation of a pie if the players' preferences are absolutely continuous with respect to each other?

We think it intriguing that there is

- a definite answer to Gale's question (*yes*) when there are two players with or without absolute continuity;
- no answer yet when there are three players without absolute continuity;
- a definite answer (*no*) when there are four or more players without absolute continuity; and
- no answer yet when there are four or more players with absolute continuity.

Conclusions

We began by describing two envy-free pie-cutting procedures for two players using diameter cuts, one of which gave an equitable allocation and the other of which gave an undominated allocation. The equitable allocation need not be undominated, and the undominated allocation need not be equitable. Moreover, even the undominated allocation need not be undominated with respect to wedge cuts. However, we showed that there exists a procedure that yields an undominated, envy-free allocation using wedge cuts, which answers Gale's question affirmatively for two-player allocations of a pie.

We next described a three-player envy-free wedge procedure, but the allocation it gives may be dominated. In fact, we gave an example showing that an envy-free allocation obtained with this procedure is dominated by both another envy-free allocation and an envy-causing allocation, both of which are undominated. Finally, we concluded with two open questions whose answers would fill major gaps in our understanding of pie-cutting, which is certainly not a piece of cake.

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