Characterizing Neutral Aggregation on Restricted Domains

Extended abstract

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Abstract

The purpose of this paper is to give an exhaustive characterization of pairs of individual preference domains and alternative-neutral Arrovian social welfare functions (SWF) that aggregate profiles of relations from the domain to transitive relations. Sen [2] characterizes such domains for majority by value restrictions on the domain, we show that these restrictions are necessary and sufficient for transitive aggregation under any neutral monotone SWF and that they are neither necessary nor sufficient for neutral non monotone SWFs. We give an exhaustive characterization through new value restrictions and a notion of embedding we introduce for simple games. We also show that Sen's value restrictions as well as a stronger restriction we introduce are preserved in the image, and that our domain restrictions are neither necessary nor sufficient for non neutral SWFs.

Sen and Pattanaik showed that for any uneven number of voters a necessary and sufficient condition for majority to produce a transitive relation is a condition on the domain they called 'value restriction' or 'acyclicity'. How important is domain acyclicity for social welfare functions (SWFs) other than majority? Our first result shows that acyclicity is necessary and sufficient for transitivity under any neutral monotone SWF. We also prove that acyclicity is preserved in the image hence for repeated aggregation: in a multi-tier voting system intransitivity is avoided as long as the voters at the base level choose preferences from an acyclic domain and on each tier the committees vote via a neutral monotone SWF.

Our second result shows that acyclicity is neither necessary nor sufficient for transitivity under a non monotone neutral SWF. We prove this through an explicit construction of a SWF defined by a voting game without dummies called antidictator and a cyclic domain called the unicycle domain. We show that antidictator is transitive for the unicyclic domain but is intransitive on certain

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acyclic domains. We introduce the notion of *strong acyclicity* and show that it ensures transitivity for any neutral SWF. These results refute a conjecture by Maskin [1] that acyclicity is necessary but insufficient for transitivity for neutral SWFs without dummies apart from majority.

We say that a SWF f' is *embedded* in another SWF f if they are equivalent when voters of f vote in a certain block pattern. We show that any non dictatorial SWF embeds majority or parity or antidictator on three voters (Maj_3, $Prty_3$, $AntiD_3$). We prove the following theorem:

Theorem 1. *Let f be a neutral non monotone SWF defined by a game G and let $\mathfrak{C}$ be a domain of linear orders.*

1. *If G embeds $Prty_3$ or Maj_3 then:*
   (a) *$\mathfrak{C}$ is transitive iff it is strongly acyclic.*
   (b) *$\mathfrak{C}$ strongly acyclic implies $Im(f)$ strongly acyclic.*
2. *If G does not embed $Prty_3$ or Maj_3 then:*
   (a) *$\mathfrak{C}$ is transitive iff it is mixed unicyclic and strongly acyclic.*
   (b) *$Im(f)$ is transitive iff $\mathfrak{C}$ is strongly acyclic.*

Our results give the following picture: dictatorial SWFs are transitive on any domain, non dictatorial monotone SWFs are transitive only on acyclic domains, non monotone SWFs embedding $Prty_3$ or Maj_3 are transitive only on strongly acyclic domains. Non monotone SWFs that do not embed $Prty_3$ or Maj_3 are transitive on mixed unicyclic and strongly acyclic domains.

In a multi-tier voting system it is possible that the domain on a high tier, namely the image of the domain under repeated aggregation, be larger than the original domain. In such a case it may hypothetically happen that on higher tiers the aggregation becomes intransitive. The theorem shows that if the domain is transitive in the first tier, it will either remain transitive for all tiers if the domain is strongly acyclic or intransitivity will show up in the second tier.

References

[1] Maskin, Eric S., (1995) 'Majority Rule, Social Welfare Functons, and Game Forms' In *Choice Welfare , and Development* Edited by K. Basu, P. Pattanik, and K. Suzumura, Clarendon Press Oxford.

[2] Sen, Amartya K., (1966) 'A possibility theorem on majority decisions', *Econometrica* **34**:491-499.