BOUNDED RATIONALITY AND REPEATED NETWORK FORMATION*

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Abstract : We define a finite-horizon repeated network formation game with consent, and study the differences induced by different levels of individual rationality. We prove that perfectly rational players will remain unconnected at the equilibrium, while this result does not hold when, following Neyman (1985), players are assumed to behave as finite automata. We define two types of equilibria, namely the Repeated Nash Network (RNN), in which the same network forms at each period, and the Repeated Nash Equilibrium (RNE), in which different networks may form. We state a sufficient condition under which a given network may be implemented as a RNN. Then, we provide structural properties of RNE. For instance, players may form totally different networks at each period, or the networks within a given RNE may exhibit a total order relationship. Finally we investigate the question of efficiency for both Bentham and Pareto criteria.

Key words : Repeated network formation game, Two-sided link formation costs, Bounded rationality, Automata.

JEL classification : C72

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1 Introduction

Both network structures and rationality of agents play a significant role in determining the outcome of many important economic relationships. A vast and recent literature examines how network structure affects economic outcomes[1] and the literature on bounded rationality has become more and more important since its introduction by Simon (1955). Our aim is to study the process of network formation in a dynamic framework in two cases related to different levels of rationality for economic agents. In the first one, the agents are perfectly rational. In the second one, some aspects of their rationality are limited.

We consider a group of agents who are initially unconnected who form or remove links with each other. A link can be removed unilaterally but agreement by both agents is needed to form a link. Precisely, a player pays an amount $c > 0$ to seek contact with an opponent and the link forms if the opponent behaves likewise. An agent's payoff is determined as in Gilles and Sarangi's (2004) connections model with consent. Agents receive the same value from all direct and indirect connections, where the cost of creating a link is greater than the reward of a single direct connection as in Watts (2002).

Since agreement is required to form links, it is crucial to distinguish an action profile, which lists the wishes or efforts of the players, from the induced network. In fact, several distinct action profiles may form an identical network. We focus on a particular subset of action profiles called *cost-efficient action profiles*. In such an action profile, if no link connects any two players i and j, then neither i nor j seeks contact with the opponent to form the link. In the static network formation game, only cost-efficient action profiles are likely to define Nash equilibria (NE) since a player incurs a cost for seeking contact with an opponent. For the same reason, only cost-efficient action profiles are likely to be efficient either in the sense of Bentham or Pareto. Precisely, we show that in a NE, perfectly rational players must choose the cost-efficient action profile that induces the empty network. In the finitely repeated network formation game, perfectly rational players also remain unconnected. The unique NE consists in a sequence of cost-efficient action profiles which form the empty network in each period.

In the current paper, we limit players' rationality by assuming that they use finite automata with a bounded number of states to play their strategies as in Neyman (1985). Players are limited in their ability to count the number

[1]Different examples may be found in consumer's theory (Ellison, Fudenberg, 1995), labor market (Calvó-Armengol, Jackson, 2004), industrial organization (Bolton, Dewatripont, 1994), or in game theory (Ellison, 1993).

of periods played. We also restrict the analysis to cost-efficient action profiles. A player cannot use a contact with an unconsenting opponent in a period as a message for a possible agreement in a subsequent period. Notice that any network is induced by a unique cost-efficient action profile. Therefore, the restriction to cost-efficient action profiles does not limit the architecture of static networks which may form in an equilibrium outcome. We show that if the size of automata is smaller than the duration of the game, then the set of NE of the repeated game is not reduced to outcomes filled with empty networks, as it is the case with perfectly rational agents. We do not explore whether equilibrium outcomes are affected when the analysis is extended to all action profiles. Nevertheless, cost-efficient equilibria induce a large variety of sequences of networks, which deserve attention.

We distinguish two types of equilibria. In the first one, the same network is formed in all periods. We refer to such an equilibrium as a Repeated Nash Network (RNN). We provide a sufficient condition for the existence of RNN based on any static network (proposition 3). Moreover, we give a practical test that determines if nonempty RNN do exist (proposition 4). In the second one, we define a Repeated Nash Equilibrium (RNE) where different networks may form at the different periods of the game. The set of RNE includes the set of RNN as a special case. We show that there exist structural relationships between the different networks that form within a given RNE. We study the intertemporal consistency between networks and identify several properties. Proposition 6 exhibits some sequences of networks that cannot be achieved as the outcome of RNE. For instance, a RNE cannot form a sequence of expanding connected networks, or it is not possible that all players remain isolated during the last two periods. Nonetheless, these restrictions allow for several RNE with nonempty outcomes. The networks within a given RNE may exhibit a total order relationship, the smaller network being formed in the last stage. In particular, there exist RNE that form sequences of contracting networks or sequences of networks can expand for all but the last period in which connections brutally run low. We also show that there are RNE in which players forget links (proposition 7). Precisely, two directly connected players in one period are not directly connected in another period. In spite of the restrictions on the intertemporal consistency between networks in equilibrium outcomes, we prove that any network can emerge in the outcome of a RNE (proposition 8). We also investigate the question of efficiency for both Bentham and Pareto criteria. In the finitely repeated game, the structure of efficient strategy profiles is closely related to the structure of static efficient networks. In addition, we prove that the sets of Bentham-efficient networks (strategy profiles) and Pareto-efficient networks (strategy profiles) are identical. The set of efficient strategy profiles of the repeated game is most often slightly reduced when players are assumed to be boundedly rational (proposition 11).

Some other papers are concerned with the structural properties of NE in repeated games with finite automata.[2] Papers related to repeated games are mainly concerned with the set of average payoffs that can be achieved in RNE. In the current work, we are more interested in the structure of RNE than in the induced average payoffs. The differences with other papers studying a dynamic network formation[3] are mostly related to the aims of the studies. These papers focus on the formation of a static network as a result of several steps of a dynamic process. They mainly focus on the study of limiting networks as in Watts (2001), or on using learning or stochastic stability to identify limiting equilibria as in Bala and Goyal (2000). By contrast, we consider a finitely repeated game that consists in the formation of a static network in each step of the process. We are interested in understanding the implication of different levels of rationality for equilibrium structures in this finite-horizon repeated setting.

The rest of the paper proceeds as follows. Section 2 presents preliminaries and notations. We also determine the set of Nash networks in the static game and in the finitely repeated game for the case of perfectly rational players. Then the machine game is introduced and studied in section 3. We start with results on RNN and continue with results regarding structural properties of more elaborated RNE. Results dealing with the efficiency of networks and strategy profiles are in section 4. Once again, we distinguish results according to players' rationality. Section 5 concludes. Proofs not given in the body of the paper appear in an appendix.

2 Preliminaries and notations

2.1 Static one-period game

Let $G = (I, A, \pi)$ be a finite n-player game in normal form. The set $I = \{1, \ldots, n\}$ is the player set. For any $i \in I$, A_i is player i's action set and $A = \prod_{i \in I} A_i$ is the set of action profiles. Let a_{-i} be the actions chosen by all players except i and $A_{-i} = \prod_{j \neq i} A_j$. Player $i \in I$ has payoff function $\pi_i : A \to I\!R$. An action profile a is a NE of G is for all $i \in I$ and all $a'_i \in A_i$,

$$\pi_i(a) \geq \pi(a'_i, a_{-i}).$$

[2] See Rubinstein (1986), Abreu and Rubinstein (1988) and Piccione and Rubinstein (1993) among others.

[3] We refer the reader to Bala and Goyal (2000), Currarini and Morelli (2000), Dutta, Ghosal and Ray (2005), Goyal and Vega-Redondo (2005), Jackson and Watts (2002) and Watts (2001,2002).

2.2 Network

The n players are connected in some network relationships. We limit our discussion to non-directed networks on the player set I. As in Jackson and Wolinsky (1996), two players are either related to each other or not, but it cannot be that one is related to the second without the second being related to the first. We write ij to describe the *link* between two players i and j.

Let $g_I = \{ij : i, j \in I, i \neq j\}$ be the set of all potential links. Any set of links $g \subseteq g_I$ defines a *network*. We apply the convention that $g = g_I$ is called the *complete* network and that $g = g_0 = \{\emptyset\}$ is the *empty* network. Any subset $g' \subset g$ is called a *subnetwork* of g.

A *path* between players i and j in a network g is a sequence of distinct players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ where $i_1 = i$ and $i_K = j$. Two such players are said to be connected. Player i is in a *cycle* of network g if there is a path with $K \geq 3$ vertices such that $i_1 = i_K = i$.

Let $n_i(g) = \{j \in I | ij \in g\}$ be the set of *neighbors* (or direct connections) of player i. Let $N_i(g)$ be the set of players to whom player i is connected in network g. Obviously, $n_i(g) \subseteq N_i(g)$. A network g is connected if there is a path between any two players. Alike, network g is said to be k-connected if there does not exist a set of $k-1$ links whose removal disconnects the network. If g is not connected, its connected subnetworks are called *components*. A connected acyclic network (or 1-connected network) is called a *tree* and a non connected network whose distinct components are trees is called a *forest*.

Let $l_{i,j}(g)$ be the *distance* between two players i and j in network g. If i and j are connected, $l_{i,j}(g)$ is the number of links in the shortest path between i and j. By convention, if i and j are not connected, $l_{i,j}(g) = \infty$. Let $L_i(g) = \max_{j \neq i} l_{i,j}(g)$ be player i's *eccentricity* in network g. The *diameter* of network g is $L(g) = \max_{i \in I} L_i(g)$. The last two definitions apply to any component $g' \subset g$.

For any two distinct players $i, j \in I$, $g + ij = g \cup \{ij\}$ is the network obtained adding link ij to network g. Likewise, let $g - ij = g \backslash \{ij\}$ be the network obtained removing link ij from network g. The intersection $g \cap g'$ defines the set of links that networks g and g' have in common.

2.3 Link formation cost and inefficient links

In this section, we present a non-cooperative model of costly network formation with consent. We assign a network $g(a) \subseteq g_I$ to every action profile

$a \in A$. Each player $i \in I$ has an action set $A_i = \{(a_{ij})_{j \neq i} : a_{ij} \in \{0,1\}\}$. Player i seeks contact with player j if $a_{ij} = 1$. Link ij forms if both players seek contact. The network induced by a is given by

$$g(a) = \{ij \in g_I : a_{ij} = a_{ji} = 1\}.$$

If player i seeks contact with j, then he supports a cost $c > 0$. As in a wide range of costly formation network models, player i's payoff consists in his value of the network minus a cost c for any attempt he made to create links. We assume that the value of network $g(a)$ for player i only depends on the number $\#N_i(g(a))$ of players to whom i is connected where $\#$ gives the dimension of the set. Thus, the distance between two players does not matter. This is true of many networks such as Internet, economic partnership or subcontracting. This results in the following payoff function

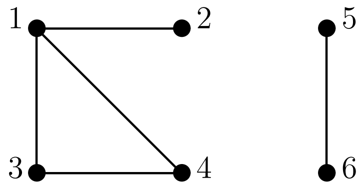
$$\pi_i(a) = v\#N_i(g(a)) - c\sum_{j \neq i} a_{ij}, \tag{1}$$

which can be seen as a particular case of the class of payoff functions investigated by Gilles and Sarangi (2004). Following Watts (2002), we assume $c > v > 0$, that is, creating a link is more costly than the reward of a single direct connection. In other words, player i needs some indirect connections to obtain a positive payoff. We also fix an upper bound $2v > c$ to avoid trivial cases due to incommensurable cost of creating a link. The next example will help discussing these assumptions.

Example 1
Consider $I = \{1,2,3,4,5,6\}$ and the network that forms as a result of the following players' choices :

a_{ij}	1	2	3	4	5	6
1	0	1	1	1	0	0
2	1	0	0	0	0	0
3	1	0	0	1	0	0
4	1	0	1	0	0	0
5	0	0	0	0	0	1
6	0	0	0	1	1	0

a



$g(a)$

Figure 1.

Network $g(a)$ consists in two components. The value of $g(a)$ is identical for players 1,2,3 and 4 in the left component and their payoffs only differ in the cost supported. Player 2's payoff is thus larger than 1's payoff since 2 intends to form a link with 1 while 1 intends to form links with 2,3 and 4. The subset $\{1,3,4\} \subset I$ defines a cycle in $g(a)$. Since the distance between two connected players has no influence on payoffs, it is not the interest of player 1 to seek contact with 3 since they are already connected by player 4. Intuitively, link 13 is superfluous. Now consider the right component of $g(a)$ involving players 5 and 6. This component is a tree, but $g(a)$ is not a forest since the right component includes a cycle. The assumption $c > v$ implies that both 5 and 6 would gain to remain isolated as it is the case in all components of diameter 1. Remark also that 6 seeks contact with 4. The link 64 fails to form since $a_{46} = 0$. As player 6's attempt is not drawn on $g(a)$ even if it affects his payoff, we will distinguish the action profile a and its induced network $g(a)$. □

Two identical networks $g(a) = g(b)$ may correspond to distinct action profiles a and b. Precisely, $g(a) = g(b)$ for two action profiles $a \neq b$ on the player set I if both networks have the same set of links and if at least one player seeks the creation of a link but does not receive the consent of an opponent. We will focus on a particular form of action profiles, namely the *cost efficient* action profiles.

Definition 1 *An action profile a is cost-efficient if there is no $i, j \in I$ such that $a_{ij} \neq a_{ji}$.*

Let A^{ce} be the set of cost-efficient action profiles. In any cost-efficient action profile, if a link between players i and j fails to form, neither i nor j seeks contact to create it. The only cost-efficient action profile that forms the empty network is denoted $a_0 = (a_{1,0}, \ldots, a_{n,0}) \in A^{ce}$ where $a_{i,0} = (0, \ldots, 0)$. Clearly, the action profile a in example 1 is not cost efficient.

If the payoff function is given by (1) and $2v > c > v > 0$, player i may obtain a larger payoff if he removes some links. Abusing notations, if $ij \in g(a)$, let $a_i(j^-)$ be the action that differs from a_i only by $a_{ij} = 0$.

Definition 2 *A link $ij \in g(a)$ is superfluous for player i in network $g(a)$ if $\pi_i(a_i(j^-), a_{-i}) > \pi_i(a)$. This inequality is satisfied in two cases :*

1. *$N_i(g(a) - ij) = N_i(g(a))$, i.e. link ij belongs to a cycle;*
2. *$n_j(g(a)) = \{i\}$, i.e. player i is j's single neighbor.*

In case 1, the removal of ij does not alter player i's connection set. This is the case of link 13 in example 1. In case 2, link ij increases the value of the network but costs player i more than it yields since it provides him a single connection and $c > v$. In example 1, this is the case of link 56 for both players 5 and 6. Let $d_i^1(a)$ and $d_i^2(a)$ denote the total gain for player i that results from removing all superfluous links in cases 1 and 2 respectively. We have

$$d_i^1(a) = c\left(\#\{j \in I : N_i(g(a) - ij) = N_i(g(a))\} - 1\right)$$

$$d_i^2(a) = (c - v)\#\left\{ij \in g(a) : n_j(g(a)) = \{i\}\right\}$$

Now let $d_i(a) = d_i^1(a) + d_i^2(a)$ be the maximal gain for player i if he deviates from action a_i against a_{-i}.

For any $i \in I$, we define $b_i \in A_i$ as player i's best response against a_{-i}, that is

$$b_i \in \arg\max_{a_i' \in A_i} \pi_i(a_i', a_{-i}).$$

When $a \in A^{ce}$, b_i is the action that satisfies

$$\pi_i(b_i, a_{-i}) - \pi_i(a) = d_i(a),$$

since a player cannot create links by his own will. All these definitions and notations will be used in the rest of the paper to determine whether action profiles are NE or efficient. Now we begin by characterizing Nash networks.

Proposition 1 *The only Nash network of G is the empty network* $g_0 = g(a_0)$.

A direct consequence of proposition 1 is that any player $i \in I$ can secure a null payoff against any opponents' behaviors by choosing $a_{i,0}$. The *minmax* payoff of each player is then 0. Now we introduce the finitely repeated game.

2.4 Finitely repeated game and network formation

In the static game G, players will not create any link even if they would be better off in some nonempty networks. In this section we assume that the players are involved in a T-period repeated game G^T that consists in $T < \infty$ repetitions of game G in period $t = 1, 2, \ldots, T$. Throughout the paper, we assume $T > 3$. It is natural to consider a dynamic process of network formation. Such a framework fits many economic situations in which relationships between agents may evolve with time.

We take the view that at the beginning of period t, all players observe a^{t-1} and not just $g(a^{t-1})$. If network $g(a)$ in example 1 forms in period $t-1$, then all players know at the beginning of period t that player 6's proposal to player 4 fails to form link 46 even if it is not drawn on the network $g(a)$.

Thus, a history of play $h^t = (a^1, \ldots, a^{t-1})$ at period t records the action profiles chosen by each player in periods $1, \ldots, t-1$. Let H^t denote the set of histories at period t and $H = \{\emptyset\} \cup (\bigcup_{t=2}^{T} H^t)$ denote the set of all possible histories of play. A repeated game strategy s_i is a sequence $s_i = \{s_i^t\}_{t=1}^{T}$ where $s_i^t : H^t \longrightarrow A_i$ models player i's action played at period t as a function of the $t-1$ previous action profiles. For any $i \in I$, S_i is the set of strategies for player

i. Let $s = (s_1, \ldots, s_n)$ denotes a strategy profile and $h^{T+1}(s) = (a^1, \ldots, a^T)$ is the repeated game outcome induced by $s \in S$, where $S = \prod_{i \in I} S_i$ is the set of all strategy profiles.

Player i's payoff function $\tilde{\pi}_i : S \longrightarrow I\!R$ from playing G^T is evaluated according to the average payoff

$$\tilde{\pi}_i(s) = \frac{1}{T} \sum_{t=1}^{T} \pi_i(a^t), \tag{2}$$

and $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$. A strategy profile $s \in S$ is a NE of the repeated game G^T if, for each $i \in I$, each $s'_i \neq s_i$,

$$\tilde{\pi}_i(s) \geq \tilde{\pi}_i(s'_i, s_{-i}).$$

Since a network is formed in each period of the repeated game, it is useful to introduce the notion of *repeated network*.

Definition 3 *A strategy profile s^* induces a repeated network based on network $g(a^*)$ if s^* induces a repeated game outcome $(a^1, \ldots, a^T)$ that satisfies $g(a^t) = g(a^*)$, $\forall t = 1, \ldots, T$.*

A repeated network is simply a network that forms in all periods as a result of players' actions. We want to highlight the robustness of a network being formed period after period. This notion is intuitively related to the robustness of a given set of relationships. We may think about situations in which trust is established among agents on a long-term basis. A Repeated Nash Equilibrium (RNE) is a NE of the repeated game. We need to define a Repeated Nash Network (RNN).

Definition 4 *A strategy profile s^* is a RNN based on network $g(a^*)$ if it is a repeated network based on $g(a^*)$ and if it is a RNE of G^T.*

Thus the set of RNE includes as a special case the set of RNN. When the horizon is finite, the only RNE induces the formation of the only Nash network in all periods, *i.e.* players always remain isolated.[4]

Proposition 2 *Assume $T < \infty$. The only RNE is the one in which players form the empty network g_0 in all periods.*

In the present section the agents are assumed to be perfectly rational. This results in an extreme conclusion: players have an incentive to remain unconnected. Now, we are going to relax the assumption of perfect rationality in order to understand the resulting differences on the types of relationships that are likely to appear.

[4] When the game is infinitely repeated, we can prove a folk theorem like result for RNE and RNN. Such a result includes a very large panel of structures even if, for instance, there cannot be a RNN based on a star network.

3 Machine game in a finite horizon setting

In this section, we suppose that players use finite automata with a limited number of states to play their strategies. We also focus on the subset of cost-efficient action profiles. This restriction may be justified by the fact that only cost-efficient action profiles are likely to define Nash networks in the static game. Moreover, for any network structure, the corresponding cost-efficient action profile is the most efficient action profile which induces the network (see section 4). In other words, only cost-efficient action profiles are likely to define efficient networks. We begin with the study of RNN. Then, we examine RNE which are not RNN. We will prove that the structure of both types of equilibria becomes non degenerate. A characterization of these structures will be stated.

3.1 Machine game

Following Neyman (1985), we focus on the repeated network formation game G^T in which player $i \in I$ chooses a *finite automaton* M_i to play his strategy. A finite automaton M_i for player i is a four-tuple $(Q_i, q_i^1, \lambda_i, \mu_i)$ where

1. Q_i is the finite set of states in M_i, with $\#Q_i = m_i$;
2. q_i^1 is the initial state ;
3. $\lambda_i : Q_i \to A_i$ is the output function which plays action $\lambda_i(q_i) \in A_i$ whenever M_i is in state q_i ;
4. $\mu_i : Q_i \times A_{-i} \to Q_i$ is the transition function. In a given period, if M_i is in state $q_i \in Q_i$ and players $-i$ choose $a_{-i} \in A_{-i}$, then the machine's next state is $\mu_i(q_i, a_{-i}) \in Q_i$.

We assume that player i's strategy space is limited to the set $\mathcal{M}_i$ of all automata of size (the number of states in the machine) $1 < m_i < T$. Before proceeding to the results, we must apply these notations to the repeated network formation game. A strategy profile of the machine game is an n-tuple $(M_1, \ldots, M_n)$ of automata. We shall also use the notation (M_i, M_{-i}) instead of $(M_1, \ldots, M_n)$. Abusing notations, we keep up writing a^t for the action profile chosen in period t instead of using the notation $(\lambda_1(q_1^t), \ldots, \lambda_n(q_n^t))$.

3.2 Existence of nonempty RNN

The main goal of this section is to provide a sufficient condition for a nonempty network to be sustained as a RNN. Before stating the result, notice that there cannot be a RNN based on a star network. In fact, the central player obtains a negative payoff as he supports a cost for creating a link with each opponent.

Proposition 3 *Consider any network $g(a^*)$ such that $a^* \in A^{ce}$ and, for any $i \in I$,*

$$0 \leq d_i(a^*) \leq \min\left\{\frac{v}{2}\#N_i(g(b_i, a^*_{-i})), 2\pi_i(a^*)\right\}. \tag{3}$$

Then there exists a RNN $(M_1, \ldots, M_n)$ such that $g(a^*)$ forms in each period.

Proof. Suppose that condition (3) is satisfied. Choose any $a^* \in A^{ce}$. We show that there exists a RNN $(M_1^*, \ldots, M_n^*)$ that induces $g(a^*)$ in all periods by constructing the required automata. For any player $i \in I$ consider the trigger strategy s_i defined for $t = 1$ by $s_i^1(\emptyset) = a_i^*$, and for $t > 1$, by

$$s_i^t(h^t) = \begin{cases} a_i^* & \text{if } a_{-i}^\tau = a_{-i}^*, \ \forall \tau = 1, \ldots, t-1, \\ a_{i,0} & \text{otherwise.} \end{cases}$$

This strategy can be implemented by the two-state automaton M_i^1 represented below :

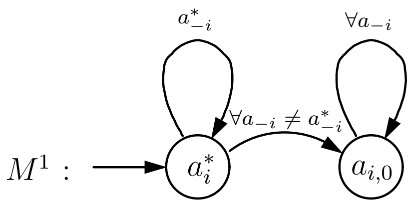


Figure 2.

If the opponents play M_{-i}^1, then player i's payoff from playing M_i^1 is

$$\tilde{\pi}_i(M_i^1, M_{-i}^1) = \pi_i(a^*) \geq 0, \tag{4}$$

since the outcome of the T-period machine game is assumed to be a repeated network.

Notice that any deviation from M_i^1 by player i releases a definitive punishment by players $j \neq i$. Recall that each player i is limited to automata of size $1 < m_i < T$. This prevents player i from using the standard best response against M_{-i}^1 that consists in playing a_i^* to form $g(a^*)$ until the last stage and then playing b_i in the round T. In fact such a strategy requires at least a T-state automaton. As a consequence, a deviation by player i must occur in a period $t < T$ and implies $T - t \geq 1$ periods of punishment. This also implies that if player i has an incentive to deviate, then this deviation must occur as late as possible in the game. However, as we will see below, this does not exactly amount to say that if player i has an incentive to deviate, he will do so in stage $T - 1$.

A deviation by player i in period t means that he uses action b_i which yields the best payoff against a_{-i}^*. Recall that the number of punishment stages is not null. Thus, player i aims at minimizing the cost for seeking contacts in the actions he plays in periods $t+1,\ldots,T$ against $a_{-i,0}$ since the value of the network in each of these periods is null. Even if the amount paid for seeking contacts in b_i is less than a_i^*, it may be very costly. Therefore, it may not be the interest of player i to play b_i at stage $t+1$ and thereafter. This is why all possible deviations for player i's may be grouped into the two following cases :

1. He plays b_i in periods $T-1$ and T,[5]
2. He plays b_i in period $k \leq T-2$ and uses a $(k+1)$th state that plays $a_{i,0}$ for the remaining stages. Clearly, player i's highest incentive to deviate is in period $k = T-2$.

We now consider these two possibilities.

Case 1.
The deviating strategy can be implemented by the following $(T-1)$-state automaton :

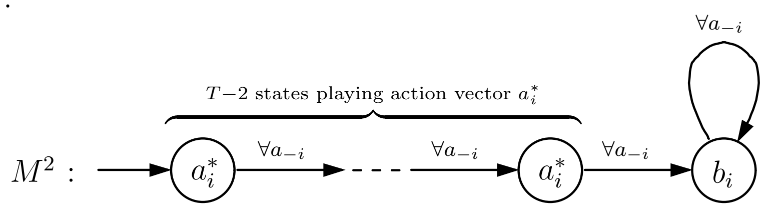


Figure 3.

Using M_i^2, player i mimics a full cooperation to form network $g(a^*)$ up to period $T-2$, and then plays b_i to obtain the best payoff against $a_{-i}^* = a_{-i}^{T-1}$.[6] He also plays b_i in period T against the punishment $a_{-i,0}$ since

$$b_i = \arg \max_{a_i \in \{b_i, a_i^*\}} \pi_i(a_i, a_{-i,0}) = \arg \min_{a_i \in \{b_i, a_i^*\}} -c \sum_{j \neq i} a_{ij}.$$

In words, player i's machine has not enough states to use another action than b_i or a_i^*. Moreover, the cost for seeking contacts in b_i cannot be more expensive than in a_i^* since the assumption $a^* \in A^{ce}$ prevents the deviating player from creating links unilateraly. Player i obtains the payoff

$$\tilde{\pi}_i(M_i^2, M_{-i}^1) = \frac{(T-2)\pi_i(a^*) + \pi_i(b_i, a_{-i}^*) + \pi_i(b_i, a_{-i,0})}{T}, \quad (5)$$

[5]Recall that using $T-2$ states for playing a_i^* and one more state for playing b_i prevent player i from playing $a_{i,0}$ in stage T as a response to the punishment.

[6]Notice that action profiles considered in the deviation tests may not be cost efficient. In fact removing links from a_i^* by playing b_i induces an action profile $(b_i, a_{-i}^*) \notin A^{ce}$ which is not cost efficient.

which has to be compared to (4). It is not the interest of player i to switch from M_i^1 to M_i^2 if and only if

$$\begin{aligned}
& \pi_i(a^*) && \geq \frac{(T-2)\pi_i(a^*) + \pi_i(b_i, a^*_{-i}) + \pi_i(b_i, a_{-i,0})}{T} \\
\Longleftrightarrow \quad & \pi_i(a^*) && \geq \frac{v}{2}\#N_i(g(b_i, a^*_{-i})) - c\sum_{j\neq i} a^{*d}_{ij} \\
\Longleftrightarrow \quad & \pi_i(a^*) && \geq \pi_i(b_i, a^*_{-i}) - \frac{v}{2}\#N_i(g(b_i, a^*_{-i})) \\
\Longleftrightarrow \quad & d_i(a^*) && \leq \frac{v}{2}\#N_i(g(b_i, a^*_{-i})). \qquad (6)
\end{aligned}$$

Case 2.
The deviating strategy can be implemented by the following $(T-1)$-state automaton :

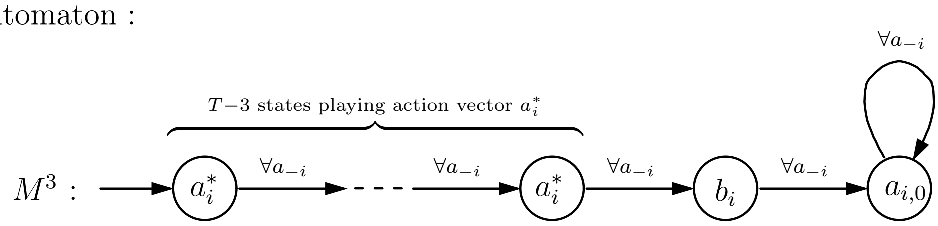


Figure 4.

Using M_i^3, player i simulates a full cooperation to form network $g(a^*)$ up to period $T-3$, and then plays in stage $T-2$ the action b_i against $a^*_{-i} = a^{T-1}_{-i}$. He uses a new state playing $a_{i,0}$ for the last two stages. Thus, player i obtains the average payoff

$$\tilde{\pi}_i(M_i^3, M_{-i}^1) = \frac{(T-3)\pi_i(a^*) + \pi_i(b_i, a^*_{-i}) + 2\pi_i(a_{i,0}, a_{-i,0})}{T},$$

which also needs to be compared to (4). It is not the interest of player i to switch from M_i^1 to M_i^3 if and only if

$$\begin{aligned}
& \pi_i(a^*) && \geq \frac{(T-3)\pi_i(a^*) + \pi_i(b_i, a^*_{-i}) + 2\pi_i(a_{i,0}, a_{-i,0})}{T} \\
\Longleftrightarrow \quad & d_i(a^*) && \leq 2\pi_i(a^*). \qquad (7)
\end{aligned}$$

Combining the two cases, player i will not deviate from strategy M_i^1 if and only if

$$d_i(a^*) \leq \min\left\{\frac{v}{2}\#N_i(g(b_i, a^*_{-i})), 2\pi_i(a^*)\right\},$$

the condition stated in the proposition. By definition, $d_i(a^*) \geq 0$ such that inequality (3) guarantees that players obtain at least the minmax payoff. Thus, the strategy profiles $(M_1^*, \ldots, M_n^*)$ is a RNN based on $g(a^*)$. ■

Remark that condition in proposition 3 does not depend on whether T is large since the average payoff function does not include a discount parameter. This condition implies that RNN must be based on networks that are not too much over-connected if players are very rancorous. Indeed, the gain each player obtains may not be too small. In static network theory, the fact that networks be too much over-connected is important too, mostly for efficiency considerations (Calvó-Armengol, 2003). The condition in proposition 3 also implies that a player must not support a too large share of the cost needed to create the network. For instance $g(a^*)$ cannot contain a star subnetwork since a player would be directly connected to each opponent and obtains in average less that the minmax payoff.

Condition (3) is necessary and sufficient to prevent any deviation from (M_i^1, M_{-i}^1). However, it is not a necessary condition to achieve a RNN based on $g(a^*)$. To see this, notice that M^1 induces the hardest possible punishment, either for the duration or for the loss of static game payoffs per period. One can think of an automaton that induces a less significant loss of payoff for a single period. If both players use such a machine and none of them has an incentive to deviate, then the necessary and sufficient condition to achieve a RNN based on network $g(a^*)$ must be less restrictive than (3). We limit our result to the sufficient condition because it is difficult to determine the action for which the punishment is minimal.

Let $g(l)$, $l \in A^{ce}$, be an n-player line network : for all $i \in I \backslash \{h, j\}$, $n_i = 2$ and $n_h = n_j = 1$ and all players have $n-1$ connections. The line begins and ends with players h and j. The next proposition gives two practical tests to determine whether the set of RNN contains nonempty repeated networks. By *(i)*, it is enough to check if a repeated line network is not a RNN to make sure that any other nonempty repeated network is not a RNN. This test proves useful if one wants to check that there cannot be a RNN based on a large and complex static network. By *(ii)* it is enough to look at the number of players to guarantee that some nonempty RNN do exist.

Proposition 4 *(i) Suppose that the repeated network based on $g(l)$ is not a RNN. Then the only RNN of G^T induced the empty network g_0 in all periods.*

(ii) Suppose $n \geq 5$. Then there exists a nonempty RNN of G^T.

Proof. *(i)* Suppose that the repeated network based on $g(l)$ is not a RNN. Firstly, observe that if the value of any network $g(a)$ is more than v for a player, then there is at least one player whose cost is $2c$ or more. Given that, $g(l)$ is the network in which the minimal payoff in the player set is maximal on the set of all networks. Secondly, all players in the line network are connected with a minimal number of links and the cost for creating links is distributed such that a player pays at most $2c$, which cannot be less in a connected network. In other words, network $g(l)$ is the unique architecture that satisfies the maxmin criterion. That is,

$$\min_{i \in I} \pi_i(l) = \max_{a \in A^{ce}} \min_{i \in I} \pi_i(a). \tag{8}$$

It follows from (8) that if a player has an incentive to deviate from the repeated network based on $g(l)$, then there is at least one player who can do so in a repeated network based on any network $a \in A^{ce}$. The assumption that the repeated network based on $g(l)$ is not a RNN implies that the only RNN of G^T induces a Nash network in all periods. Therefore, the empty network must form g_0 in all periods.

(ii) We next prove that $g(l)$ is sustained as a RNN (whatever $2v > c > v$) if $n \geq 5$. We are going to use the sufficient condition of proposition 3. In the cost-efficient action profile that induces network $g(l)$, all players except the first and the last of the line seek the creation of two links (the cost of the network is $2c$ for each of these players) while they benefit from connections with all opponents (in fact, the value of $g(l)$ is $v(n-1)$ for all players). For players h and j, the value of $g(l)$ remains $v(n-1)$ as $g(l)$ is connected but they only create a single link. As a consequence,

$$\min_{i \in I} \pi_i(l) = v(n-1) - 2c,$$

A necessary condition for the existence of a RNN based on $g(l)$ is that $\pi_i(l) \geq 0$ for all $i \in I$, that is

$$n \geq \frac{2}{v}c + 1,$$

which is always satisfied if $n \geq 5$. Notice also that $v\#N_i(b_i, l_{-i}))/2 < 2\pi_i(l)$ for all $i \in I$ and $n \geq 5$. Then using (3), a sufficient condition for the existence of a RNN based on $g(l)$ is given by

$$d_i(l) < \frac{v}{2}\#N_i(g(b_i, l_{-i})) \iff n \geq \frac{2c}{v},$$

which is guaranteed whatever $2v > c > v$ for all $n \geq 4$. As a consequence, condition $n \geq 5$ implies that a repeated network based on $g(l)$ is a RNN. ■

So far, two sufficient conditions for the existence of nonempty RNN are provided. The second one has been stated in the most general form to keep the exposition as simple as possible. In fact, we may notice in proof of *(ii)* that condition $n \geq 2c/v + 1$ is sufficient for guaranteeing the existence of a nonempty RNN. Moreover, this means that all line networks with at least

5 players can be sustained as a RNN. The results of this section enable to make precise the case in which nonempty RNN do exist. But not much is said about the structural properties of RNE which are not RNN. This is the aim of the next section.

3.3 Structural properties of RNE

In this section, we are mainly concerned with the structural properties of RNE of G^T. Proposition 5 shows that there exists RNE which are not RNN. Then, we identify a property satisfied by any RNE. This property has a crucial impact on the intertemporal consistency between networks formed in the outcome of RNE. In propositions 6 and 7, we use graph theory to characterize these restrictions and to represent the sequence of networks that can be achieved at equilibrium. We also offer economic interpretations.

To prove the existence of RNE that are not RNN, we examine the possibility that an equilibrium outcome contains the empty network $g(a_0)$ is some periods and a nonempty network $g(a^*)$, $a^* \in A^{ce}$, in others periods. This first result on non repeated NE shows that the networks can expand with time. In particular, it shows that players can establish relationships late in the game even if they choose to stay isolated until then.

Proposition 5 *Fix $0 < k < T-2$. Consider a network $g(a^*)$, $a^* \in A^{ce}$, such that for all $i \in I$, $d_i(a^*) \leq \pi_i(a^*)$. Then there is a RNE $(M_1, \ldots, M_n)$ of G^T whose outcome $(a^1, \ldots, a^k, a^{k+1}, \ldots, a^T)$ satisfies $g(a^t) = g_0$, $\forall t \leq k$ and $g(a^t) = g(a^*)$, $\forall t \geq k+1$.*

Proof. Firstly, we construct the strategy profile depicted in the proposition. Suppose that each player i uses the following $(k+1)$-state automaton :

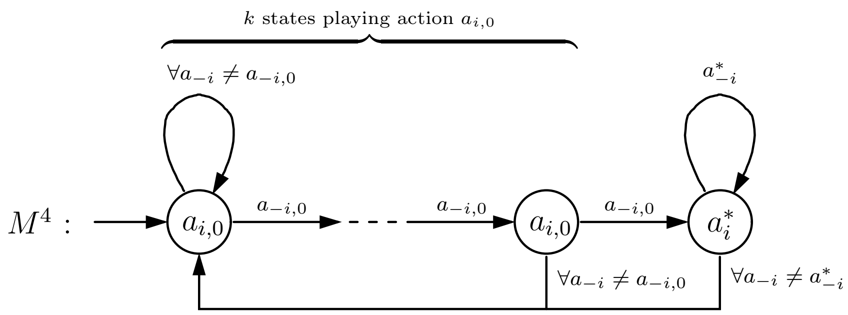


Figure 5.

The strategy profile $(M_1^4, \ldots, M_n^4)$ outputs $T-k$ networks $g(a^*)$ after the first k periods each devoted to the formation of the empty network $g(a_0)$.

Next, we determine the condition for which $(M_1^4, \ldots, M_n^4)$ is a RNE of G^T. Player i can deviate in two distinct ways.

Case 1.
By contrast to case 1 in the proof of proposition 3, player i can mimic machine M^4 during the $T-2$ first periods, then uses b_i in stage $T-1$ and finally transits for the punishment period T to a state playing $a_{i,0}$ used previously. This strategy requires $T-1$ states and can be represented as follows :

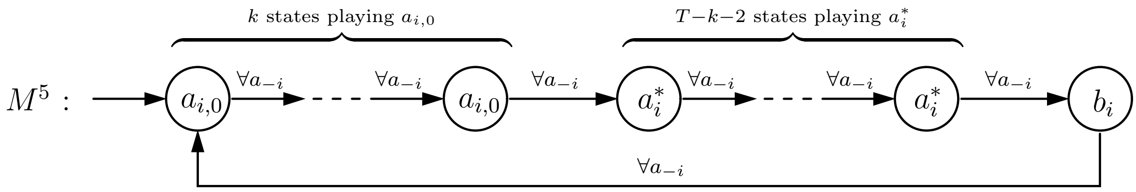


Figure 6.

It is not the interest of player i to choose machine M^5 if and only if

$$\tilde{\pi}_i(M_i^4, M_{-i}^4) \geq \tilde{\pi}_i(M_i^5, M_{-i}^4) \iff d_i(a^*) \leq \pi_i(a^*). \tag{9}$$

Case 2.
Player i can hide a deviation in period T since the opponents may not use a definitive punishment. To see this, suppose that player i uses a state playing b_i in a period $t < T$ in order to transit a second time to this state in stage T. The first use of b_i is punished by k periods in which $a_{-i,0}$ is played. Such a punishment is not definitive if $t < T-k$. In particular, the punishment is never definitive when player i chooses to use b_i in the initial period. In such a case, the number of empty networks that form is minimal since the opponents' machines simply play $a_{-i,0}$ for an additional period (as i must pass sequentially through the k states playing $a_{i,0}$ to reach the state playing a_i^*). Player i is not punished for the second time he plays b_i since this occurs in the last period. The $(T-1)$-state automaton represented below implements such a deviation :

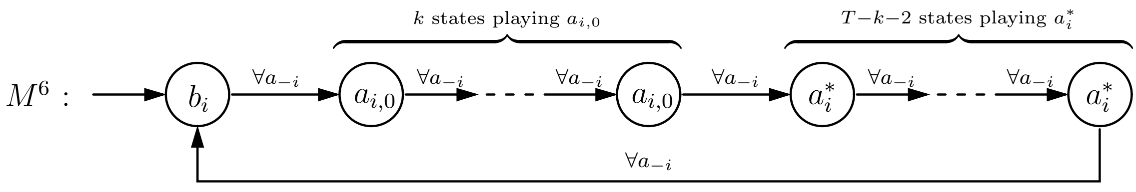


Figure 7.

Machine M^6 plays b_i in stage 1. Next it mimics machine M^4 in all other periods except at stage T in which it transits towards the initial state. Player

i will not choose M^6 if and only if

$$\tilde{\pi}_i(M_i^4, M_{-i}^4) \geq \tilde{\pi}_i(M_i^6, M_{-i}^4) \iff d_i(a^*) \leq \pi_i(a^*) + c\sum_{j\neq i} b_{ij},$$

which is verified if condition (9) is satisfied. Therefore $(M_1^4, \ldots, M_n^4)$ is a RNE of G^T. ■

Notice also that the proposition holds for both cases $k = 0$ and $k = T$: indeed $g(a_0)$ is a Nash network and proposition 3 implies the result respectively (but not with the n-tuple of automata $(M_1^4, \ldots, M_n^4)$). Proposition 5 shows that the possible structures of RNE of the repeated T-period game are more elaborate than in the case with perfectly rational players.

Next we give an intuitive property of RNE of G^T. When two different static networks form in two periods, it is often the case that a player can deviate at stage T. This is a key argument in the analysis of the structure of RNE. To see this, suppose that player i uses an action a_i^k in a period $k < T$ that is more beneficial than a_i^T against actions a_{-i}^T used by the opponents in stage T. It is the interest of player i to play this action in the last stage since he cannot be punished in a forthcoming period. Therefore, players must not have such an opportunity to deviate in any RNE.

Lemma 1 *Consider any RNE (M_i^*, M_{-i}^*) of the machine game G^T with outcome $(a^{*1}, \ldots, a^{*T})$. Then there is no network $g(a^{*k})$ that forms in period $k < T$ such that for any player $i \in I$, $\pi_i(a_i^{*k}, a_{-i}^{*T}) > \pi_i(a^{*T})$.*

Proof. The proof is by contradiction. Consider a RNE (M_i^*, M_{-i}^*) of G^T, a player i and a period $k < T$ in which network $g(a^{*k})$ forms such that $\pi_i(a_i^{*k}, a_{-i}^{*T}) > \pi_i(a^{*T})$. We prove that player i has an incentive to deviate from M_i^* towards the following $(T-1)$-state revised automaton M_i:

1. $Q_i = \left\{q_i^{s_1}, \ldots, q_i^{s_t}, \ldots, q_i^{s_{T-1}}\right\}$, $m_i = T - 1$;
2. $q_i^1 = q_i^{s_1}$;
3. $\lambda_i(q_i^{s_t}) = a_i^{*t} \in h^{T+1}(M_i^*, M_{-i}^*) = ((a_i^{*1}, a_{-i}^{*1}), \ldots, (a_i^{*T}, a_{-i}^{*T})), \forall t \leq T-1$;
4. $\mu_i(q_i^t, \lambda_{-i}(q_{-i}^t)) = \begin{cases} q_i^{s_{t+1}} & \text{if } t \leq T-2 \\ q_i^{s_k} : \lambda_i(q_i^{s_k}) = a_i^{*k} & \text{if } t = T-1 \end{cases}$

Machine M_i has $T-1$ states. The output function indicates that for each $t < T$ the state $q_i^{s_t}$ plays the action used by M_i^* against M_{-i}^* in period t. The transition function of M_i mimics the sequence of actions played by M_i^* against M_{-i}^* for all but the last period. In any periods $t < T$, the deviation towards M_i keeps player i's payoff unchanged. In stage T, M_i transits to the state $q_i^{s_k}$ that plays action a_i^{*k} used by M_i^* against M_{-i}^* in stage k. Using M_i, player i obtains the following average payoff :

$$\tilde{\pi}_i(M_i, M_{-i}^*) = \tilde{\pi}_i(M_i^*, M_{-i}^*) + \frac{\pi_i(a_i^{*k}, a_{-i}^{*T}) - \pi_i(a^{*T})}{T},$$

which is larger than $\tilde{\pi}_i(M_i^*, M_{-i}^*)$ since $\pi_i(a_i^{*k}, a_{-i}^{*T}) > \pi_i(a^{*T})$ by assumption. This contradicts the initial assumption that (M_i^*, M_{-i}^*) is a RNE. ■

The idea that a player may use in stage T a former state to deviate without being punished is central in the question of the architecture of RNE. Lemma 1 states restrictions on the intertemporal consistency between static networks that form within a given RNE of G^T. Unfortunately, these restrictions are described in terms of payoff. We are more interested in structural properties of the networks induced by such RNE. We specify some of these properties in points *(i)* and *(ii)* of the next proposition. The third point is related to both proposition 2 and lemma 1.

Proposition 6 *The outcome induced by any RNE* $(M_1, \ldots, M_n)$ *of game* G^T *must have the three following features:*

(i) there is no connected network $g(a^t) \subset g(a^T)$, $\forall t < T$,

(ii) in network $g(a^T)$, *there is no player* i *with eccentricity* $L_i(g(a^T)) = 1$,

(iii) if there is a period $t < T - 1$ *such that* $g(a^t) \neq g_0$, *then it cannot be the case that* $g(a^{T-1}) = g(a^T) = g_0$.

Proof. *(i)* The proof is by contradiction. Consider any RNE $(M_1, \ldots, M_n)$ of G^T which forms, for some $t \in \{1, \ldots, T-1\}$, a connected network $g(a^t) \subset g(a^T)$ (see definition page 4). The assumptions $g(a^t)$ connected and $g(a^t) \subset g(a^T)$ imply of course that $g(a^T)$ is also connected. By definition 2(1.), there is a player i who has some superfluous links in $g(a^T)$ that he does not have in $g(a^t)$. This player is able to play a_i^t in stage T against a_{-i}^T by a mechanism similar to that in proof of lemma 1. In network $g(a_i^t, a_{-i}^T)$ player i maintains a connection with all opponents since $g(a^t)$ is a connected subnetwork of $g(a^T)$ (in fact, $\#N_i(g(a_i^t, a_{-i}^T)) = \#N_i(g(a^T)) = n - 1$). Moreover the cost of forming links in $g(a_i^t, a_{-i}^T)$ is smaller for player i since he seeks less contacts in a_i^t than in a^T. Thus, if player i chooses the machine that simulates what plays M_i against M_{-i} for all but the last period and then transits to the state used in period t, he obtains a larger payoff in stage T. Player i obtains a larger average payoff which implies (M_i, M_{-i}) is not a RNE. This contradicts the initial assumption.

(ii) By contradiction, consider a RNE $(M_1, \dots, M_n)$ in which $L_i(g(a^T)) = 1$ for a player $i \in I$. Such an eccentricity means that player i is directly connected with each opponent. This implies that player i obtains the worst possible stage payoff $\pi_i(g(a^T)) = (n-1)(v-c)$. Since $(M_1, \dots, M_n)$ is a RNE, we know $\tilde{\pi}_i(M_1, \dots, M_n) \geq 0$. Thus player i obtains a positive payoff in some periods, that is, he does not seek contact with each opponent in these periods. Formally, there exists $t < T$ such that $L_i(g(a^t)) > 1$. Let a_i^t be the action played by player i in such a period. As in the proof of lemma 1, player i is able to deviate from M_i towards a $(T-1)$-state machine that simulates what plays M_i against M_{-i} in the first $T-1$ periods and then transits in stage T to the state playing action a_i^t. Using this altered strategy, player i must obtain in stage T a payoff $\pi_i(a_i^t, a_{-i}^T) > \pi_i(a^T)$ as he seeks less contacts and $\pi_i(a^T)$ is the worst payoff in the game. All other stage payoffs being identical, the deviating strategy yields player i a larger average payoff than M_i. This contradicts the fact that $(M_1, \dots, M_n)$ is a RNE.

(iii) Consider any RNE $(M_1, \dots, M_n)$ for which $g(a^{T-1}) = g(a^T) = g_0$ and for some periods $t \leq T-2$, $g(a^t) \neq g_0$. Let

$$t^* = \max_{t \leq T-2} \{t : g(a^t) \neq g_0\}$$

be the most remote period in which a nonempty network forms. By proposition 1, at least one player i has $d_i(a^{t^*}) > 0$. Suppose that i chooses to deviate from M_i towards a (t^*+1)-state machine that mimics M_i's behavior against machines M_{-i} up to period t^*-1, then plays action b_i in stage t^* and transits to a (t^*+1)th state playing action $a_{i,0}$ until the end of the game (see proof of lemma 1). Clearly, such a deviation yields player i a larger payoff which implies that $(M_1, \dots, M_n)$ is not a RNE. We conclude that two empty networks cannot form in the last two stages of a nonempty RNE. ■

These results lead to some conclusions. Result *(i)* has several interpretations. Firstly, the only minimal network that is likely to be connected is the last that forms. Secondly, there may be other connected networks in previous periods but this result implies that those networks must contain the last one. We may say that the formation of a connected network (if it occurs) has to be progressive. A connected network may form quickly in the process but it will be over-connected. If a connected network forms in a given period $t < T$ (of a RNE) and another one forms in the final period, the last network is more beneficial to all players and strictly more for one of them. Even a link formation process by a player generates an externality on the set of direct neighbors, this effect would be gradually internalized by some players. Third, one may also interpret the first result in proposition 6 as the impossibility that the outcome of a RNE consists in a sequence of connected networks that extends as time progresses. This once again emphasizes that too much over-connected networks fail to form in a RNE.

Result *(ii)* shows that in the last network being formed, a player cannot create a direct link with all opponents.[7] For instance, the complete network and the star network are two such networks, and cannot form in the last stage of a RNE. Recall that point *(i)* does not prevent players to form a connected network in stage T. By point *(ii)*, if players are all connected in the last network induced by a RNE, any of them avoid the burden to seek contact with all others. This shows how players learn to divide the task of connecting the network up among themselves. A consequence of this result is that the diameter of the network form in the final stage must satisfy $L(g(a^T)) > 1$. This could be interpreted as the absence of an extreme small-world effect as observed by Milgram (1967). Notice that the star network is the only tree of diameter 2. Thus, by point *(ii)*, if $L(g(a^T)) = 2$ then $g(a^T)$ is not a tree. In words, a small-world effect (diameter 2) in stage T is possible only with inefficient networks (see section 4 for efficiency considerations).

Result *(iii)* displays that if players have established relationships in the $T-2$ first periods, some old connections remain or new links are formed in at least one of the two last stages. In other words, if players create links in earlier periods, they must maintain some former links or create new links in at least one of the last two periods. For example, relationships between individuals in a connected population of agents never completely disappear with time. Consider a market represented by a network of firms. Links model competition between firms and the finite horizon of T periods indicates the lifespan of the product. A firm leaving the market is symbolized by an isolated vertex. By result *(iii)*, if some firms have competed in the market in some of the first $T-2$ years, then the market cannot become empty of firms in the final years.

Proposition 6 has determined some forbidden sequences of networks induced by RNE. In the next proposition, we consider two situations in which the structural properties of networks prevent players from deviating in stage T as in lemma 1. This results in some sequences of networks that can be achieved as a RNE provided that a condition on payoffs is satisfied.

Proposition 7 *Consider a strategy profile* $(M_1, \ldots, M_n)$ *with outcome* $(a^1, \ldots, a^T)$ *such that, for any* $i \in I$ *and any* $t \leq T$, $\pi_i(a^t) \geq 0$,

$$\pi_i(a^T) \geq \max \left\{ d_i(a^{T-1}) - c \min_{a^1_{ij}, \ldots, a^{T-1}_{ij}} \sum_{j \neq i} a^t_{ij} ; \max_{k < T-1} d_i(a^k) - \sum_{t=k+1}^{T} \pi_i(a^t) \right\} \quad (10)$$

If either,

[7] However, it is possible that a player obtains less than the minmax payoff in stage T.

(i) $g(a^t) \cap g(a^{t'}) = \emptyset$, *for any* $t, t' \in \{1, \ldots, T\}$ *or*

(ii) *there is a permutation* $p : \{1, \ldots, T-1\} \longrightarrow \{1, \ldots, T-1\}$ *such that* $g(a^T) \subseteq g(a^{p(1)}) \subseteq g(a^{p(2)}) \subseteq \cdots \subseteq g(a^{p(T-1)})$,

then $(M_1, \ldots, M_n)$ *is a RNE of* G^T.

Proof. Assume that for any $i \in I$ and any $t \leq T$, $\pi_i(a^t) \geq 0$. The proof has two parts. Firstly, we show that in both situations *(i)* and *(ii)*, a player has no incentive to deviate as in the proof of lemma 1.

(i) The assumptions that any action profile in the outcome is cost-efficient and that for any $t, t' \leq T$, $g(a^t) \cap g(a^{t'}) = \emptyset$ implies that for any $i \in I$, $N_i(g(a_i^t, a_{-i}^{t'})) = N_i(g(a_i^{t'}, a_{-i}^t)) = \{\emptyset\}$.[8] As player i may still intend to create some links (that do not form) this implies that $\pi_i(a_i^t, a_{-i}^T) \leq 0$ and $\pi_i(a_i^{t'}, a_{-i}^T) \leq 0$. By assumption, we then have $\pi_i(a_i^t, a_{-i}^T) \leq \pi_i(a^T)$ and $\pi_i(a_i^{t'}, a_{-i}^T) \leq \pi_i(a^T)$. This means that there is no period $t > T$ such that player i benefits from using the revised automaton constructed in the proof of lemma 1.

(ii) The assumption that for all $t \leq T$, $g(a^T) \subseteq g(a^t)$ implies that $g(a^T) \cap g(a^t) = g(a^T)$. Thus, for any player $i \in I$, $g(a^T) = g(a_i^t, a_{-i}^T) \subseteq g(a^t)$. This relation can be rewritten as $N_i(g(a^T)) = N_i(g(a_i^t, a_{-i}^T)) \subseteq N_i(g(a^t))$. We also know that the cost supported by player i in a^t is not less than in a^T. Therefore, for any $i \in I$ and for any $t \leq T$, $\pi_i(a^T) \geq \pi_i(a_i^t, a_{-i}^T)$. This implies once again that there is no period $t > T$ such that player i benefits from using the revised automaton constructed in the proof of lemma 1.

Secondly, we prove that condition (10) guarantees that $(M_1, \ldots, M_n)$ is a RNE. Each machine M_i in $(M_1, \ldots, M_n)$ is assumed to include a punishment state playing $a_{i,0}$ to threaten any deviation by an opponent as the one in machine M^1 in proposition 3. It is not the interest of player i to deviate in stage T in both situations *(i)* and *(ii)*. Thus, two cases similar to those in the proof of proposition 3 must be considered.

Case 1.
Player i can deviate by using a $(T-1)$-state machine that plays b_i in stage $T-1$ and $a_i = \arg\min_{t<T} c \sum_{j \neq i} a_{ij}^t$ in the last period (remember that player i cannot use a new state playing $a_{i,0}$ is stage T since $m_i < T$). Player i will

[8] As $m_i < T$, only $T-1$ totally different networks may form. We thus assume that the empty network g_0 forms in two periods. This does not contradicts condition *(i)* since trivially $g_0 \cap g_0 = \{\emptyset\}$.

not choose such an automaton if and only if

$$\pi_i(a^{T-1}) + \pi_i(a^T) \geq \pi_i(b_i^{T-1}, a_{-i}^{T-1}) - c \min_{t<T} \sum_{j \neq i} a_{ij}^t$$

$$\iff \quad \pi_i(a^T) \geq d_i(a^{T-1}) - c \min_{t<T} \sum_{j \neq i} a_{ij}^t. \tag{11}$$

Case 2.
Player i can deviate in period $k < T-1$ by playing b_i and then using a (possibly) new state playing $a_{i,0}$ until the end of the game. It is not the interest of player i to behave like this if and only if

$$\sum_{t=k}^{T} \pi_i(a^t) \geq \pi_i(b_i^k, a_{-i}^k) \iff \pi_i(a^T) \geq d_i(a^k) - \sum_{t=k+1}^{T} \pi_i(a^t). \tag{12}$$

Condition (10) in proposition 7 results from the combination of (11) and (12). By condition (10) we know that these deviations are not profitable to player i. This concludes the proof. ■

In words, if the payoff that each player obtains in the last round is sufficiently important, then two kinds of interesting structures are likely to emerge. These structures are antagonistic. In the first one, at each round a new static network forms. These networks have no common link with any network formed in previous periods. As a consequence, if any two players are direct neighbors in a given period, this is the first period for which they are direct neighbors and they would never be directly connected once again. In a sense, we can refer to such an equilibrium as one with forgotten neighbors. This may highlight that players prefer a variety of one-period direct neighbors than long term direct relations. In economic relationships, such a pattern of behavior is quite common. In a trading market, some buyers prefer visiting a variety of sellers than establishing a long term relation with a particular seller (the so-called searchers).

In the second one, networks share an identical component and the structure allows for a total order relationship among networks that form within a RNE. The sequence of static networks corresponding to the equilibrium may reveal a contraction phenomenon. At each new period, the network that forms may be more and more restricted. This is the case of many economic situations. Consider for example several firms competing in a new market. A link between two firms can represent the investment supported by both firms to differentiate their product from that of the other firm. As time goes by, least competitive firms are not strong enough to face competition. They either stop investing to differentiate their product (cuts links but preserve some) or leave the market (remain isolated). The competition network may retract progressively. Another interpretation is that a RNE may form more and more efficient networks (see next section). The network should be k-connected in the initial period. The contraction process may lead to the formation of a tree network in the last stage.

The total order may also exhibit an expansion phenomenon from the initial period to period $T-1$ and then form in the last round a network contained in all others. In all networks in the sequence are connected, one can interpret such a phenomenon as a $T-1$ periods of test before stage T in which agents choose the most valuable configuration.

In proposition 7 each player's payoff is assumed to be positive at each round. Despite the drastic restrictions on the intertemporal consistency between networks formed in any RNE, the next result shows that it is possible that any static network based on a cost-efficient action profile forms within a RNE. In the next proposition, we give a sufficient condition on the number of players for which any static network can appear in the outcome of a RNE.

Proposition 8 *Fix $n \geq 11$. There exists a RNE $(M_1, \ldots, M_n)$ whose outcome contains any network $g(a^*) \subseteq g_I$, $a^* \in A^{ce}$, at least once.*

Proof. Fix $n \geq 11$. We prove that any network $g(a^*) \subseteq g_I$, $a^* \in A^{ce}$, can form in the first period of a RNE. Denote by $g(l')$ the network that consists of the a line with $n-1$ players and an isolated player. We proceed in two steps. In a first step, we exhibit an n-tuple of automata $(M_1, \ldots, M_n)$ which forms any network $g(a^*) \neq g(l')$ in the initial period and the (n-player) maxmin line network $g(l)$ in all remaining periods. Any deviation from M_i will release a definitive minmax punishment. The choice to form $g(l)$ in each period $t > 1$ has been already justified in the proof of proposition 4. In such a network the player who obtains the worst payoff is better than in any other network. Therefore, the condition on the number of players that prevent deviations is smaller if $g(l)$ forms from period 2 than if any other network forms.

However, by lemma 1, $g(l')$ and $g(l)$ cannot form within the outcome of a RNE. In fact, $\pi_i(l_i', l_{-i}) > \pi_i(l)$ for the neighbor in $g(l)$ of the isolated player in $g(l')$. As a consequence, in a second step, we deal with the case $g(a^*) = g(l')$ and show that $g(l')$ can be sustained as a RNN.

In a first step, consider any network $g(a^*) \neq g(l')$ and suppose that each player i chooses the following automaton :

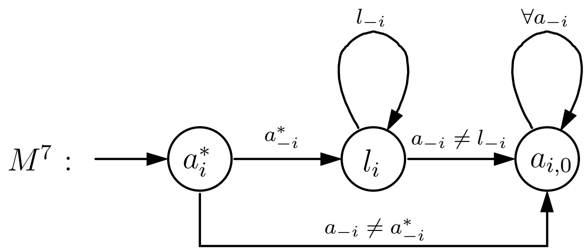


Figure 8.

As it is described in the first paragraph of the proof, the strategy profile $(M_1^7, \ldots, M_n^7)$ outputs network $g(a^*) \neq g(l')$ in period 1 and thereafter $T-1$ line networks $g(l)$. Player i obtains the average payoff

$$\tilde{\pi}_i(M_i^7, M_{-i}^7) = \frac{\pi_i(a^*) + (T-1)\pi_i(l)}{T}$$

Any deviation induces a definitive minmax punishment. Thus, deviating in the first period yields player i the average payoff $\pi_i(b_i, a_{-i}^*)/T$. It is not the interest of player i to deviate in stage 1 if and only if

$$\frac{\pi_i(a^*) + (T-1)\pi_i(l)}{T} \geq \frac{\pi_i(b_i, a_{-i}^*)}{T} \iff (T-1)\pi_i(l) \geq d_i(a^*) \tag{13}$$

Observe that $d_i(a)$ is maximal for a network $a \in A^{ce}$ in which player i seeks contact with each of the $n-1$ opponents whereas only one contact is needed to connect the network. Such a network being cost efficient, player i keeps the network connected by removing $n-2$ superfluous links and saves $(n-2)c$. That is, for any action profile $a^* \in A$,

$$\max_{a \in A^{ce}} d_i(a) = (n-2)c \geq d_i(a^*)$$

From the proof of proposition 4, we also know that the worst paid player in network $g(l)$ obtains $(n-1)v - 2c$, or equivalently,

$$\pi_i(l) \geq (n-1)v - 2c,$$

for all $i \in I$. Thus, condition (13) holds if

$$(T-1)((n-1)v - 2c) \geq (n-2)c \iff c \leq \frac{(n-1)(T-1)}{n+2T-4}v$$

As $2v > c$ and $T > 3$, the reader can check that $n \geq 11$ is enough to guarantee that player i has no incentive to deviate. Next, consider a deviation in stage $t > 1$. The outcome $(a^*, l, \ldots, l)$ satisfies the necessary condition of lemma 1, that is, for all $i \in I$, there is no stage $k < T$ such that

$\pi_i(a_i^k, l_{-i}^T) > \pi_i(l^T)$. In other words, players cannot benefit from deviating in stage T. Since we have also proved that $n \geq 11$ garantees that players won't deviate in stage 1, it remains to test deviations in stages $2, \ldots, T-1$. The line network is formed in each of these stages such that the condition which prevent deviations in these periods is identical to that in proposition 3 for a RNN based on $g(l)$. To see this, recall that if a player has an incentive to deviate, then he does so late in the game because of the definitive punishment. Condition (3) in proposition 3 is satisfied for the RNN based on a line network when $n \geq 11$. Therefore, any network $g(a^*) \neq g(l')$ is likely to form in a RNE when $n \geq 11$.

In a second step, assume that each player i chooses the following two-state automaton :

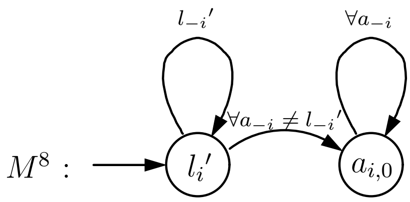


Figure 9.

The strategy profile $(M_1^8, \ldots, M_n^8)$ forms a repeated network based on $g(l')$. Once again, any deviation releases a definitive minmax punishment. This situation is a particular case of proposition 3. Therefore, it is enough to check that condition (3) is satisfied to show that $(M_1^8, \ldots, M_n^8)$ is a RNN. As $n \geq 11$, recall that only the direct neighbors of the first and last players in the line component have an incentive to deviate from l'. Let i be one of these two players. We need

$$\begin{aligned} & d_i(l') \leq \min\left\{\frac{v}{2}\#N_i(g(b_i, l_{-i}{}')), 2\pi_i(l')\right\} \\ \iff \quad & c - v \leq \min\left\{(n-2)\frac{v}{2}, 2((n-1)v - 2c)\right\} \\ \iff \quad & c - v \leq (n-2)\frac{v}{2} \\ \iff \quad & n \geq \frac{2c}{v} \qquad (14) \end{aligned}$$

which always holds when $n \geq 11$. We conclude that network $g(l')$ can also form in an equilibrium of G^T. ■

There is a contrast between this result and previous results of this section. On one hand, any static network can form if one consider just one period of the repeated game. On the other hand, the T-period outcomes of RNE must satisfy restrictive conditions. In the initial period, players can set links which form any network (proposition 8). But the formation of this first network then prevents players from creating some other networks in future periods (lemma 1 and proposition 6). For instance, if the network that forms in stage 1 is connected, then by proposition 6 this network cannot be a subnetwork of another network. Thus, the initial network conditions the structure of the whole outcome of a RNE.

4 Nework efficiency

In this section, we characterize the set of efficient static networks for both Bentham and Pareto criteria (see appendix A2 for the proofs). Next, we examine this question in the finitely repeated game and compare the results with those in the static case. Again, we consider both cases of players being perfectly rational or not, and we study the differences in the corresponding sets of efficient strategy profiles. Precisely, we give a condition on the duration of the game for which boundedly rational players can implement strictly less efficient strategy profiles than perfectly rational players.

4.1 Bentham-efficient networks and strategy profiles

We briefly recall the definition of Bentham efficiency.

Definition 5 *A network $g(a^*)$ is Bentham-efficient if*

$$a^* \in \arg\max_{a \in A} W(a),$$

where $W(a) = \sum_{i \in I} \pi_i(a)$ is the welfare of network $g(a)$. The following example underlines that the set of Bentham-efficient networks is larger than the set of Nash networks of the one-shot game.

Example 2
Fix $I = \{1, 2, 3, 4, 5\}$. Consider the cost-efficient action profiles which induce a line network $g(l)$, beginning and ending with players 1 and 3 respectively, and a star network $g(s_2)$ centered on player 2. These two networks are represented in the following figure

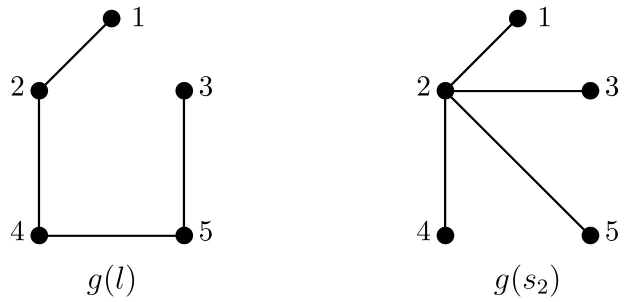


Figure 10.

By proposition 1 both $g(l)$ and $g(s_2)$ are not Nash networks. It should be clear that any Bentham-efficient network is induced by a cost-efficient strategy profile. The two networks are connected with a minimal number of 4 links. They are trees since they are also acyclic. Therefore, each player obtains the maximal value from $g(l)$ and $g(s_2)$. The criterion of Bentham considers how beneficial the network is to the whole player set. In other words, it does not matter how the cost needed to form the network is distributed among players. The two networks in this example have the same welfare but they illustrate two extreme cases. In $g(s_2)$, player 2 supports the highest possible cost (which implies that his payoff is negative) while the other players support the minimal cost. In $g(l)$, no player supports a cost more than $2c$ (which is the minimal cost required for any player to connect the whole network). The distribution of the cost among players can be represented by the diameter of the network. The diameter of a tree increases if the cost is more equitably distributed. Indeed, we have $L(g(s_2)) = 2$ and $L(g(l)) = n - 1 = 4$. □

This is the intuitive explanation that will be used to show that these two networks belong to the set of Bentham-efficient networks. We assume $n > 3$ to exclude networks for which no player obtains a positive payoff.[9]

Proposition 9 *Suppose $n > 3$. The static network $g(a^*)$ is Bentham-efficient if and only if*

1. *a^* is cost efficient,*
2. *$g(a^*)$ is a tree.*

Now we turn to the definition of Bentham efficiency in the repeated game.

[9]In fact, for $n = 2$, g_0 is the only Bentham-efficient network. For $n = 3$, g_0 is Bentham-efficient network if $3c/2 \leq v$ while the 3-player line network (which is also a star network) is Bentham-efficient if $3c/2 \geq v$.

Definition 6 *A strategy profile s^* is Bentham-efficient if*

$$s^* \in \arg\max_{s \in S} W(s),$$

where $W(s) = \frac{1}{T}\sum_{t=1}^{T} W(a^t)$. In particular, if s^* induces a repeated network based on $g(a^*)$, then s^* is Bentham-efficient if and only if the static network $g(a^*)$ is Bentham-efficient. Let A_{BE} be the set of Bentham-efficient static networks found in proposition 9 and S_{BE} be the set of Bentham-efficient strategy profiles. Then S_{BE} is characterized in the following proposition.

Proposition 10 *A strategy profile s^* is Bentham-efficient if and only if it forms in each period a Bentham-efficient static network. Formally,*

$$S_{BE} = \{s \in S : g(a^t) \in A_{BE}, a^t \in h^{T+1}(s), t \leq T\}.$$

Proof. A strategy profile s^* that forms a Bentham-efficient network in each period has a maximum average welfare $W(s^*)$ since the welfare is maximal is each period. Thus, the set of Bentham-efficient strategy profiles consists in all strategy profiles whose outcome consists in T Bentham-efficient static networks, that is $S_{BE} = \{s \in S : g(a^t) \in A_{BE}, a^t \in h^{T+1}(s), t \leq T\}$. ■

A population of perfectly rational players can implement any efficient strategy profile in S_{BE}. Now, let $\mathcal{M}_{BE}$ denote the set of Bentham-efficient strategy profiles of the machine game. A particular efficient strategy profile $s \in S_{BE}$ may induce T different efficient networks. Thus, when players are assumed to use finite automata of limited size as in section 3, one can ask whether n automata with at most $T-1$ states may form T different efficient networks. We present two examples which illustrate how boundedly rational players can implement Bentham-efficient strategy profiles. In example 3, we show that machines with at most $T-1$ states may form T distinct trees. Example 4 shows that such an operation does not work for all efficient strategy profiles. We give a sequence of T line networks which cannot be implemented by automata with at most $T-1$ states. From this example, we will state a general proposition.

Example 3
Fix $I = \{1, 2, 3, 4, 5, 6, 7\}$ and consider the n-tuple of $(T-1)$-state machines $(M_1, \ldots, M_n)$. Each machine M_i uses a new state in each period to form $T-1$ different trees in the $T-1$ first periods. Suppose that the cost-efficient action profiles a and a' played in periods t and t' induce the trees represented in the following figure:

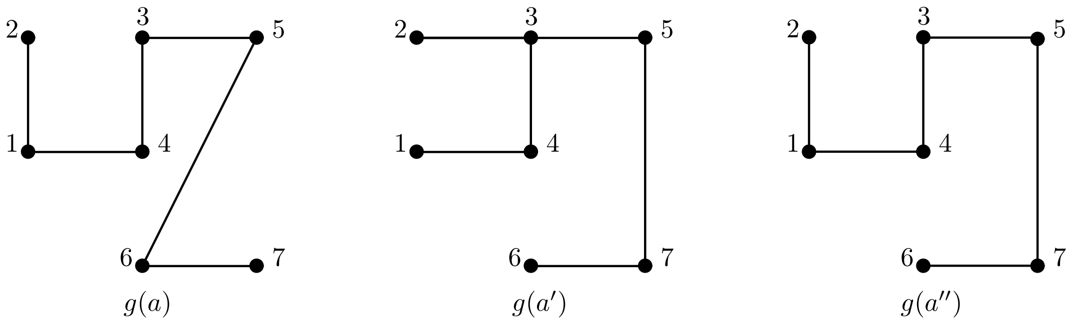


Figure 11.

Now let $g(a'') = g(a_1, a_2, a_3, a_4{}', a_5{}', a_6{}', a_7{}')$ be the static network that consists for player 1,2 and 3 in playing their actions played in stage t and for all other players in playing their actions played in stage t'. It is easy to check that network $g(a'')$ is also a tree and that a'' is cost-efficient. Assume that $g(a'')$ has not formed before stage T. Automata can use one state devoted to the formation of each of the first $T-1$ efficient networks, with transitions going from states to states until stage $T-1$. So far, we have not specified the transition in machine $M_i \in (M_1, \ldots, M_n)$ for the last stage. If for all $i = 1, 2, 3$, player i's machine transits in stage T to the state used in period t and for any $j = 4, 5, 6, 7$, player j's machine transits in stage T to the state used in period t', then network $g(a'')$ forms in round T. Therefore, the constructed strategy profile $(M_1, \ldots, M_n)$ generates an outcome of T different Bentham-efficient networks. □

Once again, consider an outcome of T distinct efficient networks. By contrast to example 3, combining actions played in the first $T-1$ stages may not allow for the creation of the Tth efficient network in the final period. This is the case in the next example.

Example 4 Fix $T = 4$ and $n = 5$. Consider the sequence of networks $(g(l^1), \ldots, g(l^4))$ represented in the following figure :

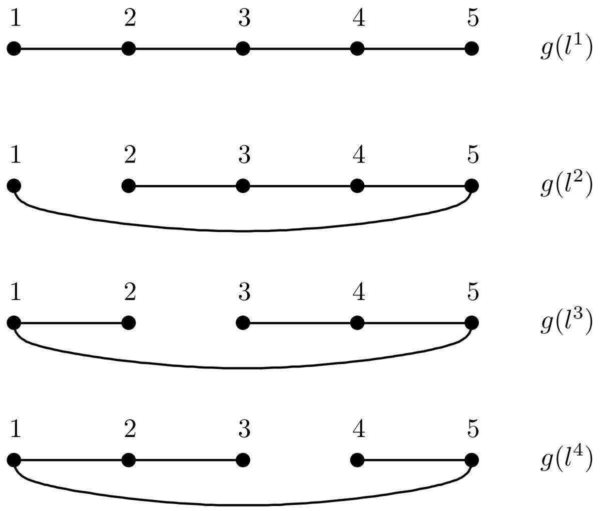


Figure 12.

A different line network forms in each period. The sequence of line networks is "circulating" : the beginning of the line moves to the right as time goes by. Such an outcome is Bentham-efficient and can be implemented by a population of perfectly rational players. However, players limited to strategies played by automata with at most three states cannot form these four networks. To see this, observe that links 34 forms in the first three networks but not in the final one. This implies that this link must belong to any combination of actions played during the first three periods. By construction, there exists an efficient outcome that cannot be formed by an automata profile whenever $n \geq T$. □

Example 4 shows that an efficient outcome may not be implemented by boundedly rational players. This example leads to the following result.

Proposition 11 *Suppose $n \geq T$. Then $\mathcal{M}_{BE} \subsetneq S_{BE}$.*

This result contrasts with those found on RNE. On the one hand, the set of RNE is more important when players are boundedly rational (see propositions 2 and 3 for instance). On the other hand, proposition 11 shows that boundedly rational players can implement less Bentham-efficient strategy profiles than perfectly rational players.

4.2 Pareto-efficient networks and strategy profiles

A network is Pareto-efficient if a player cannot obtain a larger payoff in another network without reducing an opponent's payoff.

Definition 7 *A network $g(a^*)$ is Pareto-efficient if for any $i \in I$ and any $a \neq a^*$,*

$$\pi_i(a) > \pi_i(a^*) \Rightarrow \exists j \neq i | \pi_j(a) < \pi_j(a^*).$$

Next, we provide a complete characterization of Pareto-efficient networks.

Proposition 12 *Fix $n > 4$.*[10] *The static network $g(a^*)$ is Pareto-efficient if and only if*

1. *a^* is cost efficient,*

2. *$g(a^*)$ is a tree.*

As a consequence of propositions 9 and 12, if the player set has at least 5 elements, a network is Bentham-efficient if and only if it is Pareto-efficient, that is $A_{BE} = A_{PE}$ where A_{PE} is the set of Pareto-efficient networks. This equivalence is also trivially verified when $n = 2$. Remark that the star network with $n > 4$ players is efficient for both Bentham and Pareto criteria although the player in the center of the star obtain less than the minmax payoff. This shows that a static efficient network is not always sustained as a RNN.

In the repeated game the definition of a Pareto-efficient strategy profile is as follows.

Definition 8 *A strategy s^* is Pareto-efficient if for any $i \in I$ and any $s \neq s^*$,*

$$\tilde{\pi}_i(s) > \tilde{\pi}_i(s^*) \Rightarrow \exists j \neq i | \tilde{\pi}_j(s) < \tilde{\pi}_j(s^*).$$

The characterization of the set of Pareto-efficient strategy profiles is given in the next proposition.

Proposition 13 *A strategy profile s^* is Pareto-efficient if and only if Pareto-efficient static network forms in each period. Formally*

$$S_{PE} = \{s \in S : g(a^t) \in A_{PE}, a^t \in h^{T+1}(s), t \leq T\} = S_{BE}.$$

Proof. By definitions 6 and 8 a Bentham-efficient strategy profile is Pareto-efficient too (first inclusion). By proposition 10, this implies that the set of

[10] For $n = 2$, g_0 is the only Pareto-efficient network. For $n = 3$, any tree is Pareto-efficient. For $n = 4$, any tree is Pareto-efficient but a network that consists in a 3-player line component and an isolated player may also be Pareto-efficient if $3v/2 \geq c$.

strategy profiles in which Pareto-efficient networks form at each round is included in S_{PE}. Let us prove the other inclusion. We proceed by contradiction. Consider a Pareto-efficient strategy profile s^{PE} in which at least one non Pareto-efficient networks $g(a)$ forms. It is sufficient to suppose that there is only one such network, and that it forms at round t. Thus, by definition, we know that there exists a network $g(a^{PE})$ such that, for any i, $\pi_i(a^{PE}) \geq \pi_i(a)$, and $\pi_j(a^{PE}) > \pi_j(a)$ for some j. Now consider the strategy profile s that induces the formation of network $g(a^{PE})$ at round t, and the formation of the same networks than s^{PE} at the other rounds. Then we check easily than $\tilde{\pi}_i(s) \geq \tilde{\pi}_i(s^{PE})$, and $\tilde{\pi}_j(s) > \tilde{\pi}_j(s^{PE})$. This contradicts the Pareto-efficiency of s^{PE}, and we conclude by contradiction that $S_{PE} = \{s \in S : g(a^t) \in A_{PE}, a^t \in h^{T+1}(s), t \leq T\}$. ■

When players are assumed to use finite automata of limited size as in section 3, the set of Pareto-efficient strategy profiles $\mathcal{M}_{PE}$ satisfies the same natural restriction than that highlighted for $\mathcal{M}_{BE}$ in proposition 11. Therefore, boundedly rational players can form a smaller number of efficient strategy profiles in the repeated game than the perfectly rational players.

The results regarding the question of efficiency shows that in our model Bentham-efficient and Pareto-efficient networks or strategy profiles have identical structures provided that the number of players is not too small. This correspondence between Bentham-efficiency and Pareto-efficiency is discussed in Jackson (2003) for the case of static network formation games.

5 Conclusion

Within a finite-horizon repeated game framework we study the problem of (dynamic) network formation when the players are either perfectly rational, or boundedly rational (in the sense of Neyman (1985) and by a restriction to a subset of action profiles). We prove that the set of RNE is reduced to the empty network when the agents are perfectly rational, while this set is much more elaborate when the complexity of their strategies is limited. Then we identify structural properties of RNN and RNE. In the case of RNN, we prove that each network that is sustained as a RNN cannot be too much over-connected, and that each player cannot bear a too important share of the cost of the total network. This highlights a lack of robustness of architectures such as star networks. In the case of RNE, we prove that the networks induced in any period satisfy some properties. Within a RNE, players may prefer to draw links with totally different partners at each round, or the network may retract progressively. Bounded rationality has a noticeable influence both on the existence of (non trivial) equilibria and on the dynamics of network formation. Assuming a limited ability to implement link formation seems reasonable since it is consistent with well-known economic behaviors (searchers in a trading market for instance). Finally, we make some comparisons between the sets of (Bentham and Pareto) efficient strategy profiles. In this part, the nature of results is reversed. Under some relation between the duration of the game and the number of players, it is shown that more rational players will implement a larger number of efficient strategy profiles.

One of the main assumption of the present work is that consent is needed to form links. A possible extension to this paper would be to see what happens if this assumption is relaxed. In particular, the resulting networks may be directed with the consequence that information is only one-way flow (as in Billand and Bravard, 2005). This is left for future research.

Appendix

A2 - Results with perfectly rational players

Proof. (proposition 1) Firstly, we show that in any network $g(a) \neq g_0$, at least one player has an incentive to deviate. Secondly, we prove that $g_0 = g(a_0)$ is a Nash network.

Consider any network $g(a) \neq g_0$. Two cases must be studied :

1. Suppose that a is not cost efficient, then $\exists i, j \in I$ such that $a_{ij} = 1 \neq a_{ji} = 0$. It is the interest of player i to choose action $a_i(j^-)$ that only differs from a_i by $a_{ij} = 0$. Player i saves c while $g(a_i(j^-), a_{-i}) = g(a)$. He obtains $\pi_i(a_i(j^-), a_i) = \pi_i(a) + c$ which implies $g(a)$ is not a Nash network.

2. Suppose that a is cost efficient. If $g(a)$ is a cyclic network, then there is at least two players i and j such that $N_i(g(a) - ij) = N_i(g(a))$. Link ij is superfluous. We have $d_i^1(a) > 0$ which implies that $g(a)$ is not a Nash network. If $g(a)$ is an acyclic network, then there is a player i whose connection set satisfies $\#N_i(g(a)) = 1$. Let $N_i(g(a)) = \{j\}$, then link ij is superfluous for player j. Therefore, $d_j^2(a) > 0$ which implies $g(a)$ is not a Nash network.

The empty network $g_0 = g(a_0)$ is the unique network that does not fit any of the two previous cases. In $g(a_0)$, there is no player i who has an incentive to deviate from $a_{i,0}$ since he cannot create links alone and would support a cost c for any such attempt. Then the cost-efficient empty network is the only Nash network of G. ■

Proof. (proposition 2) Consider network $g(a) \neq g_0$ and suppose $g(a^T) = g(a)$. By proposition 1 and a backward induction argument, there is at least one player say i who has interest in altering his action in the last period. Player i's opponents anticipate this behavior and also remove some links. As a consequence, the empty network necessarily forms in the last stage. A similar process leads to the formation of the empty network in all periods. ■

A3 - Static efficient networks

Proof. (proposition 9) ($\Rightarrow$) Suppose network $g(a^*)$ is Bentham-efficient. Firstly, a^* must be cost efficient. Otherwise there are players $i, j \in I$ with $a^*_{ij} = 1 \neq a^*_{ji} = 0$ such that player i saves c if he or she plays $a^*_i(j^-) = 0$. Player i's altered action does not remove any link and maintains other players' payoffs. Remark that $g(a^*)$ must be acyclic, otherwise there is player i in $g(a^*)$ who has some superfluous links or equivalently $d^1_i(a^*) > 0$. If i removes one such link, say with j, he obtains $\pi_i(a^*_i(j^-), a^*_{-i}) = \pi_i(a^*) + c$ and $\pi_h(a^*_i(j^-), a^*_{-i}) = \pi_h(a^*)$, $\forall h \neq i, j$ since $a^*_i(j^-)$ keeps unchanged all other players' connections (and payoffs).

Secondly, we have to show that $g(a^*)$ can but be a tree. To see this, we prove that the welfare of a tree is larger than in any other type of network. In a tree, the n vertices must be connected by exactly $n - 1$ links. The formation of a link costs c to two players. Thus, the total cost of a tree $g(a^*)$ is $(n - 1)2c$. The value of $g(a^*)$ for each player is $(n - 1)v$. The welfare of any tree $g(a^*)$ induced by a cost-efficient action profile a^* [11] is

$$W(a^*) = n(n-1)v - (n-1)c = (n-1)(nv - 2c). \tag{15}$$

Now consider any non connected network $g(a')$. By definition, $g(a')$ is splitted in $K > 1$ components which are connected subnetworks. If $g(a')$ is candidate to be Bentham-efficient, then each subnetwork $g(a'_k)$, $k \in \{1, \ldots, K\}$, satisfies condition 1 of the proposition, that is, $g(a')$ is a forest of K trees induced by cost-efficient action profiles. The component $g(a'_k)$ has $\#I_k = n_k$ vertices (or players). The welfare of $g(a'_k)$ is

$$W(a'_k) = (n_k - 1)(n_k v - 2c),$$

and the total welfare of a' follows,

$$W(a') = \sum_{k=1}^{K} (n_k - 1)(n_k v - 2c).$$

[11] Such a network ranges from the n-player line network to the n-player star network. These two networks exhibits extreme situations according to their diameter. A line network has a diameter of $n - 1$, the largest diameter among trees, while a star network has a diameter 2, the smallest diameter among trees.

By definition of I_k,

$$nv - 2c > n_k v - 2c \Rightarrow \sum_{k=1}^{K} (n_k - 1)(nv - 2c) > \sum_{k=1}^{K} (n_k - 1)(n_k v - 2c).$$

Furthermore, $\sum_{k=1}^{K} n_k - 1 = n - 1$ implies

$$(n-1)(nv - 2c) > \sum_{k=1}^{K} (n_k - 1)(nv - 2c),$$

and we conclude that $W(a^*) > W(a')$.

($\Leftarrow$) Suppose network $g(a^*)$ satisfies the two conditions listed in the statement of proposition 9. By the previous calculation, the welfare in $g(a^*)$ is larger than in any other network. ■

Proof. (proposition 12) ($\Leftarrow$) By proposition 9, any tree induced by a cost-efficient action profile is Bentham-efficient. And it follows from definitions 5 and 7 that any Bentham-efficient network is Pareto-efficient. Therefore, any tree induced by a cost-efficient action profile is a Pareto-efficient network.

($\Rightarrow$) Firstly, the action profile that forms any Pareto-efficient network must be cost-efficient for the same reason than in the proof of proposition 9. Secondly, to show that any Pareto-efficient network is a tree, we proceed by contradiction. Consider any Pareto-efficient network $g(a^*)$ that is not a tree. Note that $g(a^*)$ cannot be k-connected, $k > 1$ since the existence of some superfluous links would contradicts the fact that $g(a^*)$ is Pareto-efficient. Then, suppose that $g(a^*)$ is a non connected Pareto-efficient network. We group all possibilities in three cases according to the structure of $g(a^*)$:

Case 1.
Network $g(a^*)$ consists in at least 2 connected components with at least 2 players. Each component must be a tree to avoid cycles (and superfluous links). Let K be the total number of components in the forest $g(a^*)$. Construct network $g(a)$ that consists in connecting the K components all together with the creation of $K - 1$ links (see for instance the proof of proposition 9). Since $2v > c$, it is easy to check that the resulting network yields all players a larger payoff than $g(a^*)$. This proves that $g(a^*)$ cannot be Pareto-efficient.

Case 2.
Network $g(a^*)$ consists in a connected component and $k > 1$ isolated players. The component must be a tree to avoid cycles. The method described in

case 1 is still beneficial to all players whenever at least 2 isolated players can be linked to the main tree component.

Case 3.
Network $g(a^*)$ consists in a connected component and a single isolated player. The component must also be a tree to avoid cycles. If the connected component includes all but one player denoted h, it follows that the creation of link ih with any player i in the component is beneficial to all players except i who loses $v - c$. Nonetheless, it is possible to connect player h to all other opponents in a way that increase everyone's payoff. Let i be a player such that $\#n_i(g(a^*)) = 1$. Player i must exist since he belongs to a tree. Precisely let $n_i(g(a^*)) = \{j\}$. Construct network $g(a) = g(a^*) - ij + hj + hi$. The reader can check that all players except h have the same number of direct neighbors in $g(a)$ than in $g(a^*)$ and benefits from the additional connection with player h. Player h creates two links but is connected to at least 4 players such that his payoff is larger than that of a isolated player. Therefore all players obtain a larger payoff, which implies that $g(a^*)$ is not a Pareto-efficient network.

Thus, a Pareto-efficient network is a tree. This concludes the proof. ■

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