Dissolving a common value partnership in a repeated 'queto' game with incomplete information on both sides*

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Abstract

We analyse a common value, alternating ascending bid, first price auction as a repeated game of asymmetric incomplete information on both sides where the bidders own the object auctioned in equal shares. The object is indivisible and of common value. Consequently an owner can accept her partner's offer (by *quitting* the repeated game) or can *veto* a proposed settlement by submitting an own offer. We characterise the equilibrium map of this game and discuss its properties. (JEL *D44,D82,G12.* Keywords: *Repeated games, Incomplete information, Common value auctions, Partnership dissolution.*)

Introduction

We analyse a common value, alternating ascending bid, first price auction as a repeated (dynamic) game of asymmetric incomplete information as defined in Mertens, Sorin, and Zamir (1994). We discuss a setting where two players hold equal ownership titles to a single indivisible object—the partnership—which has the same objective value to both players. These ownership or property rights give the players veto power over any settlement they oppose. The value of the object is decided initially through a casting move by Nature who draws the common value from some publicly known distribution. Subsequently both players are sent private signals further asymmetrically detailing their (symmetric) prior information. Player one (P1) starts bidding and player two (P2) accepts any positive initial offer for the object, and keeps it if no such offer realises—in which latter case the game is over. The same rules apply to any successive period of the game. Any bid strictly higher than the previous bid leads to a new stage of the repeated game. Bids at or below the current price end the game. We call this game a 'queto' game because, each period, the moving player has the choice between *quitting* the auction and accepting the current offer and *vetoing* the proposed sharing rule by bidding the required compensation up.

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In the present paper we extend the analysis of queto games in Schweinzer (2003) to the case of (asymmetric) incomplete information on both sides. While in the aforementioned one sided incomplete information model the informed player's information is a superset of the uninformed players', this is not the case in the setup studied in the present paper. Here players receive partly revealing signals of commonly known accuracy which determine the players' information. In contrast to the model with incomplete information on one side the player with the higher precision signal can—in general—still gain knowledge from learning the lower accuracy signal private to the opponent. Hence both players receive and emit information in the unfolding game. Queto games are strategically very similar to the alternating-offer bargaining game of Rubinstein (1982) but lack its crucial time (and thus discounting) dimension.

To motivate our setup we use an example similar to Schweinzer (2003): A couple lives in their jointly owned couple-house and decide to separate. They could go on living together but only at an unspecified cost which would substantially diminish the utility they draw from their property.[1] The partners are asymmetrically informed about the value of their house. They sit down at a table opposite to each other in order to settle the dissolution of their partnership. With each round, the bidder puts the offered sum of money on the table. Subsequent bids add to the pile on the table—once a bill is placed there it cannot be taken back. The first player accepting an offer (by failing to bid more) gets all money on the table while the other gets the house.

The applicable literature is reviewed in detail in Schweinzer (2003). As there, the general repeated game model we employ is due to Mertens, Sorin, and Zamir (1994) based on Aumann and Maschler (1966). The newly introduced signal accuracy is a discrete version of ideas developed by Persico (2000).

1 The model

The set of players—the partners—is denoted by $N = \{1, 2\}$. Players jointly own the object in equal shares and are not wealth constrained.[2] Players are *risk-neutral* and final payoffs are given by the *undiscounted sum* of stage payoffs u^t. We denote the *common value* of the indivisible object to be shared—the partnership—by $\theta \in \Theta \equiv \{\underline{\theta}, \bar{\theta}\}$ with $\underline{\theta} < \bar{\theta}$. We always normalise the set of the object's possible value realisations $\{\underline{\theta}, \bar{\theta}\}$ to $\{0, \theta = \bar{\theta} - \underline{\theta}\}$. The players' *types* are the private *signals* $s_i \in S_i$ they receive on this value. We assume the signal's precision (or accuracy) $p_i = \text{prob}(s_i = \bar{s}|\bar{\theta}) = \text{prob}(s_i = \underline{s}|\underline{\theta}) \in [{}^1\!/_2, 1]$ to be independently and identically distributed conditional on a realised value θ of the object and denote (p_1, p_2) by $\mathbf{p}$. We define a *state* of the world as a triple $\omega = (\theta, s_1, s_2) \in (\Theta \times S_1 \times S_2) \equiv \Omega$. There is some publicly known prior (joint) probability distribution φ^0 over Ω which is refined into the player's *beliefs* about

[1] As in the case of incomplete information on one side, we assume throughout the paper that this cost is sufficient to induce *both* players to dissolve the partnership.

[2] Again—as in Schweinzer (2003)—that shares are equal is not crucial in a technical sense since all bids are made for the *whole* object. Only the unanimity of any decision, i.e. both players' veto power, is important.

the state of the world on the basis of the initial signal and the players' observed behaviour.

Stage *actions* are bids $b_i^t : \{b^\tau\}_{\tau=1}^{t-1} \times S_i \mapsto \Delta(B)$ taking values in the set of possible stage actions B. This set defines the constant minimal bidding increase ν. Bids are transfers (and therefore stage payoffs) to the opponent in exchange for possession of the object. Players can monitor their opponents' actions perfectly well and enjoy perfect recall. A player's repeated game *strategy* $\beta_i(s_i, \mathbf{p})$ is a complete, contingent plan, a profile of which is denoted by $\beta(s, \mathbf{p})$. We refer to a strategy as a *separating* strategy if it entails an action which reveals the own signal fully to the opponent before the maximum feasible bid is reached. The discrete time t *price* of the object p^{t-1} is last period's highest bid, or—since there is only one bid—just b^{t-1}. A *stage game* consists of a single bid; P1 bids at odd periods and P2 at even periods t. The repeated queto game is denoted by Γ. The rules of the game are that bids have to be strictly increasing and alternating. Initial possession is given to P2, the informed party P1 makes the first move and the game ends if a player chooses to exit in which case the current possessor becomes owner of the object. If neither player ever exits, the game continues forever.

2 Symmetric incomplete information

2.1 The analytical framework

Both players are sent partially revealing signals $s_i \in S = \{\underline{s}, \bar{s}\},\ i = \{1, 2\}$ on the true value of the object θ. The example is parameterised by the probability of the opponent receiving the opposite signal $\eta = \text{prob}(s_1 = \bar{s}, s_2 = \underline{s}) = \text{prob}(s_1 = \underline{s}, s_2 = \bar{s})$ and the following assumptions

$$\begin{gathered}\text{prob}(\theta = \bar{\theta} | s_1 = \bar{s}, s_2 = \bar{s}) = \text{prob}(\theta = \underline{\theta} | s_1 = \underline{s}, s_2 = \underline{s}) = 1,\\ \text{prob}(\theta = \bar{\theta} | s_i = \bar{s}, s_{-i} = \underline{s}) = \text{prob}(\theta = \underline{\theta} | s_i = \underline{s}, s_{-i} = \bar{s}) = {}^1\!/_2,\\ \text{prob}(s_i = \bar{s}, s_{-i} = \underline{s} | \theta = \bar{\theta}) = \text{prob}(s_i = \bar{s}, s_{-i} = \underline{s} | \theta = \underline{\theta}) = \eta.\end{gathered}$$

These—given common priors for s and θ of ${}^1\!/_2$—define all remaining probabilities in the model. We restrict attention to $\nu \in [0,\ {}^1\!/_2]$. Denote the probability of P1 bidding 1 after observing a low signal $\underline{s}_1$ by $\underline{\lambda}$ and, similarly, the probability of P2 bidding 2 after observing P1's bid of 1 and observing a high signal $\bar{s}_2$ is $\bar{\mu}$.

2.2 An example

We reduce the above model to the simplest case. Let the possible values of the partnership be $\Theta = \{0, 3\}$. We set the bidding grid to $\mathbb{N}$ which defines minimal bidding increments $\nu = 1$. Let P1 have the first move. The initial price is set to $p^0 = 0$. We illustrate the game tree for (a slightly more general version of) this example in figure 1. As it turns out it is convenient to subdivide the analysis into three intervals (a)–(c) for η. We consider the equilibrium candidate c_s^* depicted in figure 2 and start by computing the players' expected payoffs for each interval.

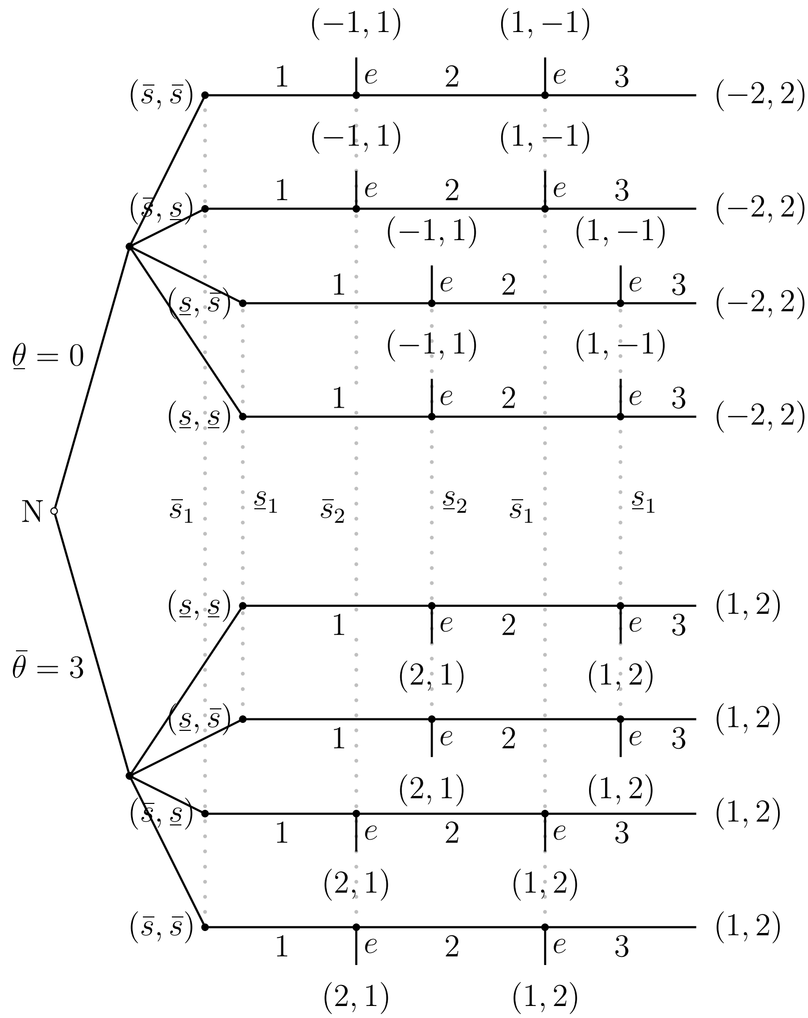


Figure 1: Extensive form for $\theta \in \{0, 3\}$ with symmetric incomplete information.

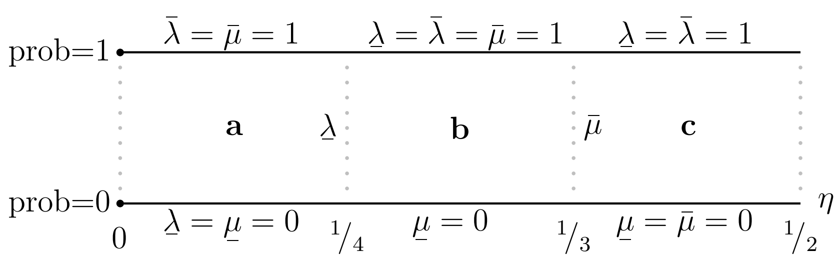


Figure 2: The equilibrium candidate c_s^* in the first example.

(a) $\eta \in [0, 1/4]:\ \underline{\lambda} = \underline{\mu} = 0,\ \bar{\lambda} = \bar{\mu} = 1$

(a) $u_1(\bar{s}_1) = \text{prob}(\bar{s}_2|\bar{s}_1)(1) + \text{prob}(\bar{s}_2, \underline{\theta}|\bar{s}_1)(-1) + \text{prob}(\bar{s}_2, \bar{\theta}|\bar{s}_1)(2) = 1 - \eta$

(b) $u_1(\underline{s}_1) = \text{prob}(\underline{s}_2|\underline{s}_1)(0) + \text{prob}(\bar{s}_2, \underline{\theta}|\underline{s}_1)(0) + \text{prob}(\bar{s}_2, \bar{\theta}|\underline{s}_1)(0) = 0$

(c) $u_2(\bar{s}_2) = \text{prob}(\bar{s}_1|\bar{s}_2)(2) + \text{prob}(\underline{s}_1, \underline{\theta}|\bar{s}_2)(0) + \text{prob}(\underline{s}_1, \bar{\theta}|\bar{s}_2)(3) = 2 - \eta$

(d) $u_2(\underline{s}_2) = \text{prob}(\underline{s}_1|\underline{s}_2)(0) + \text{prob}(\bar{s}_1, \underline{\theta}|\underline{s}_2)(1) + \text{prob}(\bar{s}_1, \bar{\theta}|\underline{s}_2)(1) = 2\eta$

(b) $\eta \in (1/4, 1/3]:\ \underline{\mu} = 0,\ \underline{\lambda} = \bar{\lambda} = \bar{\mu} = 1$

(a) $u_1(\bar{s}_1) = \text{prob}(\bar{s}_2|\bar{s}_1)(1) + \text{prob}(\bar{s}_2, \underline{\theta}|\bar{s}_1)(-1) + \text{prob}(\bar{s}_2, \bar{\theta}|\bar{s}_1)(2) = 1 - \eta$

(b) $u_1(\underline{s}_1) = \text{prob}(\underline{s}_2|\underline{s}_1)(-1) + \text{prob}(\bar{s}_2, \underline{\theta}|\underline{s}_1)(1) + \text{prob}(\bar{s}_2, \bar{\theta}|\underline{s}_1)(1) = 4\eta - 1$

(c) $u_2(\bar{s}_2) = \text{prob}(\bar{s}_1|\bar{s}_2)(2) + \text{prob}(\underline{s}_1, \underline{\theta}|\bar{s}_2)(-1) + \text{prob}(\underline{s}_1, \bar{\theta}|\bar{s}_2)(2) = 2 - 3\eta$

(d) $u_2(\underline{s}_2) = \text{prob}(\underline{s}_1|\underline{s}_2)(1) + \text{prob}(\bar{s}_1, \underline{\theta}|\underline{s}_2)(1) + \text{prob}(\bar{s}_1, \bar{\theta}|\underline{s}_2)(1) = 1$

(c) $\eta \in (^1/_3, ^1/_2] : \ \underline{\mu} = \bar{\mu} = 0, \ \underline{\lambda} = \bar{\lambda} = 1$

(a) $u_1(\bar{s}_1) = \text{prob}(\bar{s}_2|\bar{s}_1)(2) + \text{prob}(\bar{s}_2, \underline{\theta}|\bar{s}_1)(-1) + \text{prob}(\bar{s}_2, \bar{\theta}|\bar{s}_1)(2) = 2 - 3\eta$

(b) $u_1(\underline{s}_1) = \text{prob}(\underline{s}_2|\underline{s}_1)(-1) + \text{prob}(\bar{s}_2, \underline{\theta}|\underline{s}_1)(-1) + \text{prob}(\bar{s}_2, \bar{\theta}|\underline{s}_1)(2) = 3\eta - 1$

(c) $u_2(\bar{s}_2) = \text{prob}(\bar{s}_1|\bar{s}_2)(1) + \text{prob}(\underline{s}_1, \underline{\theta}|\bar{s}_2)(1) + \text{prob}(\underline{s}_1, \bar{\theta}|\bar{s}_2)(1) = 1$

(d) $u_2(\underline{s}_2) = \text{prob}(\underline{s}_1|\underline{s}_2)(1) + \text{prob}(\bar{s}_1, \underline{\theta}|\underline{s}_2)(1) + \text{prob}(\bar{s}_1, \bar{\theta}|\underline{s}_2)(1) = 1$

A straightforward check for deviations suffices to conclude that c_s^* is indeed an equilibrium. There is, however, nothing interesting or unexpected in this analysis. In order to obtain more interesting asymmetric results where a player can gain from the opponent's inferior information we need to introduce asymmetric incomplete information.

3 Asymmetric incomplete information

3.1 The analytical framework

Except for the probability which specifies the joint signal probability we leave the features of the setting with symmetric incomplete information intact. The probabilities p_1, p_2 with which the signal is correct (i.e. the signals' *accuracies*) are assumed to be iid *conditional* on the realised value θ. The possible range for the publicly known (asymmetric) idiosyncratic signal precision p_i is $[^1/_2, 1], \ i = \{1, 2\}$. In the following we will use matrices such as

$$\begin{pmatrix} 1 & \ldots & 0 \\ 0 & \ldots & 1 \end{pmatrix} \text{ to represent the strategies } \begin{pmatrix} \bar{\lambda}^1 = 1 & \ldots & \bar{\mu}^t = 0 \\ \underline{\lambda}^1 = 0 & \ldots & \underline{\mu}^t = 1 \end{pmatrix} \text{ for even } t.$$

The general game for a set of possible realisations $\{\underline{\theta}, \bar{\theta}\} = \{0, \theta\}$ unfolds as follows.

***t*=0**: By assumption we have $\text{prob}(\bar{\theta}) = \text{prob}(\underline{\theta}) = {}^1/_2$. From the assumption that signal probabilities are iid conditional on a realised value, we get

$$\begin{aligned}
&\text{prob}(\underline{s}_1, \underline{s}_2|\bar{\theta}) = (1 - p_1)(1 - p_2), \ \text{prob}(\underline{s}_1, \underline{s}_2|\underline{\theta}) = p_1 p_2, \\
&\text{prob}(\underline{s}_1, \bar{s}_2|\bar{\theta}) = (1 - p_1)p_2, \ \text{prob}(\underline{s}_1, \bar{s}_2|\underline{\theta}) = p_1(1 - p_2), \\
&\text{prob}(\bar{s}_1, \underline{s}_2|\bar{\theta}) = p_1(1 - p_2), \ \text{prob}(\bar{s}_1, \underline{s}_2|\underline{\theta}) = (1 - p_1)p_2, \\
&\text{prob}(\bar{s}_1, \bar{s}_2|\bar{\theta}) = p_1 p_2, \ \text{prob}(\bar{s}_1, \bar{s}_2|\underline{\theta}) = (1 - p_1)(1 - p_2).
\end{aligned}$$

Given the payoff uncertainty induced by the imperfectly revealing signal, the players form expectations based on the possible signal combinations

$$\mathrm{E}[\underline{s}_1, \underline{s}_2] = \bar{\theta}\frac{(1-p_1)(1-p_2)}{(1-p_1)(1-p_2)+p1p2},\ \mathrm{E}[\underline{s}_1, \bar{s}_2] = \bar{\theta}\frac{(1-p_1)p_2}{p_1+p_2-2p1p2},$$
$$\mathrm{E}[\bar{s}_1, \underline{s}_2] = \bar{\theta}\frac{p_1(1-p_2)}{p_1+p_2-2p1p2},\ \mathrm{E}[\bar{s}_1, \bar{s}_2] = \bar{\theta}\frac{p_1p_2}{p_1p_2+(1-p1)(1-p2)}.$$

The probabilities of these possible signal combinations are given by

$$\begin{aligned}
\mathrm{prob}(\underline{s}_1, \underline{s}_2) &= \mathrm{prob}(\underline{s}_1, \underline{s}_2|\bar{\theta})\,\mathrm{prob}(\bar{\theta}) + \mathrm{prob}(\underline{s}_1, \underline{s}_2|\underline{\theta})\,\mathrm{prob}(\underline{\theta}),\\
\mathrm{prob}(\bar{s}_1, \underline{s}_2) &= \mathrm{prob}(\bar{s}_1, \underline{s}_2|\bar{\theta})\,\mathrm{prob}(\bar{\theta}) + \mathrm{prob}(\bar{s}_1, \underline{s}_2|\underline{\theta})\,\mathrm{prob}(\underline{\theta}),\\
\mathrm{prob}(\underline{s}_1, \bar{s}_2) &= \mathrm{prob}(\underline{s}_1, \bar{s}_2|\bar{\theta})\,\mathrm{prob}(\bar{\theta}) + \mathrm{prob}(\underline{s}_1, \bar{s}_2|\underline{\theta})\,\mathrm{prob}(\underline{\theta}),\\
\mathrm{prob}(\bar{s}_1, \bar{s}_2) &= \mathrm{prob}(\bar{s}_1, \bar{s}_2|\bar{\theta})\,\mathrm{prob}(\bar{\theta}) + \mathrm{prob}(\bar{s}_1, \bar{s}_2|\underline{\theta})\,\mathrm{prob}(\underline{\theta}).
\end{aligned}$$

t=**1**: On the basis of the above probabilities, the players form the (ex-ante) conditional beliefs (later denoted by φ_i^t)

$$\begin{aligned}
\mathrm{prob}(\bar{s}_2|\bar{s}_1) &= \tfrac{\mathrm{prob}(\bar{s}_1,\bar{s}_2)}{p_1\,\mathrm{prob}(\bar{\theta})+(1-p_1)\,\mathrm{prob}(\underline{\theta})}, & \mathrm{prob}(\bar{s}_2|\underline{s}_1) &= \tfrac{\mathrm{prob}(\underline{s}_1,\bar{s}_2)}{(1-p_1)\,\mathrm{prob}(\bar{\theta})+p_1\,\mathrm{prob}(\underline{\theta})},\\
\mathrm{prob}(\underline{s}_2|\bar{s}_1) &= \tfrac{\mathrm{prob}(\bar{s}_1,\underline{s}_2)}{p_1\,\mathrm{prob}(\bar{\theta})+(1-p_1)\,\mathrm{prob}(\underline{\theta})}, & \mathrm{prob}(\underline{s}_2|\underline{s}_1) &= \tfrac{\mathrm{prob}(\underline{s}_1,\underline{s}_2)}{(1-p_1)\,\mathrm{prob}(\bar{\theta})+p_1\,\mathrm{prob}(\underline{\theta})},\\
\mathrm{prob}(\bar{s}_1|\bar{s}_2) &= \tfrac{\mathrm{prob}(\bar{s}_1,\bar{s}_2)}{p_2\,\mathrm{prob}(\bar{\theta})+(1-p_2)\,\mathrm{prob}(\underline{\theta})}, & \mathrm{prob}(\underline{s}_1|\bar{s}_2) &= \tfrac{\mathrm{prob}(\underline{s}_1,\bar{s}_2)}{p_2\,\mathrm{prob}(\underline{\theta})+(1-p_2)\,\mathrm{prob}(\bar{\theta})},\\
\mathrm{prob}(\bar{s}_1|\underline{s}_2) &= \tfrac{\mathrm{prob}(\bar{s}_1,\underline{s}_2)}{p_2\,\mathrm{prob}(\bar{\theta})+(1-p_2)\,\mathrm{prob}(\underline{\theta})}, & \mathrm{prob}(\underline{s}_1|\underline{s}_2) &= \tfrac{\mathrm{prob}(\underline{s}_1,\underline{s}_2)}{p_2\,\mathrm{prob}(\underline{\theta})+(1-p_2)\,\mathrm{prob}(\bar{\theta})}
\end{aligned}\tag{3.1}$$

which allow the players to calculate their expected payoffs. Hence—depending on his equilibrium strategy $\beta_1^*(s_1, \mathbf{p})$, the signal precisions $\mathbf{p}$, and his signal s_1—P1 (who is the only player to move at t=1) makes the choice of either exiting or continuing by bidding the prescribed (mixed) equilibrium action $\lambda^1(s_1)b_1^1(s_1)$. Thus P1 continues using the pure action $b_1^1(s_1)$ iff

$$u_1^1(\beta^*(s_1, \mathbf{p})) \geq u_1^1(e) = 0 \tag{3.2}$$

and quit otherwise. Moreover, he is prepared to play a mixed action iff

$$u_1^1(\beta^*(s_1, \mathbf{p})) = u_1^1(e) = 0. \tag{3.3}$$

The expected equilibrium payoff $u_1^1(\beta^*(s_1, \mathbf{p}))$ is calculated as the (undiscounted) sum of stage payoffs based on the beliefs held at t=1. For instance, upon receipt of a high signal $\bar{s}_1$, he calculates the continuation payoff *after* t=1 as

$$\begin{aligned}
\bar{u}_1^1(\beta^*(s_1, \mathbf{p})) = {} & \mathrm{prob}(\underline{s}_2|\bar{s}_1)((1-\underline{\mu}^2)(\mathrm{E}[\bar{s}_1, \underline{s}_2]-1) + \underline{\mu}^2\bar{u}_1^3(\beta^*(\mathbf{p}), \underline{s}_2))+\\
& \mathrm{prob}(\bar{s}_2|\bar{s}_1)((1-\bar{\mu}^2)(\mathrm{E}[\bar{s}_1, \bar{s}_2]-1) + \bar{\mu}^2\bar{u}_1^3(\beta^*(\mathbf{p}), \bar{s}_2)).
\end{aligned}$$

where $\bar{u}_1^3(\beta^*(\mathbf{p}), \underline{s}_2)$ is the sum of P1's t=3 expected equilibrium payoffs given that P1 received a high signal and P2 received a low signal. Similarly, $\bar{u}_1^3(\beta^*(\mathbf{p}), \bar{s}_2)$ is the sum of P1's equilibrium continuation payoffs at t=3 given that he received a high signal and P2 received a low signal.

t=2: P2 observes P1's bid of $b_1^1 = 1$ (otherwise there is nothing to do for P2 because the game is over), and revises her beliefs on the signal received by the opponent accordingly

$$
\begin{aligned}
\text{prob}(\bar{s}_1|\bar{s}_2, b_1^1 = 1) &= \frac{\bar{\lambda}^1 \text{prob}(\bar{s}_1|\bar{s}_2)}{\bar{\lambda}^1 \text{prob}(\bar{s}_1|\bar{s}_2) + \underline{\lambda}^1 \text{prob}(\underline{s}_1|\bar{s}_2)}, \\
\text{prob}(\underline{s}_1|\bar{s}_2, b_1^1 = 1) &= \frac{\underline{\lambda}^1 \text{prob}(\underline{s}_1|\bar{s}_2)}{\underline{\lambda}^1 \text{prob}(\underline{s}_1|\bar{s}_2) + \bar{\lambda}^1 \text{prob}(\bar{s}_1|\bar{s}_2)}, \\
\text{prob}(\bar{s}_1|\underline{s}_2, b_1^1 = 1) &= \frac{\bar{\lambda}^1 \text{prob}(\bar{s}_1|\underline{s}_2)}{\bar{\lambda}^1 \text{prob}(\bar{s}_1|\underline{s}_2) + \underline{\lambda}^1 \text{prob}(\underline{s}_1|\underline{s}_2)}, \\
\text{prob}(\underline{s}_1|\underline{s}_2, b_1^1 = 1) &= \frac{\underline{\lambda}^1 \text{prob}(\underline{s}_1|\underline{s}_2)}{\underline{\lambda}^1 \text{prob}(\underline{s}_1|\underline{s}_2) + \bar{\lambda}^1 \text{prob}(\bar{s}_1|\underline{s}_2)}.
\end{aligned}
\tag{3.4}
$$

Based on these beliefs, her equilibrium strategy $\beta_2^*(s_2, \mathbf{p})$, the signal accuracies $\mathbf{p}$, and her signal s_2—P2 (who is the only player to move at t=2) makes the choice of either exiting or continuing by bidding the prescribed (mixed) equilibrium action $\mu^2(s_2)b_2^2(s_2)$. She will continue using the pure action $b_2^2(s_2)$ iff

$$u_2^2(\beta^*(s_2, \mathbf{p})) \geq u_2^2(e) = 1 \tag{3.5}$$

and quit otherwise. Moreover, she is indifferent over any mixture between the two pure actions of exiting and continuing iff

$$u_2^2(\beta^*(s_2, \mathbf{p})) = u_2^2(e) = 1. \tag{3.6}$$

The expected equilibrium payoff $u_2^2(\beta^*(s_2, \mathbf{p}))$ is calculated as the (undiscounted) sum of stage payoffs based on the beliefs held at t=2. For instance, upon receipt of a low signal $\underline{s}_2$, she calculates her continuation payoff *after* t=2 as

$$
\begin{aligned}
\underline{u}_2^2(\beta^*(\mathbf{p}, s_2) = &\text{prob}(\underline{s}_1|\underline{s}_2)((1 - \underline{\lambda}^3)(\text{E}[\underline{s}_1, \underline{s}_2] - 1) + \underline{\lambda}^3 \underline{u}_2^4(\beta^*(\mathbf{p}), \underline{s}_1)) + \\
&\text{prob}(\bar{s}_1|\underline{s}_2)((1 - \bar{\lambda}^3)(\text{E}[\bar{s}_1, \underline{s}_2] - 1) + \bar{\lambda}^3 \underline{u}_2^4(\beta^*(\mathbf{p}), \bar{s}_1)).
\end{aligned}
$$

where $\underline{u}_2^4(\beta^*(\mathbf{p}), \underline{s}_1)$ is the sum of P2's t=4 expected equilibrium payoffs given that P2 received a low signal and P1 received a low signal. Similarly, $\underline{u}_2^4(\beta^*(\mathbf{p}), \bar{s}_1)$ is the sum of P2's equilibrium continuation payoffs at t=4 given that she received a low signal and P1 received a high signal.

t=3: After updating his beliefs based on an observed bid $b_2^2 = 2$, P1 is in a similar position as he was in at t=1. The game continues indefinitely until one of the two players chooses to exit.

The above sequence of unfolding events is the basis for the formulation of the equilibrium conditions of the game. The optimality criteria (3.2), (3.3) and (3.5), (3.6) generalise as naturally as do the applications of Bayes' Rule (3.1) and (3.4). Gathering these conditions together and solving for the optimal mixture probabilities λ^t and μ^t gives (after assuring that no deviations are profitable) the optimal strategies. Using these optimal mixture probabilities, solving for p_1, p_2 gives a particular region in $(p_1 \times p_2)$ where the equilibrium conditions hold.

In principle, therefore, our problem is to find areas (i.e. equilibria) in $(p_1 \times p_2)$ demarcated by polynomial inequalities (the equilibrium conditions). As the minimal bidding increment $\nu \to 0$, these conditions become numerous and of increasingly high order, hence solving for the resulting systems of equilibrium conditions becomes difficult. The field concerned with such problems in general is that of *algebraic geometry*. It has developed specialised techniques some of the most basic of which we will borrow without further mention.[3] We will now investigate the behaviour of the solution method outlined above for two particular examples.

3.2 An example with $\theta \in \{0, 3\}$

In this case, the equilibria are again determined by $\underline{\lambda}, \bar{\lambda}, \underline{\mu}, \bar{\mu}$. As it turns out, there are five distinct regions in $(p_1 \times p_2)$-space which fully determine the equilibrium candidate c_b^*—this is shown in figure 3. Solving for all combinations of (mixed) actions which solve the system described in the previous subsection gives the set of equilibrium candidates parameterised by signal accuracies. There are three pure equilibria

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and two equilibria in mixed strategies

$$\begin{pmatrix} 1 & 1 \\ m & m \end{pmatrix} \quad \underline{\lambda} = \frac{p_1 + p_1 p_2 - 2p_2}{p_1 p_2 + p_1 + p_2 - 1}, \ \underline{\mu} = \frac{p_1 p_2 + p_1 + p_2 - 2}{p_1 p_2 + p_1 + p_2 - 1}, \text{ for}$$

$$p_1 = {}^4\!/_5, \ p_2 = {}^2\!/_3, \ \text{ and } {}^4\!/_5 \leq p_1 \leq 1, \frac{2 - p_1}{1 + p_1} \leq p_2 \leq \frac{p_1}{2 - p_1}, \text{ and}$$

$$\begin{pmatrix} 1 & m \\ m & 0 \end{pmatrix} \quad \underline{\lambda} = \frac{2 - 2p_2 - 2p_1 + p_1 p_2}{p_2 + p_1 p_2 - 2p_1}, \ \bar{\mu} = \frac{2 - 3p_1}{p_2 + p_1 p + 2 - 2p_1}, \text{ for}$$

$${}^2\!/_3 \leq p_1 \leq {}^4\!/_5, \ {}^1\!/_2 \leq p_2 \leq {}^2\!/_3, \text{ and } {}^4\!/_5 < p_1 \leq 1, \ {}^1\!/_2 \leq p_2 \leq \frac{2 - p_1}{1 + p_1}.$$

The map of all equilibria parameterised by $\mathbf{p}$ is shown in figure 3.

[3] A review of the methods of algebraic geometry from the point of view of computational complexity theory is provided by Baxter and Iserles (2003). A more general discussion is Sturmfels (2002).

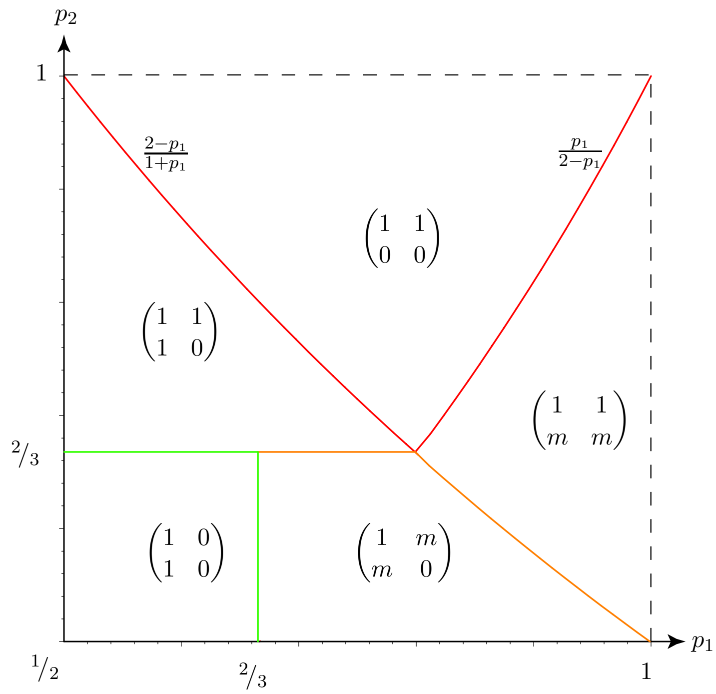


Figure 3: The candidate $c^*_{b,3}$.

3.3 An example with $\theta \in \{0, 5\}$

The extensive form for this example is shown in figure 4. As above, we use

$$\begin{pmatrix}1 & 1 & m & 0\\ 0 & m & 1 & 0\end{pmatrix} \text{ to represent } \begin{pmatrix}\bar{\lambda}^1 = 1 & \bar{\mu}^2 = 1 & \bar{\lambda}^3 \in [0,1] & \bar{\mu}^4 = 0\\ \underline{\lambda}^1 = 0 & \underline{\mu}^2 \in [0,1] & \underline{\lambda}^3 = 1 & \underline{\mu}^4 = 0\end{pmatrix}.$$

λ^5 is zero for any $\mathbf{p}$. Matrices such as the one above represent systems of polynomial inequalities solved by a system of restrictions on the constants (λ, μ) and $\mathbf{p}$. These results are summarised in figure 5 (where we do not consider measure zero equilibria).

In the following we list the strategies for which solutions in $\mathbf{p}$ (i.e. equilibrium candidates) can be found. The equilibrium candidates and their respective areas of applicability are based on solutions for the mixed action probabilities (if those are involved).[4] For convenience we

[4] In order not to break the flow of the argument but still facilitate ease of examination of our results, these are provided in the appendix.

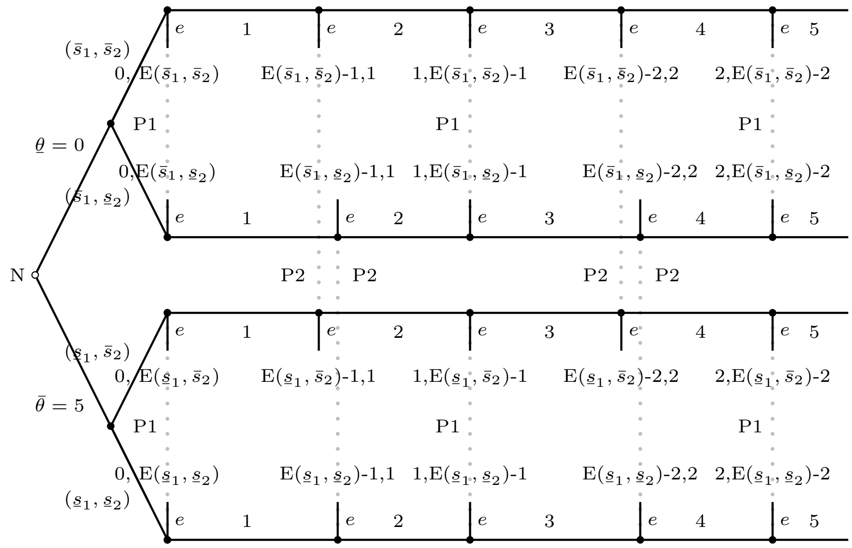


Figure 4: Extensive form for $\theta \in \{0, 5\}$.

define the following curves:

$$f_1(p_1) \equiv \frac{0.5(-11 - 2p_1 + 8p_1^2)}{-3 - 8p_1 + 3p_1^2} + 1.1180\sqrt{\frac{5 - 16p_1 + 48p_1^2 - 52p_1^3 + 20p_1^4}{(-3 - 8p_1 + 3p_1^2)^2}},$$

$$f_2(p_1) \equiv \frac{0.1(19 - 3p_1)}{2 + p_1} + 0.1\sqrt{\frac{41 - 114p_1 + 89p_1^2}{(2 + p_1)^2}},$$

$$f_3(p_1) \equiv \frac{0.1(7 + 16p_1)}{2 + p_1} - 0.1\sqrt{\frac{49 - 96p_1 + 96p_1^2}{(2 + p_1)^2}}.$$

No pure continuation for $\underline{s}$

$$\begin{pmatrix} 1 & m & 1 & m \\ m & 0 & m & 0 \end{pmatrix} \Rightarrow \ {}^4\!/_5 \leq p_1 \leq 1 \wedge 0 \leq p_2 \leq \frac{3p_1 - 4}{p_1 - 3} \tag{3.7}$$

$$\begin{pmatrix} 1 & m & 1 & m \\ m & m & m & m \end{pmatrix} \Rightarrow \ {}^4\!/_5 \leq p_1 \leq 1 \wedge p_2 = {}^1\!/_2 \tag{3.8}$$

$$\begin{pmatrix} 1 & 1 & 1 & m \\ m & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.8453 \leq p_1 \leq 0.9515 \wedge \frac{3p_1-4}{p_1-3} \leq p_2 \leq f_2(p_1) \\ (ii) & 0.9515 < p_1 \leq 1 \wedge \frac{3p_1-4}{p_1-3} \leq p_2 \leq f_1(p_1) \end{cases} \tag{3.9}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^5\!/_8 \leq p_1 \leq .9078 \wedge f_1(p_1) \leq p_2 \leq 1 \\ (ii) & 0.9078 < p_1 < 1 \wedge \frac{p_1}{4-3p_1} \leq p_2 \leq 1 \end{cases} \tag{3.10}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & m & m \end{pmatrix} \Rightarrow \left\{ \quad f_1(p_1) \leq p_2 \leq \frac{p_1}{4-3p_1} \wedge 0.9078 \leq p_1 \leq 1 \right. \tag{3.11}$$

One pure continuation for $\underline{s}$

$$\begin{pmatrix} 1 & 1 & m & 0 \\ 1 & m & 0 & 0 \end{pmatrix} \Rightarrow {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge {}^3\!/_5 \leq p_2 \leq \frac{3-2p_1}{2+p_1} \tag{3.12}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^3\!/_5 \leq p_1 \leq 0.6202 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{3p_1}{2+p_1} \\ (ii) & 0.6202 < p_1 \leq {}^4\!/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{-4+4p_1}{-4+3p_1} \end{cases} \tag{3.13}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \\ (ii) & {}^3\!/_5 < p_1 \leq {}^{16}\!/_{25} \wedge \frac{3p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \end{cases} \tag{3.14}$$

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & 0 & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.6202 \leq p_1 \leq 0.64 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq {}^{3p_1}\!/_{2p_1} \\ (ii) & 0.64 < p_1 \leq {}^4\!/_5 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq f_2(p_1) \\ (iii) & {}^4\!/_5 < p_1 \leq .8453 \wedge f_3(p_1) \leq p_2 \leq f_2(p_1) \\ (iv) & .8453 < p_1 < 0.9756 \wedge f_3(p_1) \leq p_2 \leq \frac{3p_1-4}{p_1-3} \end{cases} \tag{3.15}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^1\!/_2 \leq p_1 <= {}^5\!/_8 \wedge f_2(p_1) \leq p_2 \leq 1 \\ (ii) & {}^5\!/_8 < p_1 < {}^4\!/_5 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \\ (iii) & {}^5\!/_8 < p_1 \leq 0.9515 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \end{cases} \tag{3.16}$$

Two pure continuations for $\underline{s}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \; {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge {}^1\!/_2 \leq p_2 \leq {}^3\!/_5 \tag{3.17}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \; {}^3\!/_5 \leq p_1 \leq {}^4\!/_5 \wedge {}^1\!/_2 \leq p_2 \leq \frac{3 - 2p_1}{2 + p_1} \tag{3.18}$$

The strategies (3.7)–(3.18) fully cover our parameter space ($p_1 \times p_2$). Checking for deviations is excessively tedious but straightforward and confirms the above equilibria.

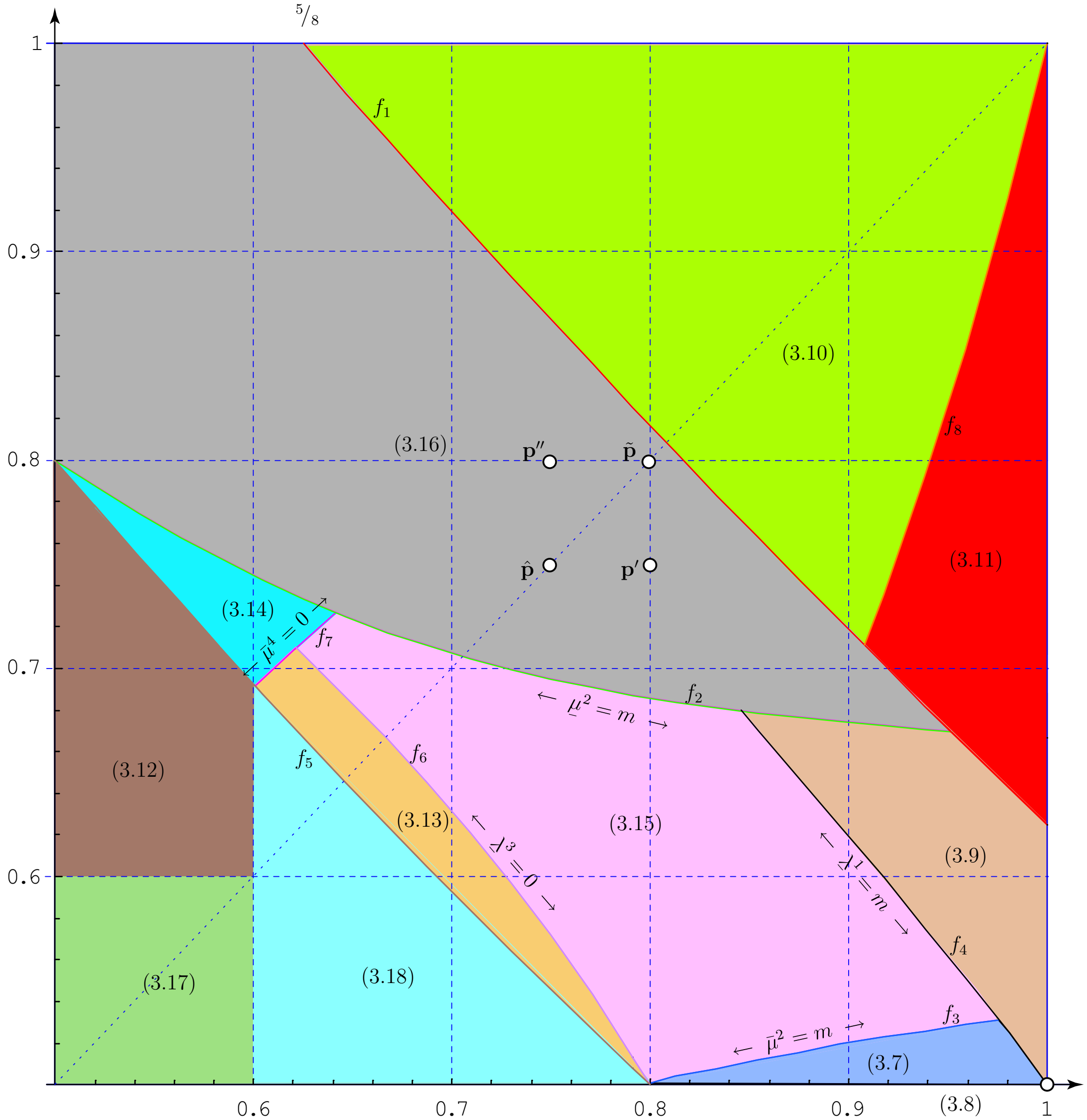
3.4 Discussion

The maps in figures 2 and 5 have a striking feature: There is a *unique* equilibrium for any $\mathbf{p}$ in full dimension.[5] These equilibria are intuitively appealing. For instance in the equilibrium region (3.17), the players have very little information and cannot effectively discriminate between the high and low signal states. Hence they bid up to the expectation of the object and quit as soon as the required bid exceeds this expectation. As expected from our previous work, (3.8), the essentially unique equilibrium of the game with incomplete information on one side can be retrieved in the more general setting of incomplete information on both sides. For $\lambda^5 = 0$, it occupies the interval $p_1 \in [{}^4\!/_5, 1]$ for $p_2 = 0$. (3.7) is the extension to this equilibrium for slightly better information of P2: As soon as she is able to separate states, she exits upon a low signal. (3.9) and (3.11) extend this interpretation for P2's increasingly precise signal. In this fashion, information-based interpretations can be given for all regions in figure 5.

The map shows both equilibria in immediately revealing (separating) and non-revealing strategies: In (3.13), P2 reveals her signal at t=2 by exiting upon receipt of a low signal. Thus if P1 observes a bid of 2 in this region, he knows that P2 received a high signal. All strategies are eventually fully revealing because to use the own information means to signal it to the opponent. The early full revelation in (3.13) suggests that P2's information is (known to be) too bad to continue while P1's information—although bad—is too good to believe cheats.

One of the more interesting results of Schweinzer (2003) is the existence of 'bubble-payoffs:' the better informed party can make a profit out of the dissolution *even if he knows the partnership to be worthless*. As shown in the final section, this result carries over to the present scenario but its intuition is somewhat blurred by the fact that now both parties face uncertainty. However, the agent receiving the higher accuracy signal has less uncertainty to bear and

[5] There are more equilibria of measure zero but we disregard them in the present discussion. There are no other equilibria for full dimension $\mathbf{p}$.



$f_1(p_1) \equiv \frac{.5(-11-2p_1+8p_1^2)}{-3-8p_1+3p_1^2} + 1.1180\sqrt{\frac{5-16p_1+48p_1^2-52p_1^3+20p_1^4}{(-3-8p_1+3p_1^2)^2}}$

$f_2(p_1) \equiv \frac{0.1(19-3p_1)}{2+p_1} + 0.1\sqrt{\frac{41-114p_1+89p_1^2}{(2+p_1)^2}}$

$f_3(p_1) \equiv \frac{0.1(7+16p_1)}{2+p_1} - 0.1\sqrt{\frac{49-96p_1+96p_1^2}{(2+p_1)^2}}$

$f_4(p_1) \equiv \frac{3p_1-4}{p_1-3}$

$f_5(p_1) \equiv \frac{3-2p_1}{2+p_1}$

$f_6(p_1) \equiv \frac{4p_1-4}{3p_1-4}$

$f_7(p_1) \equiv \frac{3p_1}{2+p_1}$

$f_8(p_1) \equiv \frac{p_1}{4-3p_1}$

Figure 5: The equilibrium candidate for $\theta \in \{0, 5\}$.

is able to exploit the partner's higher uncertainty profitably. This uncertainty is an addition over the incomplete information players face over the opponent's signal. We will investigate one particular example for the case of $\theta \in 0, 5$. Consider, for instance, the equilibrium area (3.16) of figure 5 where strategies and expected payoffs are given by

$$\beta^*_{3.16} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m & 0 \end{pmatrix}.$$

This equilibrium holds for $^1/_2 \leq p_1 \leq 0.9515$ with $0.667 \leq p_2 \leq 1$ so let's choose a pair of sample signal precisions $\mathbf{p}' = (.8, .75)$. In this setting, P1's expected utilities are $u_1(\beta^*_{3.16}, \mathbf{p}', \underline{s}_1) = 0.5085$, $u_1(\beta^*_{3.16}, \mathbf{p}', \bar{s}_1) = 1.5356$ leading to an expectation *given* a low value partnership of $u_1(\beta^*_{3.16}, \mathbf{p}', \mathrm{E}[s]) = 1.0221$. Now compare this to the diametrically opposed setting where signal accuracies are given by $\mathbf{p}'' = (.75, .8)$. Then P1's expected utilities are $u_1(\beta^*_{3.16}, \mathbf{p}'', \underline{s}_1) = 0.5316$, $u_1(\beta^*_{3.16}, \mathbf{p}'', \bar{s}_1) = 1.2625$ leading to an expectation (again *given* a low state) of $u_1(\beta^*_{3.16}, \mathbf{p}'', \mathrm{E}[s]) = 0.8971$. Hence P1 profits from an increase in signal precision. Informally speaking, in order to classify as 'bubble'-payoff, these payoffs must be higher and lower, respectively, than what P2 could expect if both players were equally well informed at $\hat{\mathbf{p}} = (.75, .75)$. As required, P1's expected payoffs there are $u_1(\beta^*_{3.16}, \hat{\mathbf{p}}, \underline{s}_1) = 0.6463$, $u_1(\beta^*_{3.16}, \hat{\mathbf{p}}, \bar{s}_1) = 1.3638$ leading to an expectation *given* a low value partnership of $u_1(\beta^*_{3.16}, \hat{\mathbf{p}}, \mathrm{E}[s]) = 1.005$.[6] What did we do? We calculated payoff expectations and their differentials leaving the opponent's signal accuracy fixed. If payoffs get lower given the true value is low or get higher given the true value is high, respectively, then the player's expectation corresponds more closely to the real value. This induces the player to take the appropriate action—exit in the case of a low value to realise the gain from a transfer from the opponent, bidding higher in the case of high value in order to obtain the object—with higher probability than with the lower precision. Since we are in an ex-post constant sum game, this is paid by the opponent and hence the better informed player can expect an information or 'bubble' rent.

4 Results

In order to generalise the above examples to arbitrarily large $\bar{\theta}$, we scale $\bar{\theta} = 1$ and let the minimal bidding increment $\nu \to 0$. Thus we normalise the possible type space to $\{0, 1\}$. In principle, the analytical method outlined above remains applicable for any $\nu > 0$. For systems where $\nu < {}^1\!/_5$ (the case analysed in subsection 3.2), though, this quickly exceeds the power of standard mathematics packages. Even specialised algebraic geometry packages on bigger machines cannot solve significantly larger systems. Hence we must confine ourselves to analytical results and are unable to extend the numerical examples of the previous section.

In general games of asymmetric incomplete information on both sides with discontinuous payoffs—such as first price auctions—equilibria may fail to exist. In finite queto games the existence of equilibria is a consequence of the application of the Fan-Glicksberg Theorem to the compact strategy sets at each stage (the basis of the 'compact' case in the Mertens-Sorin-Zamir framework). Since this already ensures the existence of a stage game equilibrium, we need not argue for the existence of a repeated game equilibrium itself although clearly a general argument—such as the one provided for the case of incomplete information on one side by Simon, Spież, and Toruńczyk (1995)—would be desirable.

We define Γ_b as a queto game with incomplete information on both sides as the game outlined

[6] It may be useful to state the payoff expectations at $\tilde{\mathbf{p}} = (0.8, 0.8)$ as well. Given a low value these are: $u_1(\beta^*_{3.16}, \tilde{\mathbf{p}}, \underline{s}_1) = 0.3903$, $u_1(\beta^*_{3.16}, \tilde{\mathbf{p}}, \bar{s}_1) = 1.4358$ and $u_1(\beta^*_{3.16}, \tilde{\mathbf{p}}, \mathrm{E}[s]) = 0.9130$.

in the model section fulfilling the same additional assumptions as in the case of incomplete information on one side, namely that

1. the bids b_i, the minimal constant bidding increment ν, and the possible values $\theta \in \{\underline{\theta}, \bar{\theta}\}$, $\underline{\theta} \leq \bar{\theta}$, take only finite values in $\mathbb{N}$,[7] and
2. we continue to view all deviations from the equilibrium path as mistakes, i.e. each deviation is believed to carry the same probability as the prescribed equilibrium action.

Appropriately redefined, the results from Schweinzer (2003) carry over; especially lemma 3—which excludes jump bidding—applies directly and players will find it optimal to use 'minimum increase' bidding strategies by including the minimal feasible bid in the support of the continuation action. An additional simplifying feature stems from the below lemma which asserts that if the players have the same information they will choose equal actions provided that the bidding grid is fine enough. Hence the first mover disadvantage disappears as the game gets longer. This idea is captured by the following definition. Notice that this symmetry is a property of the equilibrium map and *not* one applicable to the information of the agents.

Definition 1. *The equilibrium map on* $(p_1 \times p_2)$ *is called* symmetric *if Pi's,* $i \in \{1, 2\}$*, equilibrium action* ${}^*b_i^t(p_i, \cdot)$ *equals that of the opponent* ${}^*b_{-i}^{t\pm 1}(p_{-i}, \cdot)$ *provided that* $p_i = p_{-i}$ *for all* t*.*

Lemma 1. *For* $\nu \to 0$*, the equilibrium map on* $(p_1 \times p_2)$ *is symmetric.*

Proof. Since both players follow their 'minimal increase' equilibrium bidding strategies, the additional information transferred by any single continuation bid goes to zero and the sequence of play becomes immaterial. More formally, if P2 observes the bid $b_1^1 = \nu$ (otherwise the game is over), she updates her belief $\varphi_2^1 = \text{prob}(\bar{s}_1 | s_2)$ to

$$\varphi_2^2 = \text{prob}(\bar{s}_1 | s_2, b_1^1 = \nu) = \frac{\bar{\lambda}^1 \,\text{prob}(\bar{s}_1 | s_2)}{\bar{\lambda}^1 \,\text{prob}(\bar{s}_1 | s_2) + \underline{\lambda}^1 \,\text{prob}(\underline{s}_1 | s_2)}.$$

Since ν is small, the first bids will be continuation bids for generic $\mathbf{p}$ and *both* signals; hence $\underline{\lambda}^1 \approx \bar{\lambda}^1 \approx 1$ and $\varphi_2^2 \approx \varphi_2^1$ as claimed. Therefore P2's updated information at the second stage is almost equal to her prior information after receipt of her signal. If we assume the signal accuracy to be the same for both players, P1 is in a similar situation to P2. Therefore it does not matter (very much) who moves first and the players' equilibrium strategies are (nearly) the same. Since there is no more additional information introduced during the game, this symmetry must hold for any stage. Thus the equilibrium map is symmetric for $\nu \to 0$. $\square$

Lemma 2. *The equilibrium payoff in the setting with incomplete information on one side (where—by convention—P1 is fully informed and P2 not at all) suffices to characterise the payoffs in cases where P2 has arbitrary information.*

[7] This is the setup of the examples section. However, since most of the results below require $\nu \to 0$, it is occasionally more convenient to set $\nu = {}^1/_{\bar{\theta}}$ and then normalise the high value to 1. These formulations are equivalent.

Proof. For $\mathbf{p} = (1, p_2)$, P1's information necessarily includes everything P2 knows. Hence he can learn nothing from P2's behaviour and the analysis for the case with incomplete information on one side applies. In these games, P2 starts mixing only when the payoff exceeds the expectation based on her priors—which is $t \geq {}^{\bar\theta}/_4$ for $\mathbf{p} = (1, {}^1/_2)$.

In the present case this expectation is determined by the prior signal accuracy $p_2 \in [{}^1/_2, 1]$. Thus P2's prior-based expected payoff is $p_2\bar\theta$. Hence she will not be willing to mix at stages where her exit payoff is lower than $p_2\bar\theta$. Following a 'minimum increase' strategy this results in expected payoffs of $\left(\bar\theta - \frac{p_2\bar\theta}{2}, \frac{p_2\bar\theta}{2}\right)$. These expectations are obviously monotonic in p_2. □

Lemma 3. *Given p_{-i}, player i's payoff expectations for precisions $(p_i \in [{}^1/_2, 1], p_{-i})$ are monotonic in p_i.*

Proof. As pointed out for example by Sorin (2002, p16), in an ex-post constant-sum game, a player cannot loose from gaining information: She can make use of information dependent strategies, and hence her strategy set expands. A symmetric argument holds true for winning from less information. Nothing changes if the game or the argument is repeated. □

Proposition 1. *Consider a $\mathbf{p} \in (p_1 \times p_2)$ with $p_1 \geq p_2$. Leave P2's signal accuracy fixed but vary p_1 along $[{}^1/_2, 1]$. Then P1's payoffs are bounded from below by the symmetric expected payoff at the diagonal $\underline{u}(s, \mathbf{p})$ and bounded from above by his payoff in the case of incomplete information on one side $\bar u(s, \mathbf{p})$. The same is true for P2 if $p_1 < p_2$.*

Proof. A direct consequence of the above lemmata. □

Proposition 2. *The structure of equilibria β^* in Γ_b for $p_1 \geq p_2$ and $\nu \to 0$ is determined by*

$$\underline\theta \leq \mathrm{E}[\underline{s}] \leq \mathrm{E}[\underline{s}_1, s_2] \leq \mathrm{E}[s_1, \underline{s}_2] \leq \mathrm{E}[s_1, \bar s_2] \leq \mathrm{E}[\bar s_1, s_2] \leq \mathrm{E}[\bar s] \leq \bar\theta. \tag{4.1}$$

Proof. We only consider $p_1 \geq p_2$. Then the optimal continuation probabilities for bids in the respective regions are given by the below table:

	$\underline\theta \leq$	$\mathrm{E}[\underline{s}] \leq$	$\mathrm{E}[\underline{s}_1, s_2] \leq$	$\mathrm{E}[s_1, \underline{s}_2] \leq$	$\mathrm{E}[s_1, \bar s_2] \leq$	$\mathrm{E}[\bar s_1, s_2] \leq$	$\mathrm{E}[\bar s] \leq$	$\bar\theta$
P1($\bar s_1$)	1	1	1	1	1	0	0	0
P1($\underline{s}_1$)	1	1	[0,1)	[0,1)	[0,1)	0	0	0
P2($\bar s_2$)	1	1	1	1	0	0	0	0
P2($\underline{s}_2$)	1	1	1	[0,1)	0	0	0	0

Table 1: Continuation probabilities and the scope for mixed actions in Γ_b.

The above intervals are defined through the players' payoff expectations which depend on their conditional beliefs over their opponents' signal. Hence, generally, only the first intervals are static, i.e. not changed by the unfolding game where the opponents' observed actions change the players' beliefs on their opponents' signals they held initially.

The 8 ones and 9 zeroes to the left and right of the table are immediate. Within the interesting centre block we label cells (a,b,c,d) vertically and (1,2,3,4) horizontally and discuss each interval in turn. We denote the next feasible bid by b.

(a1-3) Since $b \leq \text{prob}(\underline{s}_2|\cdot)\,\mathrm{E}[\bar{s}_1, \underline{s}_2] + \text{prob}(\bar{s}_2|\cdot)\,\mathrm{E}[\bar{s}_1, \bar{s}_2]$, P1 will continue with probability 1;

(a4) since $b > \mathrm{E}[\bar{s}_1, s_2]$, P1 will exit;

(b1-3) since $b > \text{prob}(\underline{s}_2|\cdot)\,\mathrm{E}[\underline{s}_1, \underline{s}_2] + \text{prob}(\bar{s}_2|\cdot)\,\mathrm{E}[\underline{s}_1, \bar{s}_2]$, P1 expects the value of the object to be below the next bid; he can, however, pretend to have received a high signal and continue as long as the expected payoffs from exiting and continuing are equal;

(b4) since the required bid exceeds even $\mathrm{E}[\bar{s}_1, s_2]$, P1 can no longer mimic the high signal and will exit;

(c1-3) since $b \leq \text{prob}(\underline{s}_1|\cdot)\,\mathrm{E}[\underline{s}_1, \bar{s}_2] + \text{prob}(\bar{s}_1|\cdot)\,\mathrm{E}[\bar{s}_1, \bar{s}_2]$, P2 will continue with probability 1;

(c4) since $b > \mathrm{E}[\bar{s}_1, s_2]$, P2 will exit;

(d1-2) since $b > \text{prob}(\underline{s}_1|\cdot)\,\mathrm{E}[\underline{s}_1, \underline{s}_2] + \text{prob}(\bar{s}_1|\cdot)\,\mathrm{E}[\bar{s}_1, \underline{s}_2]$, P2 expects the value of the object to be below the next bid; she can, however, pretend to have received a high signal and continue as long as the expected payoffs from exiting and continuing are equal;

(d3-4) since the required bid now exceeds $\mathrm{E}[s_1, \bar{s}_2]$, which is the level where the high signal P2 quits, she can no longer mimic the high signal type and exits.

The case of $p_2 \geq p_1$ is symmetric for a chain of inequalities similar to (4.1). $\square$

Proposition 3. *Equilibrium payoffs $u(\beta^*(s, \mathbf{p}))$ for $\nu \to 0$ in Γ_b are such that the better informed player expects an information rent.*

Proof. There are two distinct sources for the players' payoffs: The value of the object and the bids of the opponent. Let the better informed player be P1; since $p_1 > p_2 \geq {}^1\!/_2$, the β^*-prescribed equilibrium action is *in expectation* the right one. Hence if (*i*) $s_1 = \bar{s}_1$, then the less precisely informed P2 will exit at the latest at the bid $\mathrm{E}[s_1, \bar{s}_2]$ which is earlier than P1. Thus she agrees to a sharing rule which gives P1 more than half the object's expectation. If P2's signal is $\underline{s}_2$, the result is reinforced because P2 starts mixing already at the earlier $\mathrm{E}[s_1, \underline{s}_2]$. If (*ii*) $s_1 = \underline{s}_1$, then the opponent will exit at the earliest at the bid $\mathrm{E}[s_1, \underline{s}_2]$ which is later than P1 starts to mix. Again, *in expectation*, P2 will pay a zero-sum transfer and P1 profits in expected terms. If P2's signal is $\bar{s}_2$, the result is reinforced as above. $\square$

Definition 2. *An equilibrium $\beta^*(s, \mathbf{p})$ is called* essentially unique *if (i) the underlying $\mathbf{p}$ is measurable and (ii) it is unique up to the final stage of the quitting game Γ_b, but has an arbitrary final action by P1 for odd $\theta = \bar{\theta}$.*

Conjecture: Equilibria are essentially unique in full dimension.

Proof. Schweinzer (2003) argues for the essential uniqueness of the equilibrium in the case of incomplete information on one side by excluding all separating equilibria. The idea for the present proof is to extend this to show that for each $\mathbf{p}$, there is a stretch of possible mixed actions in the low signal case which do not fully and immediately inform the less informed player of the true state (columns a–d of the table 1). Within these bounds, no separating equilibrium is possible, outside of these bounds, the 'first' such separating equilibrium will realise. Hence all measurable equilibria are based on incomplete information—the uncertainty based ones at both ends can only work for zero measure. To make this intuition precise, however, a general handle on the geometry of the problem is required which—at the moment—we cannot provide. □

Conclusion

We provide a full characterisation of the equilibria and expected equilibrium payoffs in a queto game with asymmetric incomplete information on both sides. We show that regardless of the bidding grid and the signal precisions, the better informed partner realises an information rent from the partnership dissolution while the less informed agent is still willing to participate.

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Appendix

We provide details of the numerical solutions we calculated for the example of $\theta \in \{0, 5\}$. By themselves, they are not interesting. Without them, however, the (considerable) effort of solving the corresponding systems of inequalities would have to be duplicated for the purpose of checking our numerical results. Hence we provide the results here.[8] For convenience we repeat the definition of some shortcuts used above:

$$f_1(p_1) \equiv \frac{0.5(-11 - 2p_1 + 8p_1^2)}{-3 - 8p_1 + 3p_1^2} + 1.1180\sqrt{\frac{5 - 16p_1 + 48p_1^2 - 52p_1^3 + 20p_1^4}{(-3 - 8p_1 + 3p_1^2)^2}},$$

$$f_2(p_1) \equiv \frac{0.1(19 - 3p_1)}{2 + p_1} + 0.1\sqrt{\frac{41 - 114p_1 + 89p_1^2}{(2 + p_1)^2}},$$

$$f_3(p_1) \equiv \frac{0.1(7 + 16p_1)}{2 + p_1} - 0.1\sqrt{\frac{49 - 96p_1 + 96p_1^2}{(2 + p_1)^2}}.$$

No pure continuation for $\underline{s}$

(3.7) analytical solution

$$\begin{pmatrix} 1 & m & 1 & m \\ m & 0 & m & 0 \end{pmatrix} \begin{cases} \underline{\lambda}^1 = \frac{7-25p_1+25p_1^2}{3-20p_1+25p_1^2} \wedge \\ \bar{\mu}^2 = \frac{5p_1-4}{5p_1-3} \wedge \\ \underline{\lambda}^3 = \frac{5p_1-4}{5\underline{\lambda}^1 p_1 - \underline{\lambda}^1} \wedge \\ \bar{\mu}^4 = \frac{5p_1-2}{5p_1-1} \text{and} \\ {}^4\!/_5 \leq p_1 \leq 1 \wedge 0 \leq p_2 \leq \frac{3p_1-4}{p_1-3} \end{cases}$$

(3.8) is the essentially unique equilibrium of the case of incomplete information on one side—although similar to the above (3.7) it does not have a structure which is directly comparable to the other equilibria because of the implicit constraint that $\mu^t = \bar{\mu}^t = \underline{\mu}^t$ for all t which stems from the fact that P2 cannot distinguish the two states *at all* because of $p_2 = {}^1\!/_2$. This has the immediate consequence that it must be an equilibrium in zero measure.

$$\begin{pmatrix} 1 & m & 1 & m \\ m & m & m & m \end{pmatrix} \begin{cases} \underline{\lambda}^1 = \frac{7-25p_1+25p_1^2}{3-20p_1+25p_1^2} \wedge \mu^2 = \frac{5p_1-4}{5p_1-3} \wedge \\ \underline{\lambda}^3 = \frac{5p_1-4}{5\underline{\lambda}^1 p_1 - \underline{\lambda}^1} \wedge \mu^4 = \frac{5p_1-2}{5p_1-1} \text{ and} \\ {}^4\!/_5 \leq p_1 \leq 1 \wedge p_2 = {}^1\!/_2 \end{cases}$$

(3.9) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & m \\ m & m & m & 0 \end{pmatrix} \begin{cases} & \underline{\mu}^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2} \wedge \\ & \underline{\lambda}^3 = \frac{-3\underline{\lambda}^1 - p_1 + 3\underline{\lambda}^1 p_1 - p_2 + 3\underline{\lambda}^1 p_2 + 2p_1p_2 - \underline{\lambda}^1 p_1 p_2}{-2\underline{\lambda}^1 + 2\underline{\lambda}^1 p_1 + 2\underline{\lambda}^1 p_2 + \underline{\lambda}^1 p_1 p_2} \wedge \\ & \bar{\mu}^4 = \frac{2\underline{\mu}^2 - 3p_1 - 2\underline{\mu}^2 p_1 + 2p_2 - 2\underline{\mu}^2 p_2 + p_1p_2 - \underline{\mu}^2 p_1 p_2}{-4p_1+p_2+3p_1p_2} \wedge \text{ and} \\ (i) & 0.8453 \leq p_1 \leq 0.9515 \wedge \frac{-4+3p_1}{-3+p_1} \leq p_2 \leq f_2(p_1) \\ (ii) & 0.9515 < p_1 \leq 1 \wedge \frac{-4+3p_1}{-3+p_1} \leq p_2 \leq f_1(p_1) \end{cases}$$

[8] If there is interest in the *mathematica*, *maple* or *macauley 2* procedures used to compute the numerical results, please contact the author.

($-$) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; p_1 = p_2 = 1$$

(3.10) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & 0 & 0 \end{pmatrix} \begin{cases} & \underline{\mu}^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2} \wedge \underline{\lambda}^1 = \frac{-p_1-p_2+2p_1p_2}{3-3p_1-3p_2+p_1p_2} \text{ and} \\ (i) & {}^5\!/_8 \leq p_1 \leq .9078 \wedge f_1(p_1) \leq p_2 \leq 1 \\ (ii) & .9078 < p_1 < 1 \wedge \frac{p_1}{4-3p_1} \leq p_2 \leq 1 \end{cases}$$

(3.11) no analytical solution; used rasterisation instead

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & m & m \end{pmatrix} \begin{cases} sols(.9, \cdot) := \emptyset \\ sols(.91, \cdot) := \{.7089 \leq p_2 \leq .7165\} \\ sols(1, \cdot) := \{.6250 \leq p_2 \leq 1\} \\ sols(\cdot, .5) := \emptyset \\ sols(\cdot, .55) := \emptyset \\ sols(\cdot, .6) := \{\cdot, 1\} \\ sols(\cdot, .65) := \{.9727 \leq p_1 \leq 1\} \\ sols(\cdot, .95) := \{.9870 \leq p_1 \leq 1\} \\ sols(\cdot, 1) := \emptyset \end{cases}$$

One pure continuation for $\underline{s}$

(3.12) analytical solution

$$\begin{pmatrix} 1 & 1 & m & 0 \\ 1 & m & 0 & 0 \end{pmatrix} \begin{cases} \underline{\mu}^2 = \frac{3-3p_1-3p_2+p_1p_2}{2p_1-3p_2+p_1p_2} \wedge \\ \bar{\lambda}^3 = \frac{3-5p_2}{2p_1-3p_2+p_1p_2} \text{ and} \\ {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge \frac{3}{5} \leq p_2 \leq \frac{3-2p_1}{2+p_1} \end{cases}$$

(3.13) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{cases} (i) & {}^3\!/_5 \leq p_1 \leq 0.6202 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{3p_1}{2+p_1} \\ (ii) & 0.6202 < p_1 \leq {}^4\!/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{-4+4p_1}{-4+3p_1} \end{cases}$$

(3.14) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & m & m & 0 \end{pmatrix} \begin{cases} & \underline{\mu}^2 = \frac{-3p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2} \wedge \underline{\lambda}^3 = \frac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2} \text{ and} \\ (i) & {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \\ (ii) & {}^3\!/_5 < p_1 \leq {}^{16}\!/_{25} \wedge \frac{3p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \end{cases}$$

(3.15) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & 0 & m & 0 \end{pmatrix} \begin{cases} & \underline{\mu}^4 = \frac{-3p_1+2p_2+p_1p_2}{4p_1+2p_2+3p_1p_2} \wedge \underline{\lambda}^3 = \frac{4-4p_1-4p_2+3p_1p_2}{-4p_1+p_2+3p_1p_2} \text{ and} \\ (i) & 0.6202 \leq p_1 \leq 0.64 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq {}^{3p_1}/_{2p_1} \\ (ii) & 0.64 < p_1 \leq {}^4/_5 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq f_2(p_1) \\ (iii) & {}^4/_5 < p_1 \leq .8453 \wedge f_3(p_1) \leq p_2 \leq f_2(p_1) \\ (iv) & .8453 < p_1 < 0.9756 \wedge f_3(p_1) \leq p_2 \leq \frac{3p_1-4}{p_1-3} \end{cases}$$

(−) no analytical solution; used rasterisation instead

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & m & m & 0 \end{pmatrix} \begin{cases} (i) & p_1 = {}^1/_2 \wedge p_2 = {}^4/_5 \wedge \bar{\mu}^4, \underline{\mu}^2, \underline{\lambda}^3 \\ (ii) & p_1 = 0.55 \wedge p_2 = 0.77 \wedge \bar{\mu}^4, \underline{\mu}^2, \underline{\lambda}^3 \\ (iii) & p_1 = 0.60 \wedge p_2 = 0.7444 \wedge \bar{\mu}^4, \underline{\mu}^2, \underline{\lambda}^3 \\ (iv) & p_1 = 0.95 \wedge p_2 = 0.6698 \wedge \bar{\mu}^4, \underline{\mu}^2, \underline{\lambda}^3 \\ (v) & p_1 > 0.95 \Rightarrow \emptyset \end{cases}$$

(3.16) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m & 0 \end{pmatrix} \begin{cases} & \underline{\mu}^2 = \frac{p_1+p_2-2p_1p_2}{-2+2p_1+2p_2+p_1p_2} \wedge \underline{\lambda}^3 = \frac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2} \text{ and} \\ (i) & {}^1/_2 \leq p_1 <= {}^5/_8 \wedge f_2(p_1) \leq p_2 \leq 1 \\ (ii) & {}^5/_8 < p_1 < {}^4/_5 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \\ (iii) & {}^5/_8 < p_1 \leq 0.9515 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \end{cases}$$

Two pure continuations for $\underline{s}$

(3.17) analytical solution

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \ {}^1/_2 \leq p_1 \leq {}^3/_5 \wedge {}^1/_2 \leq p_2 \leq {}^3/_5$$

(3.18) analytical solution

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \ {}^3/_5 \leq p_1 \leq {}^4/_5 \wedge {}^1/_2 \leq p_2 \leq \frac{3-2p_1}{2+p_1}$$