Correlated Equilibrium in Evolutionary Models with Subpopulations[*]

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Abstract

We study a version of the multipopulation replicator dynamics, where each population is comprised of multiple subpopulations. We establish that correlated equilibrium is a natural solution concept in this setting. Specifically, we show that every correlated equilibrium is equivalent to a stationary state in the replicator dynamics of some subpopulation model. We also show that every interior stationary state in a subpopulation model is equivalent to a correlated equilibrium. We find that any state that is Lyapunov stable or the limit of an interior solution is equivalent to a correlated equilibrium. We also provide an example with a Lyapunov stable limit state that is equivalent to a correlated equilibrium outside the convex hull of the set of Nash equilibria. Finally, we prove that any subpopulation setting in which the matching distribution is a product measure leads to equivalence with Nash equilibrium.

1   Introduction

The concept of correlated equilibrium has received much attention in the literature since its introduction by Aumann in 1974. An important focus of the discussion has involved the relationship and applicability of the correlated equilibrium solution concept to the rationalistic and evolutionary frameworks.[1] The correlated equilibrium is an extension of the Nash equi-

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[1]For a discussion of the epistemic foundations of correlated equilibrium see Aumann (1987). For a survey of literature in this area, see Brandenburger (1992).

librium solution concept, allowing for correlation between the actions of different players. A common interpretation for the correlating device is an exogenous instrument that sends to players potentially correlated signals, the distribution of which is common knowledge. Each player chooses an optimal strategy given the strategies of her opponents and her knowledge of the signal distribution.

A key feature of evolutionary and learning frameworks is that they permit us to relax the assumptions about knowledge and rationality of players. In fact, a useful characteristic of evolutionary models is that they provide a test for how sensitive game theoretic results are to these typical assumptions. The appropriateness of the Nash equilibrium solution concept in models with evolutionary dynamics has been well established in the literature. However, how to interpret the correlated equilibrium solution concept in a this setting is not as well understood.

We develop a dynamic evolutionary model that gives rise to correlated equilibria. As noted above, a correlated equilibrium can be thought of as a Nash equilibrium in a game where rational players observe external signals for which they know the joint distribution. Such a game is mathematically equivalent to a game where players have multiple types, the distribution of types is known to each player, and players can condition their actions on their realized types. We assert that a game with types has a natural interpretation with large populations that are partitioned into subpopulations. Rather than matching particular types of each player role, the matching occurs over the subpopulations of each player population. Correlation in how the individuals are chosen from among these subpopulations generates replicator dynamics that produce correlated equilibria.

Specifically, we show that for every correlated equilibrium, there exists some subpopulation breakdown and match distribution for which there is an equivalent stationary state under the replicator dynamics. We also show that for stationary states that are interior for all subpopulations, there exists an equivalent correlated equilibrium. Furthermore, we show that any Lyapunov stable state or any limit to an interior solution may be represented by a correlated equilibrium. We then discuss an additional restriction that generates Nash equilibria in this subpopulation model, and we discuss the relationship of this model to the standard replicator dynamics.

We believe that the subpopulation interpretation fits more naturally in many contexts. The replicator dynamics with subpopulations appears especially well suited to locational models. Players in one geographic region may be matched more frequently with opponents from nearby regions. Hence, if the populations are divided into discrete neighborhoods, the matching distribution can reflect such proximities. As we will show, correlated equilibrium

is an appropriate solution concept for such games, even if the agents are assumed to behave in a purely evolutionary manner.

This paper is certainly not the first to analyze correlated equilibria in the context of evolutionary-like behavior. Hart and Mas-Colell (2000), for example, apply correlated equilibria to adaptive models. Their work is similar to ours in that it emphasizes the dynamic mechanisms for which the play converges to a correlated equilibrium. However, agents in Hart and Mas-Colell's model exhibit learning and regret in that they weigh their past actions against alternatives. Rather than assuming specific behavior on the part of agents, we show that correlated equilibria can arise simply from the way in which individuals are matched for play. Even with the most basic evolutionary dynamics, the replicator dynamics, we obtain a representation of correlated equilibrium. The replicator dynamics have the advantage of both simplicity and widespread applicability. A model of replicator dynamics can apply to purely biological contexts, where one imagines agents as pre-programmed to play a pure strategy and hence unable to exhibit learning or regret. The replicator dynamics can also be interpreted as an aggregate representation of certain types of boundedly rational social learning. By deriving a representation of correlated equilibrium directly in terms of the replicator dynamics, we ensure that our results can be useful in a wide variety of contexts.

The model of Cripps (1991) is closer in spirit to the present work. Cripps shows that for a class of evolutionary models, strict correlated equilibria can be interpreted in a framework of evolutionary stability. In particular, the set of evolutionary stable strategies in simple contests is identical to the set of strict correlated equilibria. Cripps's use of player roles is similar to our use of subpopulations. We choose to explore these issues in a dynamic setting, rather than with a static solution such as evolutionary stability. We also offer a more detailed comparison to the standard evolutionary model, highlighting what attributes give rise correlated equilibria outside of the set of Nash equilibria. Based on the relationship between evolutionary stability and stability in replicator dynamics, we are confident that our model of subpopulation-matching in a context of evolutionary stability would generate results that are compatible with those of Cripps.

Mailath, Samuelson, and Shaked (1997) present an approach that leads to a characterization similar to ours. They generalize the concept of Nash equilibrium to an equilibrium with local interactions and demonstrate equivalencies between this equilibrium concept and correlated equilibrium. Their equilibrium with local interactions is effectively a pure strategy Nash equilibrium of a transformed version of the original game, in which they create a finite population of individuals for each player role and match individuals for

play according to exogenously given probability distribution. Correlation in how individuals are matched allows for correlation in strategies played at the aggregate level, and hence Nash equilibria of the transformed game would be correlated equilibria of the original game. Our approach is the natural analog of Mailath *et al.*'s in a dynamic evolutionary framework. Intuitively, given their main proposition, along with appropriate results for standard replicator dynamics, one would expect equivalencies similar to those that we demonstrate.[2] Instead, we prefer to establish our results directly from the mechanics of our subpopulation replicator dynamics. In addition to finding the direct approach more accessible, we find that it helps us to better demonstrate the necessity of certain assumptions to our results.

The remainder of the paper is structured as follows. In Section 2, we define a model of subpopulation replicator dynamics and develop notation that we will use throughout the paper. In Section 3, we demonstrate the connection between stationarity in the subpopulation replicator dynamics and correlated equilibrium. In Section 4, we discuss how Lyapunov stable states and the limits of interior solutions are related to correlated equilibria. We describe in Section 5 an additional restriction that generates Nash equilibria.

2   Model and Notation

Let $\Gamma = (I, (S_i)_{i \in I}, (\pi_i)_{i \in I})$ be a finite n-player normal form game. The set of players, or in the evolutionary context, the set of n player populations is $I = \{1, \ldots, n\}$. Let the finite set of pure strategies for each player population be denoted S_i and let $S \equiv \times_{i \in I} S_i$ be the set of possible pure-strategy profiles. Each player population has a payoff function $\pi_i : S \to \mathbb{R}$.

As is standard, we denote a profile of strategies excluding that of player i as $s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$. Furthermore, (k, s_{-i}) is a strategy profile in which player i plays strategy $k \in S_i$ and her opponents play $s_{-i} \in S_{-i}$. We will carry this notation over to the various functions and subpopulations defined below.

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[2]To explicitly make this connection, one would first have to allow individuals in Mailath *et al.*'s model to play mixed strategies and extend their equivalency result to mixed-strategy Nash equilibria of the transformed game.

2.1 Subpopulations and Replicator Dynamics

Each player population i can be partitioned into a finite set of subpopulations indexed by a set M_i.[3] For every subpopulation $h \in M_i$, let the SUBPOPU-LATION STATE, $x_{i,h}$ be a point in the mixed-strategy simplex ΔS_i such that $x_{i,h}^k$ is the share of the subpopulation h that plays pure strategy $k \in S_i$. The subpopulation state gives the distribution of play in subpopulation h of population i.

One subpopulation from each population is selected for play. Let $M \equiv \times_{i \in I} M_i$ be the set of possible subpopulation matches and let $m \in M$ be a vector of subpopulations that characterizes a match. Define $x_m = (x_{i,m_i})_{i \in I}$, so that x_m describes the distribution of play in the active subpopulation of each population under the match m. Then, let the POPULATION STATE be $x = (x_m)_{m \in M}$. The population state fully describes the distribution of play under all subpopulation matches. Furthermore, define $x_{m_{-i}} = (x_{j,m_j})_{j \in I: j \neq i}$.

For every subpopulation match $m \in M$, let $\eta(m)$ give the probability of that match. Note that the probability that subpopulation $m_i \in M_i$ plays is $\eta_i(m_i) = \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i})$. One can interpret $\eta_i(m_i)$ as the share of population i in subpopulation m_i. In standard models of replicator dynamics, the probability distribution of which individual in each of the populations will play is uniform, so we would have the probability that the individuals come from subpopulations $m_1, \ldots, m_n$ equal to $\eta_1(m_1)\eta_2(m_2)\cdots\eta_n(m_n)$. However, in our model there is the potential for correlation between which subpopulation of population i plays and which subpopulations of the other populations play: that is, η might not be a product measure. For any game Γ, we will call the pair (M, η) a SUBPOPULATION SETTING of Γ.

Note that for a given subpopulation match m, the conditional probability that a certain pure-strategy profile s is played is $\prod_{i \in I} x_{i,m_i}^{s_i}$. For a given subpopulation match m, the expected utility function for a member of subpopulation m_i of population i is:

$$u_i(x_m) = \sum_{s \in S} \Big[\prod_{j \in I} x_{j,m_j}^{s_j} \Big] \pi_i(s) \qquad (1)$$

Let e_i^k be the point in population i's mixed-strategy simplex that puts probability one on pure strategy k. When one fixes a pure strategy k for a subpopulation m_i, the utility function is written as $u_i(e_i^k, x_{m_{-i}})$. Note that:

$$u_i(e_i^k, x_{m_{-i}}) = \sum_{s_{-i} \in S_{-i}} \Big[\prod_{j \neq i} x_{j,m_j}^{s_j} \Big] \pi_i(k, s_{-i}) \qquad (2)$$

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[3]In many situations, it is convenient to think of a set of subpopulations as a subset of the natural numbers: $M_i = \{1, \ldots, |M_i|\}$.

Let $\dot{x}^k_{i,m_i}$ denote the derivative of the share of subpopulation m_i playing strategy k with respect to time. The η-REPLICATOR DYNAMICS for any $i \in I$, $m_i \in M_i$, and $k \in S_i$ are defined:

$$\dot{x}^k_{i,m_i} = \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \Big[u_i(e^k_i, x_{m_{-i}}) - u(x_m) \Big] x^k_{i,m_i} \tag{3}$$

One might alternatively define the replicator dynamics using the conditional distribution that an individual from population i plays against opponents from subpopulations m_{-i} given that he is from subpopulation m_i. That is, one would replace $\eta(m_i, m_{-i})$ with $\eta(m_{-i}|m_i)$ in Equation (3). We believe that using the joint distribution is preferable because it permits the replicator dynamics to capture the relative size of each of the subpopulations. Intuitively, a relatively small subpopulation would play less often and therefore evolve more slowly than a large subpopulation. In contrast, the conditional distribution would allow subpopulations to evolve at rates independent of their size.

A population state is STATIONARY in the η-replicator dynamics if $\dot{x}^k_{i,m_i} = 0$ for all $i \in I$, $m_i \in M_i$, and $k \in S_i$. A population state is INTERIOR if $x_{i,m_i} \gg 0$ for all $i \in I$, $m_i \in M_i$. One should note that this definition of interiority requires that all subpopulations play a completely mixed strategy.

2.2 Correlated Equilibrium

We will use the following definition of a correlated equilibrium:

Definition 1 *A* CORRELATED EQUILIBRIUM *in* Γ *is a probability distribution* ψ *on* S *such that for all* $i \in I$ *and for any* $f_i : S_i \to S_i$,

$$\sum_{s \in S} \psi(s) \Big[\pi_i\big(f_i(s_i), s_{-i}\big) - \pi_i(s) \Big] \leq 0. \tag{4}$$

One can conceptualize a correlated equilibrium as a random signal that recommends that the players play the strategy profile s with probability $\psi(s)$. Each player i receives the recommendation for her own strategy; however, she is unaware of the recommendations given to the other players. In this sense, the recommendation given to a player gives her information about the state of the world, but does not necessarily fully reveal the state of the world. Each player i can condition her action on this information by choosing a mapping $f_i : S_i \to S_i$. The distribution ψ constitutes a correlated equilibrium if following the recommendation s_i is weakly preferred by all players to any other strategy given by $f_i(s_i)$. Let $\psi_i(s_i)$ be the marginal probability that strategy s_i is recommended to player i, so that $\psi_i(s_i) = \sum_{s_{-i} \in S_{-i}} \psi(s_i, s_{-i})$.

Note that ψ need not be a product measure: it is not necessarily true that $\psi(s) = \prod_{i \in I} \psi_i(s_i)$. At any population state, the average play over all sub-population matches is a distribution on S of particular interest:

Definition 2 *Given a game Γ and a subpopulation setting (M, η), a probability distribution ψ on S, not necessarily a correlated equilibrium, is* EQUIV-ALENT *to a population state x if and only if for all $s \in S$:*

$$\psi(s) = \sum_{m \in M} \eta(m) \left[\prod_{i \in I} x_{i,m_i}^{s_i} \right] \tag{5}$$

3 Stationarity

An important result of standard replicator dynamics is the relationship between Nash equilibria and interior stationary states. It has been shown that every Nash equilibrium is stationary in the replicator dynamics and that any interior stationary state is a Nash equilibrium.[4] We now show that there exists a similar relationship between correlated equilibria and stationarity in our subpopulation model.

Proposition 1 *Consider a game Γ.*

  i. *For any correlated equilibrium ψ, there exists a subpopulation setting (M, η) with a stationary state that is equivalent to ψ.*

  ii. *For any subpopulation setting (M, η), if x is an interior stationary state in the η-replicator dynamics, then the equivalent probability distribution ψ is a correlated equilibrium.*

   Proof: (i) The first claim is almost trivial if we are careful in our construction of the subpopulation setting. Let $M = S$ and $\eta = \psi$. In this specification, each population is divided into subpopulations that correspond to the available pure strategies for that population. Hence, the possible subpopulation matches are the possible pure strategy profiles. Furthermore, we set the probability of each subpopulation match to the probability that the distribution ψ assigns to the strategy profile played under the match.
   Consider a population state x such that $x_{i,m_i} = e_i^{m_i}$ for all $i \in I$, $m_i \in M_i$. Here, each subpopulation is programmed to play the pure strategy to which it corresponds. Thus $\prod_{i \in I} x_{i,m_i}^{s_i} = 1_{\{m\}}(s)$ for all $m \in M$ and for all $s \in S$,

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[4]For the properties of Nash equilibria in standard replicator dynamics see Weibull (1998).

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where $1_{\{\cdot\}}$ is the indicator function. We can quickly see that x is a stationary state. For any $i \in I$, $m_i \in M_i$, if $k = m_i$ then

$$u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) = \pi_i(k, m_{-i}) - \pi_i(m) = 0, \tag{6}$$

so $\dot{x}_{i,m_i}^k = 0$. For any $i \in I$, $m_i \in M_i$, if $k \neq m_i$ then $x_{i,m_i}^k = 0$, so $\dot{x}_{i,m_i}^k = 0$. It remains only to show that x is equivalent to ψ. For any $s \in S$,

$$\sum_{m \in M} \eta(m)\left[\prod_{i \in I} x_{i,m_i}^{s_i}\right] = \sum_{m \in M} \eta(m)1_{\{m\}}(s) = \eta(s) = \psi(s) \tag{7}$$

and hence x and ψ are equivalent.

(ii) The second claim is slightly more involved. The equivalent distribution ψ is given by $\psi(s) = \sum_{m \in M} \eta(m) \prod_{i \in I} x_{i,m_i}^{s_i}$ for all $s \in S$. If x is an interior stationary state in the η-replicator dynamics, then for all $i \in I$, $m_i \in M_i$, and $k \in S_i$, we have $x_{i,m_i}^k > 0$ and therefore:

$$\sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i})\left[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m)\right] \leq 0 \tag{8}$$

While the above in fact holds with equality, it will be useful later to see that this weaker condition implies that ψ is a correlated equilibrium.

Consider any $i \in I$ and $m_i \in M_i$, and any mapping $f_i : S_i \to S_i$. For any s_i, $f_i(s_i)$ is fixed over s_{-i} and m_{-i}. Equation (8) therefore holds with $f_i(s_i)$ in place of k. Furthermore, since $\sum_{s_i \in S_i} x_{i,m_i}^{s_i} = 1$, we have:

$$\sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \sum_{s \in S} \left[\prod_{j \in I} x_{j,m_j}^{s_j}\right] \pi_i(s)$$

$$= \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) u_i(x_m)$$

$$= \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) u_i(x_m)$$

$$\geq \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) u_i(e_i^{f_i(s_i)}, x_{m_{-i}})$$

$$= \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{s_{-i} \in S_{-i}} \left[\prod_{j \neq i} x_{j,m_j}^{s_j}\right] \pi_i(f_i(s_i), s_{-i})$$

$$= \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \sum_{s \in S} \left[\prod_{j \in I} x_{j,m_j}^{s_j}\right] \pi_i(f_i(s_i), s_{-i}) \tag{9}$$

The inequality follows from Equation (8) and the subsequent equality follows from Equation (2).

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Since (9) holds for all m_i, we have:

$$0 \geq \sum_{m_i \in M_i} \left[\sum_{s \in S} \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] \left[\pi_i(f_i(s_i), s_{-i}) - \pi_i(s) \right] \right]$$

$$= \sum_{s \in S} \left[\sum_{m \in M} \eta(m) \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] \right] \left[\pi_i(f_i(s_i), s_{-i}) - \pi_i(s) \right]$$

$$= \sum_{s \in S} \psi(s) \left[\pi_i(f_i(s_i), s_{-i}) - \pi_i(s) \right] \tag{10}$$

Thus ψ is a correlated equilibrium.

<div align="right">Q.E.D.</div>

One should note that our definition of interiority in a subpopulation model is more stringent than in a model without subpopulations. In particular, we require that *all* subpopulations play every strategy with strictly positive probability. However, this stronger definition is necessary to guarantee that the equivalent distribution is a correlated equilibrium.

For example, consider a game with the following normal form representation:

	l	r
t	1, 2	0, 0
b	0, 0	2, 1

Suppose that the each of the populations has two subpopulations: that is, let $M_1 = M_2 = \{1, 2\}$. Suppose that the subpopulation matches occur with the following probabilities:

	1	2
1	1/2	0
2	0	1/2

Consider the population state x with subpopulation states

$$x_{1,1} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \qquad x_{1,2} = (1, 0)$$
$$x_{2,1} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \qquad x_{2,2} = (0, 1)$$

Notice that x is not an interior state by our definition even though the average play in each population is a completely mixed strategy. We will show that x is stationary but does not correspond to a correlated equilibrium. Similarly to the proof of Proposition 1, since the second subpopulation in each

population is programmed to play a pure strategy, we have $\dot{x}^t_{1,2} = \dot{x}^b_{1,2} = 0$ and $\dot{x}^l_{2,2} = \dot{x}^r_{2,2} = 0$. For subpopulation 1 of population 1, we have:

$$
\begin{aligned}
\dot{x}^t_{1,1} &= \frac{1}{2}\Big[u_1(e^t_1, x_{2,1}) - u_1(x_{1,1}, x_{2,1})\Big]x^t_{1,1} \\
&= \frac{1}{2}\Big[\Big(\tfrac{2}{3}\cdot\pi_1(t,l)\Big) - \Big(\tfrac{1}{3}\cdot\tfrac{2}{3}\cdot\pi_1(t,l) + \tfrac{2}{3}\cdot\tfrac{1}{3}\cdot\pi_1(b,r)\Big)\Big] \\
&= \frac{1}{2}\Big[\tfrac{2}{3} - \Big(\tfrac{2}{9}+\tfrac{4}{9}\Big)\Big] \\
&= 0
\end{aligned}
\tag{11}
$$

and analogously for $\dot{x}^b_{1,1}$, $\dot{x}^l_{2,1}$, and $\dot{x}^r_{2,1}$. Therefore, x is a stationary state.

This population state is equivalent to the following distribution over strategy profiles:

	l	r
t	1/9	5/9
b	2/9	1/9

To see that this distribution is not a correlated equilibrium, consider a rational agent as the row player. If she is given the recommendation to play t, then she knows that column player was recommended to play l with probability 1/6 and r with probability 5/6. Her payoff from following the recommendation would be 1/6, whereas if she deviated and played b, her payoff would be 5/3. Hence, this distribution is not a correlated equilibrium.

We should also discuss another important difference between our subpopulation model and the model without subpopulations. Proposition 1 (ii) states that for any interior stationary state in the η-replicator dynamics, the equivalent distribution is a correlated equilibrium. Again, in a model without subpopulations, any interior stationary state in the replicator dynamics is a Nash equilibrium. In such a model, the converse also holds: any state that is not stationary is not a Nash equilibrium. In our model Proposition 1 (i) provides a partial converse to (ii), but the complete converse is not true. It is possible to have a nonstationary state that is equivalent to a correlated equilibrium. Consider the previous game, again where each population has subpopulations $\{1,2\}$ and suppose population matches occur with the following probabilities:

	1	2
1	1/3	0
2	0	2/3

Consider the population state x with subpopulation states

$$x_{1,1} = (1,0) \qquad x_{1,2} = (0,1)$$
$$x_{2,1} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \qquad x_{2,2} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$

It is straightforward to show that $\dot{x}^l_{2,1} = 2/9$ and $\dot{x}^r_{2,2} = 4/9$, so x is not stationary. However, x is equivalent to the correlated equilibrium with the following distribution:

	l	r
t	2/9	1/9
b	4/9	2/9

Notice that this distribution is a product measure. Any correlated equilibrium that is a product measure has randomization that is independent across players and is hence a Nash equilibrium. Thus x is actually equivalent to a Nash equilibrium.

Proposition 1 (i) states that for any correlated equilibrium there exists a subpopulation setting with a stationary state equivalent to this correlated equilibrium. The full converse, which we showed does not hold, would require that, for any population setting, any state equivalent to a correlated equilibrium must be stationary. There are two main reasons for the contrast between the subpopulation model and the model without subpopulations. The first is that since the η-replicator dynamics depend on the choice of subpopulations and their distribution, the same game can have many evolutionary representations, whereas models without subpopulations have unique replicator dynamics. The second is that in our subpopulation model, multiple states can be equivalent to the same correlated equilibrium. For instance, in the current example, the state x with subpopulation states $x_{1,1} = x_{1,2} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$ and $x_{2,1} = x_{2,2} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$ is stationary and equivalent to the same correlated equilibrium.

4   Stability and Limit States

Having discussed the relationship between correlated equilibria and rest points in the replicator dynamics of subpopulation models, we now turn to stable states in these replicator dynamics. As we shall see, the relationship between correlated equilibria and stable states in the replicator dynamics of subpopulation models is similar to the that of Nash equilibria and stability under the standard replicator dynamics. Definition 3 is merely a reformulation of the standard dynamics solution concept for a model with subpopulations. With this construction, the definition of Lyapunov stability remains unchanged and is simply restated.

Definition 3 *A* SOLUTION *through a point $x \in X$ to a system of η-replicator dynamics is a function $\xi(\cdot, x) : \mathbb{R} \to X$ such that $\xi(0, x) = x$ and such that for all $t \in \mathbb{R}$, $i \in I$, $m_i \in M_i$, and $k \in S_i$:*

$$\frac{d}{dt}\xi_{i,m_i}^k(t, x) = \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \Big[u_i\big(e_i^k, \xi_{m_{-i}}(t, x)\big) - u_i\big(\xi_m(t, x)\big) \Big] \xi_{i,m_i}^k(t, x)$$

$$(12)$$

Definition 4 *A state $x \in X$ is* LYAPUNOV STABLE *if every neighborhood U of x contains a neighborhood U^0 of x such that $\xi(t, x) \in U$ for all $x \in U^0 \cap X$ and $t \geq 0$.*

The following result establishes the relationship between stability in the replicator dynamics and correlated equilibria.

Proposition 2 *Consider a game Γ. Given a subpopulation setting (M, η), if a population state $x \in X$ is Lyapunov stable in the η-replicator dynamics, then the equivalent distribution is a correlated equilibrium.*

Proof: If x is Lyapunov stable, then it is stationary. We must show that any stationary state for which the equivalent distribution is not a correlated equilibrium is not Lyapunov stable.

Suppose that $x \in X$ is stationary and the equivalent distribution is not a correlated equilibrium. Then there exist some $i \in I$, $m_i \in M_i$, and $k \in S_i$ such that:

$$g_{i,m_i}^k(x) \equiv \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i}) \Big[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) \Big] > 0 \qquad (13)$$

If this were not the case, then, as we saw in the proof of Proposition 1, x would be equivalent to a correlated equilibrium. Since x is stationary, this requires that $x_{i,m_i}^k = 0$.

The growth function g_{i,m_i}^k is multilinear in x and hence (locally) Lipschitz continuous. Thus, the solution path is well defined, and there exists a $\delta > 0$ and a neighborhood V of x such that $g_{i,m_i}^k(y) \geq \delta$ for all $y \in V \cap X$. For any $\xi(t, y) \in V \cap X$:

$$\frac{d}{dt}\xi_{i,m_i}^k(t, y) = g_{i,m_i}^k\big(\xi(t, y)\big) \xi_{i,m_i}^k(t, y) \geq \delta\,\xi_{i,m_i}^k(t, y) \qquad (14)$$

For all $y \in V \cap \text{int}(X)$ and for all $t > 0$ such that $\xi(t', y) \in V \cap \text{int}(X)$ for all $t' \in [0, t]$, we have

$$\ln\big(\xi_{i,m_i}^k(t, y)\big) - \ln(y_{i,m_i}^k) \geq \delta t$$

which implies

$$\xi_{i,m_i}^k(t, y) \geq y_{i,m_i}^k \exp(\delta t) \tag{15}$$

Therefore, there exists an $\varepsilon > 0$ such that for any neighborhood U^0 of x and $y \in U^0 \cap \text{int}(X)$, there exists $0 \leq t < \infty$ such that either $\xi_{i,m_i}^k(t, y) > \varepsilon$ or $\xi(t, y) \notin V$. That is, we can choose a neighborhood $U = \{y \in V : y_{i,m_i}^k \leq \varepsilon\}$ of x so that from every neighborhood U^0 of x and $y \in U^0 \cap \text{int}(X)$, the trajectory from y will eventually leave U. Therefore, x is not Lyapunov stable.

<div align="right">Q.E.D.</div>

Another important relationship in this context is that limit states to interior solutions can also be represented by correlated equilibria. This result complements the previous proposition in that some limit states, such as saddle points, are not Lyapunov stable and some Lyapunov stable states are not limit states.

Proposition 3 *Consider a game Γ. Given a subpopulation setting (M, η), if the population state x is the limit to some interior solution in the η-replicator dynamics, then the equivalent distribution is a correlated equilibrium.*

Proof: If $x^0 \in \text{int}(X)$ and $\xi(t, x^0)_{t \to \infty} \to x$, then x is stationary.[5] If x is not equivalent to a correlated equilibrium, then as in the proof of Proposition 2, there exist some $i \in I$, $m_i \in M_i$, $k \in S_i$, and $\delta > 0$ and some neighborhood V of x such that $g_{i,m_i}^k(y) \geq \delta$ for all $y \in V \cap X$.

However, if $\xi(t, x^0)$ converges to x, then there exists some $T > 0$ such that $\xi(t, x^0) \in V \cap \text{int}(X)$ for all $t \geq T$. Since x is stationary and $g_{i,m_i}^k(x) \geq \delta$, it must be that $x_{i,m_i}^k = 0$. This implies $\frac{d}{dt}\xi_{i,m_i}^k(t, x^0) < 0$ for some $t \geq T$, which requires that $g_{i,m_i}^k\big(\xi(t, x^0)\big) < 0$. This contradicts $g_{i,m_i}^k(y) \geq \delta$ for all $y \in V \cap X$. Thus x is equivalent to a correlated equilibrium.

<div align="right">Q.E.D.</div>

We have so far shown several sufficient conditions for a state to be equivalent to a correlated equilibrium. One may ask whether these conditions are in fact sufficient for a state to be equivalent to a Nash equilibrium. We show here that this is not the case. The following is an example of a game with

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[5]This is a straightforward result from the theory of differential equations.

a Lyapunov stable state that is also the limit of an interior solution, but is not a Nash equilibrium.

Consider the following normal form game, appearing in Aumann (1987):

	l	r
t	6, 6	2, 7
b	7, 2	0, 0

Suppose that each population has subpopulations $\{1, 2\}$ and that the population matches occur with the following probabilities:

	1	2
1	1/3	1/3
2	1/3	0

Consider the population state x with subpopulation states

$$x_{1,1} = (1, 0) \qquad x_{1,2} = (0, 1)$$
$$x_{2,1} = (1, 0) \qquad x_{2,2} = (0, 1)$$

Notice that the unplayed action of each subpopulation of population 1 has a negative growth rate:

$$
\begin{aligned}
g_{1,1}^b(x) &= \frac{1}{3}\left[u_1(e_1^b, x_{2,1}) - u_1(x_{1,1}, x_{2,1})\right] + \frac{1}{3}\left[u_1(e_1^b, x_{2,2}) - u_1(x_{1,1}, x_{2,2})\right] \\
&= \frac{1}{3}[7 - 6] + \frac{1}{3}[0 - 2] = -\frac{1}{3} < 0 \\
g_{1,2}^t(x) &= \frac{1}{3}\left[u_1(e_1^t, x_{2,1}) - u_1(x_{1,2}, x_{2,1})\right] = -\frac{1}{3} < 0
\end{aligned}
$$

Similarly, $g_{2,1}^r(x) < 0$ and $g_{2,2}^l(x) < 0$. Since g is continuous, there exists some open ball V containing x such that these conditions also hold for any $y \in V \cap X$. This implies that x is Lyapunov stable, and in fact, x is the limit of the solution through any initial state $y \in V \cap \mathrm{int}(X)$. Either of these conditions is sufficient for x to be equivalent to a correlated equilibrium. The equivalent correlated equilibrium has the distribution:

	l	r
t	1/3	1/3
b	1/3	0

However, this is clearly not a Nash equilibrium distribution. Furthermore, this distributions lies outside the convex hull of the Nash equilibria.

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5 Nash Equilibrium

Given the previous example, one might ask what conditions are sufficient for a population state in this framework to be equivalent to a Nash equilibrium. One possibility is to consider a population setting (M, η) in which η is a product measure. In this case, dividing each population into subpopulations has no effect on the distribution of opponents that a member of a population will face: the conditional distribution $\eta(m_{-i}|m_i)$ is the same for all $m_i \in M_i$. Therefore, one would expect the same results as for standard replicator dynamics. Namely, the analogs of Proposition 1 (ii), Proposition 2, and Proposition 3 should hold with Nash equilibrium substituted for correlated equilibrium. The following proposition shows that this is in fact the case.

Proposition 4 *Consider a game Γ. If (M, η) is a subpopulation setting such that η is a product measure, then for any population state x, the equivalent distribution ψ on S is also a product measure.*

As we discussed earlier, any correlated equilibrium that is product measure is also a Nash equilibrium. Thus Proposition 4, combined with the previous propositions, implies the desired results.

Proof: If η is a product measure, then for all $m \in M$, $\eta(m) = \prod_{i \in I} \eta_i(m_i)$. Therefore, for all $s \in S$,

$$
\begin{aligned}
\psi(s) &= \sum_{m \in M} \eta(m) \Big[\prod_{j \in I} x_{j,m_j}^{s_j} \Big] = \sum_{m \in M} \Big[\prod_{i \in I} \eta_i(m_i) \prod_{i \in I} x_{i,m_i}^{s_i} \Big] \\
&= \sum_{m \in M} \Big[\prod_{i \in I} \eta_i(m_i) x_{i,m_i}^{s_i} \Big] = \prod_{i \in I} \Big[\sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \Big] \\
&= \prod_{i \in I} \psi_i(s_i)
\end{aligned}
\tag{16}
$$

The last equality follows from the fact that for any η, not necessarily a product measure, we have:

$$
\begin{aligned}
\psi_i(s_i) &= \sum_{s_{-i} \in S_{-i}} \sum_{m \in M} \eta(m) x_{i,m_i}^{s_i} \Big[\prod_{j \neq i} x_{j,m_j}^{s_j} \Big] \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \sum_{s_{-i} \in S_{-i}} \Big[\prod_{j \neq i} x_{j,m_j}^{s_j} \Big] \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i}
\end{aligned}
\tag{17}
$$

Thus ψ is a product measure.

<div align="right">Q.E.D.</div>

That we obtain the standard relationship between population states and Nash equilibria when η is a product measure is rather intuitive. In such a subpopulation setting, the subpopulation to which an individual is assigned has no effect on the distribution of his opponents. Basically, the subpopulation breakdown merely labels individuals and does not introduce any substantive differences from the standard replicator dynamics. The standard replicator dynamics are clearly a special case of subpopulation replicator dynamics in which each population has only one subpopulation. Any subpopulation setting where η is a product measure is essentially the same as this standard setting. The fact that we obtain the same results in setting where η is a product measure as in settings where each population has a single subpopulation serves as a consistency check of our model in that inconsequential differences in assignments into subpopulations does not affect the results.

6   Concluding Remarks

We have established in the preceding analysis the relationship between correlated equilibria and the replicator dynamics of evolutionary games with subpopulations. We showed that every correlated equilibrium is equivalent to a stationary state in the replicator dynamics of some subpopulation model. We also showed that every interior stationary state in a subpopulation model is equivalent to a correlated equilibrium. We found that any state that is Lyapunov stable or the limit of an interior solution is equivalent to a correlated equilibrium. We also provided an example with a Lyapunov stable limit state whose equivalent correlated equilibrium was outside the convex hull of the set of Nash equilibria. Finally, we proved that any subpopulation setting in which the matching distribution is a product measure leads to equivalence with Nash equilibrium, thus demonstrating the connection to the standard replicator dynamics model.

There is, however, much more to investigate regarding this relationship. An interesting line of further research is to investigate the robustness of the correlated equilibria to perturbations in the matching distribution and in the underlying breakdown of the subpopulations. Consider a subpopulation model with a stable state, which, as we have shown, is equivalent to a correlated equilibrium. Imagine that a small change occurs in the subpopulation setting. For instance, in a locational context, suppose some individuals are exogenously moved from one subpopulation to another. One would ask

if there is then necessarily a stable state in the new subpopulation setting "near" the original stable state. If so, then this new state will be equivalent to a correlated equilibrium. The dynamics of adjustment, and whether they will converge to this new stable state, would be interesting to investigate.

We believe that the main contribution of this paper is its interpretation of the correlated equilibrium, or more precisely the identification of a class of games for which the correlated equilibrium is a natural solution concept. We would like to emphasize the subpopulations framework and its intuitive relationship to the correlated equilibrium. While we formulate our results in terms of the replicator dynamics, we believe that similar results could be obtained if one were to employ learning and more sophisticated adaptation in this same framework.

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